CHAPTER -VI

Vertex corona product of fan graph with some Classical graphs

In this Chapter, the exact value for the b-chromatic number of vertex corona product of star graph with fan graph, wheel graph with fan graph, fan graph with cycle graph, fan graph with path graph, path graph with fan graph and fan graph with double fan graph are obtained.

6.1 Introduction

Star graph [Balakrishnan, R et al., 2012]

Star graph is a special type of graph in which $n-1$ vertices have degree 1 and a single vertex have degree $n-1$. This looks like $n-1$ vertex is connected to a single central vertex.

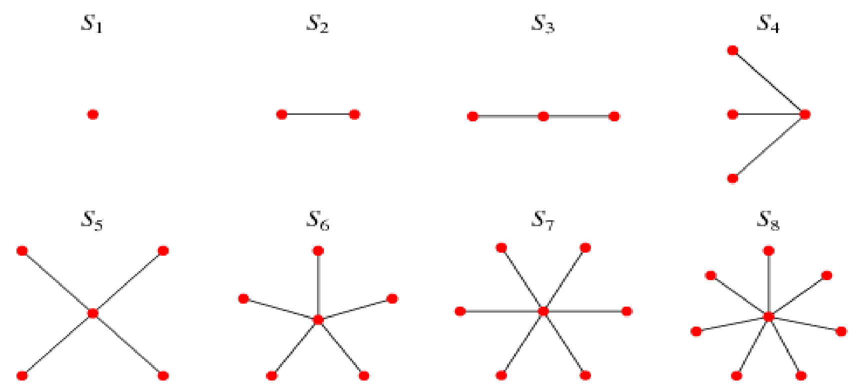

Fig 6.1: Star graph

6.2 The b-chromatic number of b-coloring of vertex corona product of Fan graph with some classical graphs.

Algorithm 6.2.1: The b-coloring of vertex corona product of star graph with fan graph

Input: The number "n" of $K_{1,n} \circ F_{1,n}$

Output: Assigning b -coloring to the vertices of $K_{1,n} \circ F_{1,n}$.

begin

for $i = 1$ to $n+1$

{

$V_1 = \{s_i\}$;

$C(s_i) = i$;

}

for $i = 1$ to $n+1$, $j = 1$ to $n+1$

{

$V_2 = \{f_j^i\}$;

If $i = 1$

$C(f_j^i) = n + 2$

else

$C(f_j^i) = k$, $k = 1$ to n

}

$V = V_1 \cup V_2$;

end.

Theorem 6.2.1: For a Star graph $K_{1,n}$ and a fan graph $F_{1,n}$ the b-chromatic number of the corona product $K_{1,n} \circ F_{1,n}$ is given by $\varphi\left[K_{1,n} \circ F_{1,n}\right] = n+2,\ n \geq 3.$

Proof:

Let $V\left(K_{1,n}\right) = \{s_i : 1 \leq i \leq n+1\}$ and $V\left(F_{1,n}\right) = \{f_i : 1 \leq i \leq n+1\}$

By the definition of corona product each vertex of $K_{1,n}$ is adjacent to every vertex of number of copies of $F_{1,n}$.

i.e., Every vertex $s_i \in V\left(K_{1,n}\right)$ is adjacent to every vertex from the set

$\{f_j^i : 1 \leq i \leq n+1,\ 1 \leq j \leq n+1\}$.

Assign a proper coloring to $V\left(K_{1,n} \circ F_{1,n}\right)$ by using the above algorithm.

From this coloring procedure we have that, the b – chromatic number of corona graph of star graph $K_{1,n}$ with fan graph $F_{1,n}$ is $n+2$.

