Chapter 7

J-Quotient Maps and J-Homeomorphisms in Topological Spaces

§ 7.1. Introduction

In this literature of Mathematics, quotient maps are generally called strong continuous maps or identification maps, because of strong conditions of continuity i.e. $f^{-1}(U)$ is open in (Y,ζ) iff U is open in (Z,σ) of a surjective function f: $(Y,\zeta) \rightarrow (Z,\sigma)$ and their importance for the philosophy of gluing. A Homeomorphism f : A$\rightarrow$ B is a bijection function where A and B are subsets of Euclidean space such that both f and f^{-1} are continuous. At the point when such f exists, A and B are supposed to be homeomorphic to one another. A property of an object which is invariant under Homeomorphism is supposed to be topological in character. The concept of generalized Homeomorphisms and gc-Homeomorphisms were presented by Maki (1991). In this chapter, J-Homeomorphisms and $J\mathcal{C}$-Homeomorphisms are initiated. The interrelationships of these newly introduced Homeomorphisms with various Homeomorphisms are examined and the relations of J-Homeomorphisms with various existing Homeomorphisms are analysed. The composition of two $J\mathcal{C}$-Homeomorphisms is a $J\mathcal{C}$-Homeomorphisms. The set $J\mathcal{C}\hbar(Y,\zeta)$ is a group under the composition of functions. The $J\mathcal{C}$-Homeomorphism f : $(Y,\zeta) \rightarrow (Z,\sigma)$ induces an isomorphism from the group $J\mathcal{C}\hbar(Y,\zeta)$ onto the group $J\mathcal{C}\hbar(Z,\sigma)$.

§ 7.2. J-Closed Functions

Definition 7.2.1. A function f : $(Y,\zeta)\rightarrow (Z,\sigma)$ is said to be **J-closed function** if the image of every closed set in (Y,ζ) is J-closed in (Z,σ).

Example 7.2.2. Let f : $(Y,\zeta)\rightarrow (Z,\sigma)$ be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p},{q},{p,q}}.Here JC(Z,σ)= {Z, ϕ, {r}, {q,r}, {p,r}}and ζ^c = {Y, ϕ, {q,r}}.Then f is a J-closed function.

Proposition 7.2.3. A closed (δ-closed, δg*-closed, g-closed) function f :(Y,ζ) →(Z,σ) is a J-closed function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a closed(resp.δ-closed, δg*-closed,g-closed) function. Let U be any closed set in (Y,ζ).Since f is a closed (resp.δ-closed, δg*-closed, g-closed) function, f(U) is closed (resp.δ-closed, δg*-closed, g-closed) in (Z,σ).By **Proposition 2.3.2.,2.3.4.,2.3.6.and 2.3.10.,**f(U) is J-closed in (Y,ζ).Hence f is a J-closed function.

The converse of the above Propositions do not hold good. It can be seen from the following Counter Examples.

Counter Example 7.2.4. Let f: (Y,ζ) → (Z,σ) be the many-one into function defined by f(p) = p, f(q) = p, f(r) = r. Consider Y = Z = {p,q,r} with ζ = {Y,ϕ,{p}} and σ = {Z, ϕ,{p,q}}.Here σ^c = {Z, ϕ,{r}} and ζ^c= {Y, ϕ, {q,r}}.Since for the closed set {q,r} in (Y,ζ),the corresponding image is {p,r}which is not closed (resp.δ-closed) in (Z,σ). As JC(Z,σ) = P(Z), it is a J-closed function but not a closed (resp.δ-closed) function.

Counter Example 7.2.5. Let f: (Y,ζ) → (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y,ϕ,{p,q}} and σ = {Z, ϕ,{p}}.Here δg*C(Z,σ) = {Z, ϕ,{q,r}} and ζ^c= {Y, ϕ, {r}}.Since for the closed set {r} in (Y,ζ),the image is {r} is not δg*-closed in (Z,σ). As JC(Z,σ) = P(Z) −{p},it is a J-closed function but not a δg*-closed function.

Counter Example 7.2.6. Let f: (Y,ζ) → (Z,σ) be the many-one into function defined by f(p) = q, f(q) = p, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y,ϕ,{p}} and σ = {Z, ϕ,{p},{p,q}}.Here gC(Z,σ) = {Z, ϕ,{r},{p,r},{q,r}} and ζ^c= {Y, ϕ, {q,r}}.Since for the closed set {q,r} in (Y,ζ),the image is{p,q} is not g-closed in (Z,σ). As JC(Z,σ) = P(Z) −{p}, it is a J-closed function but not a g-closed function.

Proposition 7.2.7. A J-closed function f :(Y,ζ) →(Z,σ) is a πg-closed(resp.πgp-closed, πgsp-closed, πgs-closed, πgα-closed) function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a J-closed function. Let U be any closed set in (Y,ζ).Since f is a J-closed function, f(U) is a J-closed set in (Z,σ).By **Proposition 2.3.29., (resp.Proposition 2.3.31., Proposition 2.3.33., Proposition 2.3.35., Proposition**

2.3.37.,) f(U) is πg-closed(resp.πgp-closed, πgsp-closed, πgs-closed, πgα-closed**)** in (Y,ζ).Hence f is a πg-closed function.

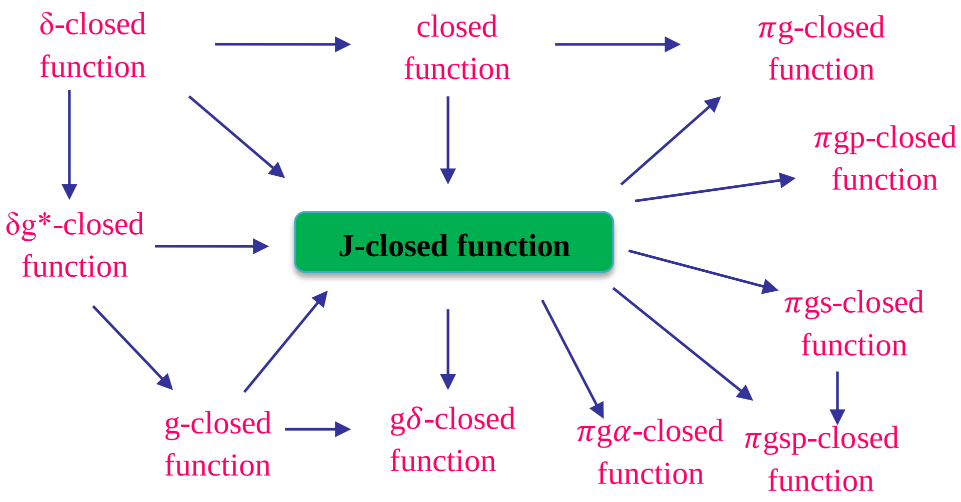
Counter Example 7.2.8. Let f: (Y,ζ) → (Z,σ) be the many-one into function defined by f(p) = q, f(q) = p, f(r) = p. Consider Y = Z = {p,q,r} with ζ = {Y,ϕ,{p}} and σ = {Z, ϕ,{p},{p,q}}.Here πgC(Z,σ) = πgpC(Z,σ) = πgspC(Z,σ) = πgsC(Z,σ) = πgαC(Z,σ) = P(Z) and ζ^c= {Y, ϕ, {q,r}}.Since for the closed set {q,r} in (Y,ζ),the image is {p} which is πg-closed(resp. πgp-closed, πgsp-closed, πgs-closed, πgα-closed) in (Z,σ).As JC(Z,σ) = P(Z) –{p}, f is not a J-closed function but f is a πg-closed(resp. πgp-closed, πgsp-closed, πgs-closed, πgα-closed) function.

