CHAPTER 3

CHAPTER 3

CARTESIAN PRODUCT OF INTUITIONISTIC FUZZY n-NORMED LINEAR SPACES

In this chapter the concepts of intuitionistic fuzzy n-normed linear space, completeness of intuitionistic fuzzy n-normed linear space and generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces are discussed.

In section one of chapter 3, intuitionistic fuzzy n-normed linear space, Cauchy sequence, convergent sequence and completeness in intuitionistic fuzzy n-normed linear space corresponding to the fuzzy n-normed linear space are studied.

In section two of chapter 3, cartssian product of two intuitionistic fuzzy n-normed linear spaces, its commutative property and its distributive property with respect to union, intersections and difference are analyzed.

SECTION: 3.1

INTUITIONISTIC FUZZY n-NORMED LINEAR SPACE

Definition: 3.1.1

An *intuitionistic fuzzy n-normed linear space (or) in short i-f-n-NLS* is an object of the form

$A = \{(X, N(x_1, x_2, \ldots, x_n, t), M(x_1, x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^n\}$

where X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and N, M are fuzzy sets on $X^n \times (0, \infty)$, N denotes the degree of membership and M denotes the degree of non-membership of $(x_1, x_2, \ldots, x_n, t) \in X^n \times (0, \infty)$ satisfying the following conditions :

$(i) N(x_1, x_2, \ldots, x_n, t) + M(x_1, x_2, \ldots, x_n, t) \leq 1;$

$(ii) N(x_1, x_2, \ldots, x_n, t) > 0;$

$(iii) N(x_1, x_2, \ldots, x_n, t) = 1$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

$(iv) N(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

$(v) N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|}), if c \neq 0, c \in F(\text{field});$

$(vi) N(x_1, x_2, \ldots, x_n, s) * N(x_1, x_2, \ldots, x'_n, t) \leq N(x_1, x_2, \ldots, x_n + x'_n, s + t);$

$(vii) N(x_1, x_2, \ldots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t;

$(viii) M(x_1, x_2, \ldots, x_n, t) > 0;$

$(ix) M(x_1, x_2, \ldots, x_n, t) = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

$(x) M(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$;

$(xi) M(x_1, x_2, \ldots, cx_n, t) = M(x_1, x_2, \ldots, x_n, \frac{t}{|c|}),$ if $c \neq 0, c \in F(\text{field})$;

$(xii) M(x_1, x_2, \ldots, x_n, s) \diamond M(x_1, x_2, \ldots, x_n, t) \geq M(x_1, x_2, \ldots, x_n + x_n, s + t);$

$(xiii) M(x_1, x_2, \ldots, x_n, t) : (0, \infty) \rightarrow [0, 1]$ is continuous in t

Example: 3.1.2

Let $(X, \|\bullet, \bullet, \ldots, \bullet\|)$ be an n-normed space. Define $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$, for all $a, b \in [0, 1]$,

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|},$$
$$M(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then $A = \{(X, N(x_1, x_2, \ldots, x_n, t), M(x_1, x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^n\}$ is an i-f-n-NLS.

Proof:

(i) Clearly $N(x_1, x_2, \ldots, x_n, t) + M(x_1, x_2, \ldots, x_n, t) \leq 1$

(ii) It is obvious that $N(x_1, x_2, \ldots, x_n, t) > 0$

The results

(iii) $N(x_1, x_2, \ldots, x_n, t) = 1$, if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent.

(iv) $N(x_1, x_2, \ldots, x_{n-1}, x_n, t) = N(x_1, x_2, \ldots, x_n, x_{n-1}, t)$

(v) $N(x_1, x_2, \ldots, cx_n) = N(x_1, x_2, \ldots, x_n, \frac{t}{|c|})$

(vi) $N(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x_n, t)\}$

vii) $N(x_1, x_2, \dots, x_n, t)$ is continuous in t.

are obtained in Chapter 2 of Example 2.1.7.

(viii) $M(x_1, x_2, \dots, x_n, t) > 0$

(ix) $M(x_1, x_2, \dots, x_n, t) = 0$

(i)if and only if $\dfrac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|} = 0,$

(ii)if and only if $\|x_1, x_2, \dots, x_n\| = 0,$

(iii)if and only if $x_1, x_2, \dots, x_n$are linearly dependent.

