CHAPTER 2

CHAPTER 2

FUZZY n-NORMED LINEAR SPACE AND QUASI α-n-NORMED LINEAR SPACE

This chapter is devoted to the study of fuzzy n-normed linear space, complete fuzzy n-normed linear space, best approximation sets in α-n-normed space, quasi α-n-normed linear space.

In section one of chapter 2, the concepts of 2-normed space, n-normed space, fuzzy normed linear space, fuzzy n-normed linear space and results on α-n-norms on X corresponding to the fuzzy n-norm on X are studied.

In section two of chapter 2, the notion of Cauchy sequence, convergent sequence and completeness in fuzzy n-normed linear space are analysed.

In section three of chapter 2, examples of α-2-normed linear space and α-n-normed linear space for the sets $D_{x_2,x_3,\ldots,x_n}(x_0, G)$ and $P_{G,x_2,x_3,\ldots,x_n}(x)$ and some characterization for these are discussed.

In section four of chapter 2, ascending family of quasi α-n-norms corresponding to fuzzy quasi n-norm are studied.

SECTION: 2.1

FUZZY n-NORMED LINEAR SPACE

Definition: 2.1.1 [20]

Let X be a real vector space of dimension greater than 1 and let $||\bullet, \bullet||$ be a real-valued function on $X \times X$ satisfying the following conditions :

(1) $||x, y|| = 0$ if and only if x and y are linearly dependent,

(2) $||x, y|| = ||y, x||$,

(3) $||\alpha x, y|| = |\alpha| \, ||x, y||$, where α is real,

(4) $\|x, y+z\| \leq \|x, y\| + \|x, z\|$.

$\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a *linear 2-normed space.*

Definition: 2.1.2 [22]

Let n $\in N$ (natural numbers) and let X be a real vector space of dimension $d \geq n$. (Here we allow d to be infinite.) A real-valued function $\|\bullet, \ldots, \bullet\|$ on $\underbrace{X \times \ldots \times X}_{n}$ satisfying the following four properties :

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2 \ldots, x_n$ are linearly dependent,

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under any permutation,

(3) $\|x_1, x_2, \ldots, \alpha x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$, for any $\alpha \in \Re$(real),

(4) $\|x_1, x_2, \ldots, x_{n-1}, y+z\| \leq \|x_1, x_2, \ldots, x_{n-1}, y\| + \|x_1, x_2, \ldots, x_{n-1}, z\|$,

is called a n-norm on X and the pair $(X, \|\bullet, \ldots, \bullet\|)$ is called an *n-normed space.*

Definition: 2.1.3 [3]

Let X be a linear space over F (field of real or complex numbers). A fuzzy subset N of X$\times\Re$ ($\Re$, set of real numbers) is called a fuzzy norm on X if and only if for all x,u $\in$ X and c$\in$ F,

(N1)For all t $\in \Re$ with t$\leq$ 0, N(x,t)=0,

(N2)For all t $\in \Re$ with t>0, N(x,t)=1 if and only if x=0,

(N3)For all t $\in \Re$ with t>0, N(cx,t)=N(x,t/|c|), if c$\neq$ 0,

(N4)For all s,t$\in \Re$, x,u$\in$X, N(x+u,s+t)> min $\{N(x,s), N(u,t)\}$,

(N5)N(x,t)is a nondecreasing function of $\Re$ and $\lim_{t\to\infty}$N(x,t)=1.

The pair (X,N) will be referred to as a *fuzzy normed linear space.*

Definition: 2.1.4 [3]

Let (X,N) be a fuzzy normed linear space. Assume further that

(N6) $N(x,t) > 0$ for all $t > 0$ implies $x = 0$.

Define $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}, \alpha \in (0,1)$. Then $\|x\|_\alpha$ is a norm on X and $\|\bullet\|_\alpha$ is called a α-norms on X.

Then $\{\|\bullet\|_\alpha : \alpha \in (0,1)\}$ is *an ascending family of norms on X (or) α-norms on X corresponding to the fuzzy norm on X.*

Definition: 2.1.5

Let X be a linear space over a real field F. A fuzzy subset N of $\underbrace{X \times \ldots \times X}_{n} \times \Re$ ($\Re$, set of real numbers) is called a fuzzy n-norm on X if and only if

(N1) For all t$\in \Re$ with t$\leq$0, N$(x_1, x_2, \ldots, x_n, t)$=0,

(N2) For all t$\in \Re$ with t>0, N$(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(N3) N$(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$,

(N4) For all t$\in \Re$ with t>0, N$(x_1, x_2, \ldots, cx_n, t) =$ N$(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$, if c$\neq$0, c$\in$F (field),

(N5) For all s,t$\in \Re$, N$(x_1, x_2, \ldots, x_n+x_n'$,s+t)> $\min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x_n', t)\}$,

(N6) N$(x_1, x_2, \ldots, x_n$,t) is a nondecreasing function of $\Re$ and $\lim_{t\to\infty}$ N$(x_1, x_2, \ldots, x_n$,t)=1.

Then (X,N) is called a *fuzzy n-normed linear space or in short f-n-NLS.*

Remark: 2.1.6

From (N3), if follows that in an f-n-NLS,

(N4) For all t$\in \Re$ with t>0, N$(x_1, x_2, \ldots, cx_i, \ldots, x_n$,t)=N$(x_1, x_2, \ldots, x_i, \ldots, x_n, \frac{t}{|c|})$, if c$\neq$0,

(N5) For all s,t$\in \Re$,

N$(x_1, x_2, \ldots, x_i+x_i', \ldots, x_n$,s+t) > $\min\{N(x_1, x_2, \ldots, x_i, \ldots, x_n, s), N(x_1, x_2, \ldots, x_i', \ldots, x_n, t)\}$

Example: 2.1.7

Let $(X, \|\bullet, \ldots, \bullet\|)$ be an n-normed space. Let

$$N(x_1, x_2, \ldots, x_n, t) = \begin{cases} \dfrac{t}{t + \|x_1, x_2, \ldots, x_n\|}, & \text{when } t > 0, t \in \Re, \\ & (x_1, x_2, \ldots, x_n) \in \underbrace{X \times \ldots, \times X}_{n} \\ 0, & \text{when } t \leq 0 \end{cases}$$

then (X,N)is an f-n-NLS.

Proof:

(N1)From the definition of $N(x_1, x_2, \ldots, x_n, t)$, we have for all t$\in \Re$ with t≤ 0,

$$N(x_1, x_2, \ldots, x_n, t) = 0$$

(N2)For all t $\in \Re$ with t>0,

$$N(x_1, x_2, \ldots, x_n, t) = 1$$

(i)if and only if $\dfrac{t}{t + \|x_1, x_2, \ldots, x_n\|} = 1$,

(ii)if and only if $t = t + \|x_1, x_2, \ldots, x_n\|$,

(iii)if and only if $\|x_1, x_2, \ldots, x_n\| = 0$,

(iv)if and only if $x_1, x_2, \ldots, x_n$are linearly dependent.

(N3)For all t$\in \Re$ with t >0,

$$\begin{aligned} N(x_1, x_2, \ldots, x_{n-1}, x_n, t) &= \frac{t}{t + \|x_1, x_2, \ldots, x_{n-1}, x_n\|} \\ &= \frac{t}{t + \|x_1, x_2, \ldots, x_n, x_{n-1}\|} \\ &= N(x_1, x_2, \ldots, x_n, x_{n-1}, t) \\ &= \ldots \end{aligned}$$

(N4)For all t$\in \Re$ with t>0 and c$\in$ F, c$\neq 0$,

$$N(x_1, x_2, \ldots, x_n, \frac{t}{|c|}) = \frac{\frac{t}{|c|}}{\frac{t}{|c|} + \|x_1, x_2, \ldots, x_{n-1}, x_n\|}$$

$$
\begin{aligned}
&= \frac{t}{t+|c|\|x_1,x_2,\ldots,x_n\|} \\
&= \frac{t}{t+\|x_1,x_2,\ldots,cx_n\|} \\
&= N(x_1,x_2,\ldots,cx_n,t)
\end{aligned}
$$

