

**A STUDY ON PYTHAGOREAN FUZZY GRAPH
STRUCTURES**

**Thesis submitted in
Partial fulfilment of the Requirements for the
Degree of Master of Science (M.Sc.)**

**By
Anni Pon Sorna M
(21PMA026)
Department of Mathematics**

**Avinashilingam Institute for Home Science and Higher Education for
Women,
Coimbatore - 641043
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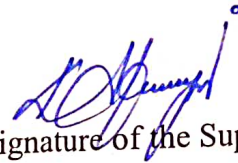
DECLARATION

DECLARATION

I declare that the thesis entitled "**A Study On Pythagorean Fuzzy Graph Structures**" submitted by me for the degree of **Master of Science (M. Sc.)** is the record of work carried out by me during the period from December 2022 to May 2023 under the guidance of **Dr.K. Akalyadevi, M.Sc., M.Phil., Ph.D.,** Assistant Professor and Head, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, and has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this institute or any other University or other similar institution of Higher Learning.



Signature of the Candidate



Signature of the Supervisor

CERTIFICATE

CERTIFICATE

I certify that the thesis entitled "A STUDY ON PYTHAGOREAN FUZZY GRAPH STRUCTURES" submitted for the degree of **Master of Science (M.Sc.)** by **Ms.M.Anni Pon Sorna**, is the record of research work carried out by her during the period from December 2022 to May 2023 under my guidance and supervision, and that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship or other Titles in this institute or any other University or institution of Higher Learning.



Signature of the
Head of the Department



Signature of the Supervisor
with designation



Signature of the Director

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Anni Pon Sorna. M

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SYNOPSIS

SYNOPSIS

Chapter-I deals with Introduction, Review of Literature and the concepts of graph theory which are used in our research work likes Fuzzy Set, Intuitionistic Fuzzy Set, Pythagorean Fuzzy Set, Fuzzy Graphs, Intuitionistic Fuzzy Graphs and Pythagorean Fuzzy Graphs.

Chapter-II deals with extended concepts of Intuitionistic Fuzzy Graph Structures and also discussed some examples for trees, cycles and complement.

Chapter-III deals with the new ideas of Pythagorean Fuzzy Graph Structures and also extended our concepts to some illustration for trees, cycles and complement.

CHAPTER 1

CHAPTER-1

1.1 INTRODUCTION

Set theory is a basis of modern mathematics, and notions of sets are used in all formal descriptions. All other notions of mathematics can be built up based on the set theory.

Fuzzy sets were introduced by Zadeh (1965). The notion of fuzzy set theory has caused great interest among both pure and applied mathematics. This day's fuzzy set hypothesis has arisen as a likely space of interdisciplinary exploration. It has fruitful applications in different fields as a phenomenal apparatus for addressing human information and discernment. Zadeh (1971) introduced a similarity relation, which is a fuzzy relation which is reflexive, symmetric, and transitive. He also introduced a fuzzy ordering, which is a fuzzy relation which is transitive. In particular, a fuzzy partial ordering, is a fuzzy ordering that is reflexive and antisymmetric.

Atanassov (1983) introduced the concept of Intuitionistic Fuzzy Set as a generalisation of fuzzy sets. Intuitionistic fuzzy models provide more precision, flexibility and compatibility to the system compared to the fuzzy models and he also added new components that determine the degree of non-membership in the definition of fuzzy set. The fuzzy sets give the degree of membership, while intuitionistic fuzzy sets give both the degree of membership and the degree of non-membership, which are more or less independent from each other; the only requirement is that the sum of these two degrees is not greater than one. Intuitionistic fuzzy sets have been applied in a wide variety of fields, including computer science, engineering, mathematics, medicine, chemistry, and economics.

Atanassov (1989) introduced new results on intuitionistic fuzzy sets and also defined two new operations on intuitionistic fuzzy sets and their basic properties. Young et al. (2005) using the notion intuitionistic fuzzy sets, the concept of intuitionistic fuzzy semi-preopen sets and intuitionistic fuzzy semi pre continuous mappings are introduced.

Yager et al. (2013) has proposed the Pythagorean Fuzzy Set (PFS) as an effective tool for handling the uncertainty or vague information more adequately in real-world situations. In PFSs, the sum of squares of the degrees

of membership and non-membership is less than or equal to 1. For example, if a decision maker provides the membership degree 0.6 and non-membership degree 0.7 in his evaluation, then this situation cannot handle by intuitionistic fuzzy set theory because of $0.6 + 0.7 > 1$. However, it is easily observed that $(0.6)^2 + (0.7)^2 < 1$, that is to say, the Pythagorean Fuzzy Set (PFS) is capable to represent this evaluation information. In other words, the PFSs are more powerful to handle problems in uncertain situations. Under PFS environment, many researchers have started work in different directions and obtained various significant results.

Yager (2014) introduces a class of nonstandard Pythagorean fuzzy subsets with membership grades of pairs (a,b) , satisfying the requirement $a^2 + b^2 \leq 1$, as well as a number of aggregation operations for these Pythagorean fuzzy subsets. Garg (2017) has proposed the improved score function for the ranking order of Interval-Valued Pythagorean Fuzzy Sets (IVPFSs). Based on it, a Pythagorean fuzzy Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method by taking the preferences of the experts in the form of interval-valued intuitionistic Pythagorean fuzzy decision matrices has been presented. Garg (2018) proposes an improved score function for solving Multi-Criteria Decision-Making (MCDM) problems that takes the form of interval valued Pythagorean fuzzy sets. Peng et al. (2019), they developed two novel algorithms for decision-making problems in a Pythagorean fuzzy environment.

In Mathematics, Graph theory is the investigation of graphs, which are numerical designs used to display the relationship between two or more objects or with set of vertices and edges. Mathematics plays an essential role in our day to day life. The graph theory origin can be traced back to Euler's work on the Konigsberg bridges problem in 1735. Denes Konig (1936) wrote the first book on graph theory. There is wide utilization of graph theory. The concepts of graph theory have applications in many areas of computer science (such as data mining, image segmentation, clustering, image capturing, networking, etc.).

The Kaufmann (1973) gave first definition of fuzzy graph based on Zadeh's fuzzy relations. But it was Rosenfeld who laid the foundations for fuzzy

graph theory in 1975. Generalization of fundamental ideas of graph theory like paths, cycles, trees, connectedness, and their properties to fuzzy graph theory have been done by Rosenfeld. Fuzzy graph models can address the complex, imprecise and uncertain problems where classical graph models may fail. Thus, a fuzzy graph representation is more appropriate to reality than crisp graph representation. Fuzzy graph theory has applications in the modern science and technology especially in the fields of information theory, neural network, expert systems, cluster analysis, medical diagnosis, control theory, etc. Bhattacharya (1987) introduced the notions of eccentricity and centre on the fuzzy graph. Myna (2015) proposed using a fuzzy graph model to represent the traffic network of a city and discussed a method to find the different type of accidental zones in traffic flows using edge colouring of a fuzzy graph. Radha et al. (2015) studied on lexicographic products of two fuzzy graphs.

Atanassov et al. (2006) discussed a new generalization of the intuitionistic fuzzy graphs, using as a basis the concepts of intuitionistic fuzzy sets, intuitionistic fuzzy relations and index matrices. Parvathi et al. (2006) gave a new definition for min-max intuitionistic fuzzy graph. Some properties of intuitionistic fuzzy graphs are analyzed through suitable illustrations. Karunambigai et al. (2007) studied shortest paths in networks by using intuitionistic fuzzy graph method. Akram et al. (2012) define and investigate the 4 properties of strong intuitionistic fuzzy graphs, and discuss some propositions of self-complementary and self-weak complementary strong intuitionistic fuzzy graphs, and introduce the concept of intuitionistic fuzzy line graphs.

Recently, Naz et al. (2018) initially presented the idea of Pythagorean Fuzzy Graph (PFG). Akram et al. (2018) introduced certain notions, including intuitionistic fuzzy Graphs of 3-Type (IFGs3T), Intuitionistic Fuzzy Graphs of 4-Type (IFGs4T), and Intuitionistic Fuzzy Graphs of n-Type (IFGsnT), and proved that every IFG (n-1)T is an IFGsnT, and also discussed the application of Pythagorean fuzzy graphs in decision making. Akram et al. (2019) apply the concept of Pythagorean fuzzy sets to graphs and then combine two Pythagorean Fuzzy Graphs (PFGs) using two new graph products, namely, the maximal product and the residue product, and also investigate the regularity of these products and properties such as strongness, connectedness, and completeness.

Pythagorean fuzzy graph is more powerful and more practical tool having the capability to deal with imprecise and incomplete information in 7 different decision-making disciplines, such as engineering, mathematics, statistics, artificial intelligence, medical and social sciences than fuzzy graphs. The Pythagorean fuzzy graphs accurately represent the complex graphic problems.

In our research work, we studied the ideas of a PFGS and also We extended our views Pythagorean fuzzy K_i -cycles, Pythagorean fuzzy K_i -trees, complement and ϕ -complement of a PFGS. In addition to self-complementary and strong self complementary, the complement of a PFGS are deliberated.

Chapter-I deals with Introduction, Review of Literature and the concepts of graph theory which are used in our research work likes Fuzzy Set, Intuitionistic Fuzzy Set, Pythagorean Fuzzy Set, Fuzzy Graphs, Intuitionistic Fuzzy Graphs and Pythagorean Fuzzy Graphs.

Chapter-II deals with extended concepts of Intuitionistic Fuzzy Graph Structures and also discussed some examples for trees, cycles and complement.

Chapter-III deals with the new ideas of Pythagorean Fuzzy Graph Structures and also extended our concepts to some illustration for trees, cycles and complement.

1.2 REVIEW OF LITERATURE

In 1965, Zadeh was introduced a fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterised by a membership (characteristic) function that assigns to each object a grade of membership ranging between zero and one. The notions of inclusion, union, intersection, complement, relation, convexity, etc., are extended to such sets, and various properties of these notions in the context of fuzzy sets are established.

In 1989, Atanassov defined new results on Intuitionistic fuzzy sets. Two new operators on intuitionistic fuzzy sets are defined and their basic properties are studied.

In 2013, Yager introduced a new class of non-standard fuzzy subsets called Pythagorean fuzzy subsets and the related idea of Pythagorean membership grades. They discussed the negation operation and its relationship to the Pythagorean Theorem and compared Pythagorean fuzzy subsets with intuitionistic fuzzy subsets.

In 2014, Yager introduced a class of nonstandard Pythagorean fuzzy subsets whose membership grades are pairs (a,b) satisfying the requirement $a^2 + b^2 \leq 1$. They also introduce a variety of aggregation operations for these Pythagorean fuzzy subsets. The issue of having to choose the best alternative in multicriteria decision making leads us to consider the problem of comparing Pythagorean membership grades.

In 2019, Xindong Peng et al. present an overview of the Pythagorean fuzzy set with the aim of offering a clear perspective on the different concepts, tools, and trends related to their extension. In particular, we provide two novel algorithms for decision-making problems in a Pythagorean fuzzy environment. It may serve as a foundation for developing more algorithms in decision-making.

In 1971, Zadeh proposed the notion of "similarity" as defined in this paper is essentially a generalisation of the notion of equivalence. In the same vein, fuzzy ordering is a generalization of the concept of ordering. More

concretely, a similarity relation, S , is a fuzzy relation that is reflexive, symmetric, and transitive. A fuzzy ordering is a fuzzy relation which is transitive. In particular, a fuzzy partial ordering, P , is a fuzzy ordering which is reflexive and antisymmetric.

In 1735, graph theory was first introduced by Swiss mathematician Leonhard Euler by solving a Konigsberg bridge problem.

In 2010, Shirinivas et al. proposed an overview of graph theory applications in heterogeneous fields, but primarily focused on computer science applications that use graph theoretical concepts. An overview of various papers based on graph theory has been studied related to scheduling concepts and computer science applications, and an overview has been presented in this article.

In 1973, Kaufmann gave first definition of fuzzy graph based on Zadeh's fuzzy relations. In 1975, Rosenfeld developed the theory of fuzzy graphs and also defined generalization of fundamental ideas of graph theory like paths, cycles, trees, connectedness, and their properties to fuzzy graph theory.

In 1987, Bhattacharya introduced the notions of eccentricity and centre on the fuzzy graph. Myna (2015) proposed using a fuzzy graph model to represent the traffic network of a city and discussed a method to find the different type of accidental zones in traffic flows using edge colouring of a fuzzy graph. Radha et al. (2015) studied on lexicographic products of two fuzzy graphs.

In 2006 Atanassov et al. discussed a new generalization of the intuitionistic fuzzy graphs, using as a basis the concepts of intuitionistic fuzzy sets, intuitionistic fuzzy relations and index matrices. Parvathi et al. (2006) gave a new definition for min-max intuitionistic fuzzy graph. Some properties of intuitionistic fuzzy graphs are analyzed through suitable illustrations. Karunambigai et al. (2007) studied shortest paths in networks by using intuitionistic fuzzy graph method. Akram et al. (2012) define and investigate the 4 properties of strong intuitionistic fuzzy graphs, and discuss some propositions of self-complementary and self-weak complementary strong intuitionistic fuzzy graphs, and introduce the concept of intuitionistic fuzzy line graphs.

In (2017) Muhamad Akram and Rabia Akmal introduce the concept of an intuitionistic fuzzy graph structure (IFGS). We discuss certain notions,

including intuitionistic fuzzy B_i -cycles, intuitionistic fuzzy B_i -trees and ϕ -complement of an intuitionistic fuzzy graph structure with several examples. and also present ϕ -complement of an intuitionistic fuzzy graph structure along with self-complementary and strong self-complementary intuitionistic fuzzy graph structures.

In 2018 Muhammad Akram, Jawaria Mohsan Dar, Sumera Naz introduce the concept of Pythagorean fuzzy sets to graphs and then combine two Pythagorean fuzzy graphs (PFGs) using two new graph products namely, maximal product and the residue product. This research paper investigates the regularity for these products. Moreover, it discusses some eminent properties such as strongness, connectedness and completeness. Further, it proposes some necessary and sufficient conditions for $G1 * G2$ and $G1 \cdot G2$ to be regular. Finally, decision-making problems concerning evaluation of best company for investment and alliance partner selection of a software company are solved to better understand PFGs.

NOTATIONS

- FS = Fuzzy Set
- IFS = Intuitionistic Fuzzy Set
- PFS = Pythagorean Fuzzy Set
- FR = Fuzzy Relation
- IFR = Intuitionistic Fuzzy Relation
- PFR = Pythagorean Fuzzy Relation
- CG = Classical Graph
- IFG = Intuitionistic Fuzzy Graph
- PFG = Pythagorean Fuzzy Graph
- IFGS = Intuitionistic Fuzzy Graph Structure
- PFGS = Pythagorean Fuzzy Graph Structure

1.3 PRELIMINARIES

Definition 1.3.1

A *Fuzzy Set (FS)* on a universe X is an object of the form,

$$J = \{ \langle u, \mu_A(x) \mid x \in X \rangle \}$$

where $\mu_A : X \rightarrow [0,1]$ represents the membership function of J .

Definition 1.3.2

An *Intuitionistic Fuzzy Set (IFS)* on an universe X is an object of the form

$$J = \{ \langle x, \mu_A(x), \nu_A(x) \mid x \in X \rangle \}$$

where $\mu_A(x) \in ([0,1])$ is called degree of membership of $x \in J$, $\beta_J(x) \in ([0,1])$

is called degree of non-membership of, $x \in J$ and α_J, β_J satisfy the following condition for all $x \in X$, $\alpha_J(x) + \beta_J(x) \leq 1$.

