

Chapter IV



CHAPTER IV

LINEAR OPERATORS PRESERVING TERM RANK AND ZERO TERM RANK OF MATRICES OVER SEMIRINGS

Notation : 4.1

The arithmetic properties of term rank and zero-term rank which are already defined in chapter II are restricted by the following list of inequalities:

- (1) $t(A+B) \leq t(A)+t(B)$;
- (2) $t(A+B) \geq \max\{t(A),t(B)\}$;
- (3) $t(AB) \leq \min\{c(A),r(B)\}$;
- (4) $t(AB) \geq t(A)+t(B)-n$;
- (5) if S is a subsemiring of the semiring of nonnegative reals, R_+ , then
 $\rho(AB) + \rho(BC) \leq t(ABC) + t(B)$;
- (6) $z(A+B) \geq 0$;
- (7) $z(A+B) \leq \min\{z(A),z(B)\}$;
- (8) $z(AB) \geq 0$;
- (9) $z(AB) \leq z(A)+z(B)$.

In order to denote the sets of matrices that arise as extremal cases in the inequalities listed above, the following notations are used :

$$\mathbf{T}_1(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(X+Y) = t(X) + t(Y)\};$$

$$\mathbf{T}_2(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(X+Y) = \max(t(X), t(Y))\};$$

$$\mathbf{T}_3(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid t(XY) = \min\{r(X), c(Y)\}\};$$

$$\mathbf{T}_4(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_n(\mathbf{S})^2 \mid t(XY) = t(X) + t(Y) - n\};$$

$$\mathbf{T}_5(\mathbf{S}) = \{(X, Y, Z) \in \mathbf{M}_{m,n}(\mathbf{S})^3 \mid t(XYZ) + t(Y) = \rho(XY) + \rho(YZ)\};$$

$$\mathbf{Z}_1(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(X+Y) = \min\{z(X), z(Y)\}\};$$

$$\mathbf{Z}_2(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(X+Y) = 0\};$$

$$\mathbf{Z}_3(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(XY) = 0\};$$

$$\mathbf{Z}_4(\mathbf{S}) = \{(X, Y) \in \mathbf{M}_{m,n}(\mathbf{S})^2 \mid z(XY) = z(X) + z(Y)\}.$$

Linear operators of special types that preserve \mathbf{T}_1

Theorem :4.2

Let \mathbf{S} be an antinegative semiring without zero divisors and $T : \mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ be a surjective linear map. Then T preserves the set $\mathbf{T}_1(\mathbf{S})$ if and only if T is a (P, Q, B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units.

Proof :

It is straight forward to see that all (P, Q, B) -operators preserve the term renk . Thus they preserve the set $\mathbf{T}_1(\mathbf{S})$.

By Theorem 3.9 we have $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all i, j , $1 \leq i \leq m$, $1 \leq j \leq n$, where $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units and σ is a permutation on the set of pairs (i, j) .

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not, say $E_{i,j}$ and $E_{k,l}$ are the cells such that $t(E_{i,j} + E_{k,l}) = 2$ and $t(T(E_{i,j} + E_{k,l})) = 1$. Then $(E_{i,j} + E_{k,l}) \in \mathbf{T}_1(\mathbf{S})$ but $(T(E_{i,j}), T(E_{k,l})) \notin \mathbf{T}_1(\mathbf{S})$, which is a contradiction. Thus T maps lines to lines. Thus, by Theorem 3.12, T is a (P, Q, B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units.

Theorem : 4.3

Let \mathbf{S} be a finite antinegative semiring or a chain semiring. A linear operator $T : \mathbf{M}_{m,n}(\mathbf{S}) \rightarrow \mathbf{M}_{m,n}(\mathbf{S})$ strongly preserves the set $\mathbf{T}_1(\mathbf{S})$ if and only if T is a (P, Q, B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are not zero divisors for all i, j .

Proof :

If $b_{i,j}$ are not zero divisors, then all (P,Q,B) -operators preserve the term rank and thus they strongly preserve the set $T_1(S)$.