(N5)We have to prove

$N(x_1,x_2,\ldots,x_n+x_n',s+t) > \min\{N(x_1,x_2,\ldots,x_n,s),N(x_1,x_2,\ldots,x_n',t)\}$

if

$(a)s+t<0$,

$(b)s=t=0$,

$(c)s+t>0; s>0,t<0; s<0,t>0$,then the above relation is true. If

(d)$s>0,t>0,s+t>0$, then

$$N(x_1,x_2,\ldots,x_n+x_n',s+t) = \frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n+x_n'\|}$$

$$\geq \frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x_n'\|}$$

If

$$\frac{s}{s+\|x_1,x_2,\ldots,x_n\|} \geq \frac{t}{t+\|x_1,x_2,\ldots,x_n'\|},$$

then

$$\frac{s}{s+\|x_1,x_2,\ldots,x_n\|} - \frac{t}{t+\|x_1,x_2,\ldots,x_n'\|} \geq 0,$$

which implies

$$s(t+\|x_1,x_2,\ldots,x_n'\|)-t(s+\|x_1,x_2,\ldots,x_n\|) \geq 0,$$

which in turn implies

$$s\|x_1,x_2,\ldots,x_n'\|-t\|x_1,x_2,\ldots,x_n\| \geq 0 \tag{2.1}$$

so

$$\frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x'_n\|}-\frac{t}{t+\|x_1,x_2,\ldots,x'_n\|}$$

$$=\frac{s\|x_1,x_2,\ldots,x'_n\|-t\|x_1,x_2,\ldots,x_n\|}{(s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x'_n\|)(t+\|x_1,x_2,\ldots,x'_n\|)}$$

By (2.1),

$$\frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x'_n\|}-\frac{t}{t+\|x_1,x_2,\ldots,x'_n\|}\geq 0,$$

which implies

$$\frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x'_n\|}\geq\frac{t}{t+\|x_1,x_2,\ldots,x'_n\|}$$

Similarly, if

$$\frac{t}{t+\|x_1,x_2,\ldots,x'_n\|}\geq\frac{s}{s+\|x_1,x_2,\ldots,x_n\|},$$

then

$$\frac{s+t}{s+t+\|x_1,x_2,\ldots,x_n\|+\|x_1,x_2,\ldots,x'_n\|}\geq\frac{s}{s+\|x_1,x_2,\ldots,x_n\|}$$

Thus,

$$N(x_1,x_2,\ldots,x_n+x'_n,s+t)>\min\{N(x_1,x_2,\ldots,x_n,s),N(x_1,x_2,\ldots,x'_n,t)\}$$

(N6) For all $t_1,t_2\in\Re$, if $t_1<t_2\leq 0$, then

$N(x_1,x_2,\ldots,x_n,t_1)=N(x_1,x_2,\ldots,x_n,t_2)=0$

Suppose $t_2>t_1>0$, then

$$\frac{t_2}{t_2+\|x_1,x_2,\ldots,x_n\|}-\frac{t_1}{t_1+\|x_1,x_2,\ldots,x_n\|}$$

$$=\frac{\|x_1,x_2,\ldots,x_n\|(t_2-t_1)}{(t_2+\|x_1,x_2,\ldots,x_n\|)(t_1+\|x_1,x_2,\ldots,x_n\|)}\geq 0,$$

For all $(x_1, x_2, \ldots, x_n) \in \underbrace{X \times \ldots \times X}_{n}$ implies

$$\frac{t_2}{t_2 + \|x_1, x_2, \ldots, x_n\|} \geq \frac{t_1}{t_1 + \|x_1, x_2, \ldots, x_n\|},$$

which in turn implies $N(x_1, x_2, \ldots, x_n, t_2) \geq N(x_1, x_2, \ldots, x_n, t_1)$

Thus $N(x_1, x_2, \ldots, x_n, t)$ is a non decreasing function. Also,

$$\begin{aligned}
\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) &= \lim_{t \to \infty} \frac{t}{t + \|x_1, x_2, \ldots, x_n\|} \\
&= \lim_{t \to \infty} \frac{t}{t(1 + (1/t)\|x_1, x_2, \ldots, x_n\|)} \\
&= 1
\end{aligned}$$

Thus (X,N) is an f-n-NLS.

Proposition: 2.1.8

Let (X,N) be an f-n-NLS. Assume the condition that

(N7) N$(x_1, x_2, \ldots, x_n, t)$ >0 for all t>0 implies $x_1, x_2, \ldots, x_n$ are linearly dependent.

Let $\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$, $\alpha \in$(0,1)

Then $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of n-norms on X. These n-norms are called α-n-norms on X corresponding to the fuzzy n-norm on X.

Proof:

Let N$(x_1, x_2, \ldots, x_n, t)$ >0 for all t>0 implies $x_1, x_2, \ldots, x_n$ are linearly dependent.

Claim: $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-n-norms on X corresponding to the fuzzy n-norm on X.

(1) $\|x_1, x_2, \ldots, x_n\|_\alpha$=0. This

(i) implies, $\inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$=0,

(ii) implies, for all t $\in \Re, t > 0, N(x_1, x_2, \ldots, x_n, t) \geq \alpha > 0, \alpha \in (0,1)$,

(iii) implies, by (N7), $x_1, x_2, \ldots, x_n$ are linearly dependent.

Conversely assume that $x_1, x_2, \ldots, x_n$ are linearly dependent. This

(i) implies, by (N2), N$(x_1, x_2, \ldots, x_n, t) = 1$ for all t > 0,

(ii) implies, for all $\alpha \in (0,1), \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = 0$,

(iii) implies $\|x_1, x_2, \ldots, x_n\|_\alpha = 0$.

(2) As N$(x_1, x_2, \ldots, x_n, t)$is invariant under any permutation, it follows that $\|x_1, x_2, \ldots, x_n\|_\alpha$ is invariant under any permutation.

(3)if c$\neq$ 0,then

$$\begin{aligned} \|x_1, x_2, \ldots, cx_n\|_\alpha &= \inf\{s : N(x_1, x_2, \ldots, cx_n, s) \geq \alpha\} \\ &= \inf\left\{s : N(x_1, x_2, \ldots, x_n, \frac{s}{|c|}) \geq \alpha\right\} \end{aligned}$$

Let t=$\frac{s}{|c|}$, then

$$\begin{aligned} \|x_1, x_2, \ldots, cx_n\|_\alpha &= \inf\{|c|t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} \\ &= |c|\inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} \\ &= |c|\|x_1, x_2, \ldots, x_n\|_\alpha \end{aligned}$$

if c=0 then

$$\begin{aligned} \|x_1, x_2, \ldots, cx_n\|_\alpha &= \|x_1, x_2, \ldots, 0\|_\alpha \\ &= 0 \\ &= 0\|x_1, x_2, \ldots, x_n\|_\alpha \\ &= |c|\|x_1, x_2, \ldots, x_n\|_\alpha, \forall c \in F \end{aligned}$$

(4)$\|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x_n'\|_\alpha$

$$\begin{aligned} &= \inf\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha\} \ + \\ &\quad \inf\{s : N(x_1, x_2, \ldots, x_n', s) > \alpha\} \\ &= \inf\{t + s : N(x_1, x_2, \ldots, x_n, t) > \alpha, N(x_1, x_2, \ldots, x_n', s) > \alpha\} \\ &> \inf\{t + s : N(x_1, x_2, \ldots, x_n + x_n', t + s) > \alpha\} \\ &> \inf\{r : N(x_1, x_2, \ldots, x_n + x_n', r) > \alpha\}, r = t + s \\ &= \|x_1, x_2, \ldots, x_n + x_n'\|_\alpha \end{aligned}$$

Therefore, $\|x_1, x_2, \ldots, x_n + x'_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, +x'_n\|_\alpha$.

Thus $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ is an $\alpha - n-$norm on X.

Let $0 < \alpha_1 < \alpha_2$.Then

$$\|x_1, x_2, \ldots, x_n\|_{\alpha_1} = \inf\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_1\}$$

$$\|x_1, x_2, \ldots, x_n\|_{\alpha_2} = \inf\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_2\}$$

As $\alpha_1 < \alpha_2$,

$$\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_2\} \subset \{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_1\}$$

implies

$$\inf\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_2\} > \inf\{t : N(x_1, x_2, \ldots, x_n, t) > \alpha_1\}$$

which implies

$$\|x_1, x_2, \ldots, x_n\|_{\alpha_2} > \|x_1, x_2, \ldots, x_n\|_{\alpha_1}$$

Hence $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of α-n-norms on X corresponding to the fuzzy n-norm on X.

Remark: 2.1.9

Every fuzzy n-norm induces an ascending family of α-n-norms.

SECTION: 2.2

COMPLETE FUZZY n-NORMED LINEAR SPACE

Definition: 2.2.1[22]

A sequence $\{x_n\}$ in an n-normed linear space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is said to *converge* to an $x \in X$ (in the n-norm) whenever $\lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_n - x\| = 0$.