Definition 1.3.3

A *Pythagorean Fuzzy Set (PFS)* on a universe X is an object of the form

$$J = \{ \langle \eta, \alpha_J(x), \beta_J(x) \mid \eta \in X \rangle \}$$

where $\alpha_J : X \rightarrow [0,1]$ and $\beta_J : X \rightarrow [0,1]$ represent the membership and non-membership function of K and α_J, β_J satisfies the condition

$$0 \leq \alpha_J^2(\eta) + \beta_J^2(\eta) \leq 1 \text{ for all } \eta \in X.$$

Definition 1.3.4

A fuzzy set on $X \times Y$ is said to be a *Fuzzy Relation (FR)* on Y , denoted by

$$J = \{ \langle xy, \mu_A(x) \mid xy \in X \times Y \rangle \}$$

where $\mu_R : X \times Y \rightarrow [0,1]$ represents the membership function of

Definition 1.3.5

An *Intuitionistic Fuzzy Relation (IFR)* $R = (\mu_R(x, y), \nu_R(x, y))$ in an universe $X \times Y (R(X \rightarrow Y))$ is an intuitionistic fuzzy set of the form

$$R = \{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \mid (x, y) \in X \times Y \rangle \}$$

where $\mu_R : X \times Y \rightarrow [0,1]$ and $\nu_R : X \times Y \rightarrow [0,1]$. The Intuitionistic fuzzy relation R satisfies $0 \leq \mu_R(x, y) + \nu_R(x, y) \leq 1$ for all $x, y \in X$.

Definition 1.3.6

A Pythagorean Fuzzy Set n on $X \times Y$ is said to be a *Pythagorean Fuzzy Relation (PFR)* on X , denoted by

$$m = \{ \langle xy, \mu_A(xy), \nu_A(xy) \rangle \mid xy \in X \times Y \}$$

where, $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ represents the membership and non-membership functions of n , and μ_A, ν_A satisfies the condition $0 \leq \mu_A^2(x) + \nu_A^2(y) \leq 1$ for all $xy \in X \times Y$.

Definition 1.3.7

A *Classical Graph (CG)* $G(V, E)$ is a combination of a set of vertices $V = v_1, v_2, \dots, v_n$ together with a set of edges $E = e_1, e_2, \dots, e_n$ such that each edge e_k is a pair of vertices (v_i, v_j) and is denoted by v_i, v_j .

Definition 1.3.8

A *Fuzzy Graph (FG)* $G=(V, \sigma, \mu)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma:V \rightarrow [0,1]$ and $\mu:E \rightarrow [0,1]$ such that for all $x, y \in V$,

$$\mu(xy) \leq \sigma(x) \wedge \sigma(y).$$

Definition 1.3.9

An *Intuitionistic Fuzzy Graph (IFG)* on a non-empty set Y is a pair $g=(m, n)$ with m an IFS on Y and n an IFR on Y such that

$$\mu_A(xy) \leq \mu_A(x) \wedge \mu_A(y), \nu_A(xy) \geq \nu_A(x) \vee \nu_A(y)$$

and $0 \leq \mu_A(xy) + \nu_A(xy) \leq 1$ for all $\eta, \lambda \in Y$, where $\mu_R : X \times Y \rightarrow [0,1]$ and $\nu_R : X \times Y \rightarrow [0,1]$ represents the membership and non-membership functions of n , respectively.

Definition 1.3.10

A *Pythagorean Fuzzy Graph (PFG)* on a non-empty set Y is a pair $g=(m, n)$ with m a PFS on Y and n a PFR on Y such that,

$$\mu_A(xy) \leq \mu_A(x) \wedge \mu_A(y), \nu_A(xy) \geq \nu_A(x) \vee \nu_A(y),$$

and $0 \leq \alpha_j^2(\eta) + \beta_j^2(\eta) \leq 1$ for all $x, y \in Y$, where, $\mu_A : X \times Y \rightarrow [0,1]$ and $\nu_A : X \times Y \rightarrow [0,1]$ represents the membership and non-membership functions of n , respectively.

Definition 1.3.11

Let $G^* = (U, E_1, E_2, \dots, E_k)$ be a graph structure and $\nu, \rho_1, \dots, \rho_n$ be the subsets U, E_1, E_2, \dots, E_n respectively such that $0 \leq \rho_i(xy) \leq \nu(x) \wedge \nu(y)$ for all $x, y \in U$ and $i = 1, 2, 3, \dots, L$. Then $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ is a fuzzy graph structure of G^* .

Definition 1.3.12

Let $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$ be a fuzzy graph structure of G^* . If $xy \in \sup p(\rho_i)$, then xy is said to be a ρ_i -edge of G .

Definition 1.3.13

The strength of a ρ_i -path $x_0 x_1 \dots x_n$ of a fuzzy graph structure G is $\bigwedge_{j=1}^n \rho_i(x_{j-1} x_j)$ for $i = 1, 2, 3, \dots, k$.

Definition 1.3.14

In any fuzzy graph structure G ,

$$\rho_i^2(xy) = \rho_i \circ \rho_i(xy) = V \{ \rho_i(xz) \wedge \rho_i(zy) \},$$

$$\rho_i^j(xy) = (\rho_i^{j-1} \circ \rho_i)(xy) = V \{ \rho_i^{j-1}(xz) \wedge \rho_i(zy) \},$$

$j = 2, 3, \dots, m$ for any $m \geq 2$. Also $\rho_i^\infty(xy) = \vee \{ \rho_i^j(xy), j = 1, 2, \dots \}$.

Definition 1.3.15

$G = (\nu, \rho_1, \rho_2, \dots, \rho_n)$ is a ρ_i cycle if $(\sup p(\nu), \sup p(\rho_1), \dots, \sup p(\rho_n))$ is a E_i cycle and there exist no unique xy in the $\sup p(\rho_i)$ such that,

$$\rho_i(xy) = \bigwedge \{ \rho_i(uv) \mid uv \in \sup p(\rho_i) \}$$

Definition 1.3.16

$G = (\nu, \rho_1, \rho_2, \dots, \rho_n)$ is a fuzzy ρ_i -tree if it has a partial fuzzy spanning sub graph structure, $F_i = (\nu, \tau_1, \tau_2, \dots, \tau_n)$ which is a τ_i -tree where for all d_i -edges not in F_i $\rho_i(xy) < \tau_i^\infty(xy)$.

CHAPTER 2

CHAPTER-2

Intuitionistic Fuzzy Graph Structure

Definition 2.1

Let $\{E_i : i = 1, 2, \dots, n\}$ be a set of irreflexive, symmetric and mutually disjoint relations on a non-empty set U . An *Intuitionistic Fuzzy Graph Structure (IFGS)* with underlying vertex set U is denoted by $\check{G}_s = (A, B_1, B_2, \dots, B_n)$, where

A intuitionistic fuzzy set of U with $\mu_A : U \rightarrow [0, 1]$ and $\nu_A : U \rightarrow [0, 1]$, namely the degree of membership and the degree of non membership of $x \in U$ respectively, such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \text{for all } x \in U.$$

Each B_i is an intuitionistic fuzzy set of E_i such that the functions $\mu_{B_i} : E_i \rightarrow [0, 1]$ and $\nu_{B_i} : E_i \rightarrow [0, 1]$ are defined by

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y) \quad \text{and} \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y)$$

and

$$0 \leq \mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1, \quad \text{for all } xy \in \subset U \times U, i = 1, 2, \dots, n.$$

Equivalently an IFGS of a graph structure may be defined in the following way,

Let $G^* = (A, E_1, E_2, \dots, E_n)$ be a graph structure and let $A, B_1, B_2, \dots, B_{n-1}$ and B_n be intuitionistic fuzzy subsets of $U, E_1, E_2, \dots, E_{n-1}$ and E_n , respectively. Then $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ is called an IFGS of G^* , if

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y)$$

For all $xy \in E_i, i = 1, 2, \dots, n$, and

$$\mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1 \quad \text{for all } xy \in U \times U$$

Example 2.2

Let $G^* = (U, E_1, E_2)$ be a graph structure such that $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_2, a_2a_3\}$ and $E_2 = \{a_3a_4, a_1a_4\}$. Let A, B_1 and B_2 be intuitionistic fuzzy subsets of U, E_1 and E_2 , respectively such that

$$A = \{(a_1, 0.5, 0.2), (a_2, 0.7, 0.3), (a_3, 0.4, 0.3), (a_4, 0.7, 0.3)\}$$

$$B_1 = \{(a_1a_2, 0.5, 0.3), (a_2a_3, 0.4, 0.3)\},$$

$$B_2 = \{(a_3 a_4, 0.4, 0.3), (a_1 a_4, 0.1, 0.2)\}.$$

Then $\check{G}_s = (A, B_1, B_2)$ is an IFGS of G^* as shown in Fig 2.1.

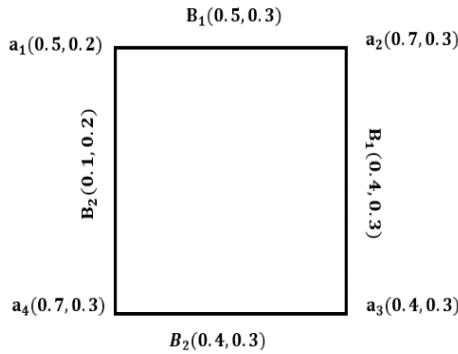


FIGURE 2.1 IFGS $\check{G}_s = (A, B_1, B_2)$

Definition 2.3

An IFGS $\check{H}_s = (C, D_1, D_2, \dots, D_n)$ is said to be an intuitionistic fuzzy subgraph structure of an IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , if $C \subseteq A$ and $D_i \subseteq C_i$ for all i , that is

$$\mu_C(x) \leq \mu_A(x) \text{ and } \nu_C(x) \geq \nu_A(x), \text{ for all } x \in U,$$

and for $i = 1, 2, \dots, n$

$$\mu_{D_i}(xy) \leq \mu_{B_i}(xy) \text{ and } \nu_{D_i}(xy) \geq \nu_{B_i}(xy), \text{ for all } xy \in U \times U,$$

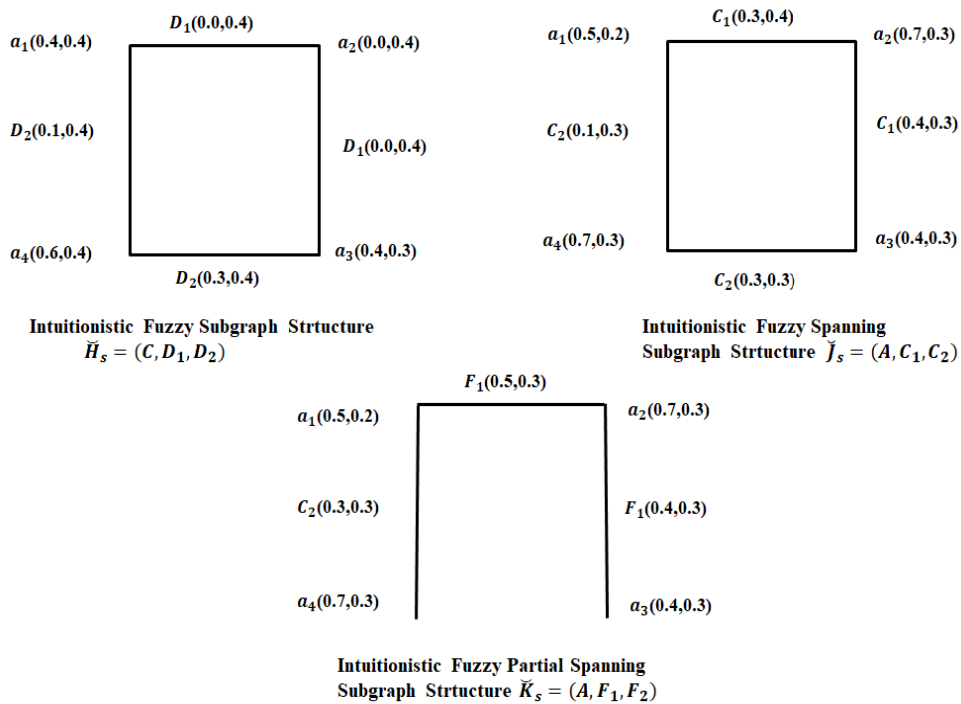


FIGURE 2.2. Intuitionistic Fuzzy Subgraph Structures

\check{H}_s is called an intuitionistic fuzzy spanning sub graph structure of an IFGS \check{G}_s , if $C = A$.

\check{H}_s is called an intuitionistic fuzzy partial spanning sub graph structure of an IFGS \check{G}_s , if it excludes some edges of \check{G}_s .

Example 2.4

Consider an IFGS $\check{G}_s = (A, B_1, B_2)$, as shown in Fig 2.2. Let

$$C = \{(a_1, 0.4, 0.4), (a_2, 0.0, 0.4), (a_3, 0.4, 0.3), (a_4, 0.6, 0.4)\}$$

$$D_1 = \{(a_1 a_2, 0, 0.4), (a_2 a_3, 0, 0.4)\}$$

$$D_2 = \{(a_3 a_4, 0.3, 0.4), (a_1 a_4, 0.1, 0.4)\}$$

$$C_1 = \{(a_1 a_2, 0.3, 0.3), (a_2 a_3, 0.4, 0.3)\}$$

$$C_2 = \{(a_3 a_4, 0.3, 0.3), (a_1 a_4, 0.1, 0.3)\}$$

$$F_1 = \{(a_1 a_2, 0.5, 0.3), (a_2 a_3, 0.4, 0.3)\}$$

$$\text{and } F_2 = \{(a_1 a_4, 0.1, 0.3)\}.$$

By routine calculations, it is easy to see that $\check{H}_s = (C, D_1, D_2)$, $J_s = (A, C_1, C_2)$ and $\check{K}_s = (A, F_1, F_2)$ are respectively the intuitionistic fuzzy subgraph structure intuitionistic fuzzy spanning subgraph structure and intuitionistic fuzzy partial spanning subgraph structure of \check{G}_s . Their respective drawings are shown in Fig 2.2.

Definition 2.5

Let $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ be an IFGS with underlying vertex set U. Then there is a B_i - edge between two vertices x and y of U, if one of the following is true:

$$(i) \quad \mu_{B_i}(xy) > 0 \text{ and } \nu_{B_i}(xy) > 0$$

$$(ii) \quad \mu_{B_i}(xy) > 0 \text{ and } \nu_{B_i}(xy) = 0$$

$$(iii) \quad \mu_{B_i}(xy) = 0 \text{ and } \nu_{B_i}(xy) > 0$$

for some i.

Definition 2.6

For an intuitionistic fuzzy graph structure $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with vertex set U , support of B_i is given by:

$$\text{sup } p(B_i) = \{xy \in U \times U : \mu_{B_i}(xy) \neq 0 \text{ or } \nu_{B_i} \neq 0\}, \quad i = 1, 2, \dots, n.$$

Definition 2.7

B_i - path of an IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , is a sequence of distinct vertices $v_1, v_2, \dots, v_m \in U$, such that $v_{j-1}v_j$ is a B_i - edge for all $j = 2, 3, \dots, m$.

Definition 2.8

In an IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , two vertices x and y of U are said to be B_i - connected, if they are joined by B_i - path, for some $i \in \{1, 2, \dots, n\}$

Definition 2.9

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , is said to be B_i - strong, if for all B_i - edges xy

$$\mu_{B_i}(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) = \nu_A(x) \vee \nu_A(y),$$

for some $i \in \{1, 2, \dots, n\}$.