Suppose that T strongly preserves $T_1(S)$ and S is a finite antinegative semiring with identity 1_S , or a chain semiring. Then it is straight forward to see that there are positive integers α, β such that $\alpha \cdot 1_S = \beta \cdot 1_S$. It is proved in [4] that for any finite semiring there is a power of T such that this power is idempotent. The same holds for chain semirings. Indeed, from the multiplication and addition rules in chain semirings it follows that entries of each power of a given matrix A are contained in the set of entries of A , where A is the matrix representation of T . Thus only a finite set of different matrices can be obtained by considering the powers of the matrix A . Hence, there are integers m and n such that $A^m = A^{m+n}$. Thus $A^{mn} = A^{2mn}$. Hence a power of any linear operator on a chain semiring is idempotent. In both cases we denote $L=T^d$ and $L^2=L$. One can easily check that L strongly preserves $T_1(S)$.

Note that if $X \in M_{m,n}(S)$ and $(X, X) \in T_1(S)$, then necessarily $X=O$. Thus, if $X \neq O$, then $L(X) \neq O$ since L strongly preserves $T_1(S)$.

Suppose that there exists i , $1 \leq i \leq m$, such that $L(R_i)$ is not dominated by R_i . Then there is a pair of indices (r,s) such that $E_{r,s}$ is not dominated by R_i and $L(R_i) \geq E_{r,s}$. Then $(R_i, E_{r,s}) \in T_1(S)$ and $L(R_i) = aE_{r,s} + X$; here $x_{r,s} = 0$.

Now,

$$\begin{aligned}
L(\beta R_i + (\alpha - \beta)aE_{r,s}) &= L(\beta R_i) + L((\alpha - \beta)aE_{r,s}) \\
&= L^2(\beta R_i) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta L(R_i)) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta(aE_{r,s} + \beta X)) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta aE_{r,s} + \beta X) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\beta aE_{r,s}) + L((\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\beta aE_{r,s} + (\alpha - \beta)aE_{r,s}) \\
&= L(\beta X) + L(\alpha aE_{r,s})
\end{aligned}$$

$$\begin{aligned}
&= L(\alpha X) + L(\alpha a E_{r,s}) \\
&= L(\alpha(X + a E_{r,s})) \\
&= L(\alpha L(R_i)) \\
&= L^2(\alpha R_i) \\
&= L(\alpha R_i) \\
&= L(\beta R_i).
\end{aligned}$$

Now,

$$(\beta R_i, (\alpha - \beta)a E_{r,s}) \in T_1(S),$$

but

$$L(\beta R_i) + L((\alpha - \beta)a E_{r,s}) = L(\beta R_i + (\alpha - \beta)a E_{r,s}) = L(\beta R_i)$$

and hence

$$(L(\beta R_i), L((\alpha - \beta)a E_{r,s})) \notin T_1(S), \text{ which is a contradiction.}$$

We have established that $L(R_i) \leq R_i$ for all i . Similarly, $L(C_j) \leq C_j$ for all j . By considering the matrix $E_{i,j}$, which is dominated by both R_i and C_j , we have $L(E_{i,j}) \leq E_{i,j}$. Since S is antinegative, we that T also maps a cell to a multiple of a cell, or $|T(E_{i,j})| = 1$ for all i, j , and $T(J)$ has all nonzero entries.

Thus, T induces a permutation σ on the set of subscripts $\{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$, i.e., $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for some scalars $b_{i,j}$.

By repeating the arguments used in the proof of Theorem 4.2, one can obtains that T is a (P, Q, B) -operator.

Linear operators of special types that preserve T_2

Theorem : 4.4

Let S be an antinegative semiring without zero divisors and $T : M_{m,n}(S) \rightarrow M_{m,n}(S)$ be a linear surjective map. Then T preserves the set $T_2(S)$ if and only if T is a (P, Q, B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in Z(S)$ are all units.

Proof :

By Theorem 3.9 we have that T is bijective and $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ for all i, j , $1 \leq i \leq m$, $1 \leq j \leq n$, where $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units and σ is a permutation on the set of pairs (i,j) .