Definition: 2.2.2[22]

A sequence $\{x_n\}$ in an n-normed linear space $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called a *Cauchy sequence* if $\lim_{n\to\infty} \|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\| = 0$.

Definition: 2.2.3[22]

An n-normed linear space is said to be *complete* if every Cauchy sequence in it is converrgent.

Definition: 2.2.4

A sequence $\{x_n\}$ in a f-n-NLS (X,N) is said to *converge* to X if given r>0, t>0, 0<r<1, there exists an integer $n_0 \in$ N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ >1-r for all $n \geq n_0$.

proposition: 2.2.5

In a f-n-NLS (X,N) a sequence $\{x_n\}$ converges to x if and only if $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.

proof:

Assume that $\{x_n\}$ converges to x.

Claim: $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to$1 as $n \to \infty$.

Fix t > 0. since $\{x_n\}$ converges to x, for a given r with 0 < r < 1, there exists an integer $n_0 \in$ N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) >$ 1-r for all $n \geq n_0$. Thus, 1-$N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) <$ r and hence $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.

Conversly, assume that for each t > 0, $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$

Claim: $\{x_n\}$ converges to x.

For every r, 0 < r < 1, there exists an integer n_0 such that

1-$N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) <$ r for all $n \geq n_0$. Thus $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) >$ 1-r for all $n \geq n_0$. Hence $\{x_n\}$ converges to x in (X,N).

Definition: 2.2.6

A sequence $\{x_n\}$ in a f-n-NLS (X,N) is said to be *Cauchy sequence* if given $\epsilon > 0$ with $0 < \epsilon < 1$, t > 0, there exists an integer $n_0 \in$ N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1-\epsilon$ for all $n, k \geq n_0$.

Definition: 2.2.7

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if $*$ satisfies the following conditions :

(1)$*$ is commutative and associative

(2)$*$ is continuous

(3)$a*1=a$ for all $a \in [0,1]$

(4)$a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

Definition: 2.2.8

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-conorm* if $\diamond$ satisfies the following conditions :

(1)$\diamond$ is commutative and associative

(2)$\diamond$ is continuous

(3)$a \diamond 0=a$ for all $a \in [0,1]$

(4)$a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

Proposition: 2.2.9

In a f-n-NLS (X,N) every convergent sequence is a Cauchy sequence.

Proof:

Assume that $\{x_n\}$ be a convergent sequence in (X,N).

Claim: $\{x_n\}$ be a Cauchy sequence.

Suppose $\{x_n\}$ converges to x. Let t>0 and ϵ ∈(0,1). Choose r∈(0,1) such that $(1-r)*(1-r) > 1-\epsilon$.

Since $\{x_n\}$ converges to x, there is an integer n_0 such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) > 1 - r$.

Now, $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t)$

$$\begin{aligned}
&= N(x_1, x_2, \ldots, x_{n-1}, x_n - x + x - x_k, t) \\
&= N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) \quad * \\
&\quad\; N(x_1, x_2, \ldots, x_{n-1}, x - x_k, \frac{t}{2}) \\
&\geq 1 - r) * (1 - r) \text{ for all } n, k \geq n_0 \\
&> 1 - \epsilon \text{ for all } n, k \geq n_0
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in (X,N).

Definition: 2.2.10

A f-n-NLS is said to be *complete* if every Cauchy sequence in it is convergent.

The following example shows that there may exists Cauchy sequence in a f-n-NLS which is not convergent.

Example: 2.2.11

Let $(X, \|\bullet, \bullet \ldots, \bullet\|)$ be an n-normed linear space and let $a * b = \min\{a, b\}$ for all a,b$\in [0, 1]$ and

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then (X,N)is a f-n-NLS.

Let $\{x_n\}$ be a sequence in f-n-NLS,then

(a)$\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$ if and only if $\{x_n\}$ is a Cauchy sequence in (X,N).

(b)$\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$ if and only if $\{x_n\}$ is a convergent sequence in (X,N).

Proof:

Assume that $\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

Claim: $\{x_n\}$ is a Cauchy sequence in (X, N).

(a)$\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

$$\Leftrightarrow \lim_{n,k \to \infty} \|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\| = 0$$

$$\Leftrightarrow \lim_{n,k \to \infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t)$$

$$= \lim_{n,k \to \infty} \frac{t}{t + \|x_1, x_2, \ldots, x_n - x_k\|} = 1$$

$$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \to 1 \text{ as } n \to \infty.$$

$$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1 - r, \text{for all } n, k \geq n_0.$$

$\Leftrightarrow$ $\{x_n\}$ is a Cauchy sequence in (X,N).

(b) Assume that $\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

Claim: $\{x_n\}$ is a convergent sequence in (X, N)

$\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

$$\Leftrightarrow \lim_{n \to \infty} \|(x_1, x_2, \ldots, x_{n-1}, x_n - x)\| = 0$$

$$\Leftrightarrow \lim_{n\to\infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$$
$$= \lim_{n\to\infty} \frac{t}{t + \|x_1, x_2, \ldots, x_{n-1}, x_n - x\|} = 1$$
$$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1 \text{ as } n \to \infty.$$
$$\Leftrightarrow N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r, \text{for all } n \geq n_0.$$

$\Leftrightarrow \{x_n\}$ is a convergent sequence in (X,N).

Thus if there exists an n-normed linear space $(X, \|\bullet, \bullet \ldots, \bullet\|)$ which is not complete, then the fuzzy n-normed induced by such a crisp n-norm $\|\bullet, \bullet \ldots, \bullet\|$ on an incomplete n-norm linear space X is an incomplete fuzzy n-normed linear space.

Proposition: 2.2.12

A f-n-NLS (X,N) in which every Cauchy sequence has a convergent subsequence is complete.

Proof:

Assume $\{x_n\}$ be a Cauchy sequence in (X,N) and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x.

Claim: $\{x_n\}$ converges to x and it is complete.

Let t>0 and $\epsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) > 1 - \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, \frac{t}{2}) > 1 - r$ for all $n, k \geq n_0$. Since $\{x_{n_k}\}$ converges to x, there is a positive integer $i_k > n_0$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) > 1 - r$

Now,

$$\begin{aligned} N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \tfrac{t}{2}) &\\ &= N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, \tfrac{t}{2} + \tfrac{t}{2}) \\ &\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, \tfrac{t}{2}) \; * \\ &\quad N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \tfrac{t}{2}) \\ &> (1 - r) * (1 - r) \end{aligned}$$

$$> \quad 1-\epsilon$$

Therefore $\{x_n\}$ converges to x in (X,N) and hence it is complete.

SECTION:2.3

BEST APPROXIMATION SETS IN α-n-NORMED SPACE

Using α-n-norm on X, the author have introduced the notion of two subsets of X, namely $D_{x_2,x_3,\ldots,x_n}(x_0, G)$ and $P_{G,x_2,x_3,\ldots,x_n}(x)$.

Definition: 2.3.1

Let $(X, \|\bullet, \bullet\|)$ be a linear 2-normed space and let G be an arbitary nonempty subset of X and $x_0 \in$ X. Then, for every $x \in$ X and for every z $\in X \setminus G$ which is independent of x and x_0,

$$d_z(x, G) \leq \|x - x_0, z\| + d_z(x_0, G), \text{where } d_z(x, G) = \inf_{g \in G} \|x - g, z\|.$$

For each $G \subset X$and $x_0 \in X$,let

$D_z(x_0, G) = \{x \in X : d_z(x, G) = \|x - x_0, z\| + d_z(x_0, G)\}$ for any z$\in X \setminus G$ which is independent of x and x_0.

Also

$$P_{G,z}(x) = \{g_0 \in G : \|x - g_0, z\| = d_z(x, G)\} \text{ and}$$

$$P_{G,z}^{-1}(x_0) = \{x \in X : \|x - x_0, z\| = d_z(x, G)\}, \text{where } x_0 \in G.$$

Definition: 2.3.2

Let $(X, \|\bullet, \bullet, \ldots, \bullet\|_\alpha)$ be an α-n-normed space corresponding to the fuzzy n-norm N on X. Let G be an arbitray nonempty subset of X and $x_0 \in X$. Then for every $x \in X$ and for every $x_2, x_3, \ldots, x_n \in X \setminus G$ which is independent of x and x_0.

$$d_{x_2,x_3,\ldots,x_n}(x, G) < \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \tag{2.2}$$

where

$$d_{x_2,x_3,\ldots,x_n}(x,G) = \inf_{g\in G} \|x-g, x_2, x_3, \ldots, x_n\|_\alpha. \tag{2.3}$$

For each $G \subset X$ and $x_0 \in X$, we define

$D_{x_2,x_3,\ldots,x_n}(x_0,G)$

$$= \{x \in X : d_{x_2,x_3,\ldots,x_n}(x,G) = \|x-x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0,G)\} \tag{2.4}$$

for any $x_2, x_3, \ldots, x_n \in X \setminus G$ which is independent of x and x_0.

