CHAPTER - IV

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ON πgb*-CLOSED SETS IN TOPOLOGICAL SPACES

In this chapter the new class of closed sets called πgb*-closed sets were studied and its properties were analyzed. Further examples were provided wherever the implications donot hold. Some characterizations of πgb*-closed sets were introduced and the idea of πgb*-$T_{1/2}$ space was discussed.

Section 4.1

Preliminaries

Definition 4.1.1

Let (X, τ) be a topological space. A subset A of (X, τ) is called a **b*-closed** [36]set if int(cl(A)) $\subset$ U, whenever A $\subset$ U and U is b-open.

The complement of b*-closed set is called **b*-open set**.

A subset A of (X,τ) is called **clopen** if it is both open and closed in (X, τ).

Theorem 4.1.2

A set A of a topological space X is called b*-open if F $\subset$ cl(int(A)) whenever F is closed and F $\subset$ A.

Theorem 4.1.3

Every b*-closed set is b-closed.

Theorem 4.1.4

Suppose that B $\subset$ A $\subset$ X, B is b*-closed set relative to A and that A is both b-open and b*-closed subset of X, then B is b*closed set relative to X.

Theorem 4.1.5

Suppose that A $\subset$ Y $\subset$ X, A is a b*-closed set relative to X. Then A is a b*closed set relative to Y.

Section 4.2

πgb*-closed sets

In this section the properties of πgb*-closed sets were discussed and counter examples were provided whereever the implications donot hold.

Definition 4.2.1

A subset A of a space (X, τ) is called **a πgb*-closed set** if int(bcl(A)) $\subset$ U whenever A $\subset$ U and U is π-open in (X, τ).

Remark 4.2.2

Finite union of πgb*-closed sets need not be πgb*-closed which can be seen from the following example.

Example 4.2.3

Let X = { a, b, c } with topology τ = { φ , {b} , {c} , {b, c} , X }.

Let A = {b} and B = {c} then both A and B are πgb*-closed.

But, A $\cup$ B = {b, c} is not πgb*-closed.

Remark 4.2.4

Finite intersection of πgb*-closed sets need not be πgb*-closed which can be seen from the following example.

Example 4.2.5

Let X = { a, b, c, d } with topology

τ = { φ , {a} , {b} , {a, b, c} , {a, b} , {a, b, d} , X }.

Let A = {a, b, c} and B = {a, b, d}.

Then both A and B are πgb*-closed.

But, A $\cap$ B = {a, b} is not πgb*-closed.

Theorem 4.2.6

Every closed set is πgb*-closed.

Proof

Let A be a closed set of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Since bcl(A) $\subset$ cl(A) = A,

int(bcl(A)) $\subset$ int(A) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.7

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.8

Let $X = \{a, b, c\}$, and $\tau = \{ \varphi, \{a\}, \{a, b\}, \{a, c\}, X \}$.

Here $A = \{a\}$ is πgb*-closed but it is not closed.

Theorem 4.2.9

Every semi-closed set is πgb*-closed.

Proof

Let A be a semi-closed set of (X, τ) such that $A \subseteq U$ and U is π-open in X.

Since $bcl(A) \subset scl(A) = A$,

$int(bcl(A)) \subset int(A) \subseteq int(U) = U$.

Hence A is πgb*-closed.

Remark 4.2.10

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.11

Let $X = \{a, b, c\}$ and $\tau = \{ \varphi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X \}$.

Let $A = \{b,c\}$. Then A is πgb*-closed but it is not semi-closed.

Theorem 4.2.12

Every pre-closed set is πgb*-closed.

Proof

Let A be a pre-closed set of (X, τ) such that $A \subseteq U$ and U is π-open in X.

Since $bcl(A) \subset pcl(A) = A$,

$int(bcl(A)) \subset int(A) \subseteq int(U) = U$.

Hence A is πgb*-closed.

Remark 4.2.13

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.14

Let $X = \{ a, b, c \}$ and $\tau = \{ \varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X \}$. Let $A = \{a,b\}$.

Then A is πgb*-closed but it is not pre-closed.

Theorem 4.2.15

Every α-closed set is πgb*-closed.