Suppose that the images of two cells are not in the same line, but the cells are in the same line, say $E_{i,j}$ and $E_{i,l}$ are the cells such that $T(E_{i,j}), T(E_{i,l})$ are not in the same line, i.e., $t(T(E_{i,j} + E_{i,l})) = 2$. Then $(E_{i,j}, E_{i,l}) \notin \mathbf{T}_2(\mathbf{S})$, which is a contradiction. Thus T^{-1} maps lines to lines, By Theorem 3.12, it follows that T^{-1} is a (P,Q,B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units. Hence, T is also of this type.

All (P,Q,B) -operators preserve the term rank. Thus they preserve the set $\mathbf{T}_2(\mathbf{S})$.

Linear operators of special types that preserve \mathbf{T}_3 **Theorem : 4.5**

Let \mathbf{S} be an antinegative semiring without zero divisors and $T : \mathbf{M}_n(\mathbf{S}) \rightarrow \mathbf{M}_n(\mathbf{S})$ be a linear surjective map. Then T preserves the set $\mathbf{T}_3(\mathbf{S})$ if and only if T is a nontransposing (P,P^t,B) -operator, where P is a permutation matrix, $B = (b_{i,j})$, and $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units.

Proof :

It is easy to see that all nontransposing (P,P^t,B) -operators preserve $t(A), c(A)$ and $r(A)$, and therefore, since \mathbf{S} is antinegative, preserve the set $\mathbf{T}_3(\mathbf{S})$.

By Theorem 3.9 one has $T(E_{i,j}) = b_{i,j} E_{\sigma(i,j)}$ in this case.

Let us show that T transforms lines to lines. For all k one has $(E_{i,j}, E_{j,k}) \in \mathbf{T}_3(\mathbf{S})$, since

$$t(E_{i,j}E_{j,k}) = t(E_{i,k}) = 1 = \min\{r(E_{i,j}), c(E_{j,k})\}.$$

Thus $t(T(E_{i,j})T(E_{j,k})) = \min\{r(T(E_{i,j})), c(T(E_{j,k}))\} = 1$, since T transforms cells to cells.

But $T(E_{i,j})T(E_{j,k})=b_{i,j}b_{j,k}E_{\sigma(i,j)}E_{\sigma(j,k)}$, so that $E_{\sigma(j,k)}$ is in the same row as $E_{\sigma(j,1)}$ for every k , i.e., T maps rows to rows. Similarly T maps columns to columns, i.e., $T(X)=P(X \circ B)Q$ for some permutation matrices P and Q .

Therefore, $T(E_{i,j})=b_{i,j}E_{\sigma(i),\tau(j)}$, where σ is the permutation corresponding to P and τ is the permutation corresponding to Q^t . But $(E_{1,i}, E_{i,1}) \in T_3(S)$. Thus $(E_{\sigma(1),\tau(i)}, E_{\sigma(i),\tau(1)}) \in T_3(S)$, and hence $\sigma \equiv \tau$, i.e., $Q=P^t$.

Linear operators of special types that preserve T_4

Lemma : 4.6

Let S be an arbitrary semiring and the linear transformation $T : M_n(S) \rightarrow M_n(S)$ preserve the set $T_4(S)$. Then T preserves the set of matrices with term rank n .

Lemma : 4.7

Let S be an antinegative semiring without zero divisors and $T : M_{m,n}(S) \rightarrow M_{m,n}(S)$ be a surjective linear transformation. Then T preserves the set of matrices with term rank n if and only if T is a (P,Q,B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in Z(S)$ are all units.

Theorem : 4.8

Let S be an antinegative semiring without zero divisors and $T : M_n(S) \rightarrow M_n(S)$ be a surjective linear operator. Then T preserves the set $T_4(S)$ if and only if T is a nontransposing (P,P^t,B) -operator, where P is a permutation matrix, $B=(b_{i,j})$, and $b_{i,j} \in Z(S)$ are all units.

