Chapter II

CHAPTER-2

REGULAR GENERALIZED STAR CLOSED SETS IN BITOPOLOGICAL SPACES

In this chapter $\tau_1\tau_2$- regular generalized star closed sets, $\tau_1\tau_2$- regular generalized star open sets in bitopological spaces and their basic properties, characterization in bitopological spaces are studied.

SECTION: 2.1

PRELIMINARIES

Definition: 2.1.1

Let (X, τ_1, τ_2) be a bitopological space .The intersection of all τ_i-semi closed sets containing A is called **τ_i-semi closure** of A, denoted by τ_i-scl(A).

Definition: 2.1.2

Let (X, τ_1, τ_2) be a bitopological space .The union of all τ_i-semi open sets containing A is called **τ_i-semi interior** of A, denoted by τ_i-sint(A).

Notation: 2.1.3

The closure and interior of B related to A with respect to topology τ_i are written as τ_i-cl_A(B) and τ_i-int_B(A) respectively.

Notation: 2.1.4

For a subset $A \subseteq X$, τ_i-rint(A) and τ_i-rcl(A) denote the regular interior and regular closure of a set A with respect to topology τ_i respectively.The regular closure and regular interior of B related to A with respect to topology τ_i are written as τ_i-$rcl_A(B)$ and τ_i-$rint_B(A)$ respectively

Definition: 2.1.5

A set A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$ - **regular closed** if τ_1 - $cl[\tau_2$- $int(A)] = A$.

Defnition: 2.1.6

A set A of a bitopological space (X, τ_1, τ_2) is called $\boldsymbol{\tau_1\tau_2}$ - **regular open** if τ_1- $int[\tau_2$- $cl(A)] = A$.

Defnition: 2.1.7

A set A of a bitopological space (X, τ_1, τ_2) is called $\boldsymbol{\tau_1\tau_2}$ **-regular generalized closed** ($\boldsymbol{\tau_1\tau_2}$ – rg closed) in X if τ_2 - $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$- regular open in X.

Defnition: 2.1.8

A set A of a bitopological space (X, τ_1, τ_2) is called $\boldsymbol{\tau_1\tau_2}$**- regular generalized open** ($\tau_1\tau_2$– rg open) in X if $F \subseteq \tau_2$- int(A) whenever $F \subseteq A$ and f is $\tau_1\tau_2$- regular closed in X.

SECTION: 2.2

$\boldsymbol{\tau_1\tau_2}$- REGULAR GENERALIZED STAR CLOSED SETS

Defnition: 2.2.1

A subset A of a bitopological space (X, τ_1, τ_2))is called $\boldsymbol{\tau_1\tau_2}$**- regular generalized star closed ($\boldsymbol{\tau_1\tau_2}$– rg* closed)** in X if and only if τ_2 - $rcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\tau_1\tau_2$- regular open in X.

Example: 2.2.2

Let $X = \{a, b, c\}$, $\tau_1 = \{\phi, X, \{a\}\}$, $\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then all subsets in P(X) are $\tau_1\tau_2$– rg* closed sets in (X, τ_1, τ_2).

Theorem: 2.2.3

Let A be a subset of a bitopological space (X, τ_1, τ_2). If A is $\tau_1\tau_2$- rg* closed then τ_2 - rcl (A) – A does not contain any non empty $\tau_1\tau_2$- regular closed sets.

Proof:

Let A be $\tau_1\tau_2$– rg* closed. Let F be a $\tau_1\tau_2$- regular closed set such that $F \subseteq \tau_2$ - rcl(A)–A.We shall show that $F=\phi$. Since $F \subseteq \tau_2$ -rcl(A)–A, $F \subseteq [\tau_2 -\text{rcl}(A)] \cap A^C$. Consequently $F \subseteq A^C$ and $F \subseteq \tau_2$ - rcl(A). Since $F \subseteq A^C$, $A \subseteq F^C$. Since F is $\tau_1\tau_2$- regular closed set, F^C is $\tau_1\tau_2$- regular open. Since A is $\tau_1\tau_2$–rg* closed, τ_2- rcl(A) $\subseteq F^C$, Thus, $F \subseteq [\tau_2- \text{rcl}(A)]^C$ $= X - [\tau_2- \text{rcl}(A)]$.Hence $F \subseteq \phi$. But $\phi \subseteq F$.

