Chapter VII

Generalizations of Soft Open and Closed Sets, Soft functions, Soft Compactness and Soft Connectedness

CHAPTER – VII
GENERALIZATIONS OF SOFT OPEN AND CLOSED SETS, SOFT FUNCTIONS, SOFT COMPACTNESS AND SOFT CONNECTEDNESS

Definition : 7.1

Let E be fixed set of parameters. Then in a soft topological space (U, τ, E),

(i) a soft set G_C is said to be a **semiopen soft set** if there exists an open soft set H_B such that $H_B \tilde{\subseteq} G_C \tilde{\subseteq}$ cl H_B ;

(ii) a soft set L_A is said to be a **semiclosed soft set** if there exists a closed soft set K_D such that int $K_D \tilde{\subseteq} L_A \tilde{\subseteq} K_D$.

Remark : 7.2

From definition of semiopen (semiclosed) soft sets it is clear that every open (closed) soft set is a semiopen (semiclosed) soft set but not conversely. Φ_E and U_E are always semiclosed and semiopen.

Notation : 7.3

The family of all semiopen soft sets (semiclosed soft sets) of a soft topological space (U, τ, E) are denoted by $SOSS(U)_E$ ($SCSS(U)_E$).

Theorem : 7.4

Arbitrary union of semiopen soft sets is a semiopen soft set and arbitrary intersection of semiclosed soft sets is a semiclosed soft set.

Theorem : 7.5

A soft set $G_C \in SOSS(U)_E$ iff for every soft point $e_C \tilde{\in} G_C$, there exists a soft set $H_B \in SOSS(U)_E$ such that $e_G \tilde{\in} H_B \tilde{\subseteq} G_C$.

Proof

By taking $H_B = G_C$, the proof follows :

$(\Leftarrow)$ $G_C = \underset{e_G \tilde{\in} G_C}{\tilde{\cup}} (e_G) \tilde{\subseteq} \underset{e_G \tilde{\in} G_C}{\tilde{\cup}} H_B \tilde{\subseteq} G_C$.

Definition : 7.6

Let (U, τ, E) be a soft topological space and G_C be a soft set over U. Then

(i) The **soft semi closure** of G_C is defined as

$$\text{ssclG}_C = \tilde{\cap}\{S_F \mid G_C \tilde{\subseteq} S_F \text{ and } S_F \in SCSS(U)_E\}$$

(ii) The **soft semi interior** of G_C is defined as

$$\text{ssintG}_C = \tilde{\cup}\{S_F \mid S_F \tilde{\subseteq} G_C \text{ and } S_F \in SOSS(U)_E\}$$

Note : 7.7

Thus ssclG_C is the smallest semiclosed soft set containing G_C and ssintG_C is the largest semiopen soft set contained in G_C.

Theorem : 7.8

Let (U, τ, E) be a soft topological space and G_C and K_D be two soft sets over U, then

(i) $G_C \in SCSS(U)_E$ iff $G_C = \text{ssclG}_C$;

(ii) $G_C \in SOSS(U)_E$ iff $G_C = \text{ssintG}_C$;

(iii) $(\text{ssclG}_C)^c = \text{ssint}(G_C^c)$;

(iv) $(\text{ssintG}_C)^c = \text{sscl}(G_C^c)$;

(v) $G_C \tilde{\subseteq} K_D \Rightarrow \text{ssclG}_C \tilde{\subseteq} \text{ssclK}_D$;

(vi) $G_C \tilde{\subseteq} K_D \Rightarrow \text{ssclG}_C \tilde{\subseteq} \text{ssclK}_D$;

(vii) $ssc l\Phi_E = \Phi_E$ and $ssclU_E = U_E$;

(viii) $ssint\Phi_E = \Phi_E$ and $ssintU_E = U_E$;

(ix) $sscl(G_C \tilde{\cup} K_D) = ssclG_C \tilde{\cup} ssclK_D$;

(x) $ssint(G_C \tilde{\cap} K_D) = ssintG_C \tilde{\cap} ssintK_D$;

(xi) $sscl(G_C \tilde{\cap} K_D) \tilde{\subseteq} ssclG_C \tilde{\cap} ssclK_D$;

(xii) $ssint(G_C \tilde{\cup} K_D) \tilde{\subseteq} ssintG_C \tilde{\cup} ssintK_D$;

(xiii) $sscl(ssclG_C) = ssclG_C$;

(xiv) $ssint(ssintG_C) = ssintG_C$.

Theorem : 7.7

If G_C is any soft set in a soft topological space (U, τ, E) then following are equivalent.

(i) G_C is semiclosed soft set ;

(ii) $int(clG_C) \tilde{\subseteq} G_C$;

(iii) $cl(int G_C^c) \tilde{\supseteq} G_C^c$;

(iv) G_C^c is semiopen soft set.