Proposition 7.2.9. A J-closed function f :(Y,ζ) →(Z,σ) is a gδ-closed function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a J-closed function. Let U be any closed set in (Y,ζ).Since f is a J-closed function, f(U) is a J-closed set in (Z,σ).By **Proposition 2.3.12.,**f(U) is gδ-closed in (Y,ζ).Hence f is a gδ-closed function.

Counter Example 7.2.10. Let f: (Y,ζ) → (Z,σ) be the many-one into function defined by f(p) = q, f(q) = p, f(r) = p. Consider Y = Z = {p,q,r} with ζ = {Y,ϕ,{p}} and σ = {Z, ϕ,{p},{p,q}}.Here gδC(Z,σ) = P(Z) and ζ^c= {Y, ϕ, {q,r}}. Therefore it is a gδ-closed function, since for the closed set {q,r} in (Y,ζ),the image of {q,r} gives {p}which is not J-closed in (Z,σ) and as JC(Z,σ) = P(Z) –{p},f is not a J-closed function.

Note 7.2.11. From the above discussions, the following diagram can be picturised as below.



Composition of J-Closed Functions

Remark 7.2.12. The composition of two J-closed functions need not be J-closed as observed from the following Counter Example.

Counter Example 7.2.13. Let f : (Y,ζ)⟶ (Z,σ) be the function defined by f(p) = q,f(q) = p, f(r)= p. Consider Y = Z = P ={p,q,r} with ζ = {Y,ϕ,{p},{q},{p,q}},σ = {Z,ϕ,{p,q}} and μ={P,ϕ,{p}}, μ^c={P,ϕ,{q,r}}.Here ζ^c = {Y,ϕ,{r},{q,r},{p,r}}.Then f is a J-closed function as JC(Z,σ)=P(Z). Let g : (Z,σ) ⟶(P,μ) be an identity function. Also g is a J-closed function as JC(P,μ) = {P,ϕ,{r},{q},{p,q},{q,r},{p,r}}.Also consider the composition function g∘f :(Y,ζ) ⟶(P,μ) such that (g∘f)(p) = g(f(p)) = q,(g∘f)(q) = g(f(q)) = p and (g∘f)(r) = g(f(r)) = p. But their composition g∘f is not a J-closed function, because for the closed set {q,r} in (Y,ζ), g∘f ({q,r}) ={p} not J-closed in (P,μ) as JC(P,μ)={P, ϕ,{r},{p,r},{q,r},{p,q},{q}}.

Proposition 7.2.14. If f : (Y,ζ)⟶ (Z,σ) is a δ-closed function and g : (Z,σ) ⟶(P,μ) is a J-closed function, then g ∘ f : (Y,ζ)⟶ (P, μ) is a J-closed function.

Proof Given f : (Y,ζ)⟶ (Z,σ) is a δ-closed function. Let U be any closed set in (Y,ζ). Then f(U) is δ-closed in (Z,σ). In general, δ-closed is closed. Hence f(U) is closed in (Z,σ). Given g : (Z,σ) ⟶(P,μ) is a J-closed function. Hence g(f(U)) is J-closed in (P,μ). So (g ∘ f)(U) = g(f(U)) is J-closed in (P,μ). Hence g ∘ f is a J-closed function.

Proposition 7.2.15. If f : (Y,ζ)→ (Z,σ) is a closed function and g : (Z,σ) →(P,μ) is a J-closed function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-closed function.

Proof Given f : (Y,ζ)→ (Z,σ) is a closed function. Let U be any closed set in (Y,ζ). Then f(U) is closed in (Z,σ).Given g : (Z,σ) →(P,μ) is a J-closed function. Hence g(f(U)) is J-closed in (P,μ). So (g ∘ f)(U) = g(f(U)) is J-closed in (P,μ). Hence g ∘ f is a J-closed function.

Remark 7.2.16. In the above Propositions, if f is a J-closed function and g is a closed (resp.δ-closed) function then their composition need not be a J-closed function which can be seen from the following Counter Example.

Counter Example 7.2.17. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = P ={p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z, ϕ,{p},{q,r}}, μ={P,ϕ,{p},{q},{p,q}}.Then f is a J-closed function as JC(Z,σ) = P(Z). Let g : (Z,σ) →(P,μ) be the many-one into function defined by g(p) = r, g(q) = r, g(r) = p. Then g is a closed (resp. δ-closed) function as μ^c={P,ϕ,{q,r},{p,r}{r}}(resp. δC(P,μ) = {P, ϕ,{r},{p,r},{q,r}}). Consider the composition function g∘f :(Y,ζ)→(P,μ) such that (g∘ f)(p) = g(f(p)) =r,(g∘ f)(q) = g(f(q)) = r and (g∘ f)(r) = g(f(r)) = p. But their composition g∘ f is not a J-closed function, because for the closed set {r} in (Y,ζ),their image is {p} not a J-closed set as JC(P,μ)={P, ϕ,{r},{p,r},{q,r}} in (P,μ).

Proposition 7.2.18. If f : (Y,ζ)→ (Z,σ) is a J-closed function and g : (Z,σ) →(P,μ) is a J-closed function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-closed function when (Z,σ) is a JTC-space.

Proof Given f : (Y,ζ)→ (Z,σ) is a J-continuous function. Let U be any closed set in (Y,ζ).Hence f(U) is J-closed in (Z,σ). When (Z,σ) is a JTC-space, f(U) is closed in (Z,σ). Given g : (Z,σ) →(P,μ) is a J-continuous function. Then g(f(U)) is J-closed in (Y,ζ). Hence g ∘ f is a J-closed function.

Proposition 7.2.19. If f : (Y,ζ)→ (Z,σ) is a J-closed function and g : (Z,σ) →(P,μ) is a J-closed function, then g ∘ f : (Y,ζ)→ (P, μ) is a J-closed function when (Z,σ) is a JTδ-space.

Proof Given f : $(Y,\zeta)\rightarrow(Z,\sigma)$ is a J-continuous function. Let U be any closed set in (Y,ζ).Hence f(U) is J-closed in (Z,σ). When (Z,σ) is a JTδ-space, f(U) is δ-closed in (Z,σ).In general δ-closed is closed.Given g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-continuous function. Then g(f(U)) is J-closed in (Y,ζ). Hence $g\circ f$ is a J-closed function.

Proposition 7.2.20. A function f : $(Y,\zeta)\rightarrow(Z,\sigma)$ is a J-closed function if g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-irresolute injective function and $g\circ f$: $(Y,\zeta)\rightarrow(P,\mu)$ is a J-closed function.

Proof Let U be any closed set in (Y,ζ). Since $g\circ f:(Y,\zeta)\rightarrow(P,\mu)$ is a J-closed function, $g\circ f(U)$ = g(f(U)) is a J-closed set in (P,μ).Given g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-irresolute function. Hence $g^{-1}(g(f(U)))$ is J-closed in (Z,σ).Since g is a injective function, $g^{-1}(g(f(U)))$ =f(U) is J-closed in (Z,σ).Hence f is a J-closed function.

Proposition 7.2.21. A function f : $(Y,\zeta)\rightarrow(Z,\sigma)$ is a J-continuous function if g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-closed injective function and $g\circ f$: $(Y,\zeta)\rightarrow(P,\mu)$ is a J-irresolute function.