i.e., $\varphi\left[K_{1,n} \circ F_{1,n}\right] = n+2,\ n \geq 3.$

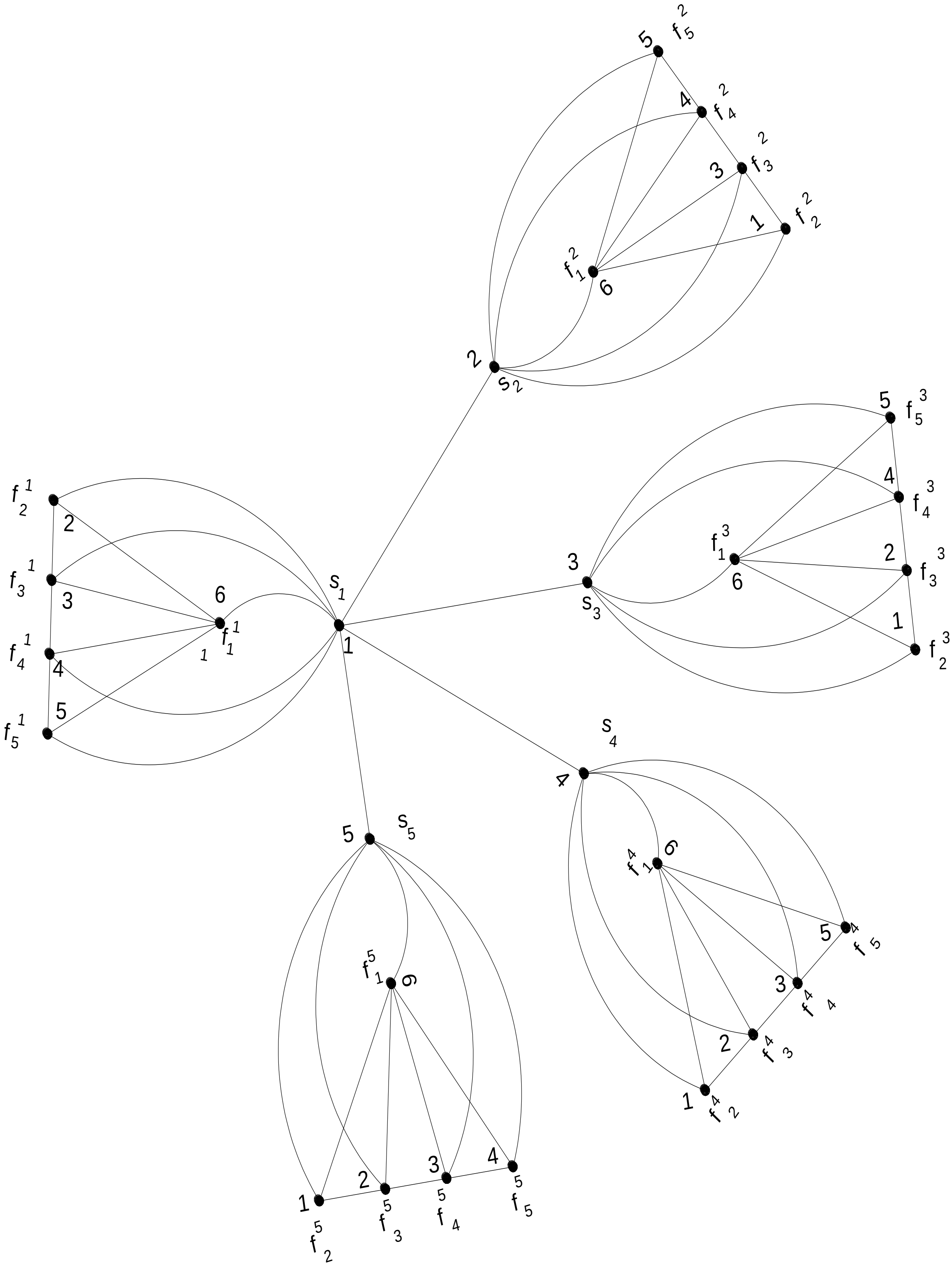

Fig 6.2: $\varphi\left[K_{1,4}\, o F_{1,4}\right]=6,\ n\geq 3.$

Algorithm 6.2.2: The b-coloring of vertex corona product of wheel graph with fan graph

Input: The number "n" of $W_n \circ F_{1,n}$.

Output: Assigning b coloring to the vertices of $W_n \circ F_{1,n}$

begin

for $i = 1$ to n

{

$V_1 = \{w_i\};$

$C(w_i) = i;$

}

for $i = 1$ to $n, \ j = \ n+1$

{

$V_2 = \{f_j^i\};$

If $i = n, \ j = 1$

$C(f_j^i) = n+1;$

else

for $i = 1$ to $n-1, \ j = \ 1$

$C(f_j^i) = n+2;$

}

{

for $i = 1$ to $n-1, \ j = \ 2$ to $n, \ k = 1$ to $n+1$

$C\left(f_j^i\right) = k;$

}

$V = V_1 \cup V_2;$

end.

Theorem 6.2.2: For a wheel graph W_n and a fan graph $F_{1,n}$ the b-chromatic number of the corona graph $W_n \circ F_{1,n}$ is given by $\varphi\left[W_n \circ F_{1,n}\right] = n+2,\ n \geq 4.$

Proof:

Let $V\left(W_n\right) = \left\{w_i : 1 \leq i \leq n\right\}$ and $V\left(F_{1,n}\right) = \left\{f_i : 1 \leq i \leq n+1\right\}$

By the definition of corona product each vertex of W_n is adjacent to every vertex of number of copies of $F_{1,n}$

i.e., Every vertex $w_i \in V\left(W_n\right)$ is adjacent to every vertex from the set

$\left\{f_j^i : 1 \leq i \leq n,\ 1 \leq j \leq n+1\right\}.$

Assign a proper coloring to $V\left(W_n \circ F_{1,n}\right)$ by using the above algorithm.