Proof

Let A be a α-closed set of (X, τ) such that A ⊆ U and U is π-open in X.

Since bcl(A) ⊂ αcl(A) = A,

int(bcl(A)) ⊂ int(A) ⊆ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.16

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.17

Let X = {a, b, c} and τ = { φ , {a} , {a, c} , X }. Let A = {a,c}.

Then A is πgb*-closed but it is not α-closed.

Theorem 4.2.18

Every b-closed set is πgb*-closed.

Proof

Let A be a b-closed subset of (X, τ) such that A ⊆ U and U is π-open in X.

Since bcl(A) = A,

int(bcl(A)) = int(A) ⊆ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.19

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.20

Let X = {a, b, c, d} with topology τ = { φ , {a} , {b} ,{a, b} , {a, b, c} ,{a, b, d} ,X }.

Let A = {a, b, c}.

Then A is πgb*-closed but it is not b-closed.

Theorem 4.2.21

Every g-closed set is πgb*-closed.

Proof

Let A be a g-closed subset of (X, τ) such that A ⊆ U and U is π-open in X.

Since every π-open set is open, cl(A) ⊂ U.

As bcl(A) $\subset$ cl(A) $\subset$ U,

int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.22

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.23

Let X = {a, b, c} and τ = { φ, {a, b}, X }. Let A = {a}.

Then A is πgb*-closed but it is not g-closed.

Theorem 4.2.24

Every gp-closed set is πgb*-closed.

Proof

Let A be a gp-closed subset of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Since every π-open set is open, pcl(A) $\subset$ U.

As bcl(A) $\subset$ pcl(A) $\subset$ U, int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.25

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.26

Let X = {a, b, c} and τ = {φ, {a, b}, X}. Let A = {a, b}.

Then A is πgb*-closed but it is not gp-closed.

Theorem 4.2.27

Every gs-closed set is πgb*-closed.

Proof

Let A be a gs-closed subset of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Since every π-open set is open, scl(A) $\subset$ U.

As bcl(A) $\subset$ scl(A) $\subset$ U,

int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.28

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.29

Let X = {a, b, c} and τ = { φ , {a} , {a, b} , {a, c} , X }. Let A = {a, c}.

Then A is πgb*-closed but it is not gs-closed.

Theorem 4.2.30

Every gα-closed set is πgb*-closed.

Proof

Let A be a gα-closed subset of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Since every π-open set is open,

αcl(A) $\subset$ U. As bcl(A) $\subset$ αcl(A) $\subset$ U,

int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.31

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.32

Let X = {a, b} and τ = { φ , {a}, X }. Let A = {a}.

Then A is πgb*-closed but it is not gα-closed.

Theorem 4.2.33

Every gb-closed set is πgb*-closed.

Proof

Let A be a gb-closed subset of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Since every π-open set is open, bcl(A) $\subset$ U.

Thus int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.34

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.35

Let X = {a, b, c} and τ = { φ , {a}, {a, c} , X }. Let A = {a,c}.

Then A is πgb*-closed but it is not gb-closed.

Theorem 4.2.36

Every b*-closed set is πgb*-closed.

Proof

Let A be a b*-closed subset of (X, τ) such that $A \subseteq U$ and U is π-open in X.

Since every π-open set is b-open and A is b*-closed,

$int(bcl(A)) \subset U$.

Hence A is πgb*-closed.

Remark 4.2.37

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.38

Let $X = \{a, b, c\}$ and $\tau = \{ \varphi , \{a\} , \{a, b\} , \{a, c\} , X \}$. Let $A = \{a\}$.

Then A is πgb*-closed but it is not b*-closed.

Theorem 4.2.39

Every πg-closed set is πgb*-closed.

Proof

Let A be a πg-closed subset of (X, τ) such that $A \subseteq U$ and U is π-open in X.

Then $cl(A) \subset U$ and as $bcl(A) \subset cl(A) \subset U$,

$int(bcl(A)) \subseteq int(U) = U$.

Hence A is πgb*-closed.