Proof :

Let us prove that transformations of the required type preserve the set $T_4(S)$. It is easy to see that all (P,Q,B) -operators preserve the term rank. Thus the

right-hand side of the equality determining $\mathbf{T}_4(\mathbf{S})$ does not change by applying T and it remains to check that $t(\mathbf{XY})=t(\mathbf{P}(\mathbf{X} \circ \mathbf{B})\mathbf{P}^t \mathbf{P}(\mathbf{Y} \circ \mathbf{B})\mathbf{P}^t)$.

But $t(\mathbf{P}(\mathbf{X} \circ \mathbf{B})\mathbf{P}^t \mathbf{P}(\mathbf{Y} \circ \mathbf{B})\mathbf{P}^t) = t((\mathbf{X} \circ \mathbf{B})(\mathbf{Y} \circ \mathbf{B}))=t(\mathbf{XY})$, since \mathbf{S} is an antinegative semiring and all elements of \mathbf{B} are units.

In order to prove that linear preservers of the set $\mathbf{T}_4(\mathbf{S})$ are $(\mathbf{P},\mathbf{Q},\mathbf{B})$ -operators, we remark that by Lemma 4.6 T preserves the set of term-rank n matrices. Thus by applying Lemma 4.7 we obtain that T is a $(\mathbf{P},\mathbf{Q},\mathbf{B})$ -operator, $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are all units.

Let us show that the transformation $T(\mathbf{X})= \mathbf{P}(\mathbf{X} \circ \mathbf{B})^t \mathbf{Q}$ does not preserve the set $\mathbf{T}_4(\mathbf{S})$. Indeed, since similarity preserves $\mathbf{T}_4(\mathbf{S})$, we may consider the transformation $T_1(\mathbf{X})= \mathbf{P}^t \mathbf{P}(\mathbf{X} \circ \mathbf{B})^t \mathbf{Q}\mathbf{P}=(\mathbf{X} \circ \mathbf{B})^t \mathbf{D}$, where $\mathbf{D}=\mathbf{Q}\mathbf{P}$ is a permutation matrix. Let $\mathbf{C}=(c_{i,j})$ and $c_{i,j}=(b_{i,j}^{-1})$ for all i,j . Then for $i \neq j$ we have $(\mathbf{U}=(\mathbf{D}^{-1})^t \mathbf{E}_{i,j}) \circ \mathbf{C}$, $\mathbf{V}=\mathbf{I} \setminus \mathbf{E}_{j,j} \in \mathbf{T}_4(\mathbf{S})$, since $t(\mathbf{E}_{j,i}((\mathbf{I} \setminus \mathbf{E}_{j,j}) \circ \mathbf{B}^t) \mathbf{D})=t(\mathbf{E}_{j,i})=1 \neq 0$.

It remains to prove that $\mathbf{P}\mathbf{Q}=\mathbf{I}$. Let us assume that a $(\mathbf{P},\mathbf{Q},\mathbf{B})$ -operator preserves the set $\mathbf{T}_4(\mathbf{S})$. Then $t(\mathbf{XY})=t(\mathbf{X} \circ \mathbf{B})\mathbf{Q}\mathbf{P}(\mathbf{Y} \circ \mathbf{B})$ for all pairs $(\mathbf{X},\mathbf{Y}) \in \mathbf{T}_4(\mathbf{S})$. The matrix $\mathbf{Q}\mathbf{P}$ is a permutation matrix as a product of two permutation matrices. Assume that $\mathbf{Q}\mathbf{P}$ permutes the i th and j th columns of \mathbf{X} . Let $\mathbf{X}=\mathbf{E}_{i,i}$ and $\mathbf{Y}=\sum_{j \neq i} \mathbf{E}_{j,j}$.