Example 2.10

Consider the IFGS $\check{G}_s = (A, B_1, B_2)$, as shown Fig 1. Then a_1a_2, a_2a_3 are B_1 - edges and a_3a_4, a_1a_4 are B_2 - edges, $a_1a_2a_3$ and $a_3a_4a_1$ are B_1 - and B_2 - Paths respectively, a_1 and a_3 are B_1 - connected vertices of U , \check{G}_s is B_1 - strong, since $\text{sup } p(B_1) = \{a_1a_2, a_2a_3\}$ and

$$\mu_{B_1}(a_1a_2) = 0.5 = (\mu_A(a_1) \wedge \mu_A(a_2)),$$

$$\nu_{B_1}(a_1a_2) = 0.3 = (\nu_A(a_1) \vee \nu_A(a_2)),$$

$$\mu_{B_1}(a_2a_3) = 0.4 = (\mu_A(a_2) \wedge \mu_A(a_3)),$$

$$\nu_{B_1}(a_2a_3) = 0.3 = (\nu_A(a_2) \vee \nu_A(a_3)).$$

Definition 2.11

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ is said to be strong, if it is B_i – strong for all $i \in \{1, 2, \dots, n\}$.

Definition 2.12

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , is called complete or B_1, B_2, \dots, B_n – complete if \check{G}_s is a strong IFGS, $\sup p(B_i) \neq \phi$ for all $i \in \{1, 2, \dots, n\}$, for each pair of vertices $x, y \in U$, xy is a B_i – edge for some i .

Example 2.13

Let $\check{G}_s = (A, B_1, B_2)$ shown in Fig 3, be IFGS of the graph structure $G^* = (U, E_1, E_2)$ where, $U = \{a_1, a_2, a_3, a_4\}$, $E_1 = \{a_1a_3, a_3a_4, a_1a_4\}$ and $E_2 = \{a_1a_2, a_2a_3, a_2a_4\}$. Then \check{G}_s is a strong IFGS since it is both B_1 – strong and B_2 – strong. Moreover $\sup p(B_1) \neq \phi$, $\sup p(B_2) \neq \phi$ every pair of vertices belonging to U , is either a B_1 – edge or an B_2 – edge so \check{G}_s is a complete or B_1B_2 –complete IFGS as well.

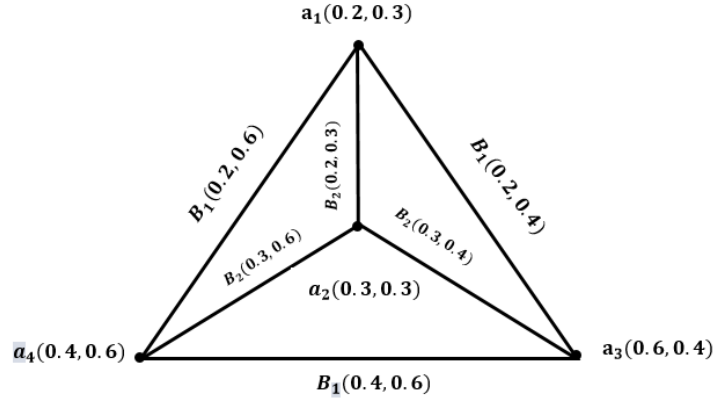


FIGURE 2.3 IFGS $\check{G}_s = (A, B_1, B_2)$

Definition 2.14

In an IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U , μ_{B_i} - and ν_{B_i} -strengths of a B_i – path “ $P_{B_i} = v_1v_2\dots v_m$ ”, are denoted by $\delta.P_{B_i}$ and $\Delta.P_{B_i}$ respectively, such that

$$\delta.P_{B_i} = \bigwedge_{j=2}^m [\mu_{B_i}(v_{j-1}v_j)] \quad \text{and} \quad \Delta.P_{B_i} = \bigvee_{j=2}^m [\nu_{B_i}(v_{j-1}v_j)].$$

Then we write, strength of the path $P_{B_i} = (\delta.P_{B_i}, \Delta.P_{B_i})$.

Example 2.15

In $\tilde{G}_s = (A, B_1, B_2)$ shown in Fig.3, $P_1 = a_1 a_3 a_4 a_1$ is a B_1 – path and $P_2 = a_3 a_2 a_4$ is a B_2 – path and

$$\delta.P_1 = \mu_{B_1}(a_1 a_3) \wedge \mu_{B_1}(a_3 a_4) \wedge \mu_{B_1}(a_4 a_1) = 0.2 \wedge 0.4 \wedge 0.2 = 0.2,$$

$$\Delta.P_1 = \nu_{B_1}(a_1 a_3) \vee \nu_{B_1}(a_3 a_4) \vee \nu_{B_1}(a_4 a_1) = 0.4 \vee 0.6 \vee 0.6 = 0.6,$$

$$\delta.P_2 = \mu_{B_2}(a_3 a_2) \wedge \mu_{B_2}(a_2 a_4) = 0.3 \wedge 0.3 = 0.3,$$

$$\Delta.P_2 = \nu_{B_2}(a_3 a_2) \vee \nu_{B_2}(a_2 a_4) = 0.4 \vee 0.6 = 0.6.$$

Thus, strength of B_1 – path $P_1 = (\delta.P_1, \Delta.P_1) = (0.2, 0.6)$, strength of B_2 – path $P_2 = (\delta.P_2, \Delta.P_2) = (0.3, 0.6)$.

Definition 2.16

In an IFGS $\tilde{G}_s = (A, B_1, B_2, \dots, B_n)$ with underlying vertex set U:

μ_{B_i} -strength of connectedness between x and y , is defined by

$$\mu_{B_i}^\infty(xy) = \bigvee_{j \geq 1} \{\mu_{B_i}^j(xy)\}, \text{ where } \mu_{B_i}^j(xy) = (\mu_{B_i}^{j-1} \circ \mu_{B_i})(xy) \text{ for } j \geq 2 \text{ and}$$

$$\mu_{B_i}^2(xy) = (\mu_{B_i} \circ \mu_{B_i})(xy) = \bigvee_z \{\mu_{B_i}(xz) \wedge \mu_{B_i}(zy)\}, \nu_{B_i} \text{-strength of connectedness}$$

between x and y , is defined by $\nu_{B_i}^\infty(xy) = \bigvee_{j \geq 1} \{\nu_{B_i}^j(xy)\}$, where

$$\nu_{B_i}^j(xy) = (\nu_{B_i}^{j-1} \circ \nu_{B_i})(xy) \text{ for } j \geq 2 \text{ and}$$

$$\nu_{B_i}^2(xy) = (\nu_{B_i} \circ \nu_{B_i})(xy) = \bigwedge_z \{\nu_{B_i}(xz) \vee \nu_{B_i}(zy)\}.$$

Example 2.17

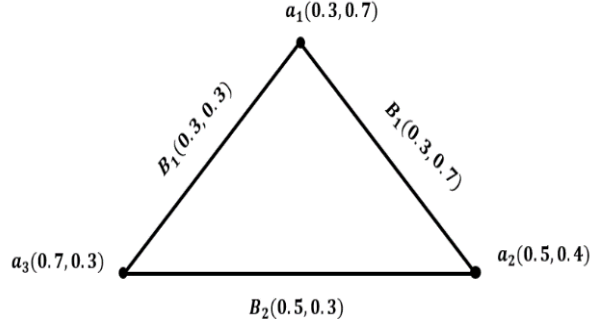
Let $\tilde{G}_s = (A, B_1, B_2)$, as shown in Fig 4. be IFGS of a graph structure $G^* = (U, E_1, E_2)$, such that $U = \{a_1, a_2, a_3\}$, $E_1 = \{a_1 a_2, a_1 a_3\}$ and $E_2 = \{a_2 a_3\}$. Since, $\mu_{B_1}(a_1 a_2) = 0.3$, $\mu_{B_1}(a_2 a_3) = 0$, therefore

$$\mu_{B_1}^2(a_1 a_2) = (\mu_{B_1} \circ \mu_{B_1})(a_1 a_2) = \mu_{B_1}(a_1 a_3) \wedge \mu_{B_1}(a_3 a_2) = 0.3 \wedge 0.0 = 0,$$

$$\mu_{B_1}^2(a_2 a_3) = (\mu_{B_1} \circ \mu_{B_1})(a_2 a_3) = \mu_{B_1}(a_2 a_1) \wedge \mu_{B_1}(a_1 a_3) = 0.3 \wedge 0.3 = 0.3,$$

$$\mu_{B_1}^2(a_1 a_3) = (\mu_{B_1} \circ \mu_{B_1})(a_1 a_3) = \mu_{B_1}(a_1 a_2) \wedge \mu_{B_1}(a_2 a_3) = 0.3 \wedge 0.0 = 0,$$

$$\mu_{B_1}^3(a_1 a_3) = (\mu_{B_1}^2 \circ \mu_{B_1})(a_1 a_3) = \mu_{B_1}^2(a_1 a_2) \wedge \mu_{B_1}(a_2 a_3) = 0.0 \wedge 0.0 = 0,$$

FIGURE 2.4 IFGS $\tilde{G}_s = (A, B_1, B_2)$

Thus we have,

$$\mu_{B_1}^\infty(a_1 a_2) = \vee\{0.3, 0.0, 0.0\} = 0.3,$$

$$\mu_{B_1}^\infty(a_2 a_3) = \vee\{0.0, 0.3, 0.0\} = 0.3,$$

$$\mu_{B_1}^\infty(a_1 a_3) = \vee\{0.3, 0.0, 0.0\} = 0.3,$$

and

Since $\nu_{B_1}(a_1 a_2) = 0.7$, $\nu_{B_1}(a_1 a_3) = 0.3$, $\nu_{B_1}(a_2 a_3) = 0$, therefore

$$\nu_{B_1}^2(a_1 a_2) = (\nu_{B_1} \circ \nu_{B_1})(a_1 a_2) = \nu_{B_1}(a_1 a_3) \vee \nu_{B_1}(a_3 a_2) = 0.3 \vee 0.0 = 0.3,$$

$$\nu_{B_1}^2(a_2 a_3) = (\nu_{B_1} \circ \nu_{B_1})(a_2 a_3) = \nu_{B_1}(a_2 a_1) \vee \nu_{B_1}(a_1 a_3) = 0.7 \vee 0.3 = 0.7,$$

$$\nu_{B_1}^2(a_1 a_3) = (\nu_{B_1} \circ \nu_{B_1})(a_1 a_3) = \nu_{B_1}(a_1 a_2) \vee \nu_{B_1}(a_2 a_3) = 0.7 \vee 0.0 = 0.7,$$

$$\nu_{B_1}^3(a_1 a_2) = (\nu_{B_1}^2 \circ \nu_{B_1})(a_1 a_2) = \nu_{B_1}^2(a_1 a_3) \vee \nu_{B_1}(a_3 a_2) = 0.7 \vee 0.0 = 0.7,$$

$$\nu_{B_1}^3(a_2 a_3) = (\nu_{B_1}^2 \circ \nu_{B_1})(a_2 a_3) = \nu_{B_1}^2(a_2 a_1) \vee \nu_{B_1}(a_1 a_3) = 0.3 \vee 0.3 = 0.3,$$

$$\nu_{B_1}^3(a_1 a_3) = (\nu_{B_1}^2 \circ \nu_{B_1})(a_1 a_3) = \nu_{B_1}^2(a_1 a_2) \wedge \nu_{B_1}(a_2 a_3) = 0.3 \vee 0.0 = 0.3,$$

and

$$\nu_{B_1}^4(a_1 a_2) = (\nu_{B_1}^3 \circ \nu_{B_1})(a_1 a_2) = \nu_{B_1}^3(a_1 a_3) \vee \nu_{B_1}(a_3 a_2) = 0.3 \vee 0.0 = 0.3,$$

$$\nu_{B_1}^4(a_2 a_3) = (\nu_{B_1}^3 \circ \nu_{B_1})(a_2 a_3) = \nu_{B_1}^3(a_2 a_1) \vee \nu_{B_1}(a_1 a_3) = 0.7 \vee 0.3 = 0.7,$$

$$\nu_{B_1}^4(a_1 a_3) = (\nu_{B_1}^3 \circ \nu_{B_1})(a_1 a_3) = \nu_{B_1}^3(a_1 a_2) \wedge \nu_{B_1}(a_2 a_3) = 0.7 \vee 0.0 = 0.7,$$

Thus we have,

$$\nu_{B_1}^\infty(a_1 a_2) = \vee\{0.7, 0.3, 0.7, 0.3\} = 0.7,$$

$$\nu_{B_1}^\infty(a_2 a_3) = \vee\{0.0, 0.7, 0.3, 0.7\} = 0.7,$$

$$\nu_{B_1}^\infty(a_1 a_3) = \vee\{0.3, 0.7, 0.3, 0.7\} = 0.7,$$

By similar calculations, it can be easily checked that

$$\begin{aligned}\mu_{B_2}^\infty(a_1a_2) &= 0, & \mu_{B_2}^\infty(a_2a_3) &= 0.5, & \mu_{B_2}^\infty(a_1a_3) &= 0, \\ \nu_{B_2}^\infty(a_1a_2) &= 0.3, & \nu_{B_2}^\infty(a_2a_3) &= 0.3, & \nu_{B_2}^\infty(a_1a_3) &= 0.3.\end{aligned}$$

Definition 2.18

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \dots, E_n)$ is a B_i -cycle, if G^* is an E_i -cycle.

Definition 2.19

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \dots, E_n)$ is an intuitionistic fuzzy B_i -cycle for some i , if following conditions hold:

- (i) \check{G}_s is a B_i -cycle
- (ii) There is no unique B_i -edge uv in \check{G}_s , such that

$$\begin{aligned}\mu_{B_i}(uv) &= \min\{\mu_{B_i}(xy) : xy \in E_i = \sup p(B_i)\} \\ \nu_{B_i}(uv) &= \max\{\nu_{B_i}(xy) : xy \in E_i = \sup p(B_i)\}\end{aligned}$$

Example 2.20

IFGS $\check{G}_s = (A, B_1, B_2)$ shown in Fig 3, is a B_1 -cycle as well as intuitionistic fuzzy B_1 -cycle, since $(\sup p(A), \sup p(B_1), \sup p(B_2))$ is an E_1 -cycle and there are B_1 -edges with minimum degree of membership and two B_1 -edges with maximum degree of non-membership of all B_1 -edges.

Definition 2.21

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \dots, E_n)$ is a B_i -tree, if $(\sup p(A), \sup p(B_1), \dots, \sup p(B_n))$ is an E_i -tree. In other words, \check{G}_s is a B_i -tree if the sub graph of \check{G}_s , induced by $\sup p(B_i)$ forms a tree.

Definition 2.22

An IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \dots, E_n)$ is an intuitionistic fuzzy B_i -tree (intuitionistic fuzzy B_i -forest), if \check{G}_s has an intuitionistic fuzzy partial spanning subgraph structure

$\check{H}_s = (A, C_1, C_2, \dots, C_n)$, such that \check{H}_s is a C_i -tree (C_i -forest) and $\mu_{B_i}(xy) < \mu_{C_i}^\infty(xy)$ and $\nu_{B_i}(xy) < \nu_{C_i}^\infty(xy)$ for all B_i -edges not in \check{H}_s .

Example 2.23

The IFGS, shown in Fig. 3, is a B_2 -tree but not an intuitionistic fuzzy B_2 -tree. While IFGS $\check{G}_s = (A, B_1, B_2)$, shown in Fig 2. 5, is not a B_1 -tree but an intuitionistic fuzzy B_1 -tree, since it has an intuitionistic fuzzy partial spanning sub graph structure (A, B_1', B_2') as a B_1 -tree, which is obtained by deleting B_1 -edge a_1a_4 from \check{G}_s , with $\mu_{B_1}(a_1a_4) = 0.3 < 0.4 = \mu_{B_1}^\infty(a_1a_4)$ and $\nu_{B_1}(a_1a_4) = 0.5 < 0.6 = \nu_{B_1}^\infty(a_1a_4)$.