From this coloring procedure, we have that the b– chromatic number of corona graph of Wheel graph W_n with Fan graph $F_{1,n}$ is $n+2$.

$$\text{i.e., } \varphi\left[W_n \circ F_{1,n}\right] = n+2,\ n \geq 4.$$

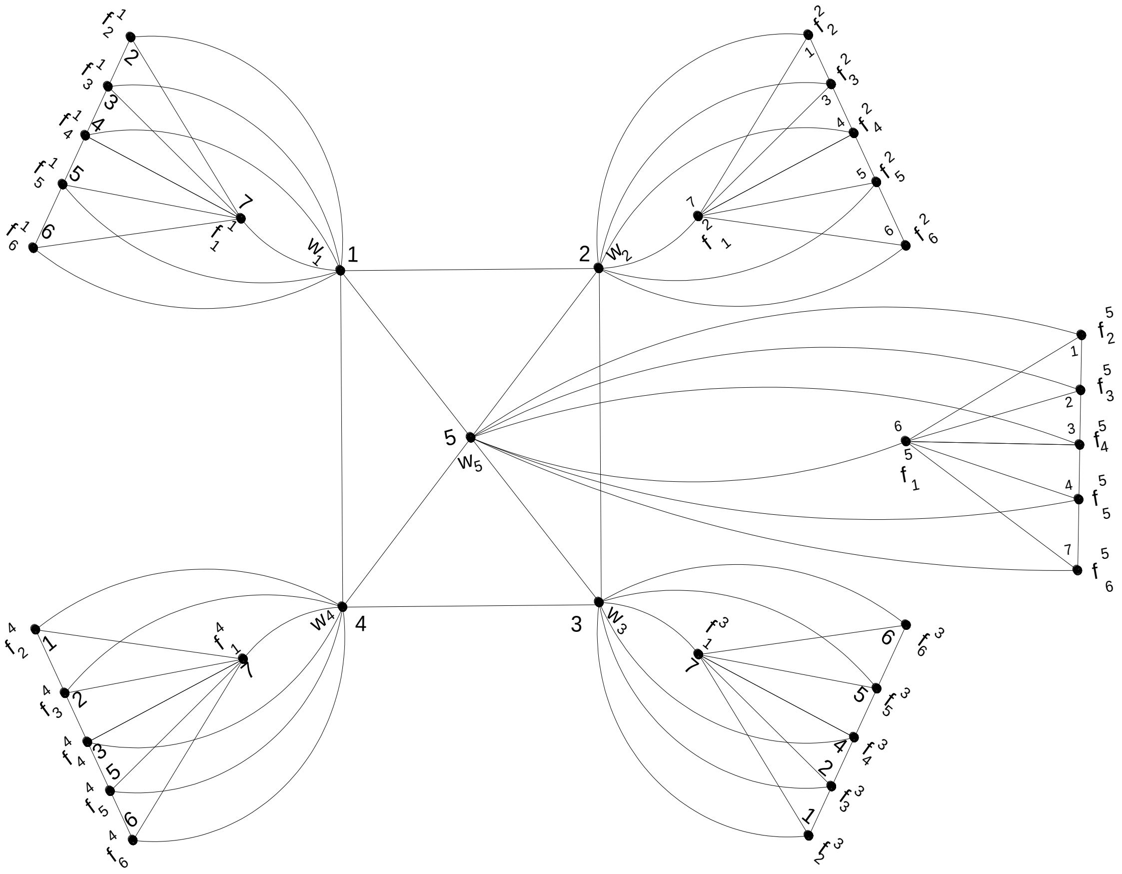

Fig 6.3 $\varphi[W_5 \circ F_{1,5}] = 7, n \geq 4$

Algorithm 6.2.3: The b-chromatic number of the corona product fan graph with cycle graph

Input: The number "n"of $F_{1,n} \circ C_n$.

Output: Assigning b coloring to the vertices of $F_{1,n} \circ C_n$.

begin

for $i = 1$ to n

{

$V_1 = \{w\};$

$C(w) = 1;$

}

{

$V_2 = \{w_i\};$

$C(w_i) = i+1;$

}

{

$V_3 = \{u_i\};$

$C(u_i) = i+1;$

}

{

$V_4 = \{\mathrm{u}_j^{\,i}\};$

$C(\mathrm{u}_j^{\,i}) = k,\ k = 1$ to $i+1;$

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4;$

end.