We denote

$$P_{G,x_2,x_3,\ldots,x_n}(x) = \{g_0 \in G : \|x-g_0, x_2, x_3, \ldots, x_n\|_\alpha = d_{x_2,x_3,\ldots,x_n}(x,G)\} \tag{2.5}$$

$$P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0) = \{x \in X : \|x-x_0, x_2, x_3, \ldots, x_n\|_\alpha = d_{x_2,x_3,\ldots,x_n}(x,G)\} \tag{2.6}$$

where $x_0 \in G$.

Example: 2.3.3 [$D_{x_2}(x_0,G)$ and $P_{G,x_2}(x)$ sets in the α-2-normed linear spaces]

Let X=$\Re^3$ be a linear space over $\Re$.

Let $\|\bullet,\bullet\| : X \times X \to \Re$ by

$$\|x_1,x_2\|_1 = \max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\}$$

$$\|x_1,x_2\|_2 = \frac{1}{2}\{\max\{|a_1b_2 - a_2b_1|, |b_1c_2 - b_2c_1|, |a_1c_2 - a_2c_1|\}\}$$

where $x_i = (a_i, b_i, c_i) \in \Re^3, i = 1, 2$. Then $(X, \|\bullet,\bullet\|_1)$ and $(X, \|\bullet,\bullet\|_2)$ are 2-normed linear spaces.

Let $N : X \times X \times \Re \to [0,1]$ by

$$N(x_1,x_2,t) = \begin{cases} 1, & \text{if } t > \|x_1,x_2\|_1 \\ 0.5, & \text{if} \|x_1,x_2\|_2 < t \leq \|x_1,x_2\|_1, \\ 0, & \text{if } t \leq \|x_1,x_2\|_2. \end{cases}$$

Then (X,N) is a fuzzy 2-normed linear space.

Let $\|x_1,x_2\|_\alpha = \inf\{t : N(x_1,x_2,t) \geq \alpha\}, \quad \alpha \in (0,1).$

The α-2-norms are given by

$$\begin{aligned}\|x_1, x_2\|_\alpha &= \|x_1, x_2\|_1, \text{ when } 1 > \alpha > 0.5,\\ &= \|x_1, x_2\|_2, \text{ when} 0 < \alpha \leq 0.5.\end{aligned}$$

Let $G = \{(a, 0, 0) : a \in \Re\}$ be a subset of X. Choose $x_0 = (0, 1, 1)$ and

$x_2 \in K = \{(0, 0, k) : k \in \Re \setminus \{0\}\}$

Then

$D_{x_2}(x_0, G) = \{x = (0, b, 0), b \in \Re^+ \setminus 0 : d_{x_2}(x, G) = \|x - x_0, x_2\|_\alpha + d_{x_2}(x_0, G)\}$,

$P_{G,x_2}(x) = \{g_0 \ = \ (a, 0, 0) \ : \ -1 \leq a \leq 1\}$.

Example: 2.3.4 [$D_{x_2,x_3,\ldots,x_n}(x_0, G)$ and $P_{G,x_2,x_3,\ldots,x_n}(x)$ sets in the α-n-normed linear spaces]

Let X=$\Re^{n+1}$ be a linear space over $\Re$. Let $\|\bullet, \bullet, \ldots, \bullet\| : \underbrace{X \times X \ldots \times X}_{n} \to \Re$ by

$$\begin{aligned}\|x_1, x_2, \ldots, x_n\|_1 &= \max\{\Delta_1, \Delta_2, \ldots, \Delta_n\}\\ \|x_1, x_2 \ldots, x_n\|_2 &= \frac{1}{2}\{\max\{\Delta_1, \Delta_2, \ldots, \Delta_n\}\}\end{aligned}$$

where

$$\Delta_1 = \begin{vmatrix} a_{12} \; a_{13} \; \cdots \; a_{1(n+1)} \\ \vdots \\ a_{n2} \; a_{n3} \; \cdots \; a_{n(n+1)} \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_{13} \; \cdots \; a_{1(n+1)} \; a_{11} \\ \vdots \\ a_{n3} \; \cdots \; a_{n(n+1)} \; a_{n1} \end{vmatrix}$$

$$\vdots$$

$$\Delta_n = \begin{vmatrix} a_{11} \; a_{12} \; \cdots \; a_{1n} \\ \vdots \\ a_{n1} \; a_{n2} \; \cdots \; a_{nn} \end{vmatrix}$$

$$\text{and } x_i = (a_{i1}, a_{i2}, \dots, a_{i(n+1)}) \in \Re^{n+1}, \quad i = 1, 2, \dots, n.$$

Then $(X, \|\bullet, \bullet, \dots, \bullet\|_1)$ and $(X, \|\bullet, \bullet, \dots, \bullet\|_2)$ are n-normed linear spaces.

Let $N : X \times X \times \dots \times X \times \Re \to [0, 1]$ by

$$N(x_1, x_2 \dots, x_n, t) = \begin{cases} 1, & \text{if } t > \|x_1, x_2, \dots, x_n\|_1 \\ 0.5, & \text{if} \|x_1, x_2, \dots, x_n\|_2 < t \leq \|x_1, x_2 \dots, x_n\|_1, \\ 0, & \text{if } t \leq \|x_1, x_2, \dots, x_n\|_2. \end{cases}$$

Then (X,N) is a fuzzy n-normed linear space. Let

$$\|x_1, x_2, \dots, x_n\|_\alpha = \inf \{t : N(x_1, x_2, \dots, x_n, t) \geq \alpha\}, \quad \alpha \in (0, 1).$$

The α-n-norms are given by

$$\begin{aligned} \|x_1, x_2, \dots, x_n\|_\alpha &= \|x_1, x_2, \dots, x_n\|_1, \quad \text{when } 1 > \alpha > 0.5, \\ &= \|x_1, x_2, \dots, x_n\|_2, \quad \text{when } 0 < \alpha < 0.5 \end{aligned}$$

Let $G = \{(a, 0, 0, \dots \text{ n times } 0) : a \in \Re\}$ be a subset of X.

Choose $x_0 = (0, 1, 1, \dots, \text{n times } 1)$ and

$x_2, \dots, x_n \in K = \left\{(0, 0, k_3^{(i)}, \dots, k_{n+1}^{(i)}) : k_3^{(i)}, \dots, k_{n+1}^{(i)} \in \Re \setminus \{0\})\right\}$

That is,

$$\begin{aligned} x_2 &= (0, 0, k_3^{(2)}, \dots, k_{n+1}^{(2)}) \\ x_3 &= (0, 0, k_3^{(3)}, \dots, k_{n+1}^{(3)}) \\ &\vdots \\ x_n &= (0, 0, k_3^{(n)}, \dots, k_{n+1}^{(n)}) \end{aligned}$$

Then

$D_{x_2, x_3, \dots, x_n}(x_0, G)$

$$= \left\{ \begin{array}{c} x = (0, b, 0, \dots, (n-1)\text{times } 0), b \in \Re^+ \setminus \{0\} : \\ d_{x_2, x_3, \dots, x_n}(x, G) = \|x - x_0, x_2, \dots, x_n\|_\alpha + d_{x_2, x_3, x_n}(x_0, G) \end{array} \right\},$$

$$\text{where } d_{x_2,x_3,\ldots,x_n}(x,G) = \max\{|b|\Delta, |a|\Delta\}$$

$$\Delta_n = \begin{vmatrix} k_3^{(2)} & k_4^{(2)} & \ldots & k_{n+1}^{(2)} \\ k_3^{(3)} & k_4^{(3)} & \ldots & k_{n+1}^{(3)} \\ & \vdots & & \\ k_3^{(n)} & k_4^{(n)} & \ldots & k_{n+1}^{(n)} \end{vmatrix}$$

$$\|x - x_0, x_2, \ldots, x_n\|_\alpha = |b-1|\Delta,$$
$$d_{x_2,x_3,\ldots,x_n}(x_0,G) = \max\{\Delta, |a|\Delta\}.$$

and also, $P_{G,x_2,x_3,\ldots,x_n}(x) = \{g' = (a, 0, \ldots, \text{n times } 0) : -1 \leq a \leq 1\}$.