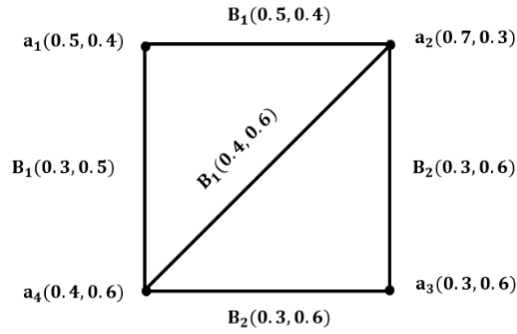


FIGURE 2.5 IFGS $\check{G}_s = (A, B_1, B_2)$

Definition 2.24

An IFGS $\check{G}_{s_1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ of a GS, $G_1^* = (U_1, E_{11}, E_{12}, \dots, E_{1n})$ is isomorphic to IFGS $\check{G}_{s_2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ of $G_2^* = (U_2, E_{21}, E_{22}, \dots, E_{2n})$, if there exist a bijection $f : U_1 \rightarrow U_2$ and a permutation ϕ on the set $\{1, 2, \dots, n\}$, such that :

$$\mu_{A_1}(u_1) = \mu_{A_2}(f(u_1)), \nu_{A_1}(u_1) = \nu_{A_2}(f(u_1)) \quad \text{for all } u_1 \in U_1$$

and for $\phi(i) = j$

$$\mu_{B_{1i}}(u_1u_2) = \mu_{B_{2j}}(f(u_1)f(u_2)), \nu_{B_{1i}}(u_1) = \nu_{B_{2j}}(f(u_1)f(u_2)),$$

for all $u_1u_2 \in E_{1i}$, $i = 1, 2, 3, \dots, n$.

Definition 2.25

An IFGS $\check{G}_{s_1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$ of GS $G_1^* = (U_1, E_{11}, E_{12}, \dots, E_{1n})$ is identical to an IFGS $\check{G}_{s_2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$ of $G_2^* = (U_2, E_{21}, E_{22}, \dots, E_{2n})$, if there exist a bijection $f : U \rightarrow U$, such that:

$$\mu_{A_1}(u) = \mu_{A_2}(f(u)), \nu_{A_1}(u) = \nu_{A_2}(f(u)) \quad \text{for all } u \in U$$

and

$$\mu_{B_{1i}}(u_1 u_2) = \mu_{B_{2i}}(f(u_1) f(u_2)), \nu_{B_{1i}}(u_1) = \nu_{B_{2i}}(f(u_1) f(u_2)),$$

for all $u_1 u_2 \in E_{1i}$, $i = 1, 2, 3, \dots, n$.

Example 2.26

\check{G}_{s_1} and \check{G}_{s_2} as shown in Fig 2.6 and Fig 2.7, are IFGSs of graph structures $G_1^* = (U, E_1, E_2, E_3, E_4)$ and $G_2^* = (U, E'_1, E'_2, E'_3, E'_4)$ respectively, where,

$$\begin{aligned} U_1 &= \{a_1, a_2, a_3, a_4, a_5\}, & E_1 &= \{a_1 a_2, a_2 a_5\} \\ E_2 &= \{a_2 a_3, a_2 a_4\}, & E_3 &= \{a_1 a_3, a_4 a_5\} \\ E_4 &= \{a_1 a_5, a_3 a_4\}, & U_2 &= \{b_1, b_2, b_3, b_4, b_5\} \\ E'_1 &= \{b_2 b_4, b_3 b_4\}, & E'_2 &= \{b_1 b_4, b_4 b_5\} \\ E'_3 &= \{b_1 b_2, b_3 b_5\}, & E'_4 &= \{b_1 b_5, b_2 b_3\} \end{aligned}$$

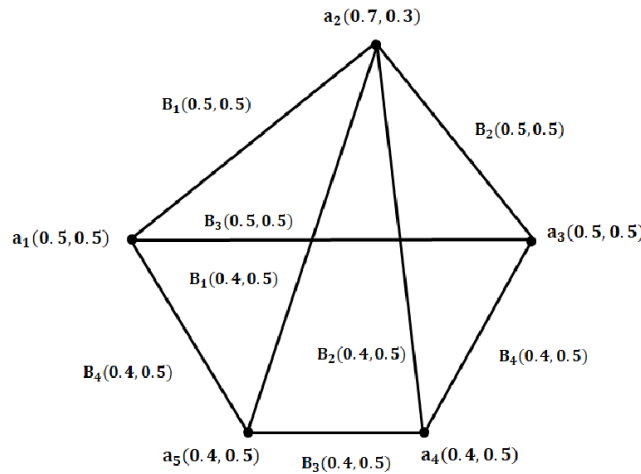


FIGURE 2.6 IFGS $\check{G}_{s_1} = (J, K_1, K_2, K_3, K_4)$

Then \check{G}_{s_1} is isomorphic to \check{G}_{s_2} under the mapping $f : U_1 \rightarrow U_2$, given by

$$f(a_1) = b_5, \quad f(a_2) = b_4, \quad f(a_3) = b_3, \quad f(a_4) = b_2, \quad f(a_5) = b_1,$$

and a permutation ϕ given by

$$\phi(1) = 2 \quad \phi(2) = 1 \quad \phi(3) = 3 \quad \phi(4) = 4,$$

such that

$$\mu_{A_i}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_i}(a_i) = \nu_{A_2}(f(a_i))$$

for all, $a_i \in U_1$, and

$$\mu_{B_k}(a_i a_j) = \mu_{B_{\phi(k)}}(f(a_i) f(a_j)), \quad \nu_{B_k}(a_i a_j) = \nu_{B_{\phi(k)}}(f(a_i) f(a_j)),$$

for all, $a_i a_j \in E_k$, $k = 1, 2, 3, 4$.

also, \tilde{G}_{s_1} is identical with \tilde{G}_{s_2} under the mapping $f : U_1 \rightarrow U_2$, given by

$$f(a_1) = b_3, \quad f(a_2) = b_4, \quad f(a_3) = b_5, \quad f(a_4) = b_1, \quad f(a_5) = b_2,$$

such that

$$\mu_{A_i}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_i}(a_i) = \nu_{A_2}(f(a_i)),$$

for all $a_i \in U_1$ and

$$\mu_{B_k}(a_i a_j) = \mu_{B_k}(f(a_i) f(a_j)), \quad \nu_{B_k}(a_i a_j) = \nu_{B_k}(f(a_i) f(a_j)),$$

for all $a_i a_j \in E_k$, $k = 1, 2, 3, 4$.

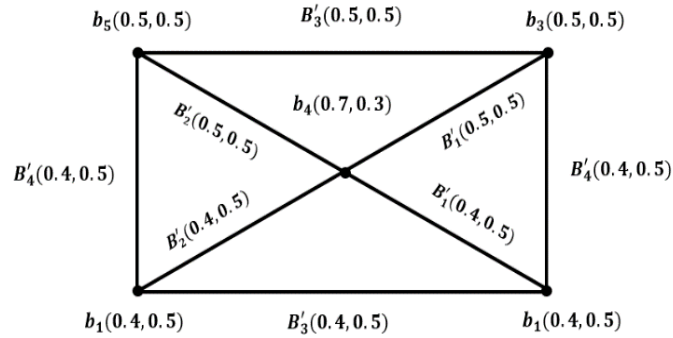


FIGURE 2.7 IFGS $\tilde{G}_{s_2} = (A_2, B'_1, B'_2, B'_3, B'_4)$

Remark 2.27

Identical IFGSs are always isomorphic but the converse is not necessarily true. As IFGS shown in Fig. 3 is isomorphic to IFGS shown in Fig. 8 but they are not identical.

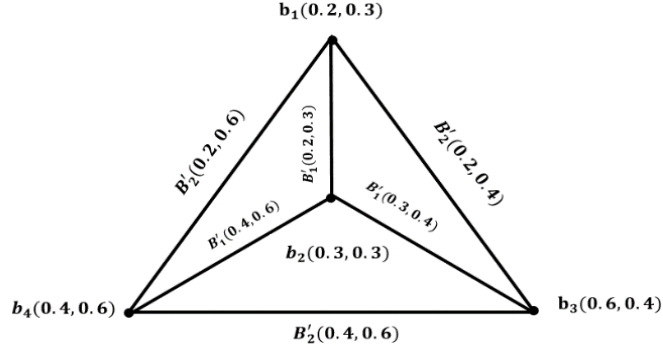


FIGURE 2.8 IFGS $\tilde{G}_s = (A, B_1, B_2)$

Definition 2.28

Let $\tilde{G}_s = (A, B_1, B_2, \dots, B_n)$ be an intuitionistic fuzzy graph structure $G^* = (U, E_1, E_2, \dots, E_n)$. Let ϕ denote a permutation on the set $\{E_1, E_2, \dots, E_n\}$ and the corresponding permutation on $\{B_1, B_2, \dots, B_n\}$, i.e., $\phi(B_i) = B_j$ iff $\phi(E_i) = E_j$ for all i .

If $xy \in B_r$ for some r and

$$\mu_{B_i^\phi}(xy) = \mu_A(x) \wedge \mu_A(y) - \bigvee_{j \neq i} \mu_{\phi B_j}(xy),$$

$$\nu_{B_i^\phi}(xy) = \nu_A(x) \vee \nu_A(y) - \bigvee_{j \neq i} \nu_{\phi B_j}(xy), \quad i = 1, 2, \dots, n,$$

Then, $xy \in B_m^\phi$; while m is chosen such that $\mu_{B_m^\phi}(xy) \geq \mu_{B_i^\phi}(xy)$ and $\nu_{B_m^\phi}(xy) \geq \nu_{B_i^\phi}(xy)$ for all i . Then, IFGS $(A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$ denoted by $\tilde{G}_s^{\phi c}$, is called the ϕ -complement of IFGS \tilde{G}_s .

Theorem 2.29

A ϕ -complement of an intuitionistic fuzzy graph structure is always a strong IFGS. Moreover if $\phi(i) = r$ for $r, i \in \{1, 2, \dots, n\}$, then all B_r -edges in IFGS $\tilde{G}_s = (A, B_1, B_2, \dots, B_n)$ become B_i^ϕ -edges in $\tilde{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$.

Proof.

First part is obvious from the definition of ϕ -complement $\tilde{G}_s^{\phi c}$ of IFGS \tilde{G}_s . Since for any B_i^ϕ -edges xy , $\mu_{B_i^\phi}(xy)$ and $\nu_{B_i^\phi}(xy)$ respectively have the maximum values of

$$[\mu_A(x) \wedge \mu_A(y)] - \bigvee_{j \neq i} \mu_{\phi B_j}(xy) \quad \text{and} \quad [\nu_A(x) \vee \nu_A(y)] - \bigvee_{j \neq i} \nu_{\phi B_j}(xy) \quad (1)$$

that is,

$$\mu_{B_i}^\phi(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}^\phi(xy) = \nu_A(x) \vee \nu_A(y), \quad (2)$$

for all edges xy in $\tilde{G}_s^{\phi c}$, hence $\tilde{G}_s^{\phi c}$ is always a strong IFGS.

Now suppose on contrary that $\phi(i) = r$ but xy is a B_s -edge in \tilde{G}_s with $s \neq r$, which implies that $\phi B_i \neq B_s$. Comparing expressions (1) and (2), we get

$$\bigvee_{j \neq i} \mu_\phi B_j(xy) = 0, \quad \bigvee_{j \neq i} \nu_\phi B_j(xy) = 0,$$

Which is not possible because $B_s = \phi B_j$ for some $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$.

So our supposition is wrong and xy must be B_r -edge. Hence we conclude that if $\phi(i) = r$ then all B_r -edges in IFGS $\tilde{G}_s = (A, B_1, B_2, \dots, B_n)$ become B_i^ϕ -edges in $\tilde{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$ for $r, i \in \{1, 2, \dots, n\}$.

Example 2.30

Consider IFGS $\tilde{G}_s = (A, B_1, B_2)$ shown in Fig. 4 and let ϕ be a permutation on the set $\{B_1, B_2\}$ such that $\phi(B_1) = B_2$ and $\phi(B_2) = B_1$.

Now for $a_1 a_2 \in B_1$,

$$\begin{aligned} \mu_{B_1}^\phi(a_1 a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 1} [\mu_\phi B_j(a_1 a_2)] = 0.3 \wedge 0.5 - [\mu_\phi B_2(a_1 a_2)] \\ &= 0.3 - \mu_{B_1}(a_1 a_2) = 0.3 - 0.3 = 0 \end{aligned}$$

$$\begin{aligned} \nu_{B_1}^\phi(a_1 a_2) &= \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j \neq 1} [\nu_\phi B_j(a_1 a_2)] = 0.7 \vee 0.4 - [\nu_\phi B_2(a_1 a_2)] \\ &= 0.7 - \nu_{B_1}(a_1 a_2) = 0.7 - 0.7 = 0 \end{aligned}$$

$$\begin{aligned} \mu_{B_2}^\phi(a_1 a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 2} [\mu_\phi B_j(a_1 a_2)] = 0.3 \wedge 0.5 - [\mu_\phi B_1(a_1 a_2)] \\ &= 0.3 - \mu_{B_2}(a_1 a_2) = 0.3 - 0 = 0.3 \end{aligned}$$

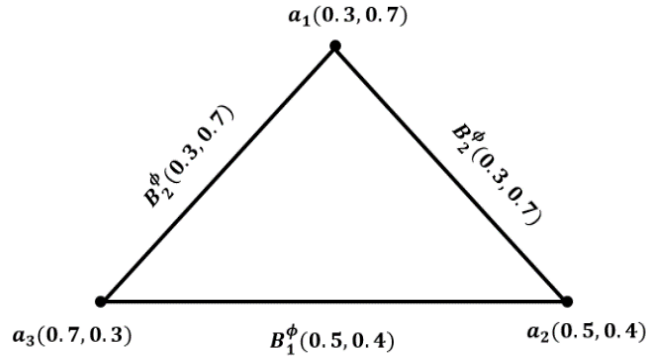


FIGURE 2.9 IFGS $\tilde{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi)$

$$\begin{aligned} \nu_{B_2}^\phi(a_1a_2) &= \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j \neq 2} [\mu_{B_j}^\phi(a_1a_2)] = 0.7 \vee 0.4 - [\nu_{B_1}^\phi(a_1a_2)] \\ &= 0.7 - \nu_{B_2}(a_1a_2) = 0.7 - 0 = 0.7 \end{aligned}$$

Clearly,

$$\mu_{B_2}^\phi(a_1a_2) = 0.3 > 0 = \mu_{B_1}^\phi(a_1a_2) \text{ and } \nu_{B_2}^\phi(a_1a_2) = 0.7 > 0 = \nu_{B_1}^\phi(a_1a_2), \text{ so } a_1a_2 \in B_2^\phi$$

Similarly for $a_1a_3 \in B_1$,

$$\mu_{B_1}^\phi(a_1a_3) = 0, \quad \nu_{B_1}^\phi(a_1a_3) = 0.4, \quad \mu_{B_2}^\phi(a_1a_3) = 0.3, \quad \nu_{B_2}^\phi(a_1a_3) = 0.7,$$

Clearly,

$$\mu_{B_1}^\phi(a_1a_3) = 0, \quad \nu_{B_1}^\phi(a_1a_3) = 0.4, \quad \mu_{B_2}^\phi(a_1a_3) = 0.3, \quad \nu_{B_2}^\phi(a_1a_3) = 0.7,$$

$$\mu_{B_2}^\phi(a_1a_3) = 0.3 > 0 = \mu_{B_1}^\phi(a_1a_3) \text{ and } \nu_{B_2}^\phi(a_1a_3) = 0.7 > 0.4 = \nu_{B_1}^\phi(a_1a_3), \text{ so } a_1a_3 \in B_2^\phi.$$

and for $a_2a_3 \in B_2$

$$\mu_{B_1}^\phi(a_2a_3) = 0.5, \quad \nu_{B_1}^\phi(a_2a_3) = 0.4, \quad \mu_{B_2}^\phi(a_2a_3) = 0, \quad \nu_{B_2}^\phi(a_2a_3) = 0.1,$$

that is,

$$\mu_{B_1}^\phi(a_2a_3) = 0.5 > 0 = \mu_{B_2}^\phi(a_2a_3) \text{ and } \nu_{B_1}^\phi(a_2a_3) = 0.4 > 0 = \nu_{B_2}^\phi(a_2a_3), \text{ so } a_2a_3 \in B_1^\phi.$$

this implies that,

$$B_1^\phi = \{a_2a_3, 0.5, 0.4\}, B_2^\phi = \{(a_1a_2, 0.3, 0.7), (a_1a_3, 0.3, 0.7)\} \text{ and } G_s^{\phi c} = (A, B_1^\phi, B_2^\phi)$$

shown in Fig 2.9 is the ϕ -complement of \tilde{G}_s .