Theorem 6.2.3: For a fan graph $F_{1,n}$ and a cycle graph C_n the b-chromatic number of the corona product $F_{1,n} \circ C_n$ is given by, $\varphi\left[F_{1,n} \circ C_n\right] = n+1,\ n \geq 1.$

Proof:

Let the vertex set of the fan graph be $V(F_{1,n}) = \{w\} \cup \{u_i : 1 \leq i \leq n\}$

and the vertex set of the cycle graph be $V(C_n) = \{w_i : 1 \leq i \leq n\}$

By the definition of corona graph, each vertex of $F_{1,n}$ is adjacent to every vertex of number of copies of C_n.

i.e., Every vertex $u_i \in V(F_{1,n})$ is adjacent to every vertex from the set

$\{u_j^{\,i} : 1 \le i \le n,\ 1 \le j \le n\}$ and $w \in V(F_{1,n})$ is adjacent to every vertex from the set $\{w_i : 1 \le i \le n\}$.

Then the vertex set $V(F_{1,n} \circ P_n)$ is

$$V(F_{1,n} \circ P_n) = \{w\} \cup \{u_i : 1 \le i \le n\} \cup \{w_i : 1 \le i \le n\} \cup \{u_j^{\,i} : 1 \le i \le n,\ 1 \le j \le n\}$$

Assign a proper coloring to $F_{1,n} \circ C_n$ by using the above algorithm.

From this coloring procedure we have that the b- chromatic number of corona graph of Fan graph $F_{1,n}$ with cycle graph C_n is greater than $n+1$.

i.e., $\varphi[F_{1,n} \circ C_n] \ge n+1$

To prove: $\varphi[F_{1,n} \circ C_n] \le n+1$

Let us assume that $\varphi[F_{1,n} \circ C_n]$ is greater than $n+1$.

i.e., $\varphi[F_{1,n} \circ C_n] = n+2$.

There must be at least $n+2$ vertices of degree $n+1$ in $F_{1,n} \circ C_n$ all with distinct colors and each adjacent to vertices all of the other colors. But there is only $n+1$ vertices with the maximum degree, this is the contradiction, b- coloring with $n+2$ colors is impossible.

Thus, we have $\varphi[F_{1,n} \circ C_n] \le n+1$

Hence $\varphi[F_{1,n} \circ C_n] = n+1$.