Remark 4.2.40

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.41

Let $X = \{a, b, c, d\}$ and $\tau = \{ \varphi , \{a\} , \{d\} , \{a, d\} , \{c, d\} , \{a, c, d\} , X \}$.

Let $A = \{c\}$.

Then A is πgb*-closed but it is not πg-closed.

Theorem 4.2.42

Every πgα-closed set is πgb*-closed.

Proof

Let A be a πgα-closed subset of (X, τ) such that $A \subseteq U$ and U is π-open in X.

Then $\alpha cl(A) \subset U$ and as $bcl(A) \subset \alpha cl(A) \subset U$, int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.43

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.44

Let X = {a, b, c, d, e} and τ = { φ , {a, b} , {c, d} , {a, b, c, d} , X }.

Let A = {a}.

Then A is πgb*-closed but it is not πgα-closed.

Theorem 4.2.45

Every πgp-closed set is πgb*-closed.

Proof

Let A be a πgp-closed subset of (X, τ) such that A $\subseteq$ U and U is π-open in X.

Then, pcl(A) $\subset$ U and as bcl(A) $\subset$ pcl(A) $\subset$ U,

int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.46

The converse of the above theorem need not be true as seen from the following example.

Example 4.2.47

Let X = {a, b, c, d, e} and τ = { φ , {a, b} , {c, d} , {a, b, c, d} , X }. Let A = {a, b}.

Then A is πgb*-closed but it is not πgp-closed.

Theorem 4.2.48

Every πgs-closed set is πgb*-closed.

Proof

Let A be a πgs-closed subset of (X, τ)such that A $\subseteq$ U and U is π-open in X.

Then scl(A) $\subset$ U and as bcl(A) $\subset$ scl(A) $\subset$ U,

int(bcl(A)) $\subseteq$ int(U) = U.

Hence A is πgb*-closed.

Remark 4.2.49

The converse of the above theorem need not be true as seen from the following example.

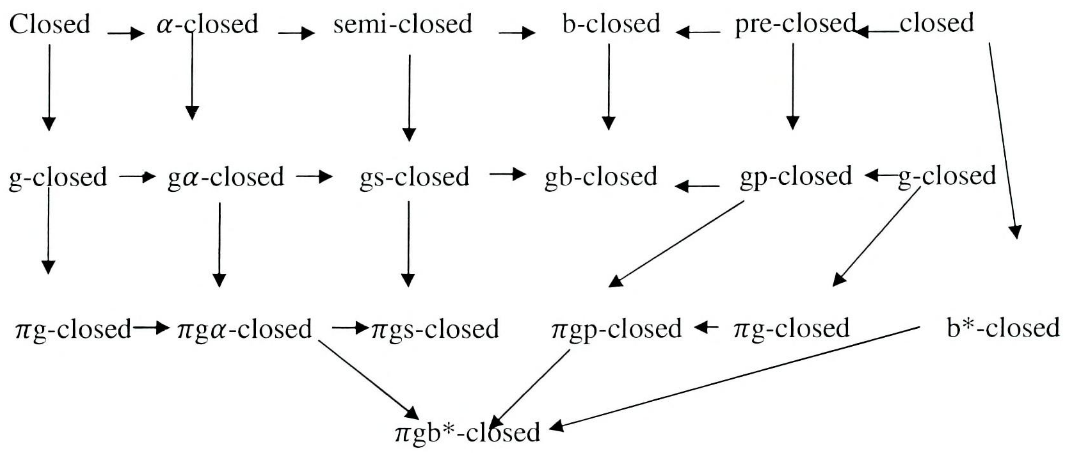
Example 4.2.50

Let X be the real numbers with the usual topology and A be the set of irrational numbers in the interval (0, 2).

Then A is πgb*-closed but it is not πgs-closed.

Remark 4.2.51

From the above results, we have the following implications diagram.



A ⟶ B means A implies B, but not conversely.

Section 4.3

Characterizations of πgb*-closed sets

In this section the characterizations and properties of πgb*-closed sets, πgb*-open sets and πgb*-$T_{1/2}$ spaces were studied.

Theorem 4.3.1

Let (X, τ) be a topological space if $A \subset X$ is πgb*-closed set then int(bcl(A)) – A does not contain any non empty π-closed set.