Definition 2.31

Let $\tilde{G}_s = (A, B_1, B_2, \dots, B_n)$ be an IFGS and ϕ be a permutation on the set $\{1, 2, \dots, n\}$. Then

- (i) \tilde{G}_s is self-complementary, if it is isomorphic to $\tilde{G}_s^{\phi c}$, the ϕ -complement of \tilde{G}_s .
- (ii) \tilde{G}_s is strong self-complementary, if it is identical to $\tilde{G}_s^{\phi c}$.
- (iii) \tilde{G}_s is totally self-complementary, if it is isomorphic to $\tilde{G}_s^{\phi c}$, the ϕ -complement of \tilde{G}_s for all permutation ϕ on the set $\{1, 2, \dots, n\}$.
- (iv) \tilde{G}_s is totally strong self-complementary, if it is identical to $\tilde{G}_s^{\phi c}$, the ϕ -complement of \tilde{G}_s for all permutation ϕ on the set $\{1, 2, \dots, n\}$.

Theorem 2.32

An IFGS \check{G}_s is strong if and only if \check{G}_s is totally self-complementary.

Proof.

Let \check{G}_s be a strong IFGS and ϕ be any permutation on the set $\{1,2,\dots,n\}$.

If $\phi^{-1}(i) = j$, then by Theorem 3.1, all B_i -edges in $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ become B_j^ϕ -edges in $\check{G}_s^{\phi^c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$. Also $\check{G}_s^{\phi^c}$ is strong, so

$$\mu_{B_i}(a_1 a_2) = \mu_A(a_1) \wedge \mu_A(a_2) = \mu_{B_j^\phi}(a_1 a_2),$$

$$\nu_{B_i}(a_1 a_2) = \nu_A(a_1) \vee \nu_A(a_2) = \nu_{B_j^\phi}(a_1 a_2).$$

Then \check{G}_s is isomorphic to $\check{G}_s^{\phi^c}$, under the identity mapping $f: U \rightarrow U$ and a permutation $\phi[\phi^{-1}(i) = j, i, j = 1, 2, \dots, n]$, such that

$$\mu_A(a) = \mu_A(f(a)), \quad \nu_A(a) = \nu_A(f(a)), \quad \text{for all } a \in U$$

and

$$\mu_{B_i}(a_1 a_2) = \mu_{B_j^\phi}(a_1 a_2) = \mu_{B_j^\phi}(f(a_1) f(a_2)),$$

$$\nu_{B_i}(a_1 a_2) = \nu_{B_j^\phi}(a_1 a_2) = \nu_{B_j^\phi}(f(a_1) f(a_2)), \quad \text{for all } a_1 a_2 \in E_i.$$

this holds for all permutation on the set $\{1, 2, \dots, n\}$. Hence \check{G}_s is totally self-complementary.

Conversely, let ϕ be any permutation on the set $\{1, 2, \dots, n\}$ and \check{G}_s and $\check{G}_s^{\phi^c}$ be isomorphic. From the definition of ϕ -complement and isomorphism of IFGSs, we have

$$\mu_{B_i}(a_1 a_2) = \mu_{B_j^\phi}(f(a_1) f(a_2)) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_A(a_1) \wedge \mu_A(a_2),$$

$$\nu_{B_i}(a_1 a_2) = \nu_{B_j^\phi}(f(a_1) f(a_2)) = \nu_A(f(a_1)) \vee \nu_A(f(a_2)) = \nu_A(a_1) \vee \nu_A(a_2),$$

for all $a_1 a_2 \in E_i, i = 1, 2, \dots, n$.

Hence, \check{G}_s is a strong IFGS.

Remark 2.33

Every self-complementary IFGS is necessarily totally self-complementary.

Theorem 2.34

If graph structure $G^* = (U, E_1, E_2, \dots, E_n)$ is totally strong self-complementary and A is an IFS of U with constant fuzzy mappings μ_A and ν_A then a strong IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of G^* is totally strong self-complementary.

Proof.

Consider a strong IFGS $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ of a graph structure $G^* = (U, E_1, E_2, \dots, E_n)$. Suppose that G^* is totally strong self-complementary and that for some constants $s, t \in [0, 1]$, $A = (\mu_A, \nu_A)$ is an IFS of U such that $\mu_A(u) = s, \nu_A(u) = t$, for all $u \in U$. Then we have to prove that \check{G}_s is totally strong self-complementary.

Let ϕ be a permutation on the set $\{1, 2, \dots, n\}$ and $\phi^{-1}(j) = i$. Since G^* is totally strong self-complementary, so there exists a bijection $f : U \rightarrow U$, such that for every E_i -edge $a_1 a_2$ in G^* , $f(a_1) f(a_2)$ (E_j -edge in G^*) is an E_i -edge in $(G^*)^{\phi^{-1}c}$. Consequently, for every B_i -edge $a_1 a_2$ in \check{G}_s , $f(a_1) f(a_2)$ (a B_j -edge in \check{G}_s) is a B_i^{ϕ} -edge in $\check{G}_s^{\phi c}$.

from the definition of A and the definition of strong IFGS \check{G}_s

$$\mu_A(u) = s = \mu_A(f(a)), \quad \nu_A(u) = t = \nu_A(f(a)), \quad \text{for all } a, f(a) \in U,$$

and

$$\mu_A(a_1 a_2) = \mu_A(a_1) \wedge \mu_A(a_2) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_{B_i^{\phi}}(f(a_1) f(a_2)),$$

$$\nu_A(a_1 a_2) = \nu_A(a_1) \vee \nu_A(a_2) = \nu_A(f(a_1)) \vee \nu_A(f(a_2)) = \nu_{B_i^{\phi}}(f(a_1) f(a_2)),$$

for all $a_1 a_2 \in B_i$, $i = 1, 2, \dots, n$, which shows \check{G}_s is strong self complementary.

Hence \check{G}_s is totally strong self-complementary, since ϕ is arbitrary.

Remark 2.35

Converse of Theorem 2.34 is not necessary, since a totally strong self-complementary and strong IFGS $\check{G}_s = (A, B_1, B_2, B_3)$ as shown in Fig. 11, has a totally strong self-complementary underlying graph structure but μ_A and ν_A are not constant fuzzy functions.

Example 2.36

The IFGS shown in Fig 8 is self-complementary, i.e., it is isomorphic to its ϕ -complement, where $\phi = (1,2)$. Also, it is self-complementary because ϕ is the only non-identity permutation on the set $\{1,2\}$.

Example 2.37

The IFGS $\check{G}_s = (A, B_1, B_2, B_3, B_4)$ shown in Fig 10, is strong self-complementary, i.e., it is identical to its ϕ -complement where the permutation ϕ is $(1,2) (3,4)$. It is not totally strong self-complementary.

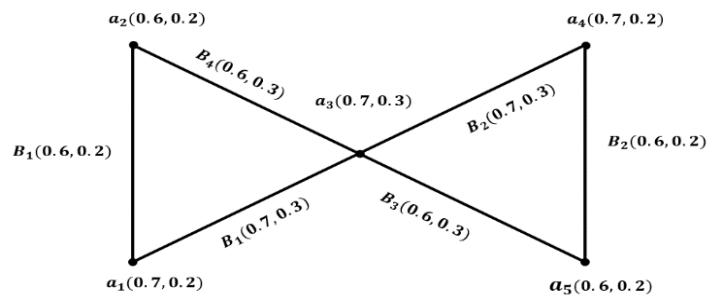


FIGURE 2.10 IFGS $\check{G}_s = (A_1, B_1, B_2, B_3, B_4)$

Example 2.38

The IFGS $\check{G}_s = (A, B_1, B_2, B_3)$, shown in Fig 2.11, is totally strong self-complementary because it is identical to its ϕ -complement for all the permutation on the ϕ on the set $\{1,2,3\}$.

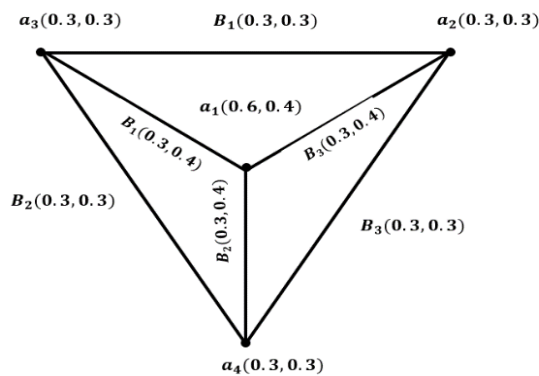


FIGURE 2.11 IFGS $\check{G}_s = (A_1, B_1, B_2, B_3)$

CHAPTER 3

CHAPTER-3

PYTHAGOREN FUZZY GRAPH STRUCTURE

Definition 3.1

Let $\{M_i : i = 1, 2, \dots, n\}$ be a set of irreflexive, symmetric and mutually disjoint relations on non empty set T . An *Pythagorean Fuzzy Graph Structure* (IFGS) with underlying vertex set T is denoted by $\overset{\cup}{G}_s = (J, K_1, K_2, \dots, K_n)$ where,

(i) J is an is an Pythagorean Fuzzy set of T with $\alpha_J : T \rightarrow [0, 1]$ and $\beta_J : T \rightarrow [0, 1]$, namely the degree of membership and the degree of non-membership of $x \in U$, respectively such that

$$0 \leq \alpha_J^2(x) + \beta_J^2(y) \leq 1, \text{ for all } x \in T$$

(ii) Each K_i is an fuzzy set of M_i such that the functions $\alpha_{K_i} : M \rightarrow [0, 1]$ and $\beta_{K_i} : M \rightarrow [0, 1]$ are defined by

$$\alpha_{K_i}(xy) \leq \alpha_J(x) \wedge \alpha_J(y), \beta_{K_i}(xy) \leq \beta_J(x) \vee \beta_J(y)$$

and

$$0 \leq \alpha_{K_i}(xy) + \beta_{K_i}(xy) \leq 1, \text{ for all } xy \in \subset T \times T, i = 1, 2, \dots, n.$$

Equivalently, an PGFS of a graph structure and let $J, K_1, K_2, \dots, K_{n-1}$ and K_n be an Pythagorean fuzzy subsets of $T, M_1, M_2, \dots, M_{n-1}$ and M_n respectively, then,

$$\alpha_{K_i}(xy) \leq \alpha_J(x) \wedge \alpha_J(y), \beta_{K_i}(xy) \leq \beta_J(x) \vee \beta_J(y)$$

for all $xy \in M_i, i = 1, 2, 3, \dots, n$ and

$$\alpha_J^2(x) + \beta_J^2(y) \leq 1 \text{ for all } xy \in T \times T.$$

Example 3.2

Let $G^* = (T, M_1, M_2)$ be a graph structure such that $T = \{a_1, a_2, a_3, a_4\}$, $M_1 = \{a_1a_2, a_2a_3, a_3a_1\}$ and $M_2 = \{a_4a_3, a_1a_4, a_2a_4\}$. Let J, K_1 and B_2 be a Pythagorean fuzzy subsets of T, M_1 and M_2 , respectively such that ,

$$J = \{(a_1, 0.7, 0.6), (a_2, 0.5, 0.6), (a_3, 0.8, 0.2), (a_4, 0.9, 0.4)\}$$

$$K_1 = \{(a_1 a_2, 0.5, 0.6), (a_2 a_3, 0.5, 0.6), (a_3 a_1, 0.7, 0.6)\}$$

$$K_2 = \{(a_4 a_3, 0.8, 0.4), (a_1 a_4, 0.7, 0.6), (a_2 a_4, 0.5, 0.6)\}$$

Then $\check{G}_s = (J, K_1, K_2)$ is a PFGS of G^* as shown in Fig 3.1.

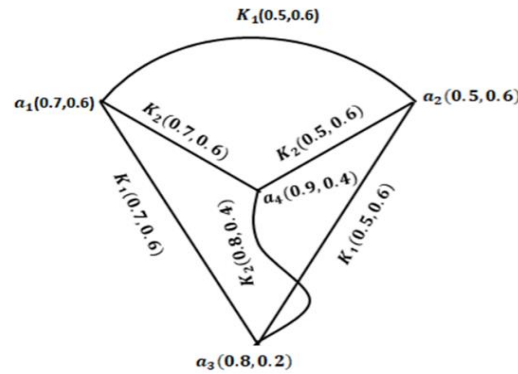


FIGURE 3.1 PFGS $\check{G}_s = (J, K_1, K_2)$

Definition 3.3

An IFGS $\check{H}_s = (V, W_1, W_2, \dots, W_n)$ is said to be Pythagorean fuzzy subgraph structure of an IFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ with underlying vertex set T , if $V \subseteq J$ and $W_i \subseteq V_i$ for all i , that is

$$\alpha_V(x) \leq \alpha_J(x), \beta_V(x) \geq \beta_J(x), \text{ for all } x \in T.$$

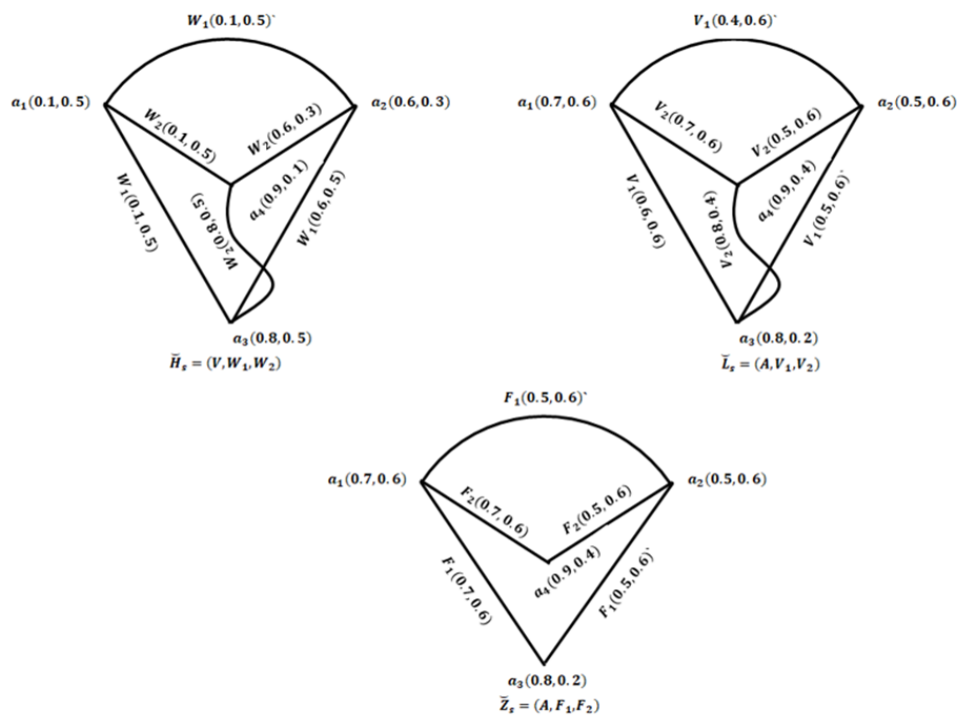


FIGURE 3.2 Pythagorean Fuzzy Subgraph Structures

and for $i = 1, 2, 3, \dots, n$

$$\alpha_{W_i}(x) \leq \alpha_{K_i}(x), \beta_{W_i}(x) \geq \beta_{K_i}(x), \text{ for all } xy \in T \times T.$$

\check{H}_s is called an Pythagorean fuzzy spanning sub graph structure of an PFGS \check{G}_s , if $V = J$.