Theorem: 2.3.5

For $x \in D_{x_2,x_3,\ldots,x_n}(x_0,G)$ and $y \in D_{x_2,x_3,\ldots,x_n}(x,G)$

$(i)\|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha$

$(ii) y - x + x_0 \in D_{x_2,x_3,\ldots,x_n}(x_0,G)$.

Proof:

(i)Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0,G)$ and $y \in D_{x_2,x_3,\ldots,x_n}(x,G)$.

Claim: $\|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha$

By (2.4) we have,

$d_{x_2,x_3,\ldots,x_n}(x,G) = \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0,G)$,

$d_{x_2,x_3,\ldots,x_n}(y,G) = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x,G)$

Consider

$$\begin{aligned} &\|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha \\ &= \|y - x_0 - x + x, x_2, x_3, \ldots, x_n\|_\alpha \\ &= \|(y-x) + (x - x_0), x_2, x_3, \ldots, x_n\|_\alpha \end{aligned}$$

$$
\begin{aligned}
&< \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + \\
&\quad \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha \\
&= (d_{x_2,x_3,\ldots,x_n}(y, G) - d_{x_2,x_3,\ldots,x_n}(x, G)) + \\
&\quad (d_{x_2,x_3,\ldots,x_n}(x, G) - d_{x_2,x_3,\ldots,x_n}(x_0, G)) \\
&= d_{x_2,x_3,\ldots,x_n}(y, G) - d_{x_2,x_3,\ldots,x_n}(x_0, G) \\
&< \|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha
\end{aligned}
$$

Therefore,

$\|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha$

(ii)Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$ and $y \in D_{x_2,x_3,\ldots,x_n}(x, G)$.

Claim: $y - x + x_0 \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$

By (2.3) we have,

$d_{x_2,x_3,\ldots,x_n}(y - x + x_0, G)$

$$
\begin{aligned}
&> d_{x_2,x_3,\ldots,x_n}(y, G) - \|y - (y - x + x_0), x_2, x_3, \ldots, x_n\|_\alpha \\
&= d_{x_2,x_3,\ldots,x_n}(y, G) - \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha \\
&= (\|y - x, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x, G)) - \\
&\quad \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha \\
&= \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)) - \\
&\quad \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha \\
&= \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \\
&= \|(y - x + x_0) - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)
\end{aligned}
$$

Again by (2.3), it follows that

$d_{x_2,x_3,\ldots,x_n}(y - x + x_0, G) = \|(y - x + x_0) - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)$

which implies $y - x + x_0 \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Theorem: 2.3.6

Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$. Then

$(i)\,[x_0, x] = \{\lambda x_0 + (1-\lambda)x : 0 \le \lambda \le 1\} \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$,

$(ii) D_{x_2,x_3,\ldots,x_n}(x, G) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Proof:

Let $y = \lambda x_0 + (1-\lambda)x$ such that $0 \le \lambda \le 1$.

Claim: $[x_0, x] = \{\lambda x_0 + (1-\lambda)x : 0 \le \lambda \le 1\} \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

$$
\begin{aligned}
& d_{x_2,x_3,\ldots,x_n}(y, G) \\
& \ge d_{x_2,x_3,\ldots,x_n}(x, G) - \|x - y, x_2, x_3, \ldots, x_n\|_\alpha \\
& = \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) - \|x - y, x_2, x_3, \ldots, x_n\|_\alpha \\
& = \|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)
\end{aligned}
$$

By (2.3) we have,

$d_{x_2,x_3,\ldots,x_n}(y, G) = \|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)$

which implies $y \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$

(ii) Let $y \in D_{x_2,x_3,\ldots,x_n}(x, G)$.

Claim: $D_{x_2,x_3,\ldots,x_n}(x, G) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Then by (2.4) and theorem (2.3.5(i))

$$
\begin{aligned}
& d_{x_2,x_3,\ldots,x_n}(y, G) \\
& = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x, G) \\
& = \|y - x, x_2, x_3, \ldots, x_n\|_\alpha + (\|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)) \\
& = \|y - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)
\end{aligned}
$$

which implies $y \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Therefore, $D_{x_2,x_3,\ldots,x_n}(x, G) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Theorem: 2.3.7

Let $x_0, y_0 \in X$ and $\lambda \neq 0$. Then

$(i) D_{x_2,x_3,\ldots,x_n}(x_0, G) + y_0 = D_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0)$,

$(ii) D_{x_2,x_3,\ldots,x_n}(x_0, \lambda G) = \lambda D_{x_2,x_3,\ldots,x_n}(x_0/\lambda, G)$.

Proof:

(i) Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$.

Claim: $D_{x_2,x_3,\ldots,x_n}(x_0, G) + y_0 = D_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0)$

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(x + y_0, G + y_0) &= d_{x_2,x_3,\ldots,x_n}(x, G) \\ &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \\ &= \|x + y_0 - (x_0 + y_0), x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \end{aligned}$$

Therefore, $x + y_0 \in D_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0)$

Conversely, let $y \in D_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0)$. Then

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(y - y_0, G) &= d_{x_2,x_3,\ldots,x_n}(y, G + y_0) \\ &= \|y - y_0 - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0) \\ &= \|(y - y_0) - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \end{aligned}$$

Therefore, $y - y_0 \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$, and so

$D_{x_2,x_3,\ldots,x_n}(x_0, G) + y_0 = D_{x_2,x_3,\ldots,x_n}(x_0 + y_0, G + y_0)$

(ii)Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, \lambda G)$.

Claim: $D_{x_2,x_3,\ldots,x_n}(x_0, \lambda G) = \lambda D_{x_2,x_3,\ldots,x_n}(x_0/\lambda, G)$

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(\frac{x}{\lambda}, G) &= \frac{1}{|\lambda|} d_{x_2,x_3,\ldots,x_n}(x, \lambda G) \\ &= \frac{1}{|\lambda|}(\|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, \lambda G)) \\ &= \|(\frac{x}{\lambda} - \frac{x_0}{\lambda}), x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(\frac{x_0}{\lambda}, G). \end{aligned}$$

Therefore, $\frac{x}{\lambda} \in D_{x_2,x_3,\ldots,x_n}(x_0/\lambda, G)$

Conversly, let $x \in D_{x_2,x_3,\ldots,x_n}(x_0/\lambda, G)$. Then

$d_{x_2,x_3,\ldots,x_n}(\lambda x, \lambda G)$

$$\begin{aligned} &= |\lambda| d_{x_2,x_3,\ldots,x_n}(x, G) \\ &= |\lambda|(\|x - \frac{x_0}{\lambda}, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(\frac{x_0}{\lambda}, G)) \\ &= \|\lambda x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, \lambda G) \end{aligned}$$

Therefore, $\lambda x \in D_{x_2,x_3,\ldots,x_n}(x_0, \lambda G)$

Thus, $D_{x_2,x_3,\ldots,x_n}(x_0, \lambda G) = \lambda D_{x_2,x_3,\ldots,x_n}(x_0/\lambda, G)$

Theorem: 2.3.8

Let G$\subset G_1$ and $x_0 \in X$, where $G_1$is a subset of X such that

$$d_{x_2,x_3,\ldots,x_n}(x_0, G) = d_{x_2,x_3,\ldots,x_n}(x_0, G_1) \tag{2.7}$$

Then $D_{x_2,x_3,\ldots,x_n}(x_0, G_1) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Proof:

Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G_1)$.

Claim: $D_{x_2,x_3,\ldots,x_n}(x_0, G_1) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Then by (2.7) we have

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(x, G) &> d_{x_2,x_3,\ldots,x_n}(x, G_1) \\ &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G_1) \\ &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \end{aligned}$$

By (2.3), it following that

$d_{x_2,x_3,\ldots,x_n}(x, G) = \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)$

which implies $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$.

Hence $D_{x_2,x_3,\ldots,x_n}(x_0, G_1) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G)$

Theorem: 2.3.9

(i)$P_{G,x_2,x_3,\ldots,x_n}(x_0) \subset P_{G,x_2,x_3,\ldots,x_n}(x)$ for every $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$,

(ii)$D_{x_2,x_3,\ldots,x_n}(x_0, G) = P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0)$ for every $x_0 \in \overline{G}$.

Proof:

(i) Let $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$ and $g \in P_{G,x_2,x_3,\ldots,x_n}(x_0)$.

Claim: $P_{G,x_2,x_3,\ldots,x_n}(x_0) \subset P_{G,x_2,x_3,\ldots,x_n}(x)$.