(x)

$$\begin{aligned} M(x_1, x_2, \dots, x_{n-1}, x_n, t) &= \frac{\|x_1, x_2, \dots, x_{n-1}, x_n\|}{t + \|x_1, x_2, \dots, x_{n-1}, x_n\|} \\ &= \frac{\|x_1, x_2, \dots, x_n, x_{n-1}\|}{t + \|x_1, x_2, \dots, x_n, x_{n-1}\|} \\ &= M(x_1, x_2, \dots, x_n, x_{n-1}, t) \\ &= \dots \end{aligned}$$

(xi)

$$\begin{aligned} M(x_1, x_2, \dots, cx_n, t) &= \frac{\|x_1, x_2, \dots, cx_n\|}{t + \|x_1, x_2, \dots, cx_n\|} \\ &= \frac{|c|\|x_1, x_2, \dots, x_n\|}{t + |c|\|x_1, x_2, \dots, x_n\|} \\ &= \frac{\|x_1, x_2, \dots, x_n\|}{\frac{t}{|c|} + \|x_1, x_2, \dots, x_n\|} \\ &= M(x_1, x_2, \dots, x_n, \frac{t}{|c|}) \end{aligned}$$

(xii)Without loss of generality assume that

$$\begin{aligned} M(x_1, x_2, \dots, x_n, s) &\leq M(x_1, x_2, \dots, x_n', t) \\ \Rightarrow \frac{\|x_1, x_2, \dots, x_n\|}{s + \|x_1, x_2, \dots, x_n\|} &\leq \frac{\|x_1, x_2, \dots, x_n'\|}{t + \|x_1, x_2, \dots, x_n'\|} \\ \Rightarrow \|x_1, x_2, \dots, x_n\|(t + \|x_1, x_2, \dots, x_n'\|) &\leq \|x_1, x_2, \dots, x_n'\|(s + \|x_1, x_2, \dots, x_n\|) \end{aligned}$$

$$\Rightarrow t\|x_1, x_2, \dots, x_n\| \leq s\|x_1, x_2, \dots, x_n'\| \qquad (3.1)$$

Now, $\dfrac{\|x_1, x_2, \ldots, x_n + x'_n\|}{s + t + \|x_1, x_2, \ldots, x_n + x'_n\|} - \dfrac{\|x_1, x_2, \ldots, x'_n\|}{t + \|x_1, x_2, \ldots, x'_n\|}$

$$\leq \frac{\|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x'_n\|}{s + t + \|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x'_n\|} - \frac{\|x_1, x_2, \ldots, x'_n\|}{t + \|x_1, x_2, \ldots, x'_n\|}$$

$$= \frac{t\|x_1, x_2, \ldots, x_n\| - s\|x_1, x_2, \ldots, x'_n\|}{(s + t + \|x_1, x_2, \ldots, x_n\| + \|x_1, x_2, \ldots, x'_n\|)(t + \|x_1, x_2, \ldots, x'_n\|)}$$

By (3.1),

$$\frac{\|x_1, x_2, \ldots, x_n + x'_n\|}{s + t + \|x_1, x_2, \ldots, x_n + x'_n\|} \leq \frac{\|x_1, x_2, \ldots, x'_n\|}{t + \|x_1, x_2, \ldots, x'_n\|}$$

Similary

$$\frac{\|x_1, x_2, \ldots, x_n + x'_n\|}{s + t + \|x_1, x_2, \ldots, x_n + x'_n\|} \leq \frac{\|x_1, x_2, \ldots, x_n\|}{s + \|x_1, x_2, \ldots, x_n\|}$$

$$\Rightarrow \quad M(x_1, x_2, \ldots, x_n + x_n, s + t) \leq \max\{M(x_1, x_2, \ldots, x_n, s), M(x_1, x_2, \ldots, x_n, t)\}$$

(vii) Clearly $M(x_1, x_2, \ldots, x_n, t)$ is continuous in t.

Thus A is an i-f-n-NLS.

Definition: 3.1.3

A sequence $\{x_n\}$ in an i-f-n-NLS A is said to *converge* to x if given r>0, t>0, 0<r<1, there exists an integer $n_0 \in$ N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ >1-r and $M(x_1, x_2, \ldots, x_{n-1}, x_n, t)$ <r, for all $n \geq n_0$.

Proposition: 3.1.4

In an i-f-n-NLS A, a sequence $\{x_n\}$ converges to x if and only if $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ and $M(x_1, x_2, \ldots, x_n - x, t) \to 0$ as n$\to \infty$.

Proof:

Assume that $\{x_n\}$ converges to x.

Claim: $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ $M(x_1, x_2, \ldots, x_n - x, t) \to 0$ as n$\to \infty$.

Let t>0. since $\{x_n\}$ converges to x, for a given r, 0<r<1, there exists an integer n_0 $\in$N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ >1-r and $M(x_1, x_2, \ldots, x_n - x, t)$ <r. Thus, 1-$N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ <r and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ <r, and hence $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 0$ as n$\to \infty$.

Conversly, assume that for each t>0, $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1$ and $M(x_1, x_2, \ldots, x_n - x, t) \to 0$ as n$\to \infty$,

Claim: $\{x_n\}$ converges to x in A.

For every r, 0<r<1, there exists an integer n_0 such that 1-$N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ <r and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ <r for all n$\geq n_0$. Thus $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ >1-r and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t)$ <r for all $n \geq n_0$. Hence $\{x_n\}$ converges to x in A.