Then $t(\mathbf{X})=1$, $t(\mathbf{Y})=n-1$, and $t(\mathbf{XY})=t(\mathbf{O})=0=t(\mathbf{X})+t(\mathbf{Y})-n$, i.e., $(\mathbf{X},\mathbf{Y}) \in \mathbf{T}_4(\mathbf{S})$. On the other hand, $(\mathbf{X} \circ \mathbf{B})\mathbf{Q}\mathbf{P}=\mathbf{b}_{i,i}\mathbf{E}_{i,j}$. Thus, $(\mathbf{X} \circ \mathbf{B})\mathbf{Q}\mathbf{P}(\mathbf{Y} \circ \mathbf{B})=\mathbf{b}_{i,j}\mathbf{b}_{j,j}\mathbf{E}_{i,j} \neq \mathbf{O}$.

Hence, $(\mathbf{P}(\mathbf{X} \circ \mathbf{B})\mathbf{Q},\mathbf{P}(\mathbf{Y} \circ \mathbf{B})\mathbf{Q}) \notin \mathbf{T}_4(\mathbf{S})$. This contradiction concludes the proof.

Linear operators of special types that preserve \mathbf{T}_5

Lemma :4.9

Let \mathbf{S} be a subsemiring of \mathbf{R}_+ and $T : \mathbf{M}_n(\mathbf{S}) \rightarrow \mathbf{M}_n(\mathbf{S})$ be a linear surjective map. If T preserves the set $\mathbf{T}_5(\mathbf{S})$, then T has the form $T(\mathbf{X})=\mathbf{P}\mathbf{D}\mathbf{X}\mathbf{E}\mathbf{P}^t$, where \mathbf{P} is a permutation matrix and \mathbf{D} and \mathbf{E} are invertible diagonal matrices in $\mathbf{M}_n(\mathbf{S})$.

Lemma : 4.10

If S is any antinegative semiring, A, B are conformal matrices over S , and D is a diagonal matrix whose diagonal has no zero divisors, then $t(ADB)=t(AB)$.

Theorem : 4.11

Let S be a subsemiring of R_+ , $n \geq 4$, and $T : M_n(S) \rightarrow M_n(S)$ be a linear surjective map. If T preserves the set $T_5(S)$ if and only if T has the form $T(X)=PDXEP^t$, where P is a permutation matrix and D and E are invertible diagonal matrices in $M_n(S)$ such that $ED=kI$ for some nonzero $k \in S$.

Proof :

Suppose that T is a surjective linear operator that preserves $T_5(S)$.

By Lemma 4.9, $T(X)=PDXEP^t$, where P is a permutation matrix and D and E are invertible diagonal matrices in $M_n(S)$. We only need to show that $ED=kI$. We denote $DE=\text{diag}\{f_1, f_2, \dots, f_n\}$. Suppose that $ED \neq kI$ for any $k \in S$, so that $f_i \neq f_{i+1}$ for some i . By permuting the i th row / column to the first and the $(i+1)^{st}$ to the second, we may assume without loss of generality that $ED=\text{diag}\{d_1, d_2, \dots, d_n\}$ and $d_1 \neq d_2$.

Let

$$X = \begin{pmatrix} 1 & 0 & 1 & & \\ & 0 & 1 & & \\ & & & O_{2,n-3} & \\ & & & & O_{n-2,n-3} \\ & & O_{n-2,3} & & \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 1 & 1 & 1 & & \\ \frac{d_1}{d_2} & \frac{d_1}{d_2} & \frac{d_1}{d_2} & & \\ 0 & 0 & 1 & & \\ & & & O_{n-4,3} & \\ & & & & O_{n-4,n-3} \end{pmatrix}$$

Now,

$$XY = XYI = \begin{bmatrix} 2 & 4 & 4 & O_{2,n-3} \\ \frac{d_1}{d_2} + 1 & \frac{d_1}{d_2} + 3 & \frac{d_1}{d_2} + 3 & \\ O_{n-2,3} & & & O_{n-2,n-3} \end{bmatrix}$$

so that $t(XYI) = 2$, $t(Y) = 3$, $\rho(XY) = 2$, and $\rho(YI) = 3$, whence $t(XYI) + t(Y) = \rho(XY) + \rho(YI)$. Thus, $(X, Y, I) \in \mathbf{T}_5(\mathbf{S})$. We see that