Proof Let U be any closed set in (Z,σ). Since g is a J-closed function, g(U) is a J-closed function in (P,μ).Since $g\circ f:(Y,\zeta)\rightarrow(P,\mu)$ is a J-irresolute function, $(g\circ f)^{-1}(g(U))$ = $f^{-1}(g^{-1}(g(U)))$= $f^{-1}(U)$ is a J-closed set in (Y,ζ) when g is a injective function. Hence f is a J-continuous function.

Proposition 7.2.22. A function g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-closed function if f : $(Y,\zeta)\rightarrow(Z,\sigma)$ is a surjective continuous function and $g\circ f$: $(Y,\zeta)\rightarrow(P,\mu)$ is a J-closed function.

Proof Let U be any closed set in (Z,σ). Since f is a continuous function, then $f^{-1}(U)$ is a closed set in (Y,ζ).Since $g\circ f:(Y,\zeta)\rightarrow(P,\mu)$ is a J-closed function, $g\circ f(f^{-1}(U))$ = g(U) is a J-closed set in (P,μ). Hence g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-closed function.

Proposition 7.2.23. A function g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-closed function when (Y,ζ) is a JTC-space if f : $(Y,\zeta)\rightarrow(Z,\sigma)$ is a surjective J-continuous function and $g\circ f$: $(Y,\zeta)\rightarrow(P,\mu)$ is a J-closed function.

Proof Let U be any closed set in (Z,σ). Since f is a J-continuous function,then $f^{-1}(U)$ is a J-closed set in (Y,ζ).Consider (Y,ζ) is a JTC-space, then $f^{-1}(U)$ is a closed set in

(Y,ζ).Since $g \circ f : (Y,\zeta) \rightarrow (P, \mu)$ is a J-closed function, $g \circ f(f^{-1}(U)) = g(U)$ is a J-closed set in (P, μ). Hence $g : (Z,\sigma) \rightarrow (P,\mu)$ is a J-closed function.

Proposition 7.2.24. A function $g : (Z,\sigma) \rightarrow (P,\mu)$ is a J-closed function when (Y,ζ) is a $T_{1/2}$-space if $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is a surjective g-continuous function and $g \circ f : (Y,\zeta) \rightarrow (P, \mu)$ is a J-closed function.

Proof Let U be any closed set in (Z,σ). Since f is a g-continuous function,then $f^{-1}(U)$ is a g-closed set in (Y,ζ).Consider (Y,ζ) is a $T_{1/2}$-space,then $f^{-1}(U)$ is a closed set in (Y,ζ).Since $g \circ f : (Y,\zeta) \rightarrow (P, \mu)$ is a J-closed function, $g \circ f(f^{-1}(U)) = g(U)$ is a J-closed set in (P, μ). Hence $g : (Z,\sigma) \rightarrow (P,\mu)$ is a J-closed function.

Proposition 7.2.25. A function $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is a closed function if $g : (Z,\sigma) \rightarrow (P,\mu)$ is a quasi J-continuous function, injective and $g \circ f : (Y,\zeta) \rightarrow (P, \mu)$ is a J-closed function.

Proof Let U be any closed set in (Y,ζ).Since $g \circ f : (Y,\zeta) \rightarrow (P, \mu)$ is a J-closed function,then g(f(U)) is a J-closed set in (P, μ).A function $g : (Z,\sigma) \rightarrow (P,\mu)$ is a quasi J-continuous function and injective, $g^{-1}(g(f(U))) = f(U)$ is a closed set in (Z,σ). Hence a function $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is a closed function.

§ 7.3. Properties of J-Closed Functions

Theorem 7.3.1. A function $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is a J-closed function and V is a closed subset of (Y,ζ),then $f|_V : (V, \zeta|_V) \rightarrow (Z,\sigma)$ is a J-closed function.

Proof Let $U \subseteq V$ be a closed set in $(V, \zeta|_V)$. Since V is closed in (Y,ζ), U is closed in (Y,ζ). Since f is a J-closed function, $f(U) = (f|_V)(U)$ is J-closed in (Z,σ). Hence $f|_V$ is a J-closed function.

Theorem 7.3.2. A function $f : (Y,\zeta) \rightarrow (Z, \sigma)$ is J-closed if and only if for each subset G of (Z, σ) and for each open set U of (Y,ζ) containing $f^{-1}(G)$, there exists a J-open set B of (Z, σ) such that $G \subseteq B$ and $f^{-1}(B) \subseteq U$.

Proof Let f be a J-closed function and let G be a subset of (Z,σ), U be an open set of (Y,ζ) containing $f^{-1}(G)$. Then $Y - U$ is closed in (Y,ζ). Since f is J-closed, $f(Y - U)$ is a J-closed set in (Z, σ). Hence $Z - f(Y - U)$ is a J-open set in (Z, σ).Take $B = Z - f(Y - U)$. Then B is J-open in (Z, σ) containing G such that $f^{-1}(B) \subseteq U$.

Conversely, Let F be a closed subset of (Y,ζ). Then $f^{-1}(Z - f(F)) \subseteq Y - F$ and $Y - F$ is open. By hypothesis, there is a J-open set B of (Z, σ) such that $Z - f(F) \subset B$ and $f^{-1}(B) \subset Y - F$. Therefore $F \subset Y - f^{-1}(B)$. Hence $Z - B \subset f(F) \subset f(Y - f^{-1}(B)) \subset Z - B$, which implies $f(F) = Z - B$ and hence f(F) is J-closed in (Z,σ). Thus f is a J-closed function.

Theorem 7.3.3. A bijection function $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is a J-closed function if and only if f(U) is J-open in (Z,σ) for every open set U in (Y,ζ).

Proof Let $f : (Y,\zeta) \rightarrow (Z, \sigma)$ be a J-closed function and U be an open set in (Y,ζ). Then $Y - U$ is a closed set in (Y,ζ). Since f is a J-closed function, $f(Y-U)$ is a J-closed set in (Z,σ). Since f is bijection, $f(Y-U) = Y - f(U)$ and hence $Y - f(U)$ is J-closed in (Z,σ). Hence f(U) is J-open in (Z,σ).

Conversely, Let U be a closed subset of (Y,ζ). Then $Y - U$ is an open set in (Y,ζ). By the hypothesis, $f(Y - U)$ is J-open in (Z,σ). Since f is bijective, $f(Y - U) = Y - f(U)$ and hence f(U) is J-closed in (Z,σ). Thus f is a J-closed function.

Remark 7.3.4. Bijection of f is necessary in the above Theorem which can be seen in the following Example.

Example 7.3.5. Let $Y = Z = \{p,q,r\}$ with $\zeta = \{Y, \phi, \{p\}\}$ and $\sigma = \{Y, \phi, \{p\}, \{p,q\}\}$. Let $f : (Y,\zeta) \rightarrow (Z, \sigma)$ be the many-one into function defined by f(p) = q, f(q) = p, f(r) = p. Here f is not bijective. Then for the only open set {p} in (Y,ζ), f({p}) is J-open in (Z,σ) but f is not a J-closed function as for the closed set {q,r} in (Y,ζ), f({q,r}) = {p} is not J-closed in (Z,σ).

§ 7.4. J-Open Functions

Definition 7.4.1. A function $f : (Y,\zeta)\rightarrow (Z,\sigma)$ is said to be **J-open function** if the image of every open set in (Y,ζ) is J-open in (Z,σ).