Definition: 3.1.5

A sequence $\{x_n\}$ in an i-f-n-NLS A, is said to be *cauchy sequence* if given ϵ >0, with 0< ϵ <1, t>0, there exists an integer $n_0 \in$ N such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t)$ >1-ϵ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) < \epsilon$ for all $n, k \geq n_0$.

Proposition: 3.1.6

In an i-f-n-NLS A, every convergent sequence is a cauchy sequence.

Proof:

Assume that $\{x_n\}$ be a convergent sequence in A.

Claim: $\{x_n\}$ be a cauchy sequence.

Suppose $\{x_n\}$ converges to x. Let t>0 and ϵ ∈(0,1). Choose r∈(0,1) such that $(1-r)*(1-r) > 1-\epsilon$ and $r \diamond r < \epsilon$.

Since $\{x_n\}$ converges to x, there is an integer n_0 such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2})$ >1-r and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2})$ <r.

Now, $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t)$

$$\begin{aligned}
&= N(x_1, x_2, \ldots, x_{n-1}, x_n - x + x - x_k, \frac{t}{2} + \frac{t}{2}) \\
&\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) \quad * \\
&\quad\; N(x_1, x_2, \ldots, x_{n-1}, x - x_k, \frac{t}{2}) \\
&> (1-r)*(1-r), \text{ for all } n, k \geq n_0 \\
&> 1-\epsilon \text{ for all } n, k \geq n_0
\end{aligned}$$

and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t)$

$$\begin{aligned}
&= M(x_1, x_2, \ldots, x_{n-1}, x_n - x + x - x_k, \frac{t}{2} + \frac{t}{2}) \\
&\leq M(x_1, x_2, \ldots, x_{n-1}, x_n - x, \frac{t}{2}) \quad \diamond \\
&\quad\; M(x_1, x_2, \ldots, x_{n-1}, x - x_k, \frac{t}{2}) \\
&< r \diamond r \text{ for all } n, k \geq n_0 \\
&< \epsilon \text{ for all } n, k \geq n_0
\end{aligned}$$

Therefore $\{x_n\}$ is a cauchy sequence in A.

Definition: 3.1.7

A i-f-n-NLS A is said to be *complete* if every cauchy sequence in it is convergent.

The following example shows that there may exists cauchy sequence in an i-f-n-NLS which is not convergent.

Example: 3.1.8

Let $(X, \|\bullet, \bullet \ldots, \bullet\|)$ be an n-normed linear space and let $a * b = \min\{a, b\}$ and $a \diamond b = \max\{a, b\}$ for all a,b$\in [0, 1], t > 0$.

$$N(x_1, x_2, \ldots, x_n, t) = \frac{t}{t + \|x_1, x_2, \ldots, x_n\|}$$
$$M(x_1, x_2, \ldots, x_n, t) = \frac{\|x_1, x_2, \ldots, x_n\|}{t + \|x_1, x_2, \ldots, x_n\|}.$$

Then A is an i-f-n-NLS.

Let $\{x_n\}$ be a sequence in A, then

(a)$\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$ if and only if $\{x_n\}$ is a Cauchy sequence in A.

(b)$\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$ if and only if $\{x_n\}$ is a convergent sequence in A.

Proof:

Assume that $\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

Claim: $\{x_n\}$ is a Cauchy sequence in A.

(a)$\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet \ldots, \bullet\|)$

$$\Leftrightarrow \quad \lim_{n,k\to\infty} \|(x_1, x_2, \ldots, x_{n-1}, x_n - x_k)\| = 0$$

$$\Leftrightarrow \quad \lim_{n,k\to\infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) = \lim_{n,k\to\infty} \frac{t}{t + \|x_1, x_2, \ldots, x_{n-1}, x_n - x_k\|} = 1$$

$$\text{and} \quad \lim_{n,k\to\infty} M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) = \lim_{n,k\to\infty} \frac{\|x_1, x_2, \ldots, x_n - x_k\|}{t + \|x_1, x_2, \ldots, x_n - x_k\|} = 0$$

$$\Leftrightarrow \quad N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \to 1$$

$$\text{and} \quad M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \to 0 \text{ as } n, k \to \infty.$$

$$\Leftrightarrow \quad N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1 - r,$$

$$\text{and} \quad M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) < r, \quad r \in (0, 1) \text{ for all } n, k \geq n_0.$$

$\Leftrightarrow \{x_n\}$ is a Cauchy sequence in A.

(b) Assume that $\{x_n\}$ is a convergent sequence in $(X, ||\bullet, \bullet \ldots, \bullet||)$

Claim: $\{x_n\}$ is a convergent sequence in A.