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(x, G) &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + \|x_0 - g_0, x_2, x_3, \ldots, x_n\|_\alpha \end{aligned}$$

By thm(2.3.5(i)) we have

$d_{x_2,x_3,\ldots,x_n}(x, G) = \|x - g_0, x_2, x_3, \ldots, x_n\|_\alpha$

which implies $g_0 \in P_{G,x_2,x_3,\ldots,x_n}(x)$,which in turn implies

$P_{G,x_2,x_3,\ldots,x_n}(x_0) \subset P_{G,x_2,x_3,\ldots,x_n}(x)$.

(ii)Let $x_0 \in \overline{G}$ and $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$.

Claim: $D_{x_2,x_3,\ldots,x_n}(x_0, G) = P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0)$.

$$\begin{aligned} d_{x_2,x_3,\ldots,x_n}(x, G) &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G) \\ &= \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha, \text{ where } x_0 \in \overline{G}. \end{aligned}$$

which implies $x \in P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0)$. So

$$D_{x_2,x_3,\ldots,x_n}(x_0, G) \subset P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0) \tag{2.8}$$

. conversly, let $x \in P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0)$.

Then $x_0 \in P_{G,x_2,x_3,\ldots,x_n}(x)$.

Since $x_0 \in \overline{G}, d_{x_2,x_3,\ldots,x_n}(x_0, G) = 0$

Hence we have

$d_{x_2,x_3,\ldots,x_n}(x, G) = \|x - x_0, x_2, x_3, \ldots, x_n\|_\alpha + d_{x_2,x_3,\ldots,x_n}(x_0, G)$.

which implies $x \in D_{x_2,x_3,\ldots,x_n}(x_0, G)$, which in turn implies

$$P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0) \subset D_{x_2,x_3,\ldots,x_n}(x_0, G) \tag{2.9}$$

From (2.8) and (2.9) we have

$D_{x_2,x_3,\ldots,x_n}(x_0, G) = P^{-1}_{G,x_2,x_3,\ldots,x_n}(x_0)$

SECTION: 2.4

QUASI α-n-NORMED LINEAR SPACE

Definition: 2.4.1

In the definition (2.1.2) if we replace (3) by (3') $\|x_1, x_2, \ldots, ax_n\| = |a|^p \|x_1, x_2 \ldots, x_n\|$, for any a$\in \Re$ (real) and $0 \leq p < 1$, then $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ is called a *quasi n-normed linear space.*

Definition: 2.4.2

In the definition (2.1.5), if we replace (N4) by (N4') for all t$\in \Re$ with t>0 $N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, cx_n, \frac{t}{|c|^p})$ if c$\neq$0, c$\in$F (field), $0 \leq p < 1$. Then (X,N) is called a *fuzzy quasi n-normed linear space or in short f-q-n-NLS.*

Theorem: 2.4.3

Let (X,N) be a f-q-n-NLS. Assume the condition that

(N7) $N(x_1, x_2, \ldots, x_n, t) > 0$ for all t>0 implies $x_1, x_2, \ldots, x_n$ are linearly dependent.

Let $\|x_1, x_2, \ldots, x_n\|_\alpha = \inf \{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}, \alpha \in (0, 1)$.

Then $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$, is an ascending family of quasi n-norms on X. We call these quasi n-norms as quasi α-n-norms on X corresponding to the fuzzy quasi n-norm on X.

Proof:

Assume the condition that (N7) $N(x_1, x_2, \ldots, x_n, t) > 0$ for all t>0 implies $x_1, x_2, \ldots, x_n$ are linearly dependent. Let $\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Claim: $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$, is an ascending family of quasi α-n-norms on X corresponding to the fuzzy quasi n-norm on X.

(1)

$$\begin{aligned} \|x_1, x_2, \ldots, x_n\|_\alpha &= 0 \\ \Rightarrow \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} &= 0 \end{aligned}$$

$\Rightarrow$ For all t$\in \Re$, t>0, $N(x_1, x_2, \ldots, x_n, t) \geq \alpha > 0,\ \alpha \in (0, 1)$.

$\Rightarrow$ By (N7) $x_1, x_2, \ldots, x_n$ are linearly dependent.

Conversly assume that $x_1, x_2, \ldots, x_n$ are linearly dependent.

$\Rightarrow$ By (N2) $N(x_1, x_2, \ldots, x_n, t) = 1$ for all t>0.

$\Rightarrow$ For all $\alpha \in (0, 1), \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} = 0$.

$\Rightarrow \|x_1, x_2, \ldots, x_n\|_\alpha = 0$.

(2) As $N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation it follows that $\|x_1, x_2, \ldots, x_n\|_\alpha$ is invariant under any permutation.

(3')For all c$\in$F, $0 \leq p < 1$, then,

$$\begin{aligned} \|x_1, x_2, \ldots, cx_n\|_\alpha &= \inf\{s : N(x_1, x_2, \ldots, cx_n, s) \geq \alpha\} \\ &= \inf\left\{s : N(x_1, x_2, \ldots, x_n, \frac{s}{|c|^p}) \geq \alpha\right\} \end{aligned}$$

Let $t = \dfrac{s}{|c|^p}$ then,

$$\begin{aligned} \|x_1, x_2, \ldots, cx_n\|_\alpha &= \inf\{|c|^p : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} \\ &= |c|^p \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} \\ &= |c|^p \|x_1, x_2, \ldots, x_n\|_\alpha. \end{aligned}$$

(4)$\|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x'_n\|_\alpha$

$$\begin{aligned} &= \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\} + \inf\{s : N(x_1, x_2, \ldots, x'_n, s) \geq \alpha\} \\ &= \inf\{t + s : N(x_1, x_2, \ldots, x_n, t) \geq \alpha,\ N(x_1, x_2, \ldots, x'_n, s) \geq \alpha\} \\ &\geq \inf\{t + s : N(x_1, x_2, \ldots, x_n + x'_n, t + s) \geq \alpha\} \\ &\geq \inf\{r : N(x_1, x_2, \ldots, x_n + x'_n, r) \geq \alpha\},\ r = t + s \\ &= \|x_1, x_2, \ldots, x_n + x'_n\|_\alpha \end{aligned}$$

Therefore, $\|x_1, x_2, \ldots, x_n + x'_n\|_\alpha \leq \|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x'_n\|_\alpha$.

Thus $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$, is a quasi α-n-norms on X.

Let $0 < \alpha_1 < \alpha_2$. Then

$$\begin{aligned} \|x_1, x_2, \ldots, x_n\|_{\alpha_1} &= \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\} \\ \|x_1, x_2, \ldots, x_n\|_{\alpha_2} &= \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\} \end{aligned}$$

As $\alpha_1 < \alpha_2$

$\Rightarrow \{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\} \subset \{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\}$

$\Rightarrow \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_2\} \geq \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha_1\}$

$\|x_1, x_2, \ldots, x_n\|_{\alpha_2} \geq \|x_1, x_2, \ldots, x_n\|_{\alpha_1}$.

Hence $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$, is an ascending family of quasi α-n-norms on X corresponding to the fuzzy quasi n-norm on X.

Remark: 2.4.4

Every fuzzy quasi n-norm induces an ascending family of quasi $\alpha-$n-norms.

Theorem: 2.4.5

Let $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of quasi n-norms corresponding to (X,N). Let $N' : X^n \times \Re \to [0, 1]$ be defined as,

$$N'(x_1,x_2,\ldots,x_n,t)=\begin{cases}\sup\{\alpha\in(0,1):\|x_1,x_2,\ldots,x_n\|_\alpha\le t\}, & \text{when}\\ & x_1,x_2,\ldots,x_n \text{ are linearly independent, } t\neq 0.\\ 0, & \text{otherwise.}\end{cases}$$

Then (X,N') is a f-q-n-NLS.

Proof:

Let $\{\|\bullet,\bullet,\ldots,\bullet\|_\alpha:\alpha\in(0,1)\}$ be an ascending family of quasi n-norms corresponding to (X,N).

Claim: (X,N') is a f-q-n-NLS.