Therefore, $F = \phi$

Theorem: 2.2.4

(a) Suppose that a subset A is $\tau_1\tau_2$ – rg closed and it is $\tau_1\tau_2$- semi open. i.e) $A \subseteq \tau_2$ - cl[τ_1 - int(A)], then A is $\tau_2\tau_1$- regular closed in X if and only if τ_2 -cl[τ_1- int(A)] – A is $\tau_1\tau_2$- regular closed in X.

(b) Let a subset A be a $\tau_1\tau_2$–rg* closed set. Then A is τ_2 - closed in X if and only if τ_2 - cl(A) – A is $\tau_1\tau_2$- regular closed in X.

Proof:

(a) Let A be $\tau_2\tau_1$- regular closed in X. Then A = τ_2 - cl[τ_1 -int(A)]. Consequently τ_2- cl [τ_1 - int(A)]–A = ϕ Therefore, τ_2- cl[τ_1- int(A)]–Ais $\tau_1\tau_2$- regular closed in X.

Conversely, let τ_2 - cl[τ_1- int(A)] – A be $\tau_1\tau_2$- regular closed in X. We shall show that A is $\tau_2\tau_1$- regular closed in X. Obviously, τ_1 - int (A) $\subseteq$ A

Consequently τ_2 - cl[τ_1 - int(A)] $\subseteq$ τ_2- cl(A).

Hence τ_2- cl[τ_1- int(A)] – A $\subseteq$ τ_2- cl(A) – A. Since A is $\tau_1\tau_2$– rg closed in X, we have τ_2 - cl(A) – A does not contain non empty $\tau_1\tau_2$- regular closed set. Hence τ_2- cl [τ_1- int(A)] – A = ϕ. Therefore, τ_2- cl [τ_1- int(A)] $\subseteq$ A. Since A is $\tau_1\tau_2$ - semi open, A $\subseteq$ τ_2- cl[τ_1- int(A)]. Hence τ_2- cl [τ_1- int(A)] = A.Therefore, A is $\tau_2\tau_1$- regular closed.

(b) Let A be $\tau_1\tau_2$– rg* closed. Let A be τ_2 - closed. We shall show that τ_2 - cl (A)–A is $\tau_1\tau_2$- regular closed in X. Since A is τ_2 - closed, τ_2- cl(A) = A. Consequently, τ_2-cl(A)–A = ϕ. Therefore, τ_2 - cl(A) - A is $\tau_1\tau_2$- regular closed in X.

Conversely, Let τ_2 - cl(A) – A be $\tau_1\tau_2$ - regular closed in X. We shall show that A is τ_2 - closed. Since τ_2 - cl(A) $\subseteq$ τ_2 - rcl(A), τ_2-cl(A)–A $\subseteq$ τ_2 - rcl(A)–A for any subset A of X. Since A is $\tau_1\tau_2$–rg* closed, τ_2 - cl(A) – A = ϕ. Hence τ_2 - cl(A) = A. Consequently, A is τ_2 - closed.

Remark: 2.2.5

The semi openess of A can not be removed from Theorem 2.2.4(a) in general as can be seen from the following example.

Example: 2.2.6

Let X = {a, b, c}, τ_1= {ϕ, X, {a}, {a, b}}, τ_2= { ϕ,X, {b}, {b, c}}. Then A = {a, c} is $\tau_1\tau_2$- rg closed but not $\tau_1\tau_2$- semi open in X. Also

τ_2 -cl[τ_1- int (A)] – A = ϕ is $\tau_1\tau_2$- regular closed. But A is not $\tau_2\tau_1$ - regular closed set in X.

Theorem: 2.2.7

If A and B are $\tau_1\tau_2$– rg* closed sets then A $\cup$ B is $\tau_1\tau_2$– rg*closed.