Example 7.4.2. Let $f : (Y,\zeta)\rightarrow (Z,\sigma)$ be the identity function. Consider $Y = Z = \{p,q,r\}$ with $\zeta = \{Y, \phi, \{p\}\}$ and $\sigma = \{Z,\phi,\{p,q\}\}$.Here $JO(Z,\sigma)= P(Z)$.Then f is a J-open function.

Proposition 7.4.3. If $f : (Y,\zeta) \rightarrow (Z,\sigma)$ is any function, $g : (Z,\sigma) \rightarrow (P,\mu)$ is an injective function and also a J-irresolute function, their composite function $g \circ f : (Y,\zeta) \rightarrow (P,\mu)$ is a J-open function then f is a J-open function in (Z,σ).

Proof Let U be an open set in (Y,ζ). Since $g \circ f : (Y,\zeta) \rightarrow (P,\mu)$ is a J-open function, $(g \circ f)(U)$ is a J-open set in (P,μ). Since $g : (Z,\sigma) \rightarrow (P,\mu)$ is a J-irresolute function implies $g^{-1}[(g \circ f)(U)]$ is a J-open set in (Z,σ) and g is an injective function which gives $g^{-1}[(g \circ f)(U)] = f(U)$ is a J-open set in (Z,σ). Hence f is a J-open function.

Proposition 7.4.4. For a bijective function $f : (Y,\zeta) \rightarrow (Z,\sigma)$,the following statements are equivalent.

- **(a) $f^{-1} : (Z,\sigma) \rightarrow (Y,\zeta)$ is a J-continuous function**
- **(b) f is a J-open function**
- **(c) f is a J-closed function.**

Proof: (a) ⇒(b) Let U be an open set in (Y,ζ). Since f^{-1} is a J-continuous function, $(f^{-1})^{-1}(U)$ is a J-open set in (Z,σ). Since f is a bijective function, $(f^{-1})^{-1}(U) = f(U)$ which is a J-open set in (Z,σ). Hence f is a J-open function.

(b) ⇒ (c) Let V be a closed set in (Y,ζ). Then $Y-V$ is an open set in (Y,ζ).Since f is J-open, $f(Y-V)$ is a J-open set in (Z,σ). Since f is a bijective function, $f(Y-V) = Y - f(V)$ which is a J-open set in (Z,σ). This implies that $f(V)$ is a J-closed set in (Z,σ).Hence f is a J-closed function.

(c)⇒(a) Let V be a closed set in (Y,ζ). Since f is a J-closed function, $f(V)$ is a J-closed set in (Z,σ). Since f is a bijective function, $f(V) = (f^{-1})^{-1}(V)$ is a J-closed set in (Z,σ). Hence f^{-1} is a J-continuous function.

§ 7.5. J-Quotient Maps

Definition 7.5.1. A surjective function $f: (Y,\zeta) \rightarrow (Z,\sigma)$ is said to be a **J-quotient map** if f is a J-continuous function and $f^{-1}(U)$ is open in (Y,ζ) implies U is J-open in (Z,σ).

Example 7.5.2. Let $f : (Y,\zeta)\rightarrow (Z,\sigma)$ be the identity function. Consider $Y = Z = \{p,q,r\}$ with $\zeta = \{Y, \phi, \{p\},\{p,q\}\}$ and $\sigma = \{Z,\phi,\{p,q\}\}$.Then a surjective function f is a J-continuous function as $JO(Y,\zeta)= P(Y) - \{q,r\}$ and $f^{-1}(U) = \{p\} \Rightarrow U = f(\{p\}) = \{p\}$,

$f^{-1}(U)$ = {p,q} ⇒ U = f({p,q}) = {p,q} is a J-open set in (Z,σ) as JO(Z,σ)= P(Z).Hence the function f is a J-quotient map.

Definition 7.5.3. A function f : (Y,ζ)→ (Z,σ) is said to be **strongly J-open function** if the image of every J-open set in (Y,ζ) is J-open in (Z,σ).

Example 7.5.4. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ ={Y, ϕ, {p},{q},{p,q}} and σ = {Z,ϕ,{p,q}}.Here JO(Y,ζ)= {Y, ϕ, {p},{q},{p,q}} and JO(Z,σ)= P(Z).Then f is a strongly J-open function.

Proposition 7.5.5. A strongly J-open function f : (Y,ζ)→ (Z,σ) is a J-open function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a strongly J-open function. Let U be an open set in (Y,ζ).By **Theorem 2.3.75.**,U is a J-open set in (Y,ζ).Since f is a strongly J-open function, f(U) is J-open in (Z,σ). Hence f is a J-open function.

Counter Example 7.5.6. Let f : (Y,ζ)→ (Z,σ) be an identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p,q}}.Here JO(Z,σ) = {Y, ϕ, {p},{q},{p,q}} and JO(Y,ζ) = P(Y) –{q,r}.Then f is a J-open function but not a strongly J-open function. Because for a J-open set {r} in (Y,ζ), the image of {r} is not a J-open set in (Z,σ).

Proposition 7.5.7. If f : (Y,ζ) → (Z,σ) is any function, g : (Z,σ) → (P,μ) is an injective function and also a J-irresolute function, their composite function g ∘ f : (Y,ζ) → (P,μ) is a strongly J-open function then f is a strongly J-open function in (Z,σ).

Proof The proof follows from **Proposition 7.5.5.** and **Proposition 7.4.3.**

Proposition 7.5.8. If a surjective function f :(Y,ζ) →(Z,σ) is a J-continuous function and J-open function, then f is a J-quotient map.

Proof Consider $f^{-1}(U)$ is open in (Y,ζ). Since f is a J-open function and surjective, $f(f^{-1}(U))$ = U is J-open in (Z,σ) and since f is J-continuous. f is a J-quotient map.

Proposition 7.5.9. If a surjective function f :(Y,ζ) →(Z,σ) is a J-continuous function and J-closed function, then f is a J-quotient map.

Proof Consider f $^{-1}$(U) is open in (Y,ζ). Then Y – f $^{-1}$(U) is closed in (Y,ζ). Since f is a J-closed function,Y −f(f $^{-1}$(U)) is J-closed in (Z,σ).This implies that f(f $^{-1}$(U)) = U is J-open in (Z,σ),since f is surjective. Therefore f is a J-quotient map.

Proposition 7.5.10. Every ĝ-quotient map is a J-quotient map but not conversely.

Proof By hypothesis, f is surjective. We know that every ĝ-continuous function is a J-continuous function (By **Proposition 5.2.11.**).Let f $^{-1}$(U) is open in (Y,ζ).Since f is a ĝ -quotient map, U is ĝ-open in (Z,σ).By **Proposition 2.3.84.** as every ĝ-open is J-open, U is J-open in (Z,σ).Hence f is J-quotient .

Counter Example 7.5.11. Let f : (Y,ζ)→ (Z,σ) be the surjective function defined by f(p) = p,f(q) = r and f (r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{p,q}} and σ = {Z,ϕ,{p,q}}. Here f $^{-1}$(U) = {p} ⇒ U = f({p}) = {p}, f $^{-1}$(U) = {p,q} ⇒ U = f({p,q}) = {p,r}.Then f is a J-quotient map as JO(Z,σ) = P(Z).But f is not a ĝ-quotient map, since {p,r} is not a ĝ-open set in (Z,σ) as ĝO(Z,σ) ={Z,ϕ,{p},{q},{p,q}}.

§ 7.6. Strong Form of J-Quotient Maps

Definition 7.6.1. A surjective function f: (Y,ζ) → (Z,σ) is said to be a **strongly J-quotient map** provided a subset U of (Z,σ) is open ⇔ f $^{-1}$(U) is J-open in (Y,ζ).