(N1)For all t∈ $\Re$ with t<0

$$\begin{aligned}N'(x_1,x_2,\ldots,x_n,t) &= 0, \text{ for all}(x_1,x_2,\ldots,x_n)\in X^n, \text{ as}\\ \{\alpha:\|x_1,x_2,\ldots,x_n\|_\alpha\le t\} &= \phi \text{ when} t<0.\end{aligned}$$

For t=0 and $x_1,x_2,\ldots,x_n$ are linearly independent,

$$\begin{aligned}\{\alpha:\|x_1,x_2,\ldots,x_n\|_\alpha\le t\} &= \phi.\\ \Rightarrow N'(x_1,x_2,\ldots,x_n,t) &= 0\end{aligned}$$

When $x_1,x_2,\ldots,x_n$ are linearly dependent and t=0 then

$$N'(x_1,x_2,\ldots,x_n,t) = 0.$$

Thus for all t∈ $\Re$ with t≤0, $N'(x_1,x_2,\ldots,x_n,t)=0$.

(N2) Let $N'(x_1,x_2,\ldots,x_n,t)=1$.

(i.e.,) for t>0, $N'(x_1,x_2,\ldots,x_n,t)=1$.

Choose any $\epsilon\in(0,1)$. Then for t>0, there exists $\alpha_t\in(\epsilon,1]$ such that

$$\begin{aligned}\|x_1,x_2,\ldots,x_n\|_{\alpha t} &\le t\\ \text{and hence } \|x_1,x_2,\ldots,x_n\|_\epsilon &\le t.\end{aligned}$$

Since t>0 is arbitary, this implies that

$$\|x_1,x_2,\ldots,x_n\|_\epsilon = 0.$$

$\Rightarrow x_1, x_2, \ldots, x_n$ are linearly dependent.

Conversely, if $x_1, x_2, \ldots, x_n$ are linearly dependent, then for t>0,

$$\begin{aligned} N'(x_1, x_2, \ldots, x_n, t) &= \sup\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\} \\ &= \sup\{\alpha : \alpha \in (0,1)\} \\ &= 1. \end{aligned}$$

Thus for all t>0, $N'(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

(N3)As $\|x_1, x_2, \ldots, x_n\|_\alpha$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$ we have $N'(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$.

(N4)' For all t$\in \Re$ with t>0 , c$\in$F, $0 \leq p < 1$,

$$\begin{aligned} N'(x_1, x_2, \ldots, cx_n, t) &= \sup\{\alpha : \|x_1, x_2, \ldots, cx_n\|_\alpha \leq t\} \\ &= \sup\left\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq \frac{t}{|c|^p}\right\} \\ &= N'(x_1, x_2, \ldots, x_n, \frac{t}{|c|^p}). \end{aligned}$$

(N5)We have to prove that for all s,t$\in \Re$,

$N'(x_1, x_2, \ldots, x_n + x_n', s + t) \geq \min\{N'(x_1, x_2, \ldots, x_n, s), N'(x_1, x_2, \ldots, x_n, t)\}$.

if (a) s+t<0

(b) s=t=0

(c) s+t>0; s>0,t<0; s<0,t>0,

then in these cases the relation is obvious. if

(d) s>0,t>0, let p= $N'(x_1, x_2, \ldots, x_n, s)$, q=$N'(x_1, x_2, \ldots, x_n, t)$ and p$\leq$q.

If p=0 and q=0 then obviously (N5) holds.

Let $0 < r < p \leq q$. Then there exists $\alpha > r$ such that $\|x_1, x_2, \ldots, x_n\|_\alpha \leq s$ and there exists $\beta > r$ such that $\|x_1, x_2, \ldots, x_n'\|_\alpha \leq t$. Let $\gamma = \min\{\alpha, \beta\} > r$. Thus

$$\|x_1, x_2, \ldots, x_n\|_\gamma \leq \|x_1, x_2, \ldots, x_n\|_\alpha \leq s$$

and

$$\|x_1, x_2, \ldots, x'_n\|_\gamma \leq \|x_1, x_2, \ldots, x'_n\|_\alpha \leq t.$$

Now,

$$\|x_1, x_2, \ldots, x_n + x'_n\|_\gamma \leq \|x_1, x_2, \ldots, x_n\|_\alpha + \|x_1, x_2, \ldots, x'_n\|_\alpha \leq s + t$$

Therefore,

$$N'(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq \gamma > r$$

Since $0 < r < \gamma$ is arbitary,

$$N'(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq p = \min\{N'(x_1, x_2, \ldots, x_n, s), N'(x_1, x_2, \ldots, x_n, t)\}.$$

Similarly if $p \geq q$, then also the relation holds. Thus,

$N'(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq \min\{N'(x_1, x_2, \ldots, x_n, s), N'(x_1, x_2, \ldots, x_n, t)\}$.

(N6)Let $(x_1, x_2, \ldots, x_n) \in X^n$ and $\alpha \in (0, 1)$. Now $t > \|x_1, x_2, \ldots, x_n\|_\alpha$

$$\Rightarrow N'(x_1, x_2, \ldots, x_n, t) = \sup\{\beta : \|x_1, x_2, \ldots, x_n\|_\beta \leq t\} \geq \alpha.$$

So, $\lim_{t\to\infty} N'(x_1, x_2, \ldots, x_n, t) = 1$.

If $t_1 < t_2 \leq 0$ then

$N'(x_1, x_2, \ldots, x_n, t_1) = N'(x_1, x_2, \ldots, x_n, t_2) = 0$ for all $(x_1, x_2, \ldots, x_n) \in X^n$.

If $t_2 > t_1 > 0$ then

$\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t_1\} \subset \{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t_2\}$

$$\Rightarrow \sup\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t_1\} \leq \sup\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t_2\}$$
$$\Rightarrow N'(x_1, x_2, \ldots, x_n, t_1) \leq N'(x_1, x_2, \ldots, x_n, t_2).$$

Thus $N'(x_1, x_2, \ldots, x_n, t)$ is a non-decreasing function of $t \in \Re$. Hence (X,N') is a f-q-n-NLS.

Remark: 2.4.6

Assume further that for $x_1, x_2, \ldots, x_n$ are linearly independent,

(N8)$N(x_1, x_2, \ldots, x_n, t)$ is a continuous function of t$\in \Re$ (set of real numbers) and strictly increasing in the subset $\{t : 0 < N(x_1, x_2, \ldots, x_n, t) < 1\}$ of $\Re$.

Theorem: 2.4.7

Let (X,N) be a f-q-n-NLS satisfying the conditions (N7) and (N8) and $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X,N). Then for $(y_1, y_2, \ldots, y_n) \in X^n$ with $y_1, y_2, \ldots, y_n$ are linearly independent, $N(y_1, y_2, \ldots, y_n, \|y_1, y_2, \ldots, y_n\|_\alpha) \geq \alpha, \alpha \in (0,1)$.

Proof:

Let (X,N) be a f-q-n-NLS satisfying the conditions (N7) and (N8) and $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X,N).

Claim: $N(y_1, y_2, \ldots, y_n, \|y_1, y_2, \ldots, y_n\|_\alpha) \geq \alpha, \alpha \in (0,1)$.

Let $\|y_1, y_2, \ldots, y_n\|_\alpha = T$, then T>0.

Also ther exists a sequence $\{t_n\}_{n=1}^{\infty}$ such that $N(y_1, y_2, \ldots, y_n, t_n) \geq \alpha$ and $\lim_{n\to\infty} t_n = T$.

so, $\lim_{n\to\infty} N(y_1, y_2, \ldots, y_n, t_n) \geq \alpha$

$\Rightarrow$ By (N8) $N(y_1, y_2, \ldots, y_n, \lim_{n\to\infty} t_n) \geq \alpha$.

$\Rightarrow N(y_1, y_2, \ldots, y_n, \|y_1, y_2, \ldots, y_n\|_\alpha) \geq \alpha, \ \alpha \in (0,1)$.

Theorem: 2.4.8

Let (X,N) be a f-q-n-NLS satisfying the conditions (N7) and (N8) and $\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X,N). Then for $y_1, y_2, \ldots, y_n \in X^n$ with $y_1, y_2, \ldots, y_n$ are linearly independent, $\alpha \in (0,1)$ and $t'(>0) \in \Re$, $\|y_1, y_2, \ldots, y_n\|_\alpha = t'$ if and only if $N(y_1, y_2, \ldots, y_n, t') = \alpha$.

Proof:

Let (X,N) be a f-q-n-NLS satisfying the conditions (N7) and (N8) and $\{\|\bullet,\bullet,\ldots,\bullet\|_\alpha : \alpha \in (0,1)\}$ be an ascending family of quasi n-norms corresponding to (X,N).

Claim: $\|y_1, y_2, \ldots, y_n\|_\alpha = t'$ if and only if $N(y_1, y_2, \ldots, y_n, t') = \alpha$.