$\{x_n\}$ is a convergent sequence in $(X, ||\bullet, \bullet \ldots, \bullet||)$

$$
\begin{aligned}
\Leftrightarrow \quad & \lim_{n\to\infty} ||(x_1, x_2, \ldots, x_{n-1}, x_n - x)|| = 0 \\
\Leftrightarrow \quad & \lim_{n\to\infty} N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\
& \quad = \lim_{n\to\infty} \frac{t}{t + ||x_1, x_2, \ldots, x_n - x||} \quad = 1 \\
\text{and} \quad & \lim_{n\to\infty} M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\
& \quad = \lim_{n\to\infty} \frac{||x_1, x_2, \ldots, x_n - x||}{t + ||x_1, x_2, \ldots, x_n - x||} \quad = 0 \\
\Leftrightarrow \quad & N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to \ 1 \\
\text{and} \quad & M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to \ 0 \ \text{ as } n \to \infty. \\
\Leftrightarrow \quad & N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r, \\
\text{and} \quad & M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r, \quad r \in (0, 1) \text{ for all } n \geq n_0.
\end{aligned}
$$

$\Leftrightarrow \{x_n\}$ is a convergent sequence in A.

Thus if there exists an n-normed linear space $(X, ||\bullet, \bullet \ldots, \bullet||)$ which is not complete, then the intuitionistic fuzzy n-normed induced by such a crisp n-norm $||\bullet, \bullet \ldots, \bullet||$ on an incomplete n-normed linear space X is an incomplete intuitionistic fuzzy n-normed linear space.

Proposition:3.1.9

Let A be an i-f-n-NLS, such that every Cauchy sequence in A has a convergent subsequence. Then A is complete.

Proof:

Assume $\{x_n\}$ be a Cauchy sequence in A and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x.

Claim: $\{x_n\}$ converges to x and it is complete.

Let t>0 and $\epsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r)*(1-r) > 1-\epsilon$ and $r \diamond r < \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $n_0 \in N$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, \frac{t}{2}) > 1-r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, \frac{t}{2}) < r$, for all $n, k \geq n_0$. Since $\{x_{n_k}\}$ converges to x, there is a positive $i_k > n_0$ such that $N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) > 1-r$ and $M(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) < r$.

Now,

$$\begin{aligned} & N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\ &= N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}) \\ &\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, \frac{t}{2}) \quad * \\ &\quad N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) \\ &> (1-r)*(1-r) \\ &> 1-\epsilon \end{aligned}$$

and

$$\begin{aligned} & M(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \\ &= M(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, \frac{t}{2} + \frac{t}{2}) \\ &\leq M(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, \frac{t}{2}) \quad \diamond \\ &\quad M(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, \frac{t}{2}) \\ &< r \diamond r \\ &< \epsilon \end{aligned}$$

Therefore $\{x_n\}$ converges to x in A and hence it is complete.

SECTION: 3.2

GENERALISED CARTESIAN PRODUCT OF INTUITIONISTIC FUZZY n-NORMED LINEAR SPACES

Definition: 3.2.1

Let A and B be intuitionistic fuzzy sets in X and Y respectively. Then the *Generalized cartesian product*
$A \times_{*,\diamond} B = \{((x,y), \mu_A(x) * \mu_B(y), \nu_A(x) \diamond \nu_B(y)) : x \in X \text{ and } y \in Y\}$, $*$ denotes t-norm and $\diamond$ denotes the t-co-norm.

We now proceed to our new notion of generalised cartisean product of the intuitionistic fuzzy n-normed linear spaces in the following theorem.

Proposition: 3.2.2

Let A, B be two intuitionistic fuzzy n-normed linear spaces. Then $A \times_{*,\diamond} B$ is an intuitionistic fuzzy n-normed linear space where $*$ is a t-norm and $\diamond$ is a t-co-norm.

Proof:

Let $A = \{(X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^n\}$
and $B = \{(Y, N_2(y_1, y_2, \ldots, y_n, t), M_2(y_1, y_2, \ldots, y_n, t)) : (y_1, y_2, \ldots, y_n) \in Y^n\}$
be two intuitionistic fuzzy n-normed linear spaces.

Claim:

$A \times_{*,\diamond} B = \{(X \times Y, N(z_1, z_2, \ldots, z_n, t), M(z_1, z_2, \ldots, z_n, t)) : (z_1, z_2, \ldots . z_n \in (X \times Y)^n)\}$
is an i-f-n-NLS with

$$N(z_1, z_2, \ldots, z_n, t) = N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

and

$$M(z_1, z_2, \ldots, z_n, t) = M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t)$$

where

$$z_i = (x_i, y_i), \quad i = 1, 2, \ldots, n.$$

(i)As

$$N_1(x_1,x_2,\ldots,x_n,t)+M_1(x_1,x_2,\ldots,x_n,t) \leq 1 \text{ and} \tag{3.2}$$