Example 7.6.2. Let f : (Y,ζ)→ (Z,σ) be the many one onto function defined by f(p) = p = f(r),f(q) = q, f(s) = r.Consider Y = {p,q,r,s}, Z = {p,q,r} with ζ = {Y, ϕ, {r},{p,q},{p,q,r}} and σ = {Z,ϕ,{p},{q},{p,q}}.Then a surjective function f is a J-continuous function as JO(Y,ζ)={ Y, ϕ, {p},{q},{r},{p,q},{q,r},{p,r},{p,q,r}} and f $^{-1}$(U) = {p} ⇒ U = f({p}) = {p}, f $^{-1}$(U) = {q} ⇒ U = f({q}) = {q}, f $^{-1}$(U) = {r} ⇒ U = f({r}) = {p}, f $^{-1}$(U) = {p,q} ⇒ U = f({p,q}) = {p,q,r}=Z, f $^{-1}$(U) = {q,r} ⇒ U = f({q,r}) = {p,q}, f $^{-1}$(U) = {p,r} ⇒ U = f({p,r}) = {p}, f $^{-1}$(U) = {p,q,r} ⇒ U = f({p,q,r}) = {p,q} are open sets in (Z,σ).Hence a function f is a strongly J-quotient map.

Proposition 7.6.3. A strongly J-quotient map f :(Y,ζ) →(Z,σ) is a J-open function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a strongly J-quotient map. Let U be any open set in (Y,ζ). By **Theorem 2.3.75.,**as every open set is J-open, U is a J-open set in (Y,ζ). Since f is a surjective function, f^{-1}(f(U)) = U is a J-open set in (Y,ζ). By the given condition, f is Strongly J-quotient map, f(U) is an open set in (Z,σ). **By Theorem 2.3.75.,** f(U) is a J-open set in (Z,σ).Hence f is a J-open function.

Counter Example 7.6.4. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p,q}}.Here JO(Z,σ)= P(Z).Then f is a J-open function but not a strongly J-quotient map. Because f^{-1}(U) = {p,r} ⇒ U = f({p,r}) = {p,r} is not open in (Z,σ).

Proposition 7.6.5. A strongly J-quotient map f :(Y,ζ) →(Z,σ) is a strongly J-open function but the converse is not true.

Proof Consider f : (Y,ζ) →(Z,σ) is a strongly J-quotient map. Let U be any J-open set in (Y,ζ). Since f is a surjective function, f^{-1}(f(U)) = U and so it is a J-open set in (Y,ζ). By the given condition, f is strongly J-quotient map, f(U) is an open set in (Z,σ). By **Theorem 2.3.75.**, f(U) is J-open set in (Z,σ). Hence f is a strongly J-open function.

Counter Example 7.6.6. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p},{q},{p,q}} and σ = {Z,ϕ,{p,q}}.Here JO(Y,ζ)= {Y, ϕ, {p},{q},{p,q}} and JO(Z,σ)= P(Z).Then f is a strongly J-open function but not a strongly J-quotient map. Because f^{-1}(U) = {p} ⇒ U = f({p}) = {p} is not open in (Z,σ).

Proposition 7.6.7. A strongly J-quotient map is a J-continuous function.

Proof In **Definition 7.6.1.,** a subset U of (Z,σ) is open ⇒ f^{-1}(U) is J-open in (Y,ζ) gives that f is a J-continuous function.

Counter Example 7.6.8. Let f : (Y,ζ)→ (Z,σ) be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z, ϕ, {p}, {q}, {p,q}}.Here σ^c = {Z, ϕ, {r}, {q,r}, {p,r}} and ζ^c= {Y, ϕ, {r}}.Then f is J-continuous as JC(Y,ζ)= P(Y) but not a strongly J-quotient map. Because for the J-open set f^{-1}(U) = {q,r} in (Y,ζ), U ={q,r} is not open in (Z,σ).

Definition 7.6.9. A surjective function f: $(Y,\zeta) \rightarrow (Z,\sigma)$ is said to be a **[J]-quotient map** if f is a J-irresolute function and $f^{-1}(U)$ is J-open in (Y,ζ) implies U is open in (Z,σ).

Example 7.6.10. Let f : $(Y,\zeta)\rightarrow (Z,\sigma)$ be the many one onto function defined by f(p) = p = f(r),f(q) = q, f(s) = r.Consider Y = {p,q,r,s}, Z = {p,q,r} with ζ = {Y, ϕ, {r},{p,q},{p,q,r}} and σ = {Z,ϕ,{p},{q},{p,q}} = JO(Z,σ).Then a surjective function f is a J-irresolute function as JO(Y,ζ)={ Y, ϕ, {p},{q},{r},{p,q},{q,r},{p,r},{p,q,r}} and $f^{-1}(U)$ = {p} ⇒ U = f({p}) = {p}, $f^{-1}(U)$ = {q} ⇒ U = f({q}) = {q}, $f^{-1}(U)$ = {r} ⇒ U = f({r}) = {p}, $f^{-1}(U)$ = {p,q} ⇒ U = f({p,q}) = {p,q,r}=Z, $f^{-1}(U)$ = {q,r} ⇒ U = f({q,r}) = {p,q}, $f^{-1}(U)$ = {p,r} ⇒ U = f({p,r}) = {p}, $f^{-1}(U)$ = {p,q,r} ⇒ U = f({p,q,r}) = {p,q} is a open set in (Z,σ).Hence a function f is a [J]-quotient map.

Proposition 7.6.11. A [J]-quotient map f :$(Y,\zeta) \rightarrow (Z,\sigma)$ is a J-irresolute function but the converse is not true.

Proof The proof follows from the **Definition 7.6.9.**

Counter Example 7.6.12. Let f : $(Y,\zeta)\rightarrow (Z,\sigma)$ be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z, ϕ,{p},{q},{p,q}} = JO(Z,σ).Here JO(Y,ζ)= P(Y) –{q,r}.Then f is a J-irresolute function but not a [J]-quotient map , since $f^{-1}(U)$ = {p,r} ⇒ U = f({p,r}) = {p,r} is not an open set in (Z,σ).

Proposition 7.6.13. A [J]-quotient map f :$(Y,\zeta) \rightarrow (Z,\sigma)$ is a strongly J-open function but the converse is not true.

Proof Let U be an open set in (Z,σ). By **Theorem 2.3.75.**, U is a J-open set in (Z,σ). Since f is [J]-quotient, $f^{-1}(U)$ is a J-open set in (Y,ζ).Let $f^{-1}(U)$ is a J-open set in (Y,ζ). By hypothesis, f is [J]-quotient, U is an open set in (Z,σ). Hence f is a strongly J-open function.

Counter Example 7.6.14. In **Counter Example 7.6.6.,** f is a strongly J-open function but not a [J]-quotient map,since f is not a J-irresolute function. Because for a J-open set {r} in (Z,σ),its inverse image {r} is not a J-open set in (Y,ζ).

Proposition 7.6.15. A [J]-quotient map f :(Y,ζ) →(Z,σ) is a strongly J-quotient map but the converse is not true.

Proof From **Proposition 6.2.4.**,every J-irresolute function is a J-continuous function which is required to prove the implication U is an open set in (Z,σ) implies $f^{-1}(U)$ is J-open in (Y,ζ).The other implication $f^{-1}(U)$ is J-open in (Y,ζ) implies U is open in (Z,σ) follows from the definition of [J]-quotient map. Hence the result follows.