Let $\alpha \in (0,1), y_1, y_2, \ldots, y_n$ are linearly independent and $t' = \|y_1, y_2, \ldots, y_n\|_\alpha = \inf\{s : N(y_1, y_2, \ldots, y_n, s) \geq \alpha\}$. Since $N(y_1, y_2, \ldots, y_n, t)$ is continuous (by (N8)), we have by theorem(2.4.7)

$$N(y_1, y_2, \ldots, y_n, t') \geq \alpha \tag{2.10}$$

Also,$N(y_1, y_2, \ldots, y_n, t') \leq N(y_1, y_2, \ldots, y_n, s)$ if $N(y_1, y_2, \ldots, y_n, s) \geq \alpha$.
If possible, let $N(y_1, y_2, \ldots, y_n, t') > \alpha$, then again by (N8), there exists $t'' < t'$ such that $N(y_1, y_2, \ldots, y_n, t'') > \alpha$ which is impossible, since $t' = \inf\{s : N(y_1, y_2, \ldots, y_n, s) \geq \alpha\}$. Thus

$$N(y_1, y_2, \ldots, y_n, t') \leq \alpha \tag{2.11}$$

By (2.10) and (2.11) we get $N(y_1, y_2, \ldots, y_n, t') = \alpha$. Thus

$$t' = \|y_1, y_2, \ldots, y_n\|_\alpha \Rightarrow N(y_1, y_2, \ldots, y_n, t') = \alpha \tag{2.12}$$

Next if $N(y_1, y_2, \ldots, y_n, t') = \alpha$, then from the definition

$$\|y_1, y_2, \ldots, y_n\|_\alpha = \inf\{t : N(y_1, y_2, \ldots, y_n, t) \geq \alpha\} = t' \tag{2.13}$$

(Since $N(x_1, x_2, \ldots, x_n, t)$ is strictly increasing in $\{t : 0 < N(x_1, x_2, \ldots, x_n, t) < 1\}$.)
From (2.12) and (2.13) we have, for $y_1, y_2, \ldots, y_n$ are linearly independent, $\alpha \in (0,1)$ and $t'(>0) \in \Re$, $\|y_1, y_2, \ldots, y_n\|_\alpha = t'$ if and only if $N(y_1, y_2, \ldots, y_n, t') = \alpha$.

Theorem: 2.4.9

Let (X,N) be a f-q-n-NLS satisfying the conditions (N7) and (N8). Let $\|x_1, x_2, \ldots, x_n\|_\alpha = \inf \{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\}, \alpha \in (0, 1)$ and $N' : X^n \times \Re \to [0, 1]$ is defined by,

$$N'(x_1, x_2, \ldots, x_n, t) = \begin{cases} \sup \{\alpha \in (0,1) : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\}, \text{ when} \\ \quad x_1, x_2, \ldots, x_n \text{ are linearly independent, } t \neq 0. \\ 0, \text{ otherwise.} \end{cases}$$

Then (a)$\{\|\bullet, \bullet, \ldots, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of quasi α-n-norms corresponding to (X,N).

(b)(X,N') is a f-q-n-NLS.

(c)$N' = N$.

Proof:

(a) and (b) follows from Theorem(2.4.3) and Theorem(2.4.5)

Claim: $N' = N$.

(c) Let $(y_1, y_2, \ldots, y_n, t_0) \in X^n \times \Re$ and $N(y_1, y_2, \ldots, y_n, t) = \alpha_0$.

Case(i):

If $y_1, y_2, \ldots, y_n$ are linearly dependent and $t_0 \leq 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N'(y_1, y_2, \ldots, y_n, t_0) = 0$.

Case(ii):

If $y_1, y_2, \ldots, y_n$ are linearly dependent and $t_0 > 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N'(y_1, y_2, \ldots, y_n, t_0) = 1$.

Case(iii):

If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0 \leq 0$, then $N(y_1, y_2, \ldots, y_n, t_0) = N'(y_1, y_2, \ldots, y_n, t_0) = 0$.

Case(iv):

Suppose $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(>0) \in \Re$ such that $N(y_1, y_2, \ldots, y_n, t_0) = 0$. For $\alpha \in (0,1)$, $\|y_1, y_2, \ldots, y_n\|_\alpha = \inf\{t : N(y_1, y_2, \ldots, y_n, t_0) \geq \alpha\}$
By theorem (2.4.7),
$N(y_1, y_2, \ldots, y_n, \|y_1, y_2, \ldots, y_n\|_\alpha) \geq \alpha,\ \alpha \in (0,1)$. Since $N(y_1, y_2, \ldots, y_n, t_0) = 0 < \alpha$ it follows that $t_0 < \|y_1, y_2, \ldots, y_n\|_\alpha$, for all $\alpha > 0$. So,

$$\begin{aligned} N'(y_1, y_2, \ldots, y_n, t_0) &= \sup\{\alpha : \|y_1, y_2, \ldots, y_n\|_\alpha \leq t_0\} \\ &= \sup\{\phi\} \\ &= 0. \end{aligned}$$

Therefore $N(y_1, y_2, \ldots, y_n, t_0) = N'(y_1, y_2, \ldots, y_n, t_0)$.
Case(v):

If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(>0) \in \Re$ such that $0 < N(y_1, y_2, \ldots, y_n, t_0) < 1$. Let $N(y_1, y_2, \ldots, y_n, t_0) = \alpha_0$. Then $0 < \alpha_0 < 1$.
Now $N'(x_1, x_2, \ldots, x_n, t) = \sup\{\alpha : \|x_1, x_2, \ldots, x_n\|_\alpha \leq t\}$, when $x_1, x_2, \ldots, x_n$ are linearly independent,

$$t \neq 0 \tag{2.14}$$

and

$$\|x_1, x_2, \ldots, x_n\|_\alpha = \inf\{t : N(x_1, x_2, \ldots, x_n, t) \geq \alpha\},\ \alpha \in (0,1). \tag{2.15}$$

Since $N(y_1, y_2, \ldots, y_n, t_0) = \alpha_0$, we have from (2.15),

$$\|y_1, y_2, \ldots, y_n\|_{\alpha_0} \leq t_0. \tag{2.16}$$

Using (2.16) we get from (2.14)

$$N'(y_1, y_2, \ldots, y_n, t_0) \geq \alpha_0 \Rightarrow N'(y_1, y_2, \ldots, y_n, t_0) \geq N(y_1, y_2, \ldots, y_n, t_0). \tag{2.17}$$

Now from theorem (2.4.8),

$$\begin{aligned} N(y_1, y_2, \ldots, y_n, t_0) &= \alpha_0 \\ \Leftrightarrow \|y_1, y_2, \ldots, y_n\|_{\alpha_0} &= t_0. \end{aligned}$$

Now for $1 > \alpha > \alpha_0$, let $\|y_1, y_2, \ldots, y_n\|_\alpha = t'$, then $t' \geq t_0$. Then by theorem (2.4.8), $N(y_1, y_2, \ldots, y_n, t') = \alpha$. So, $N(y_1, y_2, \ldots, y_n, t') = \alpha > \alpha_0 = N(y_1, y_2, \ldots, y_n, t_0)$. Since $N(y_1, y_2, \ldots, y_n, t)$ is strictly increasing and $N(y_1, y_2, \ldots, y_n, t') > N(y_1, y_2, \ldots, y_n, t_0)$, it follows that $t' > t_0$. So for $1 > \alpha > \alpha_0$, $\|y_1, y_2, \ldots, y_n\|_\alpha = t' > t_0$. Hence

$$N'(y_1, y_2, \ldots, y_n, t_0) \leq \alpha_0 = N(y_1, y_2, \ldots, y_n, t_0). \tag{2.18}$$

By (2.17) and (2.18) we have $N'(y_1, y_2, \ldots, y_n, t_0) = N(y_1, y_2, \ldots, y_n, t_0)$.

Case(vi):

If $y_1, y_2, \ldots, y_n$ are linearly independent and $t_0(> 0) \in \Re$, such that $N(y_1, y_2, \ldots, y_n, t_0) = 1$. Then by (2.14) and (2.15) it follows that $\|y_1, y_2, \ldots, y_n\|_\alpha \leq t_0 \ \Rightarrow N'(y_1, y_2, \ldots, y_n, t_0) = 1$.

Thus $N(y_1, y_2, \ldots, y_n, t_0) = N'(y_1, y_2, \ldots, y_n, t_0) = 1$. Hence $N(x_1, x_2, \ldots, x_n, t) = N'(x_1, x_2, \ldots, x_n, t)$ for all $(x_1, x_2, \ldots, x_n) \in X^n \times \Re$.