Proof : (i) ⇒ (ii)

If G_C is semiclosed soft set, then there exists closed soft set H_B such that $intH_B \tilde{\subseteq} G_C \tilde{\subseteq} H_B \Rightarrow int\ H_B \tilde{\subseteq} G_C \tilde{\subseteq} clG_C \tilde{\subseteq} H_B$. By the property of interior we then have $int(clG_C) \tilde{\subseteq} intH_B \tilde{\subseteq} G_C$.

(ii) ⇒ (iii)

$Int(clG_C) \tilde{\subseteq} G_C \Rightarrow G_C^c \tilde{\subseteq} (int(clG_C))^c = cl(int G_C^c) \tilde{\supseteq} G_C^c$.

(iii) $\Rightarrow$ (iv)

$H_B = \text{int}\,G_C^c$ is an open soft set such that $\text{int}\,G_C^c \tilde{\subseteq} G_C^c \tilde{\subseteq} G_C^c \tilde{\subseteq} \text{cl}(\text{int}\,G_C^c)$, hence G_C^c is semiopen.

(iv) $\Rightarrow$ (i)

As G_C^c is semiopen there exists an open soft set H_B such that $H_B \tilde{\subseteq} G_C^c \tilde{\subseteq} \text{cl}H_B \Rightarrow H_B^c$ is a closed soft set such that $G_C \tilde{\subseteq} H_B^c$ and $G_C^c \tilde{\subseteq} \text{cl}H_B \Rightarrow \text{int}\,H_B^c \tilde{\subseteq} G_C$, hence G_C is semiclosed soft set.

Definition : 7.8

Let (U, τ, E) and (V, δ, E') be two soft topological spaces. A soft function $f : SS(U)_E \to SS(V)_{E'}$ is said to be

(i) **soft semicontinuous** if for each soft open set G_C of $V_{E'}$, the inverse image $f^{-1}(G_C)$ is a semiopen soft set of U_E ;

(ii) **soft irresolute** if for each semiopen soft set G_C of $V_{E'}$, the inverse image $f^{-1}(G_C)$ is a semiopen soft set of U_E ;

(iii) **soft semiopen function** if for each open soft set L_A of U_E, the image $f(L_A)$ is a semiopen soft set of $V_{E'}$;

(iv) **soft semiclosed function** if for each closed soft set K_D of U_E, the image $f(K_D)$ is semiclosed soft set of $V_{E'}$.

Remark : 7.9

(i) A soft function $f : SS(U)_E \to SS(V)_{E'}$ is soft semicontinuous if for each closed soft set $K_{D'}$ of $V_{E'}$, the inverse image $f^{-1}(K_{D'})$ is a semiclosed soft set of U_E.

(ii) A soft semicontinuous function is soft irresolute.

Theorem : 7.10

A soft function $f : SS(U)_E \to SS(V)_{E'}$ is soft semicontinuous iff $f(ssclL_A) \tilde{\subseteq} cl(f(L_A))$ for every soft set L_A of U_E.

Proof

Let $f : SS(U)_E \to SS(V)_{E'}$ be a soft semicontinuous function. Now $cl(f(L_A))$ is a soft closed set of $V_{E'}$, so by soft semicontinuity of f, $f^{-1}(cl(fL_A)))$ is soft semiclosed and $L_A \tilde{\subseteq} f^{-1}(cl(f(L_A)))$. But $ssclL_A$ is the smallest semiclosed set containing L_A, hence $ssclL_A \tilde{\subseteq} f^{-1}(cl(f(L_A))) \Rightarrow f(ssclL_A) \tilde{\subseteq} cl(f(L_A))$.

Conversely, let G_C be any soft closed set of $V_{E'} \Rightarrow f^{-1}(G_C) \tilde{\in} U_E \Rightarrow f(sscl(f^{-1}(G_C))) \tilde{\subseteq} cl(f(f^{-1}(G_C))) \Rightarrow f(sscl(f^{-1}(G_C))) \tilde{\subseteq} clG_C = G_C \Rightarrow sscl(f^{-1}(G_C)) = f^{-1}(G_C)$. Hence $f^{-1}(G_C)$ is semiclosed.

Theorem : 7.11

A soft function $f : SS(U)_E \to SS(V)_{E'}$ is soft semicontinuous iff $f^{-1}(intG_C) \tilde{\subseteq} ssint(f^{-1}(G_C))$ for every soft set G_C of $V_{E'}$.