Proposition 7.6.16. Every strongly J-quotient map is a J-quotient map but not conversely.

Proof Let U be an open set in (Z,σ).Since f is strongly J-quotient map, $f^{-1}(U)$ is J-open in (Y,ζ) gives that f is a J-continuous function. Let $f^{-1}(U)$ is open in (Y,ζ). By **Theorem 2.3.75.,** $f^{-1}(U)$ is J-open in (Y,ζ). Since f is strongly J-quotient map,U is open in (Z,σ). By **Theorem 2.3.75.,** U is J-open in (Z,σ).Hence f is a J-quotient map.

Counter Example 7.6.17. In **Example 7.5.2.**, is a J-quotient map but not a strongly J-quotient map. Because $f^{-1}(U)$ = {p,r}implies U = f({p,r}) = {p,r} is not open in (Z,σ).

Proposition 7.6.18. Every quotient map is a J-quotient map but not conversely.

Proof By hypothesis, f is surjective. We know that every continuous function is a J-continuous function (By **Proposition 5.2.3.**).Since f is a quotient map, $f^{-1}(U)$ is open in (Y,ζ) implies U is open in (Z,σ).By **Theorem 2.3.75.** as every open is J-open, U is J-open in (Z,σ).Hence f is J-quotient.

Counter Example 7.6.19. In the above Example 7.5.2., $f^{-1}(U)$ = {p} ⇒ U = f({p}) = {p} is not an open set in (Z,σ).Hence f is a J-quotient but not a quotient map.

Proposition 7.6.20. A [J]-quotient map f :(Y,ζ) →(Z,σ) is a J-quotient map but the converse need not be true.

Proof By **Proposition 7.6.15.** and **Proposition 7.6.16.,**we get the proof.

Counter Example 7.6.21. By **Example 7.5.2.,**f is a J-quotient map but not a [J]-quotient map.Because for a J-open set,$f^{-1}(U)$ = {p,r} ⇒ U = f({p,r}) = {p,r} is not an open set in (Z,σ).

Proposition 7.6.22. If an open surjective function f: $(Y,\zeta)\rightarrow(Z,\sigma)$ is a J-irresolute function and g : $(Z,\sigma)\rightarrow(P,\mu)$ is a J-quotient map, then $g\circ f : (Y,\zeta)\rightarrow(P,\mu)$ is a J-quotient map.

Proof Let U be any open subset in (P,μ).Hence $g^{-1}(U)$ is J-open in (Z,σ) as g is a J-quotient map. Since f is J-irresolute, therefore $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ).Therefore $g\circ f$ is J-continuous -----(1). Consider $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in (Y,ζ). Since f is open surjective, $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is open in (Z,σ).By the given condition that g is J-quotient map, U is J-open in (P,μ) -----(2).From (1) and (2), $g\circ f : (Y,\zeta)\rightarrow(P,\mu)$ is a J-quotient map.

Proposition 7.6.23. If a map h: $(Y,\zeta)\rightarrow(Z,\sigma)$ is a J-quotient map and g : $(Y,\zeta)\rightarrow(P,\mu)$ is a continuous function that is constant on each set $h^{-1}(y)$,for $y\in Z$,then g induces a J-continuous function f : $(Z,\sigma)\rightarrow(P,\mu)$ such that $f\circ h = g$.

Proof For the given functions h and g, we have to produce a J-continuous function f : $(Z,\sigma)\rightarrow(P,\mu)$,using the fact that g is constant on $h^{-1}(y)$ whenever $y\in Z$, we get $g(h^{-1}(y))$ is a singleton set in (P,μ).Then f is well-defined whenever f(y) denotes this set for every y. Then for each $x\in Y$, $f(h(x)) = g(x)$. For if, for an open set U in (P,μ), since g is continuous, we get $g^{-1}(U)$ is an open set in (Y,ζ).We have $g^{-1}(U)= h^{-1}(f^{-1}(U))$.Using these facts, we get $f^{-1}(U)$ is a J-open set by using h is being J-quotient. Hence f is a J-continuous function from Z to P.

Proposition 7.6.24. If a surjective function f: $(Y,\zeta)\rightarrow(Z,\sigma)$ is strongly J-open, J-irresolute and g : $(Z,\sigma)\rightarrow(P,\mu)$ is a strongly J-quotient map, then $g\circ f$: $(Y,\zeta)\rightarrow(P,\mu)$ is a strongly J-quotient map.

Proof Given g: $(Z,\sigma)\rightarrow(P,\mu)$ is a strongly J-quotient map. Let U be any open subset in (P,μ).Hence $g^{-1}(U)$ is J-open in (Z,σ).Since f is J-irresolute, therefore $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ) -----(1). Thus we get $g\circ f$ is J-continuous. Now, take $f^{-1}(g^{-1}(U))$ is J-open in (Y,ζ).Since f is strongly J-open, $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z,σ).By the given condition that g is strongly J-quotient map, U is open in (P,μ) -----(2).From (1) and (2),$g\circ f : (Y,\zeta)\rightarrow(P,\mu)$ is a strongly J-quotient map.

Proposition 7.6.25. If a surjective function f: (Y,ζ)→ (Z,σ) is strongly J-open, J-irresolute and g : (Z,σ) →(P,μ) is a [J]-quotient map , then g ∘ f : (Y,ζ)→ (P, μ) is a [J]-quotient map.

Proof Given g: (Z,σ)→(P,μ) is a [J]-quotient map. Let U be any J-open subset in (P,μ) and using the continuity of g, we get g^{-1}(U) is J-open in (Z,σ).Since f is J-irresolute, $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ). Since f is surjective and strongly J-open, $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z,σ).By the given condition that g is [J]-quotient map, U is open in (P,μ).Hence g ∘ f : (Y,ζ)→ (P, μ) is a [J]-quotient map.

Proposition 7.6.26. If a surjective function f: (Y,ζ)→ (Z,σ) is strongly J-open, J-irresolute and g : (Z,σ) →(P,μ) is a J-quotient map, then g ∘ f : (Y,ζ)→ (P, μ) is a J-quotient map.

Proof Given g : (Z,σ)→(P,μ) is a J-quotient map. Let U be any open subset in (P,μ).Hence g^{-1}(U) is J-open in (Z,σ).Since f is J-irresolute, therefore $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ).Therefore g∘f is J-continuous -----(1).Conversely, take $f^{-1}(g^{-1}(U))$ is open in (Y,ζ).By **Theorem 2.3.75.**, $f^{-1}(g^{-1}(U))$ is J-open in (Y,ζ). Since f is strongly J-open, $f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is J-open in (Z,σ).By the given condition that g is a J-quotient map, U is open in (P,μ) -----(2).From (1) and (2),g ∘ f : (Y,ζ)→ (P, μ) is a J-quotient map.

Proposition 7.6.27. If a function f: (Y,ζ)→ (Z,σ) is strongly J-quotient, J-irresolute and g : (Z,σ) →(P,μ) is a [J]-quotient map, then g ∘ f : (Y,ζ)→ (P, μ) is a [J]-quotient map.

Proof Given g: (Z,σ)→(P,μ) is a [J]-quotient map. Let U be any J-open subset in (P,μ).Since g is [J]-quotient,using the continuity part, g^{-1}(U) is J-open in (Z,σ) and using the part that f is J-irresolute, $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ). By the given condition that f is a strongly J-quotient map, g^{-1}(U) is open in (Z,σ) which will give that g^{-1}(U) is J-open in (Z,σ) by **Theorem 2.3.75.** Since g is [J]-quotient map, U is open in (P,μ). We arrive at g ∘ f : (Y,ζ)→ (P, μ) is a [J]-quotient map.