$$\begin{aligned} t(T(X)T(Y)T(I)) + t(T(Y)) &= t(\text{PD}(\text{XEDY})\text{EDEP}^t) + t(\text{PDYEP}^t) \\ &= t(\text{XEDY}) + t(Y) = t(XY) + t(Y) \end{aligned}$$

$$= 2 + 3 = 5 \text{ in view of lemma 4.10 and the fact}$$

that multiplication with permutation and invertible diagonal matrices on both sides does not change the term rank. Now,

$$\text{EDY} = \begin{bmatrix} d_1 & d_1 & d_1 & \\ d_1 & d_1 & d_1 & O_{4,n-3} \\ d_3 & 3d_3 & 3d_3 & \\ 0 & 0 & d_4 & \\ O_{n-4,3} & & & O_{n-4,n-3} \end{bmatrix}$$

$$\text{and } \text{XEDY} = \begin{bmatrix} d_1 + d_3 & d_1 + 3d_3 & d_1 + 3d_3 & O_{2,n-3} \\ d_1 + d_3 & d_1 + 3d_3 & d_1 + 3d_3 & \\ O_{n-2,3} & & & O_{n-4,n-3} \end{bmatrix}$$

Then $\rho(T(X)T(Y)) + \rho(T(Y)T(I))$

$$= \rho(\text{PD}(\text{XEDY})\text{EP}^t) + \rho(\text{PDYEDEP}^t)$$

$$= \rho(\text{XEDY}) + \rho(\text{YED})$$

$$= \rho(\text{XEDY}) + \rho(Y)$$

$$= 1 + 3 = 4, \text{ since if } \rho(Y) = n, \text{ then } \rho(XY) = \rho(YX) = \rho(X).$$

This implies that $t(T(X)T(Y)T(I)) + t(T(Y)) \neq \rho(T(X)T(Y)) + \rho(T(Y)T(I))$.

That is, $(T(X), T(Y), T(I)) \notin T_5(S)$, which is a contradiction. Thus, $ED = kI$ for some nonzero $k \in S$.

It is easily checked that if $T(X) = PDXEP^t$, where P is a permutation matrix and D and E are invertible diagonal matrices in $M_n(S)$ such that $ED = kI$ for some nonzero $k \in S$, then T preserves $T_5(S)$.

Linear operators of special types that preserve Z_1

Theorem : 4.12

Let S be an antinegative semiring without zero divisors and $T : M_{m,n}(S) \rightarrow M_{m,n}(S)$ be a linear surjective map. Then T preserves the set $Z_1(S)$ if and only if T is a (P, Q, B) -operator, where P is a permutation matrix, $B = (b_{i,j})$, and $b_{i,j} \in Z_1(S)$ are invertible for all i and j .

Proof :

By Theorem 3.9 we have $T(E_{i,j}) = b_{i,j}$ for all i, j , $1 \leq j \leq m$, $1 \leq j \leq n$, where $b_{i,j} \in Z_1(S)$ are all units and σ is a permutation on the set of pairs (i, j) .

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are in the same line, say $E_{i,j}$ and $E_{i,k}$ are the cells such that $T(E_{i,j})$ and $T(E_{i,k})$ are not in the same line.

$$\text{Then one has } z((J \setminus E_{i,j} \setminus E_{i,k}) + E_{i,k}) = 1 = z(J \setminus E_{i,j} \setminus E_{i,k}),$$

$$\text{i.e., } (J \setminus E_{i,j} \setminus E_{i,k}, E_{i,k}) \in Z_1(S),$$

$$\text{while } z(T(J \setminus E_{i,j} \setminus E_{i,k}) + T(E_{i,k})) = 1 < 2 = \min \{z(T(J \setminus E_{i,j} \setminus E_{i,k})), z(T(E_{i,k}))\},$$

i.e., $T(J \setminus E_{i,j} \setminus E_{i,k}), T(E_{i,k}) \notin Z_1(S)$, which is a contradiction. Thus T maps lines to lines.