$$N_2(x_1,x_2,\ldots,x_n,t)+M_2(x_1,x_2,\ldots,x_n,t) \leq 1, \tag{3.3}$$

It follows that

$$M_1(x_1,x_2,\ldots,x_n,t) \leq 1-N_1(x_1,x_2,\ldots,x_n,t)$$

and

$$M_2(y_1,y_2,\ldots,y_n,t) \leq 1-N_2(y_1,y_2,\ldots,y_n,t)$$

By definition (2.2.8),

$(1-N_1(x_1,x_2,\ldots,x_n,t))\diamond(1-N_2(y_1,y_2,\ldots,y_n,t))$

$$\geq M_1(x_1,x_2,\ldots,x_n,t)\diamond M_2(y_1,y_2,\ldots,y_n,t)$$

Then,

$N_1(x_1,x_2,\ldots,x_n,t)\diamond N_2(y_1,y_2,\ldots,y_n,t)$

$$=1-\Big((1-N_1(x_1,x_2,\ldots,x_n,t))\diamond(1-N_2(y_1,y_2,\ldots,y_n,t))\Big)$$

$$\leq 1-(M_1(x_1,x_2,\ldots,x_n,t)\diamond M_2(y_1,y_2,\ldots,y_n,t)) \tag{3.4}$$

where $a\diamond b=1-\Big((1-a)\diamond(1-b)\Big)$ is defined as the dual t-norm with respect to $\diamond$.

So, if

$N_1(x_1,x_2,\ldots,x_n,t)*N_2(y_1,y_2,\ldots,y_n,t)\leq N_1(x_1,x_2,\ldots,x_n,t)\diamond N_2(y_1,y_2,\ldots,y_n,t)$

then by (3.4) we've

$N_1(x_1,x_2,\ldots,x_n,t)*N_2(y_1,y_2,\ldots,y_n,t)\leq 1-(M_1(x_1,x_2,\ldots,x_n,t)\diamond M_2(y_1,y_2,\ldots,y_n,t))$

$\Rightarrow N_1(x_1,x_2,\ldots,x_n,t)*N_2(y_1,y_2,\ldots,y_n,t)+$

$M_1(x_1,x_2,\ldots,x_n,t)\diamond M_2(y_1,y_2,\ldots,y_n,t)\leq 1.$

$\Rightarrow N(z_1,z_2,\ldots,z_n,t)+M(z_1,z_2,\ldots,z_n,t) \leq 1$

(ii)Clearly $N(z_1, z_2, \ldots, z_n, t) > 0$

(iii)we have to prove $N(z_1, z_2, \ldots, z_n, t) = 1$ if and only if $z_1, z_2, \ldots, z_n$ are linearly dependent.

Assume that $N(z_1, z_2, \ldots, z_n, t) = 1$

$\Leftrightarrow N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) = 1$

$\Leftrightarrow N_1(x_1, x_2, \ldots, x_n, t) = 1$ and $N_2(y_1, y_2, \ldots, y_n, t) = 1$

$\Leftrightarrow x_1, x_2, \ldots, x_n$ are linearly dependent and $y_1, y_2, \ldots, y_n$ are linearly dependent.

$\Leftrightarrow (x_i, y_i)$ are linearly dependent. $\forall i = 1, 2, \ldots, n.$

$\Leftrightarrow z_i's$ are linearly dependent.

(iv)we have to prove $N(z_1, z_2, \ldots, z_n, t)$ is invariant under any permutation of $z_1, z_2, \ldots, z_n$.

We know $N_1(x_1, x_2, \ldots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \ldots, x_n$. and $N_2(y_1, y_2, \ldots, y_n, t)$ is invariant under any permutation of $y_1, y_2, \ldots, y_n$.

$$\begin{aligned} N(z_1, z_2, \ldots, z_{n-1}, z_n, t) &= N_1(x_1, x_2, \ldots, x_{n-1}, x_n, t) * N_2(y_1, y_2, \ldots, y_{n-1}, y_n, t) \\ &= N_1(x_1, x_2, \ldots, x_n, x_{n-1}, t) * N_2(y_1, y_2, \ldots, y_n, y_{n-1}, t) \\ &= N(z_1, z_2, \ldots, z_n, z_{n-1}, t) \end{aligned}$$

Therefore, $N(z_1, z_2, \ldots, z_n, t)$ is invariant under any permutation.

(v)We have to prove, $N(z_1, z_2, \ldots, cz_n, t) = N(z_1, z_2, \ldots, z_n, \frac{t}{|c|})$.

consider $N(z_1, z_2, \ldots, cz_n, t)$

$$\begin{aligned} &= N_1(x_1, x_2, \ldots, cx_n, t) * N_2(y_1, y_2, \ldots, cy_n, t) \\ &= N_1(x_1, x_2, \ldots, x_n, \frac{t}{|c|}) * N_2(y_1, y_2, \ldots, y_n, \frac{t}{|c|}) \\ &= N(z_1, z_2, \ldots, z_n, \frac{t}{|c|}) \text{ if} c \neq 0,\ c \in F \end{aligned}$$

(vi)we have to prove

$N(z_1, z_2, \ldots, z_n, s) * N(z_1, z_2, \ldots, z_n, t) \leq N(z_1, z_2, \ldots, z_n + z_n, s + t)$