Proposition 7.6.28. If f : (Y,ζ)→ (Z,σ) and g : (Z,σ) →(P,μ) are [J]-quotient maps, then g ∘ f : (Y,ζ)→ (P, μ) is a [J]-quotient map.

Proof Let U be a J-open set in (P,μ).By the given condition, g is a [J]-quotient map, $g^{-1}(U)$ is a J-open set in (Z,σ).Since f is a [J]-quotient map, $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ).This implies $(g\circ f)$ is a J-irresolute function -----(1).Consider $(g\circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is J-open in (Y,ζ) -----(2). Since f is a [J]-quotient map, $g^{-1}(U)$ is an open set in (Z,σ). By **Theorem 2.3.75.**,every open is J-open. $g^{-1}(U)$ is a J-open set in (Z,σ).Since g is a [J]-quotient map ,U is an open set in (P,μ) -----(3).From (1),(2) and (3), $g \circ f : (Y,\zeta)\rightarrow (P, \mu)$ is a [J]-quotient map.

Proposition 7.6.29. If a function f: $(Y,\zeta)\rightarrow (Z,\sigma)$ is a J-quotient map where (Y,ζ) and (Z,σ) are JTC-spaces. Then g : $(Z,\sigma) \rightarrow(P,\mu)$ is quasi J-continuous ⇔ the composite map $g \circ f : (Y,\zeta)\rightarrow (P, \mu)$ is quasi J-continuous.

Proof Let U be any J-open set in (P,μ).Using the quasi J-continuity of g, $g^{-1}(U)$ is open in (Z,σ). Now $f^{-1}(g^{-1}(U)) = (g\circ f)^{-1}(U)$ is J-open in (Y,ζ),using the J-continuity part of f being J-quotient.Since (Y,ζ) is a JTC -space, $f^{-1}(g^{-1}(U))$ is open in (Y,ζ).Thus $g \circ f$ is quasi J-continuous.

Conversely, let $g \circ f$ be quasi J-continuous, then for every J-open set U in (P,μ). $f^{-1}(g^{-1}(U))$ is open in (Y,ζ).Since f is a J-quotient map, $g^{-1}(U)$ is J-open in (Z,σ).Since (Z,σ) is a JTC -space, $g^{-1}(U)$ is open in (Z,σ).Hence g is quasi J-continuous.

Proposition 7.6.30. An injective [J]-quotient map f :$(Y,\zeta) \rightarrow(Z,\sigma)$ is a J-open function.

Proof Let U be an open set in (Y,ζ). By **Theorem 2.3.75.**, U is an J-open set in (Y,ζ).As f is injective [J]-quotient, $f^{-1}(f(U)) =U$ is an J-open set in (Y,ζ) implies that f(U) is an open set in (Z,σ).Again By **Theorem 2.3.75.**, f(U) is a J-open set in (Z,σ).Therefore f is a J-open function.

Theorem 7.6.31. If f :$(Y,\zeta) \rightarrow(Z,\sigma)$ is any function from a JTC-space to another JTC-space, then the following conditions are equivalent.

(i) f is a [J]-quotient map

(ii) f is a strongly J-quotient map

(iii) f is a J-quotient map.

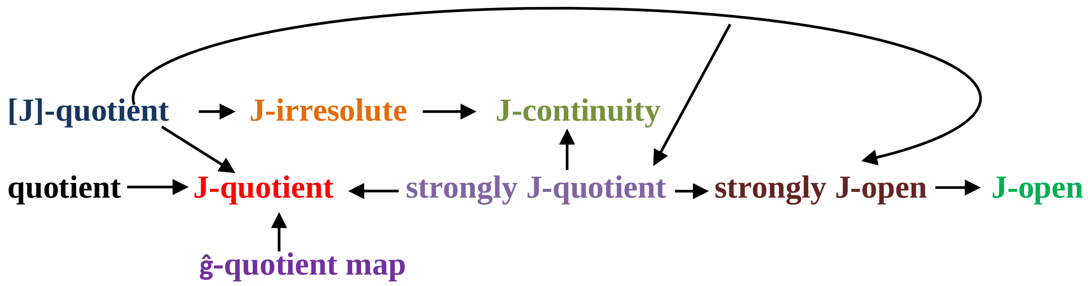
Proof (i) ⇒ (ii),(ii) ⇒(iii) by **Proposition 7.6.15., Proposition 7.6.16.**Now to prove **(iii) ⇒ (i).**Since f is a J-quotient map, f is a J-continuous function in (Z,σ). (Z,σ) is a

JTC-space, by **Theorem 6.2.40.,** f is a J-irresolute function in (Z,σ).Let $f^{-1}(U)$ be a J-open set in (Y,ζ).Since (Y,ζ) is a JTC-space, $f^{-1}(U)$ is an open set in (Y,ζ).By **(iii),**U is a J-open set in (Z,σ). (Z,σ) is JTC-space, U is an open set in (Z,σ).Hence **(iii) ⇒ (i).**

Proposition 7.6.32. An injective [J]-quotient map f :$(Y,\zeta) \rightarrow (Z,\sigma)$ is a J-closed function.

Proof Let U be a closed set in (Y,ζ). By **Proposition 2.3.2.**, U is a J-closed set in (Y,ζ).Then Y –U is a J-open set in (Y,ζ).As f is injective [J]-quotient, $f^{-1}(f(Y - U))$ = Y –U is an J-open set in (Y,ζ) implies that f(Y –U) is an open set in (Z,σ).Again By **Theorem 2.3.75.**, f(Y –U) is a J-open set in (Z,σ) implies f(U) is a J-closed set in (Z,σ). Therefore f is a J-closed function.

From the above delibrations to get the following picture.



§ 7.7. J-Homeomorphisms

Definition 7.7.1. A bijective function f: $(Y,\zeta) \rightarrow (Z,\sigma)$ is said to be **J-Homeomorphism** if f is both a J-continuous function and a J-open function.

Example 7.7.2. Let f : $(Y,\zeta) \rightarrow (Z,\sigma)$ be the identity function. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p,q}},σ^c= {Z,ϕ,{r}.Then a bijective function f is both a J-continuous function as JC(Y,ζ)= P(Y) – {p} and a J-open function as JO(Z,σ)= P(Z).Hence a function f is a J-Homeomorphism.

Proposition 7.7.3. A Homeomorphism f : $(Y,\zeta) \rightarrow (Z,\sigma)$ is a J-Homeomorphism but the converse is not true.

Proof Given f : $(Y,\zeta) \rightarrow (Z,\sigma)$ is a Homeomorphism.Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a continuous function, $f^{-1}(U)$ is closed in

(Y,ζ).By **Proposition 2.3.2.**,f $^{-1}$(U) is J-closed in (Y,ζ).Hence f is J-continuous ----- (2). Moreover, let U be any open set in (Y,ζ). Since f is an open function, the image f(U) is an open set in (Z,σ).By **Theorem 2.3.75.**, the image f(U) is a J-open set in (Z,σ)----- (3).From (1),(2) and (3),we get f is a J-Homeomorphism.

Remark 7.7.4. A J-Homeomorphism is not a Homeomorphism. This can be seen from the Counter Example.