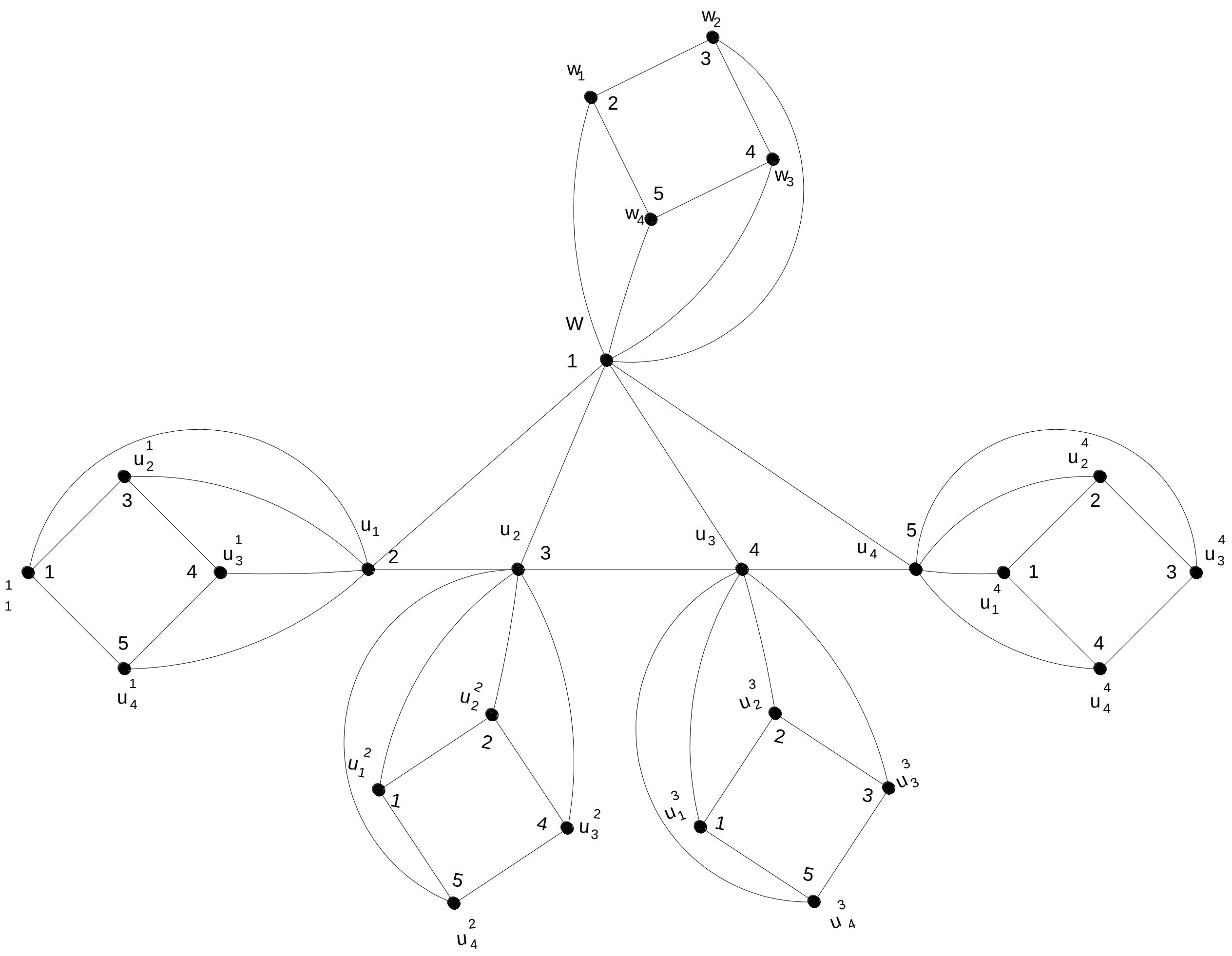

Fig 6.4: $\varphi\left[F_{1,4} \circ C_4\right] = 5,\ n \geq 1.$

Algorithm 6.2.4: The b-chromatic number of corona product of fan graph with path graph

Input: The number "n" of $F_{1,n} \circ P_n$.

Output: Assigning b coloring to the vertices of $F_{1,n} \circ P_n$.

begin

for $i = 1$ to n

{

$V_1 = \{u_i\};$

$C(u_i) = i+1;$

}

{

$V_2 = \{u^i{}_j\};$

$C(u^i{}_j) = s,\ 1 \le s \le i+1;$

}

{

$V_3 = \{w\};$

$C(w) = 1;$

}

{

$V_4 = \{w_i\};$

$C(w_i) = i+1;$

}

$V = V_1 \cup V_2 \cup V_3 \cup V_4;$

end.

Theorem 6.2.4: For a fan graph $F_{1,n}$ and path graph P_n, the b-chromatic number of the corona product $F_{1,n} \circ P_n$ is given by, $\varphi\left[F_{1,n} \circ P_n\right] = n+1,\ n \ge 1.$

Proof:

Let the vertex set of the fan graph be $V(F_{1,n}) = \{w\} \cup \{u_i : 1 \le i \le n\}$ and

the vertex set of the path graph be $V(P_n) = \{w_i : 1 \le i \le n\}$.

By the definition of corona product each vertex of $F_{1,n}$ is adjacent to every vertex of number of copies of P_n.

i.e., every vertex $u_i \in V(F_{1,n})$ is adjacent to every vertex from the set

$\{u^i{}_j : 1 \le i \le n,\ 1 \le j \le n\}$ and $w \in V(F_{1,n})$ is adjacent to every vertex from the set $\{w_i : 1 \le i \le n\}$.

Then the vertex set of the corona product $F_{1,n} \circ P_n$ is

$$V(F_{1,n} \circ P_n) = \{w\} \cup \{u_i : 1 \le i \le n\} \cup \{w_i : 1 \le i \le n\} \cup \{u_j{}^i : 1 \le i \le n, 1 \le j \le n\}$$

Assign the coloring to $F_{1,n} \circ P_n$ as per the algorithm.

By this coloring procedure, we get that

$$\varphi[F_{1,n} \circ P_n] \ge n+1$$

To prove the lower bound

i.e., $\varphi[F_{1,n} \circ P_n] \le n+1$

Let us assume that, the b-chromatic number of corona product of fan graph $F_{1,n}$ with path cycle graph P_n is greater than $n+1$.

That is the b-chromatic number of $F_{1,n} \circ P_n$ is equal to $n+2$.

There must be at least $n+2$ vertices of degree $n+1$ in $F_{1,n} \circ P_n$, all with distinct colors, and each adjacent to vertices of all of the other colors. But there is only $n+1$vertices with maximum degree which is the contradiction.

Therefore, assigning $n+2$ colors is impossible.

Thus, we have $\therefore\ \varphi[F_{1,n} \circ P_n] \le n+1$.

Hence $\varphi[F_{1,n} \circ P_n] = n+1$.