$N(z_1, z_2, \dots, z_n + z_n, s+t)$

$$\begin{aligned}
&= N_1(x_1, x_2, \dots, x_n + x_n, s+t) * N_2(y_1, y_2, \dots, y_n + y_n, s+t) \\
&\geq \Big(N_1(x_1, x_2, \dots, x_n, s) * N_1(x_1, x_2, \dots, x_n, t)\Big) \\
&\quad * \Big(N_2(y_1, y_2, \dots, y_n, s) * N_2(y_1, y_2, \dots, y_n, t)\Big) \\
&= \Big(N_1(x_1, x_2, \dots, x_n, s) * N_1(x_1, x_2, \dots, x_n, t)\Big) \\
&\quad * \Big(N_2(y_1, y_2, \dots, y_n, t) * N_2(y_1, y_2, \dots, y_n, s)\Big) \\
&= N_1(x_1, x_2, \dots, x_n, s) * \Big(N_1(x_1, x_2, \dots, x_n, t) \\
&\quad * (N_2(y_1, y_2, \dots, y_n, t)\Big) * N_2(y_1, y_2, \dots, y_n, s) \\
&= N_1(x_1, x_2, \dots, x_n, s) * N(z_1, z_2, \dots, z_n, t) \\
&\quad * N_2(y_1, y_2, \dots, y_n, s) \\
&= (N_1(x_1, x_2, \dots, x_n, s) * N_2(y_1, y_2, \dots, y_n, s)) \\
&\quad * N(z_1, z_2, \dots, z_n, t) \\
&= N(z_1, z_2, \dots, z_n, s) * N(z_1, z_2, \dots, z_n, t)
\end{aligned}$$

(vii)Clearly $N(z_1, z_2, \dots, z_n, t)$ is continuous in t, since $N_1(x_1, x_2, \dots, x_n, t) * N_2(y_1, y_2, \dots, y_n, t)$ is continuous in t.

similarly we can prove the other axioms.

Proposition: 3.2.3

The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is commutative.

Proof:

Let A, B are two intuitionistic fuzzy n-normed linear spaces.

Claim: $A = B \;\Rightarrow\; A \times_{*,\diamond} B \;=\; B \times_{*,\diamond} A$

Assume $A = B$; $(x_1, x_2, \ldots, x_n, t), (y_1, y_2, \ldots, y_n, t) \in X^n$

Then

$$N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) = N_2(x_1, x_2, \ldots, x_n, t) * N_1(y_1, y_2, \ldots, y_n, t)$$

and

$$M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t) = M_2(x_1, x_2, \ldots, x_n, t) \diamond M_1(y_1, y_2, \ldots, y_n, t)$$

Thus, $A \times_{*,\diamond} B = B \times_{*,\diamond} A$.

However the converse is not true. For example, Let

$$A = \left\{ \begin{array}{c} (X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) : N_1(x_1, x_2, \ldots, x_n, t) = a, \\ M_1(x_1, x_2, \ldots, x_n, t) = b, (x_1, x_2, \ldots, x_n) \in X^n \end{array} \right\}$$

and

$$B = \left\{ \begin{array}{c} (X, N_2(x_1, x_2, \ldots, x_n, t), M_2(x_1, x_2, \ldots, x_n, t)) : N_2(x_1, x_2, \ldots, x_n, t) = c, \\ M_2(x_1, x_2, \ldots, x_n, t) = d, (x_1, x_2, \ldots, x_n) \in X^n \end{array} \right\}$$

$a, b, c, d \in [0, 1]$

Then

$$\begin{aligned} N_1(x_1, x_2, \ldots, x_n, t) * N_2(x_1, x_2, \ldots, x_n, t) &= a * c \\ &= c * a \\ &= N_2(x_1, x_2, \ldots, x_n, t) * N_1(x_1, x_2, \ldots, x_n, t) \end{aligned}$$

and

$$\begin{aligned} M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(x_1, x_2, \ldots, x_n, t) &= b \diamond d \\ &= d \diamond b \\ &= M_2(x_1, x_2, \ldots, x_n, t) \diamond M_1(x_1, x_2, \ldots, x_n, t) \end{aligned}$$

So we obtain $A \times_{*\diamond} B = B \times_{*\diamond} A$, but $A \neq B$ if $a \neq c$ *(or)* $b \neq d$.

Proposition: 3.2.4

The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is distributive with respect to union and intersections.

Proof:

Let

$$A = \{(X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^n\}$$

and

$$B = \{(Y, N_2(y_1, y_2, \ldots, y_n, t), M_2(y_1, y_2, \ldots, y_n, t)) : (y_1, y_2, \ldots, y_n) \in Y^n\}$$
$$C = \{(Y, N_3(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)) : (y_1, y_2, \ldots, y_n) \in Y^n\}$$

are the intuitionistic fuzzy n-normed linear spaces.