Proof

Let $f : (SS(U)_E \to SS(V)_{E'}$ is soft semicontinuous. Now $int(f(G_C))$ is a soft open set of $V_{E'}$, so by soft semocontinuity of f, $f^{-1}(int(f(G_C)))$ is soft semiopen and $f^{-1}(int(f(G_C))) \tilde{\subseteq} G_C$. As $ssintG_C$ is the largest soft semiopen set contained in G_C, $f^{-1}(int(f(G_C))) \tilde{\subseteq} ssintG_C$.

Conversely, take a soft open set $G_C \Rightarrow f^{-1}(intG_C) \tilde{\subseteq} ssint(f^{-1}(G_C)) \Rightarrow f^{-1}(G_C) \tilde{\subseteq} ssint(f^{-1}(G_C)) \Rightarrow f^{-1}(G_C)$ is soft semiopen.

Theorem : 7.12

Let $f : SS(U)_E \to SS(V)_{E'}$ be soft semiopen. If $K_{D'}$ is a soft set in $V_{E'}$ and L_C is closed soft set containing $f^{-1}(K_{D'})$ then there exists a semiclosed soft set H_B such that $K_{D'} \tilde{\subseteq} H_B$ and $f^{-1}(H_B) \tilde{\subseteq} L_C$.

Proof

Take $H_B = (f(L_C^c))^c$. Now $f^{-1}(K_{D'}) \tilde{\subseteq} L_C \Rightarrow f(L_C^c) \tilde{\subseteq} K_{D'}^c$. Then L_C^c is soft open $\Rightarrow f(L_C^c)$ is semiopen, so H_B is semiclosed and $K_{D'} \tilde{\subseteq} H_B$ and $f^{-1}(H_B) \tilde{\subseteq} L_C$.

Definition : 7.13

A cover of a soft set is said to be a **semiopen soft cover** if every member of the cover is a semiopen soft set.

Definition : 7.14

A soft topological space (U, τ, E) is said to be **soft semicompact** if each semiopen soft cover of U_E has a finite subcover.

Remark : 7.15

Every compact soft topological space is also semicompact.

Theorem : 7.16

A soft topological space (U, τ, E) is semicompact iff each family of semiclosed soft sets in U_E with the finite intersection property has a nonempty intersection.

Proof

Let $\{(L_A)_\lambda \mid \lambda \in \Lambda\}$ be a collection of semiclosed soft sets with the finite intersection property. If possible, assume $\underset{\lambda \in \Lambda}{\tilde{\cap}} (L_A)_\lambda = \Phi_E \Rightarrow \underset{\lambda \in \Lambda}{\tilde{\cup}} ((L_A)_\lambda)^c = U_E$. So, the collection $\{((L_A)_\lambda)^c \mid \lambda \in \Lambda\}$ forms a soft semiopen cover of U_E, which is semicompact. So, there exists a finite subcollection Δ of Λ which also covers U_E. i.e., $\underset{\lambda \in \Delta}{\tilde{\cup}} ((L_A)_\lambda)^c = U_E \Rightarrow \underset{\lambda \in \Delta}{\tilde{\cap}} (L_A)_\lambda = \Phi_E$, a contradiction.

For the converse, if possible, let (U, τ, E) be not semicompact. Then there exists a semiopen cover $\{(G_C)_\lambda \mid \lambda \in \Lambda\}$ of U_E, such that for every finite subcollection Δ of Λ we have $\underset{\lambda \in \Delta}{\tilde{\cup}} (G_C)_\lambda \neq U_E \Rightarrow \underset{\lambda \in \Delta}{\tilde{\cap}} ((G_C)_\lambda)^c \neq \Phi_E$. Hence $\{((G_C)_\lambda)^c \mid \lambda \in \Lambda\}$ has the finite intersection property. So, by hypothesis $\underset{\lambda \in \Lambda}{\tilde{\cap}} ((G_C)_\lambda)^c \neq \Phi_E \Rightarrow \underset{\lambda \in \Lambda}{\tilde{\cup}} (G_C)_\lambda \neq U_E$, a contradiction.

Theorem : 7.17

Semicontinuous image of a soft semicompact space is soft compact.

Proof

Let $f : SS(U)_E \rightarrow SS(V)_{E'}$ be a semicontinuous function where (U, τ, E) is a semicompact soft topological space and (V, δ, E') is another soft topological space. Take a soft open cover $\{(G_C)_\lambda \mid \lambda \in \Lambda\}$ of $V_{E'} \Rightarrow \{f^{-1}((G_C)_\lambda) \mid \lambda \in \Lambda\}$ forms a soft semiopen cover of $U_E \Rightarrow$ there exists a finite subset Δ of Λ such that $\{f^{-1}((G_C)_\lambda) \mid \lambda \in \Delta\}$ forms a semiopen cover of $U_E \Rightarrow \{(G_C)_\lambda) \mid \lambda \in \Delta\}$ forms a finite soft opencover of $V_{E'}$.