Proof:

Let A and B are $\tau_1\tau_2$– rg* closed sets. We shall show that A $\cup$ B is $\tau_1\tau_2$– rg* closed. Let A $\cup$ B $\subseteq$ U and U is $\tau_1\tau_2$- regular open. Since A $\cup$ B $\subseteq$ U, A $\subseteq$ U and B $\subseteq$ U. Since A $\subseteq$ U and U is $\tau_1\tau_2$- regular open, τ_2 - rcl(A) $\subseteq$ U. {since A is $\tau_1\tau_2$– rg* closed}. Since B $\subseteq$ U and U is $\tau_1\tau_2$- regular open, τ_2 - rcl(B) $\subseteq$ U. {since B is $\tau_1\tau_2$– rg* closed}. Therefore, {τ_2 - rcl(A)} $\cup$ {τ_2 - rcl(B)} $\subseteq$ U $\cup$ U. Since [τ_2 - rcl(A)] $\cup$ [τ_2- rcl(B)]=τ_2- rcl(A $\cup$ B), τ_2- rcl(A $\cup$B) $\subseteq$ U. Hence A $\cup$ B is $\tau_1\tau_2$– rg* closed.

Remark: 2.2.8

The intersection of two $\tau_1\tau_2$– rg* closed sets need not be a $\tau_1\tau_2$– rg* closed set in general as can be seen from the following example.

Example: 2.2.9

In Example 2.2.6, A = {a, b}, B = {a, c} are $\tau_1\tau_2$– rg* closed sets, but A$\cap$B = {a} is not $\tau_1\tau_2$– rg* closed set in X.

Lemma: 2.2.10

Let A be a τ_1- open set in (X, τ_1, τ_2) and let U be $\tau_1\tau_2$- regular open in A. Then U = A $\cap$W for some $\tau_1\tau_2$- regular open set W in X.

Proof:

Let A be a τ_1- open set in (X, τ_1, τ_2) and let U be $\tau_1\tau_2$- regular open in A. We shall show that $U = A \cap W$ for some $\tau_1\tau_2$- regular open set W in X. Since U is $\tau_1\tau_2$- regular open in A, we have

$$\begin{aligned} U &= \tau_1 - int_A[\tau_2 - cl_A(U)] \\ &= \tau_1 - int_A[A \cap \tau_2 - cl(U)] \\ &= A \cap \{\tau_1 - int[A \cap \tau_2 - cl(U)]\} \\ &= A \cap \{\tau_1 - int(A) \cap [\tau_1 - int\{\tau_2 - cl(U)\}]\} \\ &= A \cap \{A \cap [\tau_1 - int\{\tau_2 - cl(U)]\}, \text{ since A is } \tau_1 \text{ - open} \\ &= A \cap A \cap [\tau_1 - int\{\tau_2 - cl(U)\}] \\ &= A \cap [\tau_1 - int\{\tau_2 - cl(U)\}] \\ &= A \cap W. \end{aligned}$$

where $W = [\tau_1 - int\{\tau_2 - cl(U)\}]$. Then $U = A \cap W$ for some $\tau_1\tau_2$- regular open set W in X

Lemma: 2.2.11

$x \in \tau_2$- rcl (A) if and only if $U \cap A \neq \phi$ for every $\tau_1\tau_2$- regular open set U containing x

Proof:

Let $x \in \tau_2$- rcl (A).We shall show that $U \cap A \neq \phi$ for every $\tau_1\tau_2$ - regular open set U containing x. Suppose that there exists a $\tau_1\tau_2$ - regular open set U containing x such that $U \cap A = \phi$.Then $A \subseteq U^C$ and U^C is $\tau_1\tau_2$– regular closed set. Since $A \subseteq U^C$, τ_2- rcl (A) $\subseteq$ τ_2- rcl (U^C). Since $x \in \tau_2$ - rcl (A), $x \in \tau_2$- rcl (U^C). Since U^C is $\tau_1\tau_2$- regular closed set, $x \in U^C$. Hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \phi$. Hence $U \cap A \neq \phi$ for every $\tau_1\tau_2$- regular open set U containing x.

Conversely, suppose that $U \cap A \neq \phi$. for every $\tau_1\tau_2$- regular open set U containing x. We shall show that $x \in \tau_2$- rcl (A). Suppose that $x \notin \tau$ - rcl (A). Then there exists a $\tau_1\tau_2$- regular open set U containing x such that $U \cap A = \phi$. This is a contradiction to $U \cap A \neq \phi$. Hence $x \in \tau_2$- rcl (A).