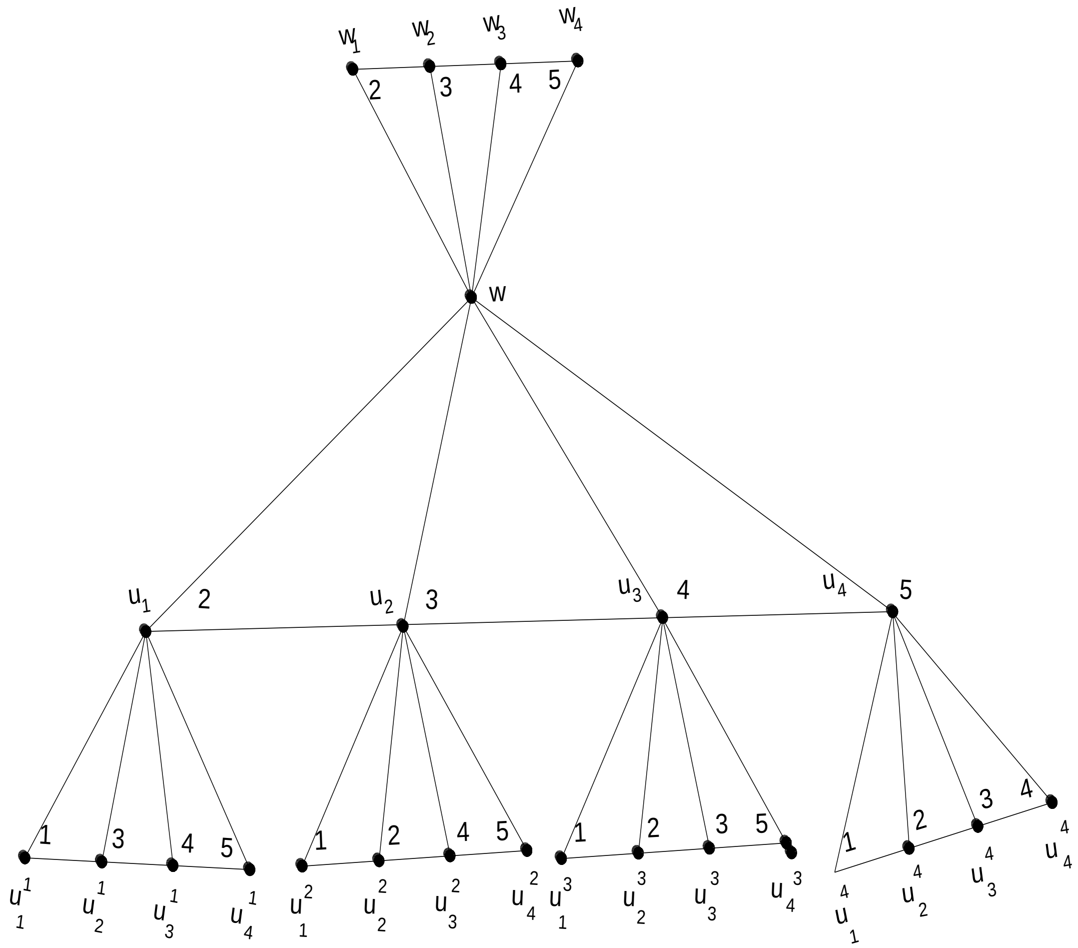

Fig 6.5: $\varphi\left[F_{1,4} \circ P_4\right] = 5,\ n \geq 1$.

Theorem 6.2.5: For any positive number n, the b-chromatic number of the corona product of path graph with fan graph is $\varphi[P_n \circ F_{1,n}] = \begin{cases} n+2, & n=2 \\ n+1, & n>2 \end{cases}$

Proof:

Let $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $V(F_{1,n}) = \{w\} \cup \{v^i{}_j : 1 \leq i \leq n,\ 1 \leq j \leq n\}$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of number of copies of $F_{1,n}$.

i.e., every vertex $v_i \in V(P_n)$ is adjacent to every vertex from the set

$\{u^i{}_j : 1 \leq i \leq n,\ 1 \leq j \leq n\}$ and $w \in V(F_{1,n})$ is adjacent to every vertex from the $\{w_i : 1 \leq i \leq n\}$

Let $V\left(P_n \circ F_{1,,n}\right)=\{v_i : 1\le i\le n\} \cup \{w_i : 1\le i\le n\} \cup \{v_j^{\mathrm{i}} : 1\le i\le n,\ 1\le j\le n\}$

Consider the following cases.

Case 1: For $n=2$

Assign the following $n+2$ colors as b-chromatic number for $P_2 \circ F_{1,2}$

- For v_1 , assign the color c_1
- For v_2 , assign the color c_2
- For w_1 , assign the color c_3
- For w_2 , assign the color c_3
- For ${v^1}_1$, assign the color c_2
- For ${v^1}_2$, assign the color c_4
- For ${v^2}_1$, assign the color c_2
- For ${v^2}_2$, assign the color c_4

The above shows that this coloring is a b- coloring.