\check{H}_s is called an Pythagorean fuzzy partial spanning sub graph structure of an PFGS \check{G}_s , if it excludes some edges of \check{G}_s .

Example 3.4

Consider a PFGS $\check{G}_s = (J, K_1, K_2)$, as shown in Fig 3.1. Let,

$$V = \{(a_1, 0.1, 0.5), (a_2, 0.6, 0.3), (a_3, 0.8, 0.5), (a_4, 0.9, 0.1)\}$$

$$W_1 = \{(a_1 a_2, 0.1, 0.5), (a_2 a_3, 0.6, 0.5), (a_1 a_3, 0.1, 0.5)\}$$

$$W_2 = \{(a_3 a_4, 0.8, 0.5), (a_1 a_4, 0.1, 0.5), (a_2 a_4, 0.6, 0.3)\}$$

$$V_1 = \{(a_1 a_2, 0.4, 0.6), (a_2 a_3, 0.5, 0.6), (a_1 a_3, 0.6, 0.6)\}$$

$$V_2 = \{(a_3 a_4, 0.8, 0.4), (a_1 a_4, 0.7, 0.6), (a_2 a_4, 0.4, 0.6)\}$$

$$F_1 = \{(a_1 a_2, 0.5, 0.6), (a_2 a_3, 0.5, 0.6), (a_1 a_3, 0.7, 0.6)\}$$

$$\text{and } F_2 = \{(a_1 a_4, 0.7, 0.6), (a_2 a_4, 0.5, 0.6)\}$$

By routine calculations, it is easy to see that $\check{H}_s = (V, W_1, W_2)$, $\check{L}_s = (J, V_1, V_2)$ and $\check{Z}_s = (J, F_1, F_2)$ are respectively the Pythagorean fuzzy sub graph structure, Pythagorean fuzzy spanning sub graph structure and Pythagorean fuzzy partial spanning sub graph structure of \check{G}_s . Their respective drawings are shown in Fig 3.2.

Definition 3.5

Let $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ be a PFGS with underlying vertex set T. Then there is a K_i - edge between two vertices x and y of V if one of the following is true:

- (i) $\alpha_{K_i}(xy) > 0$ and $\beta_{K_i}(xy) > 0$,
- (ii) $\alpha_{K_i}(xy) > 0$ and $\beta_{K_i}(xy) = 0$,
- (iii) $\alpha_{K_i}(xy) = 0$ and $\beta_{K_i}(xy) > 0$, for some i.

Definition 3.6

For a Pythagorean fuzzy graph structure $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ with vertex set T, support of K_i is given by,

$$\sup p(K_i) = \{xy \in T \times T : \alpha_{K_i}(xy) \neq 0 \text{ or } \beta_{K_i}(xy) \neq 0\}, \quad i = 1, 2, \dots, n.$$

Definition 3.7

K_i -edge of an PFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ with underlying vertex set T, is a sequence of distinct vertices $\beta_1, \beta_2, \dots, \beta_m \in T$ (except the choice $\beta_m = \beta_1$ such that $\beta_{j-1}\beta_j$ is a K_i -edge for all $j = 2, 3, \dots, m$).

Example 3.8

$$M_1 = \{a_1a_2, a_2a_3, a_3a_1\} \text{ and } M_2 = \{a_4a_3, a_1a_4, a_2a_4\}$$

- (i) a_1a_2, a_2a_3, a_1a_3 are K_1 -edges and a_3a_4, a_1a_4, a_2a_4 are K_2 -edges ;
- (ii) $a_1a_2a_3$ and $a_1a_3a_4$ are K_1 and K_2 -paths respectively;
- (iii) a_1 and a_3 are connected by vertices of T;
- (iv) \check{G}_s is a K_1 -strong , since $\sup p(K_1) = \{a_1a_2, a_2a_3, a_1a_3\}$ and

$$\alpha_{K_1}(a_1a_2) = 0.5 = (\alpha_J(a_1) \wedge \alpha_J(a_2)),$$

$$\beta_{K_1}(a_1a_2) = 0.6 = (\beta_J(a_1) \vee \beta_J(a_2)),$$

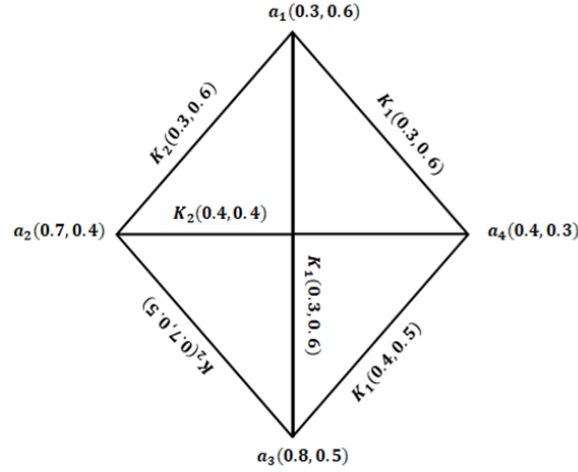
$$\alpha_{K_1}(a_2a_3) = 0.5 = (\alpha_J(a_2) \wedge \alpha_J(a_3)),$$

$$\beta_{K_1}(a_2a_3) = 0.6 = (\beta_J(a_2) \vee \beta_J(a_3)),$$

$$\alpha_{K_1}(a_1a_3) = 0.7 = (\alpha_J(a_1) \wedge \alpha_J(a_3)),$$

Example 3.9

Let $\check{G}_s = (J, K_1, K_2)$ shown in Fig 2, be IFGS of the graph structure $G_s^* = (T, M_1, M_2)$ where $T = \{a_1, a_2, a_3, a_4\}$, $M_1 = \{a_1a_4, a_3a_4, a_1a_3\}$ and $M_2 = \{a_1a_2, a_2a_3, a_2a_4\}$. Then \check{G}_s is a strong IFGS since it is both B_1 -strong and B_2 -strong . Moreover $\sup p(K_1) \neq \emptyset, \sup p(K_2) \neq \emptyset$, every pair of vertices belong to T, is either a K_1 -edge or a K_2 -edge, so \check{G}_s is a complete or $K_1 K_2$ -complete PFGS as well.

FIGURE 3.3 PFGS $\check{G}_s = (J, K_1, K_2)$ **Definition 3.10**

If an PFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ with underlying vertex set T , α_{K_i} and β_{K_i} - strengths of a B_i - path " $R_{K_i} = \beta_1, \beta_2, \dots, \beta_m$ ", are denoted by $\diamond.R_{K_i}$ and $\nabla.K_i$ respectively such that,

$$\diamond.K_i = \wedge_{j=2}^m [\alpha_{K_i}(\beta_{j-1}\beta_j)] \text{ and } \nabla.K_i = \vee_{j=2}^m [\beta_{K_i}(\beta_{j-1}\beta_j)]$$

Then, we write the strength of the path, $R_{K_i} = (\diamond.R_{K_i}, \nabla.R_{K_i})$.

Example 3.11

In $\check{G}_s = (J, K_1, K_2)$ shown in Fig 3.3, $R_1 = a_1a_3a_4$ is a K_1 - path and $R_2 = a_1a_2a_3$ is a K_2 -path and

$$\diamond R_1 = \alpha_{K_1}(a_1a_4) \wedge \alpha_{K_1}(a_1a_3) \wedge \alpha_{K_1}(a_1a_3) = (0.2 \wedge 0.4 \wedge 0.3) = 0.3$$

$$\nabla R_1 = \beta_{K_1}(a_1a_4) \vee \beta_{K_1}(a_1a_3) \vee \beta_{K_1}(a_1a_3) = (0.6 \vee 0.5 \vee 0.6) = 0.6$$

$$\diamond R_2 = \alpha_{K_2}(a_1a_2) \wedge \alpha_{K_2}(a_2a_3) \wedge \alpha_{K_2}(a_2a_4) = (0.3 \wedge 0.7 \wedge 0.4) = 0.3$$

$$\nabla R_2 = \beta_{K_2}(a_1a_2) \vee \beta_{K_2}(a_2a_3) \vee \beta_{K_2}(a_2a_4) = (0.6 \vee 0.5 \vee 0.4) = 0.6$$

The strength of K_1 -path $R_{K_1} = (\diamond.R_{K_1}, \nabla.R_{K_1}) = (0.3, 0.6)$, The strength of K_2 -path $R_{K_2} = (\diamond.R_{K_2}, \nabla.R_{K_2}) = (0.3, 0.6)$.

Definition 3.12

If a PFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ with underlying vertex set T :

- (i) α_{K_i} – Strength of connectedness between x and y, is defied by
 $\alpha_{K_i}^\infty(xy) = V_{j \geq 1} \{\alpha_{K_i}^j(xy)\}$ where $\alpha_{K_i}^\infty(xy) = \{\alpha_{K_i}^{j-1} \circ \alpha_{K_i}\}(xy)$ for $j \geq 2$ for
 $\alpha_{K_i}^2(xy) = \{\alpha_{K_i} \circ \alpha_{K_i}\}(xy) = V_z \{\alpha_{K_i}(xy) \wedge \alpha_{K_i}(zy)\}$;
- (ii) β_{K_i} – Strength of connectedness between x and y, is defied by
 $\beta_{K_i}^\infty(xy) = V_{j \geq 1} \{\beta_{K_i}^j(xy)\}$ where $\beta_{K_i}^\infty(xy) = \{\beta_{K_i}^{j-1} \circ \beta_{K_i}\}(xy)$ for $j \geq 2$ for
 $\beta_{K_i}^2(xy) = \{\beta_{K_i} \circ \beta_{K_i}\}(xy) = \Lambda_z \{\beta_{K_i}(xy) \wedge \beta_{K_i}(zy)\}$.

Example 3.13

$\tilde{G}_s = (J, K_1, K_2)$ as shown in Fig 3.4, be IFGS of graph structure
 $G^* = (T, M_1, M_2)$ such that $T = \{a_1, a_2, a_3, a_4\}$, $M_1 = \{a_1a_2, a_2a_4, a_1a_4\}$ and
 $M_2 = \{a_1a_3, a_3a_4\}$.

Since $\alpha_{K_1}(a_1a_2) = 0.1$, $\alpha_{K_1}(a_2a_4) = 0.2$, $\alpha_{K_1}(a_1a_4) = 0.1$, therefore

$$\alpha_{K_1}^2(a_1a_2) = (\alpha_{K_1} \circ \alpha_{K_1})(a_1a_2) = \alpha_{K_1}(a_1a_4) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.2 = 0.1$$

$$\alpha_{K_1}^2(a_2a_4) = (\alpha_{K_1} \circ \alpha_{K_1})(a_2a_4) = \alpha_{K_1}(a_1a_2) \wedge \alpha_{K_1}(a_1a_4) = 0.1 \wedge 0.1 = 0.1$$

$$\alpha_{K_1}^2(a_1a_4) = (\alpha_{K_1} \circ \alpha_{K_1})(a_1a_4) = \alpha_{K_1}(a_1a_2) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.2 = 0.1$$

$$\alpha_{K_1}^3(a_1a_2) = (\alpha_{K_1}^2 \circ \alpha_{K_1})(a_1a_2) = \alpha_{K_1}^2(a_1a_4) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.1 = 0.1$$

$$\alpha_{K_1}^3(a_2a_4) = (\alpha_{K_1}^2 \circ \alpha_{K_1})(a_2a_4) = \alpha_{K_1}^2(a_1a_2) \wedge \alpha_{K_1}(a_1a_4) = 0.1 \wedge 0.1 = 0.1$$

$$\alpha_{K_1}^3(a_1a_4) = (\alpha_{K_1}^2 \circ \alpha_{K_1})(a_1a_4) = \alpha_{K_1}^2(a_1a_2) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.2 = 0.1$$

$$\alpha_{K_1}^4(a_1a_2) = (\alpha_{K_1}^3 \circ \alpha_{K_1})(a_1a_2) = \alpha_{K_1}^3(a_1a_4) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.2 = 0.1$$

$$\alpha_{K_1}^4(a_2a_4) = (\alpha_{K_1}^3 \circ \alpha_{K_1})(a_2a_4) = \alpha_{K_1}^3(a_1a_2) \wedge \alpha_{K_1}(a_1a_4) = 0.1 \wedge 0.1 = 0.1$$

$$\alpha_{K_1}^4(a_1a_4) = (\alpha_{K_1}^3 \circ \alpha_{K_1})(a_1a_4) = \alpha_{K_1}^3(a_1a_2) \wedge \alpha_{K_1}(a_2a_4) = 0.1 \wedge 0.2 = 0.1$$

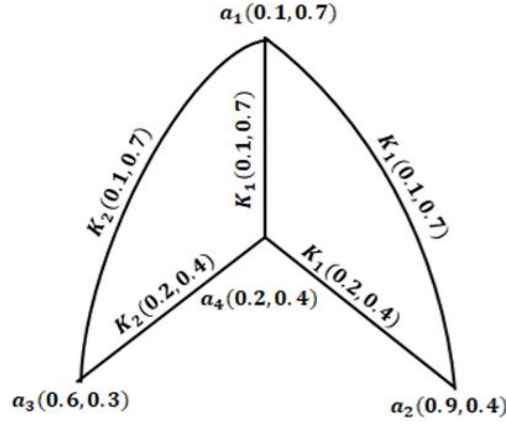


FIGURE 3.4 PFGS $\check{G}_s = (J, K_1, K_2)$

Thus we have,

$$\alpha_{K_1}^\infty(a_1 a_2) = \vee \{0.1, 0.1, 0.1, 0.1\} = 0.1$$

$$\alpha_{K_1}^\infty(a_2 a_4) = \vee \{0.2, 0.1, 0.1, 0.1\} = 0.2$$

$$\alpha_{K_1}^\infty(a_1 a_4) = \vee \{0.1, 0.1, 0.1, 0.1\} = 0.1$$

Since $\beta_{K_1}(a_1 a_2) = 0.1$, $\beta_{K_1}(a_2 a_4) = 0.2$, $\beta_{K_1}(a_1 a_4) = 0.1$, therefore

$$\beta_{K_1}^2(a_1 a_2) = (\beta_{K_1} \circ \beta_{K_1})(a_1 a_2) = \beta_{K_1}(a_1 a_4) \vee \beta_{K_1}(a_2 a_4) = 0.1 \vee 0.2 = 0.2$$

$$\beta_{K_1}^2(a_2 a_4) = (\beta_{K_1} \circ \beta_{K_1})(a_2 a_4) = \beta_{K_1}(a_1 a_2) \vee \beta_{K_1}(a_1 a_4) = 0.1 \vee 0.1 = 0.1$$

$$\beta_{K_1}^2(a_1 a_4) = (\beta_{K_1} \circ \beta_{K_1})(a_1 a_4) = \beta_{K_1}(a_1 a_2) \vee \beta_{K_1}(a_2 a_4) = 0.1 \vee 0.2 = 0.2$$

$$\beta_{K_1}^3(a_1 a_2) = (\beta_{K_1}^2 \circ \beta_{K_1})(a_1 a_2) = \beta_{K_1}^2(a_1 a_4) \vee \beta_{K_1}(a_2 a_4) = 0.2 \vee 0.2 = 0.2$$

$$\beta_{K_1}^3(a_2 a_4) = (\beta_{K_1}^2 \circ \beta_{K_1})(a_2 a_4) = \beta_{K_1}^2(a_1 a_2) \vee \beta_{K_1}(a_1 a_4) = 0.1 \vee 0.1 = 0.1$$

$$\beta_{K_1}^3(a_1 a_4) = (\beta_{K_1}^2 \circ \beta_{K_1})(a_1 a_4) = \beta_{K_1}^2(a_1 a_2) \vee \beta_{K_1}(a_2 a_4) = 0.1 \vee 0.2 = 0.2$$

$$\beta_{K_1}^4(a_1 a_2) = (\beta_{K_1}^3 \circ \beta_{K_1})(a_1 a_2) = \beta_{K_1}^3(a_1 a_4) \vee \beta_{K_1}(a_2 a_4) = 0.2 \vee 0.2 = 0.2$$

$$\beta_{K_1}^4(a_2 a_4) = (\beta_{K_1}^3 \circ \beta_{K_1})(a_2 a_4) = \beta_{K_1}^3(a_1 a_2) \vee \beta_{K_1}(a_1 a_4) = 0.1 \vee 0.1 = 0.1$$

$$\beta_{K_1}^4(a_1 a_4) = (\beta_{K_1}^3 \circ \beta_{K_1})(a_1 a_4) = \beta_{K_1}^3(a_1 a_2) \vee \beta_{K_1}(a_2 a_4) = 0.1 \vee 0.2 = 0.2$$