Claim: $A \times_{*,\diamond} (B \bigcap C) = (A \times_{*,\diamond} B) \bigcap (A \times_{*,\diamond} C)$

and $A \times_{*,\diamond} (B \bigcup C) = (A \times_{*,\diamond} B) \bigcup (A \times_{*,\diamond} C)$.

consider

$A \times_{*,\diamond} (B \bigcap C)$

$$= \left\{ \begin{array}{c} (X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\}, \\ M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}) : \\ (z_1, z_2, \ldots, z_n) \in (X \times Y)^n \end{array} \right\}$$

and

$(A \times_{*,\diamond} B) \bigcap (A \times_{*,\diamond} C)$

$$= \left\{ \begin{array}{c} (X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), \\ M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t) : (z_1, z_2, \ldots, z_n) \in (X \times Y)^n \end{array} \right\} \bigcap$$
$$\left\{ \begin{array}{c} (X \times Y, N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t), \\ M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) : (z_1, z_2, \ldots, z_n) \in (X \times Y)^n \end{array} \right\}$$

$$
= \left\{ \begin{array}{c} (X \times Y, \\ \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \\ \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}) \\ : (z_1, z_2, \ldots, z_n) \in (X \times Y)^n \end{array} \right\}
$$

So it is enough to prove that,

$$N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \quad (3.5)$$

and

$$M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \quad (3.6)$$

Let

$$N_2(y_1, y_2, \ldots, y_n, t) \leq N_3(y_1, y_2, \ldots, y_n, t) \tag{3.7}$$

Then by definition (2.2.7),

$$N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

$$\leq N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t) \tag{3.8}$$

Therefore, by (3.7) and (3.8)

$L.H.S$ of (3.5)

$$
\begin{aligned}
&= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \\
&= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \\
&= R.H.S of (3.5)
\end{aligned}
$$

Let

$$N_2(y_1, y_2, \ldots, y_n, t) > N_3(y_1, y_2, \ldots, y_n, t) \tag{3.9}$$

Then by definition (2.2.7),

$N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$

$$> N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t) \qquad (3.10)$$

Therefore, by (3.9) and (3.10)

$L.H.S$ of (3.5)

$$\begin{aligned} &= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\} \\ &= N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t) \\ &= \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\} \\ &= R.H.Sof(3.5) \end{aligned}$$

Thus equality holds in (3.5).

Let

$$M_2(y_1, y_2, \ldots, y_n, t) \leq M_3(y_1, y_2, \ldots, y_n, t) \qquad (3.11)$$

Then by definition (2.2.8),

$M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t)$

$$\leq M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) \qquad (3.12)$$

Therefore by (3.11) and (3.12)

$L.H.S$ of (3.6)

$$\begin{aligned} &= M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\ &= M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) \\ &= \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \\ &= R.H.Sof(3.6) \end{aligned}$$

Let

$$M_2(y_1, y_2, \ldots, y_n, t) > M_3(y_1, y_2, \ldots, y_n, t) \qquad (3.13)$$

Then by definition (2.2.8),

$M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t)$

$$> M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) \quad (3.14)$$

Therefore by (3.13) and (3.14)

$L.H.S$ of (3.6)

$$\begin{aligned}
&= M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\} \\
&= M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t) \\
&= \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\} \\
&= R.H.Sof(3.6)
\end{aligned}$$

Thus equality holds in (3.6).

Finally from (3.5) and (3.6) we have $A \times_{*,\diamond} (B \bigcap C) = (A \times_{*,\diamond} B) \bigcap (A \times_{*,\diamond} C)$.

Similarly we can prove, $A \times_{*,\diamond} (B \bigcup C) = (A \times_{*,\diamond} B) \bigcup (A \times_{*,\diamond} C)$.

Proposition: 3.2.5

The generalized cartesian product of the intuitionistic fuzzy n-normed linear spaces is distributive with respect to difference.

Proof:

Let

$$A = \{(X, N_1(x_1, x_2, \ldots, x_n, t), M_1(x_1, x_2, \ldots, x_n, t)) : (x_1, x_2, \ldots, x_n) \in X^n\}$$

and

$$\begin{aligned}
B &= \{(Y, N_2(y_1, y_2, \ldots, y_n, t), M_2(y_1, y_2, \ldots, y_n, t)) : (y_1, y_2, \ldots, y_n) \in Y^n\} \\
C &= \{(Y, N_3(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)) : (y_1, y_2, \ldots, y_n) \in Y^n\}
\end{aligned}$$

are the intuitionistic fuzzy n-normed linear spaces.

Claim: $A \times_{*,\diamond} (B \setminus C) \subseteq (A \times_{*,\diamond} B) \setminus (A \times_{*,\diamond} C)$.

If $B = \{(Y, N_2(y_1, y_2, \ldots, y_n, t) = 1, M_2(y_1, y_2, \ldots, y_n, t) = 0) : (y_1, y_2, \ldots, y_n) \in Y^n\}$, $c \subseteq A, * = \min$, $\diamond = \max$ then equality holds.