Case 2: For $n\ge 3$

Assign the following $n+1$colors as b-chromatic number for $P_n \circ F_{1,n}$

- For $v_i : 1\le i\le n$,assign the color c_i
- For $w_i : 1\le i\le n$,assign the color c_{n+1}
- For ${v^i}_j$,assign the colors $c_i : 1\le i\le n+2$, except the colors of vertices

w_i & $v_i,\ 1\le i\le n.$

It shows that this coloring is a b-coloring.

The b-coloring procedure depends the number of vertices having maximum degree in the path graph P_n which has degree n. so that, we can assign maximum $n+1$ colors to get b- coloring.

Hence the proof.

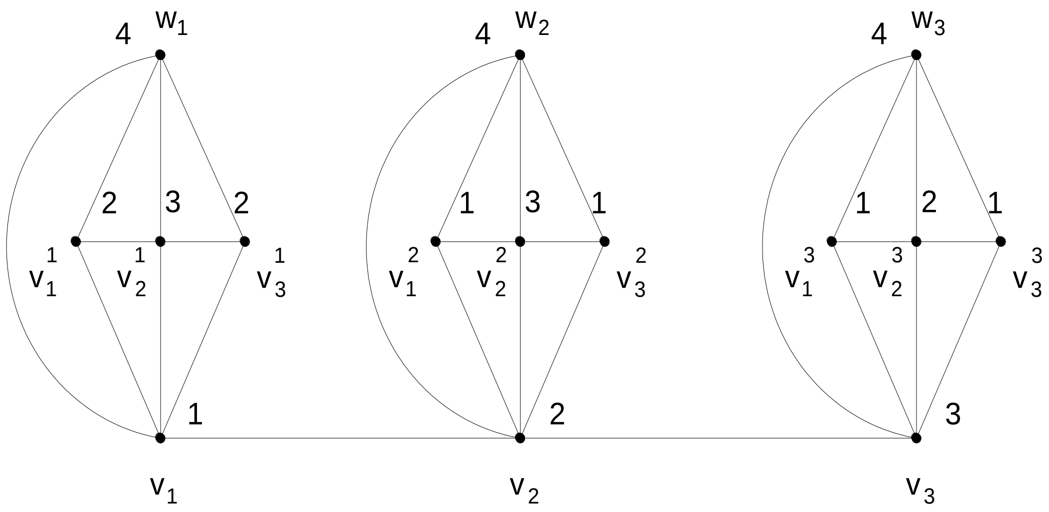

Fig 6.6: $\varphi\left(P_3 \circ F_{1,3}\right) = 4$

Theorem 6.2.6: For any positive number n, the b-chromatic number of the corona graph of path graph with double fan graph is,

$$\varphi\left[P_n \circ F_{2,n}\right] = \begin{cases} n+3, & n \le 3 \\ n+2, & n \ge 4 \end{cases}$$

Proof:

Let $V\left(P_n\right) = \{p_i : 1 \le i \le n\}$ and $V\left(F_{2,n}\right) = \left\{x_i \cup y_i \cup x^i{}_j : 1 \le i \le n,\ 1 \le j \le n\ \right\}$

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of number of copies of $F_{2,n}$.

i.e., every vertex $p_i \in V\left(P_n\right)$ is adjacent to every vertex from the set

$\left\{x_i, y_i, x^i{}_j : 1 \le i \le n,\ 1 \le j \le n\ \right\} \in V\left(F_{2,n}\right)$

Let

$V\left(P_n \circ F_{2,,n}\right) = \{p_i : 1 \le i \le n\} \cup \{x_i : 1 \le i \le n\} \cup \{y_i :\ 1 \le j \le n\} \cup \left\{x_{\mathrm{j}}^{\mathrm{i}} : 1 \le i \le n,\ 1 \le j \le n\right\}$

Consider the following cases.

Case 1: For $n \leq 3$

Assign the following $n+3$ colors as b-chromatic number for $\left(P_n \circ F_{2,n}\right)$.

Now assign the following $n+3$ colors to $P_n \circ F_{2,n}$

- For $p_i : 1 \leq i \leq n$, assign the color c_i
- For $x_i : 1 \leq i \leq n-1$, assign the color c_{n+3}
- For x_n, assign the color c_{n+2}
- For $y_i : 1 \leq i \leq n$, assign the color c_3
- For $x^i_j : 1 \leq i \leq n$, $1 \leq j \leq n$, assign the color c_k, $1 \leq k \leq n+3$

It follows that $\varphi\left[P_n \circ F_{2,n}\right] \geq n+3$.To prove the lower bound, let us assume that $\varphi\left[P_n \circ F_{2,n}\right] > n+3$.

To assign $n+4$ colors, there should be $n+4$ vertices with $n+3$ distinct colors which are adjacent to each other. But there is no such possibility in the graph, which is the contradiction.

Thus $\varphi\left[P_n \circ F_{2,n}\right] \leq n+3$. Hence $\varphi\left[P_n \circ F_{2,n}\right] = n+3$.