Lemma: 2.2.12

If A is $\tau_1\tau_2$- open and U is $\tau_1\tau_2$- regular open in X then $U \cap A$ is $\tau_1\tau_2$- regular open in A.

Proof:

Let A be $\tau_1\tau_2$- open and U is $\tau_1\tau_2$- regular open in X. We shall show that $U \cap A$ is $\tau_1\tau_2$- regular open in A.

Now,

$$\begin{aligned}
\tau_1- int_A[\tau_2- cl_A(U \cap A)] &= \tau_1- int[\tau_2- clA(U \cap A)] \cap A \\
&\supseteq \tau_1- int[\tau_2- cl_A(U \cap A) \cap A] \cap A \\
&= \tau_1- int[\tau_2- cl(U \cap A)] \cap A \\
&\supseteq \tau_1- int[\tau_2- cl(U) \cap A] \cap A \\
&= \tau_1- int[\tau_2- cl(U)] \cap \tau_1- int(A) \cap A \\
&= \tau_1- int[\tau_2- cl(U)] \cap A \cap A \\
&= U \cap A
\end{aligned}$$

since $U = \tau_1- int[\tau_2- cl(U)]$. Hence $U \cap A \subseteq \tau_1$ - $intA[\tau_2$ - $clA(U \cap A)]$.

Now,

$$\begin{aligned}
U \cap A &= \tau_1- int[\tau_2- cl(U)] \cap \tau_1- int(A) \\
&= \tau_1- int[\tau_2- cl(U) \cap A)] \\
&\supseteq \tau_1- int[\tau_2- cl(U \cap A) \cap A] \text{ since } U \cap A \subseteq A \\
&= \tau_1- int[\tau_2- cl_A(U \cap A)] \\
&\supseteq \tau_1- int[\tau_2- cl_A(U \cap A)] \cap A \\
&= \tau_1- int_A[\tau_2- cl(U \cap A)]
\end{aligned}$$

Hence τ_1- $int_A[\tau_2$- $cl_A(U \cap A)] \subseteq U \cap A$.

Therefore, τ_1- $int_A[\tau_2$- $cl_A(U \cap A)] = U \cap A$.

Hence $U \cap A$ is $\tau_1\tau_2$- regular open in A.

Lemma: 2.2.13

If A is $\tau_1\tau_2$- open in (X, τ_1, τ_2) then τ_2- $rcl_A(B) \subseteq A \cap \tau_2$- rcl(B) for any subset B of A.

Proof:

Let A be $\tau_1\tau_2$- open in (X, τ_1, τ_2) We shall show that τ_2 - $rcl_A(B) \subseteq A \cap \tau_2$- rcl(B) for any subset B of A. Let $B \subseteq A$ and $x \in \tau_2$ - $rcl_A(B)$. Since τ_2 -$rcl_A(B) \subseteq A$, $x \in A$. Let U be a $\tau_1\tau_2$- regular open in X such that $x \in U$.Then by Lemma 3.12, $A \cap U$ is $\tau_1\tau_2$- regular open in A such that $x \in U \cap A$. Since $x \in \tau_2$ - $rcl_A(B)$, $(U \cap A) \cap B \neq \phi$ {by Lemma 2.2.11}. Hence $U \cap B \neq \phi$. {since $B \subseteq A$}. Therefore, $U \cap B \neq \phi$ for every $\tau_1\tau_2$– regular open in U of X containing x. Hence $x \in \tau_2$ - rcl(B). Therefore $x \in A \cap \tau_2$ - rcl(B). Consequently, τ_2- $rcl_A(B) \subseteq A \cap \tau_2$- rcl(B) for any subset B of A.

Lemma : 2.2.14

If A is $\tau_1\tau_2$- open in (X, τ_1, τ_2) then $A \cap \tau_2$ - rcl(B) $\subseteq \tau_2$- $rcl_A(B)$ for any subset B of A.