Proof

Let A be a πgb*-closed set in (X, τ) and $F \subset$ int(bcl(A)) – A such that F is π-closed in X.

Then (X – F) is π-open in X and $A \subseteq (X - F)$.

Since A is πgb*-closed, int(bcl(A)) $\subset$ (X - F) $\Rightarrow F \subset$ (X - int(bcl(A)))

Therefore $F \subset (int(bcl(A)) - A) \cap (X - int(bcl(A))) \Rightarrow F = \varphi$.

Therefore int(bcl(A)) – A does not contain any non empty π-closed set.

Theorem 4.3.2

Let $B \subseteq A \subseteq X$ where A is πgb*-closed and π-open in X, then B is πgb*-closed relative to A if and only if B is πgb*-closed in X.

Proof

Let $B \subseteq A \subseteq X$ where A is a πgb*-closed and π-open set.

Therefore $int(bcl(A)) \subseteq A$.

Since $B \subseteq A$, $int(bcl(B)) \subseteq int(bcl(A)) \subseteq A$.

Let B be πgb*-closed in A and let $B \subseteq U$ where U is π-open in X,

then $B = B \cap A \subset U \cap A$, which is π-open in A.

Therefore $(int(bcl(B)))_A \subset U \cap A$.

Also, $(int(bcl(B)))_A = (int(bcl(B))) \cap A = (int(bcl(B)))$.

Thus $(int(bcl(B))) \subset U \cap A \subset U$.

Hence B is πgb*-closed in X.

Conversely, let B be πgb*-closed in X.

Let $B \subset O$ where O is π-open in A.

Then $O = U \cap A$ where U is π-open in X.

Therefore $B \subset O = U \cap A \subset U$.

Since B is πgb*-closed in X,

$int(bcl(B)) \subset U$.

Hence $(int(bcl(B)))_A = A \cap int(bcl(B)) \subset U \cap A = O$.

Hence B is πgb*-closed relative to A.

Theorem 4.3.3

If A is a πgb*-closed set and B is any set such that $A \subseteq B \subseteq int(bcl(A))$, then B is a πgb*-closed set.

Proof

Let $B \subseteq U$ and U be π-open.

Since $A \subseteq B \subseteq U$ and A is πgb*-closed,

$int(bcl(A)) \subseteq U$.

Now $int(bcl(B)) \subseteq int(bcl(A)) \subseteq U$.

Hence B is a πgb*-closed set.

Theorem 4.3.4

Let (X, τ) be a topological space if $A \subset X$ is nowhere dense then A is πgb*-closed.

Proof

Let $A \subseteq U$ where U is π-open in X.

Since A is nowhere dense, $int(cl(A)) = \varphi$.

Now $int(bcl(A)) \subseteq int(cl(A)) = \varphi \subseteq U$.

Therefore A is πgb*-closed in X.

Theorem 4.3.5

In a topological space(X, τ) for each $x \in X$, $X \setminus \{x\}$ is either πgb*-closed or π-open in X.

Proof

Suppose $X \setminus \{x\}$ is not π-open then X is the only π-open set containing $X\setminus\{x\}$.

Hence $int(bcl(X\setminus\{x\})) \subseteq X \Rightarrow X\setminus \{x\}$ is πgb*-closed.

Definition 4.3.6

A set $A \subseteq X$ is called **πgb*-open** if its complement is πgb*-closed in X.

Theorem 4.3.7

A subset $A \subseteq X$ is πgb*-open if and only if $F \subseteq cl(bint(A))$ whenever F is π-closed and $F \subseteq A$.

Proof

Assume that $A \subseteq X$ is πgb*-open.

Let F be π-closed such that $F \subseteq A$.

Then, $(X - A) \subseteq (X - F)$.

Since $(X - A)$ is πgb*-closed and $(X - F)$ is π-open,

$int(bcl(X - A)) \subseteq (X - F) \Rightarrow (X - cl(bint(A))) \subseteq (X - F)$.

Hence $F \subseteq cl(bint(A))$.