Case 2: For $n \geq 4$.

Assign the following $n+2$ colors as b-chromatic number for $P_n \circ F_{2,n}$

- For $p_i : 1 \leq i \leq n$, assign the color c_i
- For $x_i : 1 \leq i \leq n-1$, assign the color c_{n+2}
- For $y_i : 1 \leq i \leq n-1$, assign the color c_{n+2}
- For x_n assign the color c_{n+1}
- For y_n assign the color c_{n+1}
- For $x^i_j : 1 \leq i \leq n-1$, $1 \leq j \leq n-1$, assign the color c_k, $1 \leq k \leq n$.

This shows that this coloring is a b- coloring.

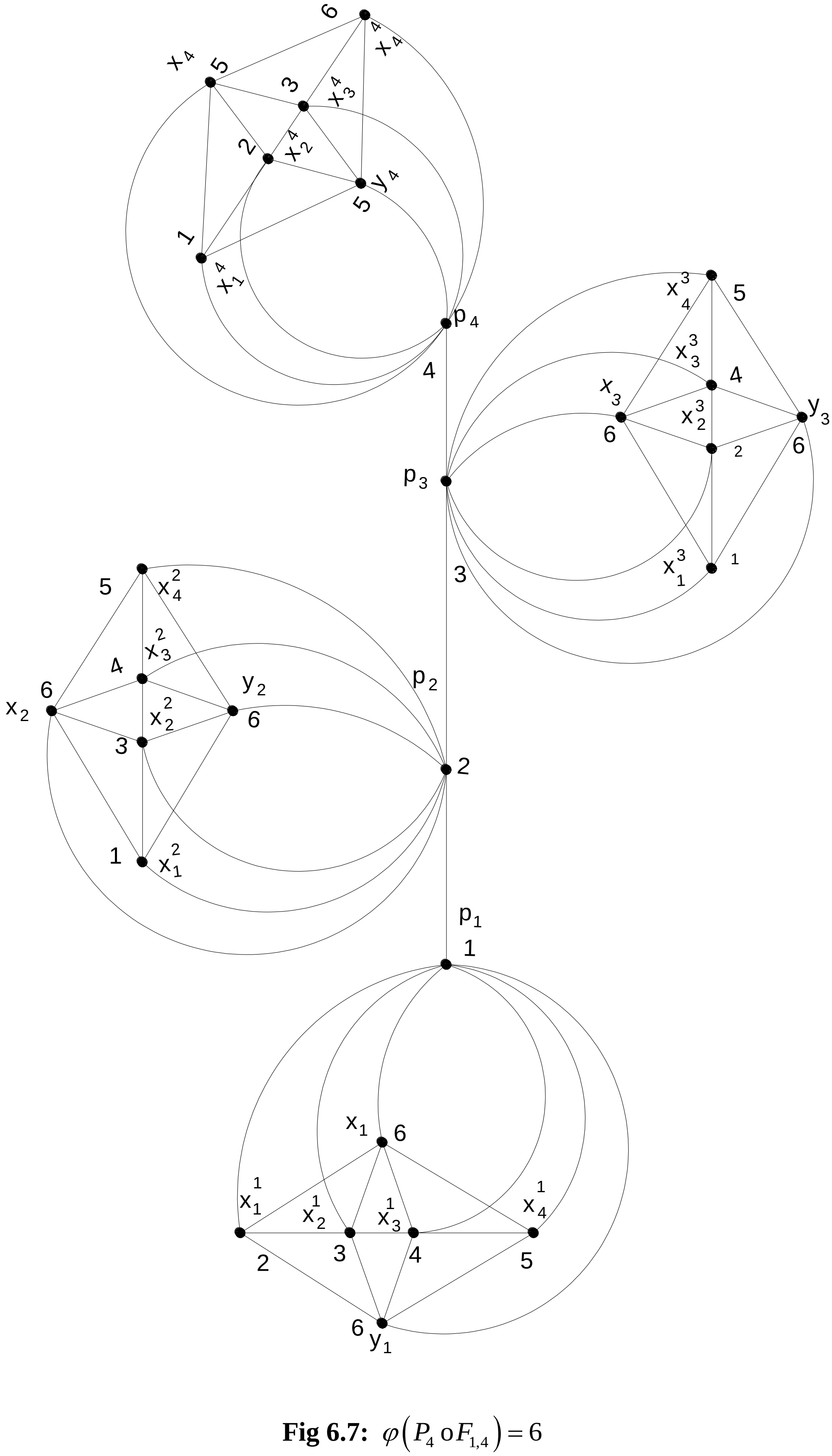

Fig 6.7: $\varphi\left(P_4 \circ F_{1,4}\right) = 6$