Definition : 7.18

Two soft sets L_A and H_B are said to be **disjoint** if $L_A(a) \cap H_B(b) = \Phi$, $\forall\, a \in A, b \in B$.

Definition : 7.19

A soft **semiseparation** of soft topological space (U, τ, E) is a pair L_A, H_B of disjoint nonnull semiopen sets whose union is U_E.

If there doesn't exists a soft semiseparation of U_E, then the soft topological space is said to be **soft semiconnected**, otherwise **soft semidisconnected**.

Theorem : 7.20

If the soft sets L_A and G_C form a soft semiseparation of U_E, and if V_B is a soft semiconnected subspace of U_E, then $V_B \tilde{\subseteq} L_A$ or $V_B \tilde{\subseteq} G_C$.

Proof

Since L_A and G_C are disjoint semiopen soft sets, so are $L_A \tilde{\cap} V_B$ and $G_C \tilde{\cap} V_B$ and their soft union gives V_B, i.e., they would constitute a soft semiseparation of V_B, a contradiction. Hence, one of $L_A \tilde{\cap} V_B$ and $G_C \tilde{\cap} V_B$ is empty and so V_B is entirely contained in one of them.

Theorem : 7.21

Let V_B be a soft semiconnected subspace of U_E and K_D be a soft set in U_E such that $V_B \tilde{\subseteq} K_D \tilde{\subseteq} cl(V_B)$, then K_D is also soft semiconnected.

Proof

Let the soft set K_D satisfies the hypothesis. If possible, let F_A and G_C form a soft semiseparation of K_D. Then, by Theorem 7.20, $V_B \tilde{\subseteq} F_A$ or $V_B \tilde{\subseteq} G_C$. Let $V_B \tilde{\subseteq} F_A \Rightarrow sscl(V_B) \tilde{\subseteq} ssclF_A$; since $ssclF_A$ and G_C and disjoint, V_B cannot intersect G_C. This contradicts the fact that G_C is a nonempty subset of $V_B \Rightarrow$ there does not exists a soft semiseparation of K_D and hence is soft semiconnected.

Theorem : 7.22

A soft topological space (U, τ, E) is soft semidisconnected iff there exists a nonnull proper soft subset of U_E which is both soft semiopen and soft semiclosed.

Proof

Let K_D be a nonnull proper soft subset of U_E which is both semiopen and semiclosed. Now $H_C = (K_D)^c$ is nonnull proper subset of U_E which is also both semiopen and semiclosed $\Rightarrow$ $ssclK_D = K_D$ and $ssclH_C = H_C \Rightarrow U_E$ can be expressed as the soft union of two semiseparated soft sets K_D, H_C and so, is semidisconnected.

Conversely, let U_E be semidisconnected $\Rightarrow$ there exists nonnull soft subsets K_D and H_C such that $ssclK_D \tilde{\cap} H_C = \Phi_E$, $K_D \tilde{\cap}$ $ssclH_C = \Phi_E$ and $K_D \tilde{\cup} H_C = U_E$. Now $K_D \tilde{\subseteq}$ $ssclK_D$ and $ssclK_D \tilde{\cap} H_C = \Phi_E \Rightarrow K_D \tilde{\cap} H_C = \Phi_E \Rightarrow H_C = (K_D)^c$. Then $K_D \tilde{\cup}$ $ssclH_C = U_E$ and $K_D \tilde{\cap}$ $ssclH_C = \Phi_E \Rightarrow K_D = (ssclH_C)^c$ and similarly $H_C = (ssclK_D)^c \Rightarrow K_D$. H_C are semiopen sets being the complements of semiclosed soft sets. Also $H_C = (K_D)^c \Rightarrow$ they are also semiclosed.

Theorem : 7.23

Semicontinuous image of a soft semiconnected soft topological space is soft conneted.

Proof

Let f : $SS(U)_E \rightarrow SS(V)_{E'}$ be a semicontinuous function where (U, τ, E) a semiconnected soft topological space and (V, δ, E′) is a soft topological space. We show that $f(U_E)$ is soft connected. Suppose $f(U_E) = K_D \tilde{\cup} H_C$ be a soft separation. i.e., K_D and H_C are disjoint soft open sets whose union is

$f(U_E) \Rightarrow f^{-1}(K_D)$ and $f^{-1}(H_C)$ are disjoint soft semiopen sets whose union is U_E. So, $f^{-1}(K_D)$ and $f^{-1}(H_C)$ form a soft semiseparation of U_E, a contradiction.

Theorem : 7.24

Irresolute image of a soft semiconnected soft topological space is soft semiconnected.

Proof

Similar to that of Theorem 7.23.