Conversely, assume that F is π-closed and $F \subseteq A$ such that $F \subseteq cl(bint(A))$.

Let $(X - A) \subseteq U$, where U is π-open.

Then $(X - U) \subseteq A$ and since $(X - U)$ is π-closed,

$(X - U) \subseteq cl(bint(A)) \Rightarrow int(bcl(X-A)) \subseteq U$.

Hence $(X - A)$ is πgb*-closed and A is πgb*-open.

Theorem 4.3.8

If $cl(bint(A)) \subseteq B \subseteq A$ and A is πgb*-open, then B is πgb*-open.

Proof

Let F be a π-closed set such that $F \subseteq B$.

Since $B \subseteq A$ we get $F \subseteq A$.

Given A is πgb*-open thus $F \subseteq cl(bint(A)) \subseteq cl(bint(B))$.

Therefore B is πgb*-open.

Definition 4.3.9

A space (X, τ) is called a **πgb*-$T_{1/2}$ space** if every πgb*-closed set is b*-closed.

Theorem 4.3.10

For a topological space (X, τ) the following are equivalent

1) X is πgb*-$T_{1/2}$
2) $\forall$ subset $A \subseteq X$, A is πgb*-open if and only if A is b*- open.

Proof

(1) $\Rightarrow$ (2)

Let $A \subseteq X$ be πgb*-open.

Then (X - A) is πgb*-closed and by (1) (X - A) is b*- closed $\Rightarrow$A is b*-open.

Conversely assume A is b*-open.

Then (X - A) is b*-closed.

As every b*-closed set is πgb*-closed,

(X - A) is πgb*-closed $\Rightarrow$A is πgb*-open.

(2) $\Rightarrow$ (1)

Let A be a πgb*-closed set in X.

Then (X - A) is πgb*-open.

Hence by (2) (X - A) is b*-open $\Rightarrow$A is b*-closed.

Hence X is πgb*-$T_{1/2}$.

Theorem 4.3.11

Let (X, τ) be a πgb^*-$T_{1/2}$ space then every singleton set is either π-closed or b*-open.

Proof

Let $x \in X$ suppose $\{x\}$ is not π-closed.

Then $X - \{x\}$ is not π-open.

Hence $X - \{x\}$ is trivially πgb^*-closed.

Since X is πgb^*-$T_{1/2}$ space,

$X - \{x\}$ is b*-closed $\Rightarrow \{x\}$ is b*-open.

Definition 4.3.12

The intersection of all πgb^*-closed set containing A is called the **πgb*-closure of A** denoted by πgb^*-cl(A).

Theorem 4.3.13

Let $A \subseteq (X, \tau)$ and $x \in X$. Then $x \in \pi gb^*\text{-cl}(A)$ if and only if $V \cap A \neq \varphi$ for every πgb^*- open set V containing x.

Proof

Suppose $x \in \pi gb^*\text{-cl}(A)$ and let V be an πgb^*-open set such that $x \in V$.

Assume $V \cap A = \varphi$,

then $A \subset X\backslash V \Rightarrow \pi gb^*\text{-cl}(A) \subset X\backslash V \Rightarrow x \in X\backslash V$, a contradiction.

Thus $V \cap A \neq \varphi$ for every πgb^*-open set V containing x.

To prove the converse suppose $x \notin \pi gb^*\text{-cl}(A) \Rightarrow x \in X \setminus \pi gb^*\text{-cl}(A) = V$ (say).

Then V is a πgb^*-open and $x \in V$.

Also since $A \subseteq \pi gb^*\text{-cl}(A) \Rightarrow A \not\subset V \Rightarrow V \cap A = \varphi$.

Hence the theorem.

Theorem 4.3.14

For a set $A \subseteq (X, \tau)$ if A is π-clopen then A is π-open, Q-set, πgb^*-closed set.

Proof

Let A be π-clopen.

Then A is both π-open and π-closed.

Hence A is both open and closed

Therefore, cl(int(A)) = int(cl(A)) ,

thus A is a Q-set.

As bcl (A) $\subseteq$ cl(A) = A .

int(bcl (A)) $\subseteq$ int(A) = A.