By Theorem 3.12, it follows that T is a (P, Q, B) -operator where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in Z_1(S)$ are all units.

It is straightforward to check that all (P, Q, B) -operators preserve zero-term rank. Thus they preserve the set $Z_1(S)$.

Linear operators of special types that preserve Z_2

Theorem : 4.13

Let S be an antinegative semiring without zero divisors and $T : M_{m,n}(S) \rightarrow M_{m,n}(S)$ be a linear surjective map. Then T preserves the set $Z_2(S)$ if and only if $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all i,j , $1 \leq i \leq m, 1 \leq j \leq n$, where $b_{i,j} \in Z_2(S)$ are all units and σ is a permutation on the set of pairs (i,j) .

Proof :

By Theorem 3.9 we have $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all i,j , $1 \leq i \leq m, 1 \leq j \leq n$, where $b_{i,j} \in Z_2(S)$ are all units and σ is a permutation on the set of pairs $\{(i,j) | 1 \leq i \leq m, 1 \leq j \leq n\}$.

From antinegativity it follows that if $(X + Y)$ has no zero elements, then the sets of zero cells in X and Y are disjoint. Thus the same holds for $T(X)$ and $T(Y)$ since σ is a permutation. Hence in $(T(X)+T(Y))$ there are no zero elements. Thus all such transformations preserve the set $Z_2(S)$.

Linear operators of special types that preserve Z_3

Theorem : 4.14

Let S be an antinegative semiring without zero divisors and $T : M_n(S) \rightarrow M_n(S)$ be a linear surjective map. Then T preserves the set $Z_3(S)$ if and only if T is a nontransposing (P, P^t, B) -operator, where P is a permutation matrix, $B = (b_{i,j})$, and $b_{i,j} \in Z(S)$ are invertible for all i and j .

Proof :

It is straightforward to check that nontransposing (P, P^t, B) -operators preserve the set $Z_3(S)$.

By Theorem 3.9 we have that T is bijective and $T(E_{i,j}) = b_{i,j} E_{\sigma(i),j}$ for all i, j , $1 \leq i, j \leq n$, where $b_{i,j} \in Z(S)$ are all units and σ is a permutation on the set of pairs (i, j) .

Let us show that T maps lines to lines. Suppose that the images of two cells are in the same line, but the cells are not in the same line, say $E_{i,j}$ and $E_{i,k}$ are the cells such that $T^{-1}(E_{i,j})$ and $T^{-1}(E_{i,k})$ all not in the same line. Let us consider $A = T^{-1}(J \setminus R_i)$. Then there are no zero rows of A , since T is bijective on the set of cells and not all elements of the preimage of the i^{th} row lie in one row by the choice of i . Hence AJ does not have zero elements by the antinegativity and $z(AJ) \neq 0$. Thus $(A, J) \in Z_3(S)$, while $(T(A), T(J)) = (J \setminus R_i, T(J)) \notin Z_3(S)$, which is a contradiction. Thus T^{-1} maps lines to lines. Hence T maps lines to lines.

By Theorem 3.12, it follows that T is a (P, Q, B) -operator where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in Z(S)$ are all units.

In order to prove that the operator $T(X) = (X \circ B)^t D$, where D is a permutation matrix, does not preserve $Z_3(S)$, it suffices to take the pair of matrices $\hat{A} = ((D^{-1})^t C_1) \circ C$, where $C = (c_{i,j})$, $c_{i,j} = b_{i,j}^{-1}$, and $\hat{B} = R_1$. Then $z(\hat{A}\hat{B}) = z((C_1 \circ C) \cdot R_1) = 0$, since D is a permutation matrix. On the other hand, $T(\hat{A}) = (((D^{-1})^t C_1) \circ C) \circ B)^t D = R_1 D^{-1} D = R_1$, $T(\hat{B}) = (R_1 \circ B)^t D = (C_1 \circ B^t) D$. Since $R_1 (C_1 \circ B^t) = \sum_{i=1}^n b_{1,i} E_{1,i}$ has zero-term rank n , it follows that T does not preserve $Z_3(S)$. Hence, $T(X) = P(X \circ B)^t Q$ does not preserve $Z_3(S)$, since similarity preserves $Z_3(S)$.