Proof:

Let A be $\tau_1\tau_2$- open in (X, τ_1, τ_2) We shall show that $A \cap \tau_2$- rcl(B) $\subseteq \tau_2$- $rcl_A(B)$ for any subset B of A. Let $B \subseteq A$ and $x \in A \cap \tau_2$- rcl(B). Then $x \in A$ and $x \in \tau_2$- rcl(B). Let U be a $\tau_1\tau_2$- regular open subset of A such that $x \in U$.Then by Lemma 2.2.10, there exists a $\tau_1\tau_2$- regular open subset W of X such that $U = A \cap W$.

Since $x \in U$, $x \in A \cap W$. Hence , $x \in A$ and $x \in W$.Since $x \in \tau_2$-rcl(B) and W is $\tau_1\tau_2$- regular open subset in X, we have $W \cap B \neq \phi$. Now, $U \cap B = (A \cap W) \cap B = W \cap (A \cap B) = W \cap B \neq \phi$. {since $B \subseteq A$}. Hence $U \cap B \neq \phi$ for any $\tau_1\tau_2$- regular open subset U of A such that $x \in U$. Therefore, $x \in \tau_2$- $rcl_A(B)$. Hence $A \cap \tau_2$- rcl(B) $\subseteq \tau_2$- $rcl_A(B)$ for any subset B of A.

Theorem: 2.2.15

Let $B \subseteq A$ where A is $\tau_1\tau_2$- regular open, $\tau_2\tau_1$- regular open and $\tau_1\tau_2$– rg* closed. Then B is $\tau_1\tau_2$– rg* closed relative to A if and only if B is $\tau_1\tau_2$– rg* closed in X.

Proof:

Let $B \subseteq A$ where A is $\tau_1\tau_2$- regular open, $\tau_2\tau_1$- regular open and $\tau_1\tau_2$–rg*closed. Suppose that B is $\tau_1\tau_2$– rg* closed relative to A. We shall show that B is $\tau_1\tau_2$– rg* closed in X. Let $B \subseteq U$ and U is $\tau_1\tau_2$- regular open in X. Since A is $\tau_1\tau_2$- regular open and $\tau_2\tau_1$- regular open in X, A is $\tau_1\tau_2$ - open in X. Since U is $\tau_1\tau_2$- regular open in X, $A \cap U$ is $\tau_1\tau_2$- regular open in A {by Lemma 2.2.12}. Since $B \subseteq U$ and $B \subseteq A$, we have $B = B \cap B \subseteq U \cap A$. Hence $B \subseteq U \cap A$ and $A \cap U$ is $\tau_1\tau_2$- regular open in A. Since B is $\tau_1\tau_2$– rg* closed relative to A,

$$\tau_2\text{– rclA(B)} \subseteq A \cap U \qquad (1)$$

Since $A \subseteq A$ and A is $\tau_1\tau_2$- regular open in X,

$$\tau_2\text{– rcl(A)} \subseteq A \qquad (2)$$

Since A is $\tau_1\tau_2$–rg* closed in X. Since $B \subseteq A$,τ_2- rcl(B) $\subseteq \tau_2$-rcl(A). Hence τ_2- rcl(B)$\subseteq$A {by (2)}.Therefore,

$$\tau_2\text{– rcl(B)} \cap A = \tau_2 – \text{rcl(B)}. \qquad (3)$$

Since A is $\tau_1\tau_2$- open in X,

τ_2- rcl (B) $\cap$ A = τ_2- rcl_A(B) {by Lemma 2.2.13, Lemma 2.2.14}.

Therefore, τ_2 - rcl(B) = τ_2 - rcl_A(B).

Hence τ_2 - rcl(B) $\subseteq$ A $\cap$ U. Therefore, B is $\tau_1\tau_2$– rg* closed in X.

Conversely, let B be $\tau_1\tau_2$– rg* closed in X. We shall show that B is $\tau_1\tau_2$– rg* closed relative to A. Let B $\subseteq$ U and U is $\tau_1\tau_2$- regular open in A. Since A is $\tau_1\tau_2$- regular open and $\tau_2\tau_1$- regular open in X,A is $\tau_1\tau_2$- open in X. Since A is τ_1- open in X and U is $\tau_1\tau_2$- regular open in A, U = A $\cap$ W for some $\tau_1\tau_2$- regular open set W in X {By Lemma 5.2.10}. Since A is $\tau_1\tau_2$- open in X and W is $\tau_1\tau_2$- regular open in X, U = A $\cap$W is $\tau_1\tau_2$- regular open set in X {by Lemma 5.2.12}. Hence B $\subseteq$ U and U is $\tau_1\tau_2$- regular open set in X. Since B is $\tau_1\tau_2$– rg* closed in X, τ_2- rcl(B) $\subseteq$U.