Thus we have,

$$\beta_{k_1}^\infty(a_1a_2) = \vee \{0.7, 0.7, 0.7, 0.7\} = 0.7$$

$$\beta_{k_1}^\infty(a_2a_4) = \vee \{0.4, 0.7, 0.7, 0.7\} = 0.7$$

$$\beta_{k_1}^\infty(a_1a_4) = \vee \{0.7, 0.7, 0.7, 0.7\} = 0.7$$

By similar calculations, it can be easily checked that

$$\alpha_{K_2}^\infty(a_1a_3) = 0.1 \qquad \beta_{K_2}^\infty(a_1a_3) = 0.7$$

$$\alpha_{K_2}^\infty(a_3a_4) = 0.2 \qquad \beta_{K_2}^\infty(a_3a_4) = 0.7$$

$$\alpha_{K_2}^\infty(a_1a_4) = 0.1 \qquad \beta_{K_2}^\infty(a_1a_4) = 0.7$$

Example 3.14

A PFGS shown in Fig 3.3, is a B_2 -tree but not an Pythagorean fuzzy B_2 -tree. While $\check{G}_s = (J, K_1, K_2)$ shown in Fig 3.5, is not a B_1 -tree but an Pythagorean fuzzy B_1 -tree, since it has an Pythagorean fuzzy partial spanning sub graph structure (J, K'_1, K'_2) as a B_1 - tree ,which is obtained by deleting B_1 -edge (a_2a_3) from \check{G}_s with $\alpha_{K_1}(a_1a_4) = 0.2 < 0.4 = \alpha_{K'_1}(a_1a_4)$ and $\beta_{K_1}(a_1a_4) = 0.5 < 0.6 = \beta_{K'_1}(a_2a_3)$.

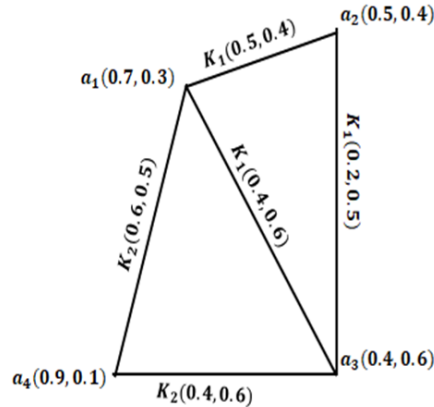


FIGURE 3.5 PFGS $\check{G}_s = (J, K_1, K_2)$

Example 3.15

\check{G}_{s_1} and \check{G}_{s_2} as shown in Fig 3.6 and Fig 3.7 are PFGS of graph structures $G_{s_1}^* = (T_1, M_1, M_2, M_3M_4)$ and $G_{s_2}^* = (T'_2, M'_1, M'_2, M'_3M'_4)$, respectively,

Where,

$$T_1 = \{a_1, a_2, a_3, a_4, a_5, a_6\} \qquad M_1 = \{a_2a_3, a_3a_5\}$$

$$\begin{aligned}
M_2 &= \{a_3a_4, a_4a_5\} & M_3 &= \{a_1a_2, a_5a_6\} \\
M_4 &= \{a_2a_5, a_1a_6\} & T_2 &= \{b_1, b_2, b_3, b_4, b_5, b_6\} \\
M'_1 &= \{b_1b_3, b_1b_2\} & M'_2 &= \{b_3b_4, b_4b_5\} \\
M'_3 &= \{b_5b_6, b_2b_3\} & M'_4 &= \{b_3b_5, b_2b_6\}
\end{aligned}$$

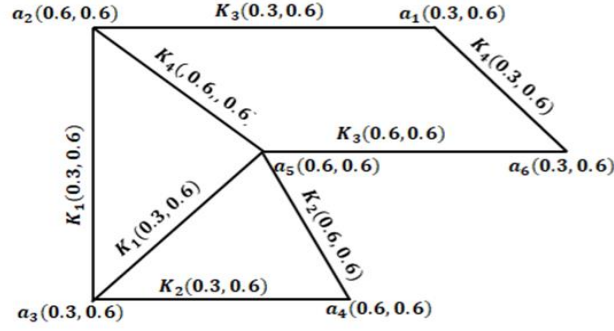


FIGURE 3.6 PFGS $\tilde{G}_{s_1} = (J, K_1, K_2, K_3, K_4)$

Then \tilde{G}_{s_1} is isomorphic to \tilde{G}_{s_2} under the mapping $f : T_1 \rightarrow T_2$, is given by

$$f(a_1) = b_1, f(a_2) = b_2, f(a_3) = b_3, f(a_4) = b_4, f(a_5) = b_6, f(a_6) = b_5$$

and a permutation ϕ is given by

$$\phi(1) = 2, \phi(2) = 1, \phi(3) = 3, \phi(4) = 4$$

Such that,

$$\alpha_{J_1}(a_i) = \alpha_{J_2}(f(a_i)), \quad \beta_{J_1}(a_i) = \beta_{J_2}(f(a_i))$$

For all $a_i \in T_1$, and

$$\alpha_{K_p}(a_i a_j) = \alpha_{K_{\phi(p)}}(f(a_i) f(a_j)), \quad \beta_{K_p}(a_i a_j) = \beta_{K_{\phi(p)}}(f(a_i) f(a_j)),$$

for all $a_i a_j \in M_p, P=1,2,3,4$.

Also, \tilde{G}_{s_1} is identical with \tilde{G}_{s_2} under the mapping $f : T_1 \rightarrow T_2$, given by

$$f(a_1) = b_5, f(a_2) = b_4, f(a_3) = b_3, f(a_4) = b_1, f(a_5) = b_1, f(a_6) = b_6,$$

such that

$$\alpha_{J_1}(a_i) = \alpha_{J_2}(f(a_i)), \beta_{J_1}(a_i) = \beta_{J_2}(f(a_i)), \text{ for all } a_i \in T_1,$$

and

$$\alpha_{K_p}(a_i a_j) = \alpha'_{K_p}(f(a_i) f(a_j)), \quad \beta_{K_p}(a_i a_j) = \beta_{K_p}(f(a_i) f(a_j))$$

for all $a_i a_j \in E_p, P=1,2,3,4$.

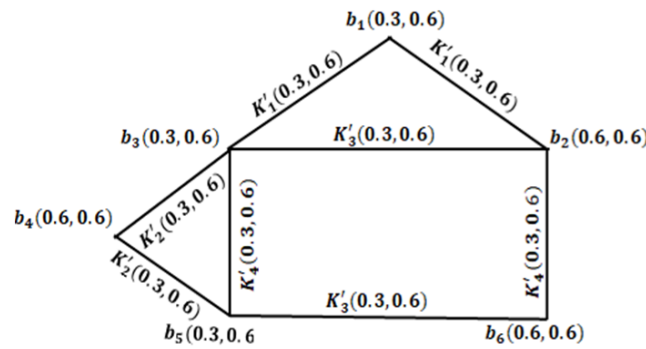


FIGURE 3.7 PFGS $\check{G}_{s_2} = (J, K_1, K_2, K_3, K_4)$

Remark 3.16

Identical PFGS are always isomorphic but the converse is not necessarily true. As PFGS Shown in Fig 3.3 is isomorphic to PFGS shown in Fig 3.8 but they are not identical.

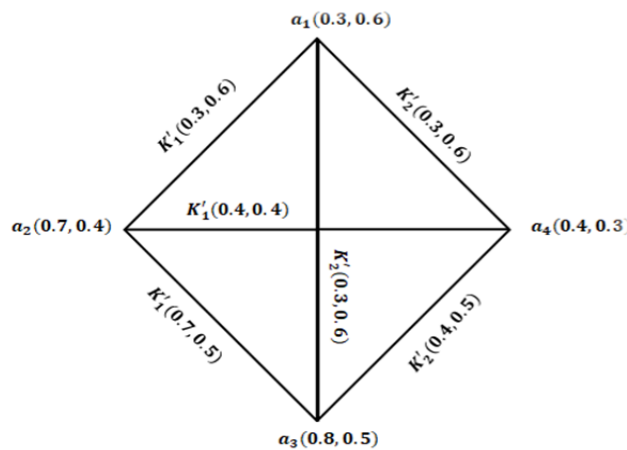


FIGURE 3.8 PFGS $\check{G}_s = (J, K_1, K_2)$

Definition 3.17

Let $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ be an Pythagorean structure of a graph structure $G^* = (T, M_1, M_2, M_3, \dots, M_n)$. Let ϕ denote a permutation on the set $\{M_1, M_2, \dots, M_n\}$. and the corresponding permutation on $\{K_1, K_2, \dots, K_n\}$, i.e., $\phi(K_i) = K_j$ iff $\phi(M_i) = M_j$ for all i

If $xy \in K_r$ for some r and

$$\alpha_{B_i^\varphi}(xy) = \alpha_A(x) \wedge \alpha_A(y) - \bigvee_{j \neq i} \alpha_{\varphi B_j}(xy),$$

$$\beta_{B_i^\varphi}(xy) = \beta_A(x) \wedge \beta_A(y) - \bigvee_{j \neq i} \beta_{\varphi B_j}(xy), \quad i=1, 2, \dots, n,$$

Then $xy \in B_m^\varphi$, while m is chosen such that $\alpha_{B_m^\varphi}(xy) \geq \alpha_{B_i^\varphi}(xy)$ and

$$\beta_{B_m^\varphi}(xy) \geq \beta_{B_i^\varphi}(xy) \text{ for all } i.$$

Then PFGS $(J, K_1^\varphi, K_2^\varphi, \dots, K_n^\varphi)$ denoted by $\tilde{G}_s^{\phi c}$, is called the φ -complement of PFGS \tilde{G}_s .

Theorem 3.18

A ϕ -complement of a Pythagorean fuzzy graph structure is always a strong PFGS. Moreover if $\varphi(i) = r$ for $r, i \in \{1, 2, \dots, n\}$, then all B_r -edges in PFGS $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ become B_j^φ -edges in $\tilde{G}_s^{\phi c} = (J, K_1^\varphi, K_2^\varphi, \dots, K_n^\varphi)$.

Proof

First part is obvious from the definition of ϕ -complement $\tilde{G}_s^{\phi c}$ of PFGS of \tilde{G}_s , since for any B_j^φ -edge xy , $\alpha_{B_i^\varphi}(xy)$ and $\beta_{B_i^\varphi}(xy)$ respectively have the maximum values of

$$\alpha_A(x) \wedge \alpha_A(y) - \bigvee_{j \neq i} \alpha_{\varphi B_j}(xy), \text{ and } \beta_A(x) \wedge \beta_A(y) - \bigvee_{j \neq i} \beta_{\varphi B_j}(xy), \quad (1)$$

that is,

$$\alpha_{B_i^\varphi}^\varphi(xy) = \alpha_A(x) \wedge \alpha_A(y), \beta_{B_i^\varphi}^\varphi(xy) = \beta_A(x) \wedge \beta_A(y), \quad (2)$$

for all edges xy in $\tilde{G}_s^{\phi c}$, hence $\tilde{G}_s^{\phi c}$ is always a strong PFGS.

Now suppose on contrary that $\varphi(i) = r$ but xy is K_s -edge in \tilde{G}_s with $s \neq r$, which implies that $\varphi B_i \neq B_s$. Comparing expressions (1) and (2), we get

$$\bigvee_{j \neq i} \alpha_{\varphi B_j}(xy), \bigvee_{j \neq i} \beta_{\varphi B_j}(xy) = 0$$

Which is not possible because $\varphi(K_i) \neq K_j$ for some $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$.

so our supposition is wrong and xy must be a K_r -edge. Hence we can conclude that if $\varphi(i) \neq r$, then all K_r -edges in PFGS $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ become K_i^φ -edges in $\tilde{G}_s^{\phi c} = (J, K_1^\varphi, K_2^\varphi, \dots, K_n^\varphi)$ for $r, i \in \{1, 2, \dots, n\}$.

Example 3.19

Consider PFGS $\tilde{G}_s = (J, K_1, K_2)$ shown in Fig 3.4 and let φ be a permutation on the set $\{K_1, K_2\}$ such that $\varphi(K_1) = K_2$ and $\varphi(K_2) = K_1$.