Let us now show that $Q = P^t$. Suppose that, on the contrary, $PQ \neq I$. Then there exist indices i and j such that PQ transforms the i^{th} column into the j^{th} column. In this case we take the matrices

$\tilde{A} = J \setminus (E_{1,1} + \dots + E_{1,n}) + E_{1,i}$ and $\tilde{B} = J \setminus E_{j,1}$. Then $\tilde{A}\tilde{B}$ has no zero elements, i.e., $z(\tilde{A}\tilde{B}) = 0$. However, the $(1,1)^{\text{th}}$ element of $T(\tilde{A})T(\tilde{B})$ is zero, i.e., $z(T(\tilde{A})T(\tilde{B})) \neq 0$. This contradiction concludes the proof.

Linear operators of special types that preserve \mathbf{Z}_4

Theorem : 4.15

Let \mathbf{S} be an antinegative semiring without zero divisors and $T : \mathbf{M}_n(\mathbf{S}) \rightarrow \mathbf{M}_n(\mathbf{S})$ be a linear surjective map. If T preserves the set $\mathbf{Z}_4(\mathbf{S})$, then T is a non transposing (P, P^t, B) -operator, where P is a permutation matrix, $B = (b_{i,j})$, and $b_{i,j} \in \mathbf{Z}(\mathbf{S})$ are invertible for all i and j .

Proof:

It is straightforward to check that nontransposing (P, P^t, B) -operators preserve the set $\mathbf{Z}_4(\mathbf{S})$.

By Theorem 3.9 we have $T(E_{i,j}) = b_{i,j}E_{\sigma(i,j)}$ for all i, j , $1 \leq i, j \leq n$, where $b_{i,j}$ are all units and σ is a permutation on the set of pairs (i,j) .

Let us show that T maps lines to lines. Suppose that the images of two cells are not in the same line, but the cells are in the same line, $E_{i,j}$ and $E_{i,k}$ are the cells such that $T(E_{i,j})$ and $T(E_{i,k})$ are not in the same line. Note that $z((J \setminus R_1)J) = z(J \setminus R_1) = 1 = 1 + 0 = z(J \setminus R_1) + z(J)$. Thus $(J \setminus R_1, J) \in \mathbf{Z}_4(\mathbf{S})$. On the other hand, $(T(J \setminus R_1), T(J)) \notin \mathbf{Z}_4(\mathbf{S})$ by the arguments used in the proof of Theorem 4.14. This contradiction shows that T maps lines to lines.

By Theorem 3.12, it follows that T is a (P, Q, B) -operator, where P and Q are permutation matrices of appropriate sizes and elements $b_{i,j} \in \mathbf{S}$ are all units.

In order to prove that the operator $T(X) = (X \circ B)^t D$, where D is a permutation matrix, does not preserve $\mathbf{Z}_4(\mathbf{S})$, it suffices to consider the pair of

matrices $\tilde{A} = \left((D^{-1}) (J \setminus R_i) \right) \circ C$, where $C = (c_{i,j})$, $c_{i,j} = b_{i,j}^{-1}$, and $\tilde{B} = J \setminus C_i$. Then $T(X) = P(X \circ B)' Q$ does not preserve $\mathbf{Z}_4(\mathbf{S})$, since similarity preserves $\mathbf{Z}_4(\mathbf{S})$.

Suppose that $Q \neq P'$; then $C_j Q P = C_i$ for some $j \neq i$. But then $z((J \setminus C_i) R_i) = z(O) = n = z(J \setminus C_i) + z(R_i)$, so that $((J \setminus C_i), R_i) \in \mathbf{Z}_4(\mathbf{S})$, while $z((J \setminus C_i) Q P R_i) = z((J \setminus C_j) R_i) = z(J) = 0$, so that $((J \setminus C_i) Q P, R_i) \notin \mathbf{Z}_4(\mathbf{S})$. This contradiction completes the proof.