Therefore τ_2- rcl(B) $\cap$ A $\subseteq$ A $\cap$ U. Since U $\subseteq$ A,

$$\tau_2\text{– rcl(B)} \cap A \subseteq U. \qquad (4)$$

Since A is $\tau_1\tau_2$- open in X,τ_2- rcl(B) $\cap$A = τ_2- rcl_A(B){ by Lemma 5.2.13, Lemma 5.2.14}. Hence τ_2- rcl_A(B) $\subseteq$ U {by (4)}. Therefore B is $\tau_1\tau_2$– rg* closed relative to A.

Theorem: 2.2.16

Let A and B be subsets such that A $\subseteq$B $\subseteq$ τ_2- rcl(A). If A is $\tau_1\tau_2$– rg* closed, then B is $\tau_1\tau_2$– rg* closed.

Proof:

Let A and B be subsets such that A $\subseteq$ B $\subseteq$ τ_2- rcl(A). Suppose that A is $\tau_1\tau_2$– rg* closed. We shall show that B is $\tau_1\tau_2$– rg* closed. Let B $\subseteq$ U and U is $\tau_1\tau_2$- regular open in X. Since A $\subseteq$B and B $\subseteq$U; we have A$\subseteq$U.

Hence A $\subseteq$U and U is $\tau_1\tau_2$- regular open in X. Since A is $\tau_1\tau_2$– rg* closed, we have

$$\tau_2\text{– rcl}(A) \subseteq U. \tag{5}$$

Since B $\subseteq$ τ_2- rcl(A); we have τ_2- rcl(B) $\subseteq$ τ_2- rcl [τ_2- rcl (A)] = τ_2– rcl (A) $\subseteq$ U { by (5)}. Hence τ_2- rcl (B) $\subseteq$U. Therefore, B is $\tau_1\tau_2$–rg* closed.

Notation : 2.2.17

The set of all τ_2- regular closed sets in X is denoted by τ_2- R.C (X, τ_1, τ_2).

The set of all τ_1 τ_2- regular open sets in X is denoted by $\tau_1\tau_2$- R.O (X, τ_1,τ_2).

Theorem: 2.2.18

Suppose that $\tau_1\tau_2$- R.O (X, τ_1, τ_2) $\subseteq$ τ_2- R.C (X, τ_1, τ_2) Then every subset of X is $\tau_1\tau_2$– rg* closed.

Proof:

Suppose that $\tau_1\tau_2$- R.O (X, τ_1, τ_2) $\subseteq$ τ_2- R.C (X, τ_1, τ_2). Let A be a subset of X. We shall show that A is $\tau_1\tau_2$– rg* closed. Let A $\subseteq$ U and U is $\tau_1\tau_2$- regular open in X. Since $\tau_1\tau_2$- R.O (X, τ_1, τ_2) $\subseteq$ τ_2- R.C (X, τ_1, τ_2), U is τ_2- regular closed in X. Then τ_2- rcl (U) = U. Since A $\subseteq$ U, τ_2- rcl(A) $\subseteq$ τ_2- rcl (U) = U. Therefore, τ_2- rcl (A) $\subseteq$ U.
Hence A is $\tau_1\tau_2$– rg* closed.

SECTION: 2.3

$\tau_1\tau_2$- REGULAR GENERALIZED STAR OPEN SETS

Defnition: 2.3.1

A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$- **regular generalized star open** ($\tau_1\tau_2$–rg* open) in X if and only if its complement is $\tau_1\tau_2$- regular generalized star closed ($\tau_1\tau_2$– rg* closed) in X

Example: 2.3.2

Let X = {a, b, c}, τ_1= {ϕ, X , {a}}, τ_2= { ϕ,X, {a}, {b, c}} Then all subsets in P(X) are $\tau_1\tau_2$– rg* open sets in (X, τ_1, τ_2)

Theorem: 2.3.3

A subset A of a bitopological space (X, τ_1, τ_2) is $\tau_1\tau_2$–rg* open if and only if F $\subseteq$ τ_2- rint (A) whenever F $\subseteq$ A and F is $\tau_1\tau_2$- regular closed in X.