Now for $a_1, a_2 \in K_1$,

$$\alpha_{K_1}^\varphi(a_1 a_2) = \alpha_J(a_1) \wedge \alpha_J(a_2) - V_{j \neq 1} [\alpha_{\varphi K_j}(a_1 a_2)] = 0.1 - 0.1 = 0$$

$$\beta_{K_1}^\varphi(a_1 a_2) = \beta_J(a_1) \vee \beta_J(a_2) - V_{j \neq 1} [\beta_{\varphi K_j}(a_1 a_2)] = 0.7 - 0.7 = 0$$

$$\alpha_{K_2}^\varphi(a_1 a_2) = \alpha_J(a_1) \wedge \alpha_J(a_2) - V_{j \neq 2} [\alpha_{\varphi K_j}(a_1 a_2)] = 0.1 - 0 = 0.1$$

$$\beta_{K_2}^\varphi(a_1 a_2) = \beta_J(a_1) \vee \beta_J(a_2) - V_{j \neq 2} [\beta_{\varphi K_j}(a_1 a_2)] = 0.7 - 0 = 0.7$$

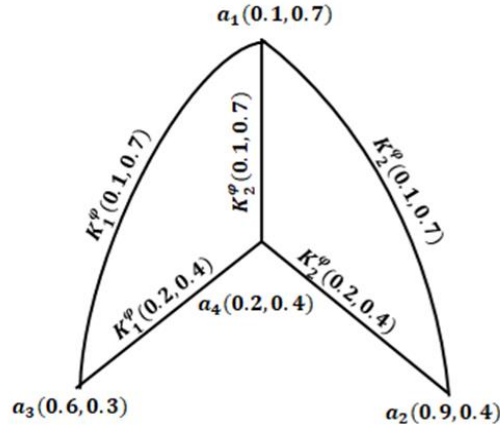


FIGURE 3.9 PFGS $\tilde{G}_s^{\varphi c} = (J, K_1^\varphi, K_2^\varphi)$

Clearly,

$$\alpha_{K_2}^\varphi(a_1 a_2) = 0.1 > 0 = \alpha_{K_1}^\varphi(a_1 a_2) \quad \text{and} \quad \beta_{K_2}^\varphi(a_1 a_2) = 0.7 > 0 = \beta_{K_1}^\varphi(a_1 a_2)$$

Similarly, for $a_2, a_4 \in K_1$,

$$\alpha_{K_1}^\varphi(a_2 a_4) = \alpha_J(a_2) \wedge \alpha_J(a_4) - V_{j \neq 1} [\alpha_{\varphi K_j}(a_2 a_4)] = 0.2 - 0.2 = 0$$

$$\beta_{K_1}^\varphi(a_2 a_4) = \beta_J(a_2) \vee \beta_J(a_4) - V_{j \neq 1} [\beta_{\varphi K_j}(a_2 a_4)] = 0.4 - 0.4 = 0$$

$$\alpha_{K_2}^\varphi(a_2 a_4) = \alpha_J(a_2) \wedge \alpha_J(a_4) - V_{j \neq 2} [\alpha_{\varphi K_j}(a_2 a_4)] = 0.2 - 0 = 0.2$$

$$\beta_{K_2}^\varphi(a_2 a_4) = \beta_J(a_2) \vee \beta_J(a_4) - V_{j \neq 2} [\beta_{\varphi K_j}(a_2 a_4)] = 0.4 - 0 = 0.4$$

Clearly,

$$\alpha_{K_2}^\varphi(a_2 a_4) = 0.2 > 0 = \alpha_{K_1}^\varphi(a_2 a_4) \quad \text{and} \quad \beta_{K_2}^\varphi(a_2 a_4) = 0.4 > 0 = \beta_{K_1}^\varphi(a_2 a_4)$$

for, $a_1, a_4 \in B_1$

$$\alpha_{K_1}^\varphi(a_1a_4) = \alpha_J(a_1) \wedge \alpha_J(a_4) - V_{j \neq 1} [\alpha_{\varphi K_j}(a_1a_4)] = 0.1 - 0.1 = 0$$

$$\beta_{K_1}^\varphi(a_1a_4) = \beta_J(a_1) \vee \beta_J(a_4) - V_{j \neq 1} [\beta_{\varphi K_j}(a_1a_4)] = 0.7 - 0.7 = 0$$

$$\alpha_{K_2}^\varphi(a_1a_4) = \alpha_J(a_1) \wedge \alpha_J(a_4) - V_{j \neq 2} [\alpha_{\varphi K_j}(a_1a_4)] = 0.1 - 0 = 0.1$$

$$\beta_{K_2}^\varphi(a_1a_4) = \beta_J(a_1) \vee \beta_J(a_4) - V_{j \neq 2} [\beta_{\varphi K_j}(a_1a_4)] = 0.7 - 0 = 0.7$$

Clearly,

$$\alpha_{K_2}^\varphi(a_1a_4) = 0.1 > 0 = \alpha_{K_1}^\varphi(a_1a_4) \quad \text{and} \quad \beta_{K_2}^\varphi(a_1a_4) = 0.7 > 0 = \beta_{K_1}^\varphi(a_1a_4)$$

For $a_1a_3 \in B_2$

$$\alpha_{K_1}^\varphi(a_1a_3) = \alpha_J(a_1) \wedge \alpha_J(a_3) - V_{j \neq 1} [\alpha_{\varphi K_j}(a_1a_3)] = 0.1 - 0 = 0.1$$

$$\beta_{K_1}^\varphi(a_1a_3) = \beta_J(a_1) \vee \beta_J(a_3) - V_{j \neq 1} [\beta_{\varphi K_j}(a_1a_3)] = 0.7 - 0 = 0.7$$

$$\alpha_{K_2}^\varphi(a_1a_3) = \alpha_J(a_1) \wedge \alpha_J(a_3) - V_{j \neq 2} [\alpha_{\varphi K_j}(a_1a_3)] = 0.1 - 0.1 = 0$$

$$\beta_{K_2}^\varphi(a_1a_3) = \beta_J(a_1) \vee \beta_J(a_3) - V_{j \neq 2} [\beta_{\varphi K_j}(a_1a_3)] = 0.7 - 0.7 = 0$$

Clearly,

$$\alpha_{K_1}^\varphi(a_1a_3) = 0.1 > 0 = \alpha_{K_2}^\varphi(a_1a_3) \quad \text{and} \quad \beta_{K_1}^\varphi(a_1a_3) = 0.7 > 0 = \beta_{K_2}^\varphi(a_1a_3)$$

For $a_3a_4 \in K_2$,

$$\alpha_{K_1}^\varphi(a_3a_4) = \alpha_J(a_3) \wedge \alpha_J(a_4) - V_{j \neq 1} [\alpha_{\varphi K_j}(a_3a_4)] = 0.2 - 0 = 0.2$$

$$\beta_{K_1}^\varphi(a_3a_4) = \beta_J(a_3) \vee \beta_J(a_4) - V_{j \neq 1} [\beta_{\varphi K_j}(a_3a_4)] = 0.4 - 0 = 0.4$$

$$\alpha_{K_2}^\varphi(a_3a_4) = \alpha_J(a_3) \wedge \alpha_J(a_4) - V_{j \neq 2} [\alpha_{\varphi K_j}(a_3a_4)] = 0.2 - 0.2 = 0$$

$$\beta_{K_2}^\varphi(a_3a_4) = \beta_J(a_3) \vee \beta_J(a_4) - V_{j \neq 2} [\beta_{\varphi K_j}(a_3a_4)] = 0.4 - 0.4 = 0$$

Clearly,

$$\alpha_{K_1}^\varphi(a_3a_4) = 0.2 > 0 = \alpha_{K_2}^\varphi(a_3a_4) \quad \text{and} \quad \beta_{K_1}^\varphi(a_3a_4) = 0.4 > 0 = \beta_{K_2}^\varphi(a_3a_4)$$

This shows that, $K_1^\varphi = \{(a_1a_3, 0.1, 0.7), (a_3a_4, 0.2, 0.4)\}$

$$K_2^\varphi = \{(a_1a_2, 0.1, 0.7), (a_1a_4, 0.1, 0.7), (a_2a_4, 0.2, 0.4)\} \quad \text{and} \quad \tilde{G}_s^\varphi = (J, K_1^\varphi, K_2^\varphi)$$

shown in Fig 3.9 is the φ -complement of \tilde{G}_s .

Theorem 3.20

A PFGS \tilde{G}_s is strong if and only if \tilde{G}_s is total self-complementary.

Proof:

Let \tilde{G}_s be a strong PFGS and φ be any permutation on the set $\{1, 2, \dots, n\}$.

If $\varphi^{-1}(i) = j$, Then by Theorem 3.18, all K_i - edges in $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ become K_j^φ - edges in $\tilde{G}_s^{\varphi c} = (J, K_1^\varphi, K_2^\varphi, \dots, K_n^\varphi)$.

Also $\tilde{G}_s^{\varphi c}$ is strong, so

$$\alpha_{K_i}(a_1 a_2) = \alpha_j(a_1) \wedge \alpha_j(a_2) = \alpha_{K_j^\varphi}(a_1 a_2)$$

$$\beta_{K_i}(a_1 a_2) = \beta_j(a_1) \wedge \beta_j(a_2) = \beta_{K_j^\varphi}(a_1 a_2)$$

Then \tilde{G}_s is isomorphic to $\tilde{G}_s^{\varphi c}$, under the identity mapping $f: T \rightarrow T$ and an permutation $\varphi[\varphi^{-1}(i) = j, i, j = 1, 2, \dots, n]$ such that

$$\alpha_j(a) = \alpha_j(f(a)), \beta_j(a) = \beta_j(f(a)) \text{ for all } a \in T$$

And $\alpha_{K_i}(a_1 a_2) = \alpha_{K_j^\varphi}(a_1 a_2) = \alpha_{K_j^\varphi}(f(a_1) f(a_2))$

$$\beta_{K_i}(a_1 a_2) = \beta_{K_j^\varphi}(a_1 a_2) = \beta_{K_j^\varphi}(f(a_1) f(a_2))$$

This holds for all permutations on the set $\{1, 2, \dots, n\}$. Hence \tilde{G}_s is totally self-complementary.

Conversely, let φ be any permutation on the set $\{1, 2, \dots, n\}$ and \tilde{G}_s and $\tilde{G}_s^{\varphi c}$ be isomorphic. From the definition of φ - complement and isomorphism of PFGSs, we have

$$\alpha_{K_i}(a_1 a_2) = \alpha_{K_j^\varphi}(f(a_1) f(a_2)) = \alpha_j(f(a_1)) \wedge \alpha_j(f(a_2)) = \alpha_j(a_1) \wedge \alpha_j(a_2)$$

$$\beta_{K_i}(a_1 a_2) = \beta_{K_j^\varphi}(f(a_1) f(a_2)) = \beta_j(f(a_1)) \vee \beta_j(f(a_2)) = \beta_j(a_1) \wedge \beta_j(a_2)$$

for all $a_1 a_2 \in M_i, i = 1, 2, \dots, n$.

Hence, \tilde{G}_s is a strong PFGS.

Remark 3.21

Every self-complementary PFGS is necessarily totally self-complementary.

Theorem 3.22

If a graph structure $G^* = (T, M_1, M_2, \dots, M_n)$ is totally strong self-complementary and J is a PFS of T with constant mappings α_J and β_J then a strong PFGS $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ of G^* is totally strong self-complementary.

Proof:

Consider a strong PFGS $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ of a graph structure $\tilde{G}_s = (T, M_1, M_2, \dots, M_n)$. Suppose that G^* is totally strong self-complementary that for some constants $s, t \in [0, 1]$, $A = (\alpha_A, \beta_A)$ is a PFS of T such that $\alpha_A(u) = s, \beta_A(u) = t, \text{ for all } u \in T$. Then we have for all $u \in T$. Then we have to prove that \tilde{G}_s is totally strong self-complementary. Let φ be an arbitrary permutation on the set $\{1, 2, \dots, n\}$ and $\varphi^{-1}(j) = i$. Since G^* is totally strong self-complementary, so there exists a bijection $f : T \rightarrow T$ such that for every M_i -edge $a_1 a_2$ in G^* , $f(a_1) f(a_2)$ is an M_i -edge in $(G^*)^{\varphi^{-1}c}$. Consequently, for every K_i -edge $a_1 a_2$ in \tilde{G}_s , $f(a_1) f(a_2)$ is an K_i^φ -edge in $\tilde{G}_s^{\varphi c}$.

$$\alpha_A(a) = s = \alpha_A(f(a)), \beta_A(a) = t = \beta_A(f(a)) \text{ for all } a, f(a) \in T,$$

and

$$\alpha_{K_i}(a_1 a_2) = \alpha_J(a_1) \wedge \alpha_J(a_2) = \alpha_A(f(a_1)) \wedge \alpha_A(f(a_2)) = \alpha_{K_i^\varphi}(f(a_1) f(a_2)),$$

$$\beta_{K_i}(a_1 a_2) = \beta_J(a_1) \vee \beta_J(a_2) = \beta_A(f(a_1)) \vee \beta_A(f(a_2)) = \beta_{K_i^\varphi}(f(a_1) f(a_2)),$$

for all $a_1 a_2 \in K_i, i = 1, 2, \dots, n$, which shows \tilde{G}_s is strong self-complementary.

Hence \tilde{G}_s is totally strong self-complementary, since φ is arbitrary.

Remark 3.23

Converse of Theorem 3.22 is not necessary, since a totally strong self-complementary and strong PFGS $\tilde{G}_s = (J, K_1, K_2, \dots, K_n)$ as shown in Fig 3.11, has a totally strong self-complementary underlying graph structure but α_A and β_A are not constant fuzzy functions.

Example 3.24

The PFGS shown in Fig 3.8 is self-complementary, i.e., it is isomorphic to its φ -complement, where α_A . Also, it is totally self-complementary because φ is the only non-identity permutation on the set $\{1, 2\}$.

Example 3.25

The PFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ shown in Fig 3.10, is strong self-complementary, i.e., it is identical to its φ -complement where the permutation φ is $(1\ 2)(3\ 4)$. It is not totally strong self-complementary.

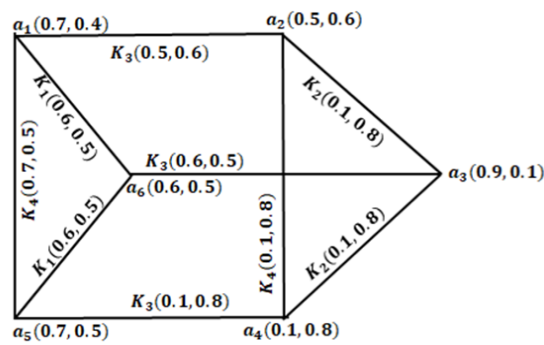


FIGURE 3.10 PFGS $\check{G}_s = (J, K_1, K_2, K_3, K_4)$

Example 3.26

The PFGS $\check{G}_s = (J, K_1, K_2, \dots, K_n)$ shown in Fig 3.11, is totally strong self-complementary, it is identical to its φ -complement for all the permutations φ on the set $\{1, 2, \text{ and } 3\}$.

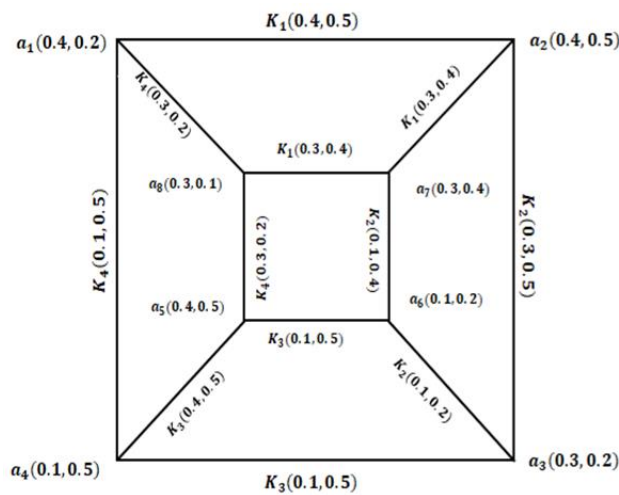


FIGURE 3.11 PFGS $\check{G}_s = (J, K_1, K_2, K_3, K_4)$

*SUMMARY AND
CONCLUSION*

SUMMARY AND CONCLUSION

In this thesis the idea of graph theoretical findings in intuitionistic fuzzy structure is discussed. It also includes some graphical structures for trees, cycles, sub graphs, self-complement, strong self-complement.

Further, we developed Pythagorean fuzzy graph structure when describing the uncertainty in numerous combinatorial issues across various fields Pythagorean fuzzy graphical models offer greater accuracy, adaptability and compatibility. We also extended our views to Pythagorean fuzzy trees, cycles and complement of a PFGS, including self complement and strong self-complement of PFGSs.

In future, we would like to extend our work to

- (i) Bipolar Environment
- (ii) Spherical Fuzzy Graph
- (iii) Neutrosophic Cubic Graph

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PRESENTATION

DETAILS OF PAPER PRESENTATION

Paper Presentation

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CERTIFICATE

This is to Certify that Prof. / Dr. / Mr. / Ms. M. ANNI PON SORNA, I M.Sc. MATHEMATICS of
AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN COE has Participated / Presented a paper titled
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 March 29th & 30th, 2023, at Dr. SNS Rajalakshmi College of Arts and Science, Coimbatore, Tamil Nadu, India.


Dr. M. UMAMAHESWARI
 Research Coordinator
 Dr.SNSRCAS


Prof. Dr. MUDIARASAN KUPPUSAMY
 Dean - Faculty of Business & Technology
 University of Cyberjaya, Malaysia


Dr. P. NARESHKUMAR
 Vice Principal
 Dr.SNSRCAS


Dr. R. ANITHA
 Principal
 Dr.SNSRCAS