It is enough to prove,

$N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$

$\leq \min\{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}$ (3.15)

and

$M_1(x_1, x_2, \ldots, x_n, t) \diamond \max\{M_2(y_1, y_2, \ldots, y_n, t), N_3(y_1, y_2, \ldots, y_n, t)\}$

$\geq \max\{M_1(x_1, x_2, \ldots, x_n, t) \diamond M_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) * N_3(y_1, y_2, \ldots, y_n, t)\}$ (3.16)

Case (i): Let

$$N_2(y_1, y_2, \ldots, y_n, t) < M_3(y_1, y_2, \ldots, y_n, t) \quad (3.17)$$

and using the fact

$$a * b \leq \min\{a, b\} \leq a \leq \max\{a, c\} \leq a \diamond c \quad (3.18)$$

Then by definition (2.2.7) and (3.18)

$$\begin{aligned} N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) &< N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \\ &\leq M_3(y_1, y_2, \ldots, y_n, t) \\ &\leq N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t) \quad (3.19) \end{aligned}$$

$\Rightarrow N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) < N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)$

Therefore by (3.17) and (3.19)

$L.H.S$ of (3.15)

$$= N_1(x_1, x_2, \ldots, x_n, t) * \min\{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

$$= \min \{N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t), N_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= R.H.Sof(3.15)$$

Thus equality holds in (3.15).

Case(ii): Let

$$N_2(y_1, y_2, \ldots, y_n, t) \geq M_3(y_1, y_2, \ldots, y_n, t) \quad (3.20)$$

By definition (2.2.7)

$$N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t) \geq N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \quad (3.21)$$

Therefore by (3.20) and (3.21)

$$N_1(x_1, x_2, \ldots, x_n, t) * \min \{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \quad (3.22)$$

$$\leq N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

$$\Rightarrow N_1(x_1, x_2, \ldots, x_n, t) * \min \{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$\leq N_1(x_1, x_2, \ldots, x_n, t) * N_2(y_1, y_2, \ldots, y_n, t)$$

By (3.18) and (3.20)

$$N_1(x_1, x_2, \ldots, x_n, t) * \min \{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$$= N_1(x_1, x_2, \ldots, x_n, t) * M_3(y_1, y_2, \ldots, y_n, t) \quad (3.23)$$

$$\leq M_3(y_1, y_2, \ldots, y_n, t)$$

$$\leq M_1(x_1, x_2, \ldots, x_n, t) \diamond M_3(y_1, y_2, \ldots, y_n, t)$$

From (3.22) and (3.23) we have

$$N_1(x_1, x_2, \ldots, x_n, t) * \min \{N_2(y_1, y_2, \ldots, y_n, t), M_3(y_1, y_2, \ldots, y_n, t)\}$$

$\leq \min\{N_1(x_1,x_2,\ldots,x_n,t) * N_2(y_1,y_2,\ldots,y_n,t), M_1(x_1,x_2,\ldots,x_n,t) \diamond M_3(y_1,y_2,\ldots,y_n,t)\}$

Thus we have proved (3.15) and (3.16) can be proved similarly. So,

$$A \times_{*,\diamond} (B \setminus C) \subseteq (A \times_{*,\diamond} B) \setminus (A \times_{*,\diamond} C)$$

Let $B = \{(Y, N_2(y_1,y_2,\ldots,y_n,t) = 1, M_2(y_1,y_2,\ldots,y_n,t) = 0) : (y_1,y_2,\ldots,y_n) \in Y^n\}$,

$c \subseteq A$, $* = \min$, $\diamond = \max$.

L.H.S of (3.15)

$$\begin{aligned}
&= N_1(x_1,x_2,\ldots,x_n,t) * \min\{N_2(y_1,y_2,\ldots,y_n,t), M_3(y_1,y_2,\ldots,y_n,t)\} \\
&= N_1(x_1,x_2,\ldots,x_n,t) * \min\{1, M_3(y_1,y_2,\ldots,y_n,t)\} \\
&= N_1(x_1,x_2,\ldots,x_n,t) * M_3(y_1,y_2,\ldots,y_n,t) \\
&= \min\{N_1(x_1,x_2,\ldots,x_n,t) * M_3(y_1,y_2,\ldots,y_n,t)\}
\end{aligned}$$

R.H.S of (3.15)

$$\begin{aligned}
&= \min\{N_1(x_1,x_2,\ldots,x_n,t) * N_2(y_1,y_2,\ldots,y_n,t), M_1(x_1,x_2,\ldots,x_n,t) \diamond M_3(y_1,y_2,\ldots,y_n,t)\} \\
&= \min\{\min\{N_1(x_1,x_2,\ldots,x_n,t), 1\}, \max\{M_1(x_1,x_2,\ldots,x_n,t) \diamond M_3(y_1,y_2,\ldots,y_n,t)\}\} \\
&= \min\{N_1(y_1,y_2,\ldots,y_n,t), M_3(y_1,y_2,\ldots,y_n,t)\}
\end{aligned}$$

Thus equality holds in (3.15)

Similarly we can prove the equality in (3.16).