Proof:

Suppose that A is $\tau_1\tau_2$– rg* open. We shall show that F $\subseteq$ τ_2- rint (A) whenever F $\subseteq$ A and F is $\tau_1\tau_2$- regular closed in X. Let A $\subseteq$ F and F is $\tau_1\tau_2$ regular closed in X. Then $A^C \subseteq F^C$ and F^C is $\tau_1\tau_2$- regular open in X. Since A is $\tau_1\tau_2$– rg* open, we have A^C is $\tau_1\tau_2$– rg* closed. Hence τ_2- rcl (A^C) $\subseteq F^C$. Consequently, [τ_2- rint $(A)]^C \subseteq F^C$. Therefore F $\subseteq$ τ_2- rint (A). Conversely, suppose that F $\subseteq$ τ_2- rint (A) whenever F $\subseteq$ A and F is $\tau_1\tau_2$- regular closed in X. We shall show that A is $\tau_1\tau_2$– rg* open. Let $A^C \subseteq$ U and U is $\tau_1\tau_2$- regular open in X. Then $U^C \subseteq$ A and U^C is $\tau_1\tau_2$- regular closed in X. By our assumption, we have

$U^C \subseteq \tau_2$- rint (A). Hence $[\tau_2$- rint $(A)]^C \subseteq U$. Therefore τ_2- rcl $(A^C) \subseteq U$. Consequently A^C is $\tau_1\tau_2$– rg* closed. Hence A is $\tau_1\tau_2$– rg* open.

Theorem: 2.3.4

Let A and B be subsets such that τ_2- rint (A) $\subseteq$ B$\subseteq$ A. If A is $\tau_1\tau_2$– rg* open, then B is $\tau_1\tau_2$– rg* open.

Proof:

Suppose that A and B are subsets such that τ_2- rint (A) $\subseteq$ B $\subseteq$ A. Let A be $\tau_1\tau_2$–rg* open. We shall show that B is $\tau_1\tau_2$–rg* open. Let F $\subseteq$ B and F is $\tau_1\tau_2$- regular closed in X. Since F $\subseteq$ B and B $\subseteq$ A, F $\subseteq$ A. Therefore,

F $\subseteq$ τ_2- rint (A) {Since A is $\tau_1\tau_2$– rg*open}. Since τ_2- rint (A) $\subseteq$ B,

$\Rightarrow$ τ_2- rint [τ_2- rint (A)] $\subseteq$ τ_2- rint (B)

$\Rightarrow$ τ_2- rint (A) $\subseteq$ τ_2- rint(B).

$\Rightarrow$ F $\subseteq$ τ_2- rint (B).

$\Rightarrow$ B is $\tau_1\tau_2$– rg* open.

Theorem: 2.3.5

If a subset A is $\tau_1\tau_2$–rg* closed, then τ_2- rcl (A) – A is $\tau_1\tau_2$– rg* open.

Proof:

Suppose that A is $\tau_1\tau_2$– rg* closed. We shall show τ_2- rcl (A)–A is $\tau_1\tau_2$– rg* open. Let F $\subseteq$ τ_2- rcl (A) – A and F is $\tau_1\tau_2$- regular closed. Since A is $\tau_1\tau_2$– rg* closed, we have τ_2- rcl (A) – A does not contain nonempty $\tau_1\tau_2$- regular closed {by Theorem 5.2.3}.

$\Rightarrow F = \phi$

$\Rightarrow \phi \subseteq \tau_2$- rcl (A) – A

$\Rightarrow \tau_2$- rint (Á) $\subseteq \tau_2$- rint [τ_2- rcl (A) – A]

$\Rightarrow \phi \subseteq \tau_2$- rint [$\tau_2$- rcl (A) – A]

$\Rightarrow$F $\subseteq \tau_2$- rint [τ_2- rcl (A) – A]

$\Rightarrow \tau_2$- rcl (A) – A is $\tau_1\tau_2$– rg* open.