Counter Example 7.7.5. In the above **Example 7.7.2.**,it is J-Homeomorphism but not Homeomorphism . Because for the closed set {r} in (Z,σ),the inverse image f $^{-1}$({r}) = {r} is not a closed set in (Y,ζ).Also for the open set {p} (Y,ζ),the image f({p}) ={p} is not an open set in (Z,σ).

Proposition 7.7.6. A δg*-Homeomorphism (resp.g-Homeomorphism) f :$(Y,\zeta) \rightarrow (Z,\sigma)$ is a J-Homeomorphism but the converse is not true.

Proof Similar as the above **Proposition 7.7.3.**

Remark 7.7.7. A J-Homeomorphism is not a δg*-Homeomorphism. This can be seen from the Counter Example.

Counter Example 7.7.8. In the above **Example 7.7.2.**,it is J-Homeomorphism but not a δg*-Homeomorphism. Because for the closed set {r} in (Z,σ),the inverse image f $^{-1}$({r}) = {r} is not a δg*-closed set in (Y,ζ).

Remark 7.7.9. A J-Homeomorphism is not a g-Homeomorphism. This can be seen from the Counter Example.

Counter Example 7.7.10. Let f : $(Y,\zeta) \rightarrow (Z,\sigma)$ be the function defined by f(p) = r, f(q) = p, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z,ϕ,{p},{p,q}},σ^c= {Z, ϕ, {r},{q,r}}.Then f is a bijective function, f is both a J-continuous function as JC(Y,ζ)= P(Y) and a J-open function as JO(Z,σ)= P(Z) −{q,r}.Hence a function f is a J-Homeomorphism but not a g-Homeomorphism. Because for the closed set {r} in (Z,σ),the inverse image f $^{-1}$({r}) = {p} is not a g-closed set in (Y,ζ).

Proposition 7.7.11. A J-Homeomorphism f : (Y,ζ) →(Z,σ) is a gδ-Homeomorphism but the converse is not true.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism . Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a J-continuous function, f^{-1}(U) is J-closed in (Y,ζ).By **Proposition 2.3.12.**,f^{-1}(U) is gδ-closed in (Y,ζ).Hence f is gδ-continuous ----- (2). Moreover, let U be any open set in (Y,ζ). Since f is an open function, the image f(U) is a J-open set in (Z,σ).By **Proposition 2.3.84.**,the image f(U) is a gδ-open set in (Z,σ) ----- (3).From (1),(2) and (3),we get f is a gδ-Homeomorphism .

Remark 7.7.12. A gδ-Homeomorphism is not a J-Homeomorphism. This can be seen from the Counter Example.

Counter Example 7.7.13. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = r, f(q) = p, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p,q}},σ^c= {Z, ϕ, {r}}.Then f is a bijective function, f is both a gδ-continuous function as gδC(Y,ζ)= P(Y) and a gδ-open function as gδO(Z,σ)= P(Z).Hence a function f is a gδ-Homeomorphism but not a J-Homeomorphism as JC(Y,ζ) = P(Y) −{p}.Because for the closed set {r} in (Z,σ),the inverse image f^{-1}({r}) = {p} is not a J-closed set in (Y,ζ).

Proposition 7.7.14. A J-Homeomorphism f : (Y,ζ) →(Z,σ) is a πg (πgp ,πgsp, πgs, πgα,rwg, gpr and rg respectively) -Homeomorphism but the converse is not true.

Proof Same that of **Proposition 7.7.11.**

Remark 7.7.15. A πg (πgp ,πgsp, πgs, πgα,rwg, gpr and rg respectively)-Homeomorphism is not a J-Homeomorphism. This can be seen from the Counter Example.

Counter Example 7.7.16. Same as the above **Counter Example 7.7.13.**

Remark 7.7.17. A gs-Homeomorphism and a J-Homeomorphism are independent of each other as shown in the following Counter Examples.

Counter Example 7.7.18. Let f : (Y,ζ)→ (Z,σ) be an identity function. Consider Y = Z = {p,q,r,s} with ζ = {Y,ϕ,{r},{p,q},{p,q,r}} and σ = {Z,ϕ,{p},{r},{p,q},{p,r},{p,q,r},{p,r,s}},σ^c={Z,ϕ,{q},{s},{q,s},{r,s},{q,r,s},{p,q,s}}. Then f is a bijective function, f is both a gs-continuous function as gsC(Y,ζ)= P(Y) –

{{p,r},{q,r},{p,q,r}}and a gs-open function as gsO(Z,σ)= {Z,ϕ,{p},{r}, {p,q},{p,r},{p,s}, {r,s},{p,q,s},{p,q,r},{p,r,s}}.Hence a function f is a gs-Homeomorphism but not a J-Homeomorphism. Because for the closed set {q} in (Z,σ),the inverse image f^{-1}({q}) = {q} is not a J-closed set in (Y,ζ) as JC(Y,ζ) = {Y,ϕ,{s},{r,s},{p,s},{q,s},{q,r,s},{p,r,s},{p,q,s}}.

Counter Example 7.7.19. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = r, f(q) = p, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p,q}} and σ = {Z,ϕ,{p},{p,q}},σ^c= {Z, ϕ, {r},{q,r}}.Then f is a bijective function, f is both a J-continuous function as JC(Y,ζ)= P(Y) and a J-open function as JO(Z,σ)= P(Z) −{q,r}.Hence a function f is a J-Homeomorphism but not a gs-Homeomorphism. Because for the closed set {r} in (Z,σ),the inverse image f^{-1}({r}) = {p} is not a gs-closed set in (Y,ζ) as gsC(Y,ζ)= {Y,ϕ,{r},{p,r},{q,r}}.

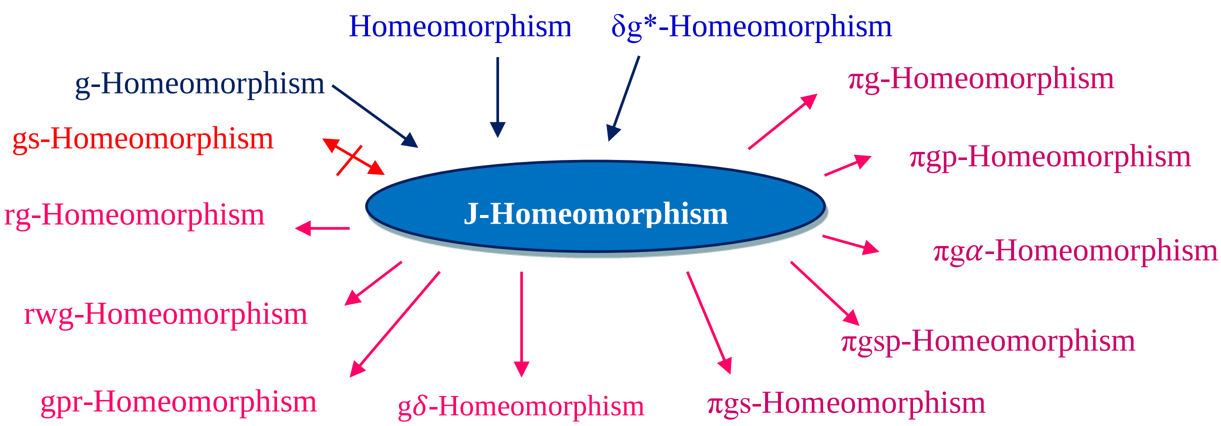
Proposition 7.7.20. If a function f : (Y,ζ) → (Z,σ) is a bijective function and a J-continuous function. Then f : (Y,ζ) → (Z,σ) is a J-open function if and only if f is a J-Homeomorphism.

Proof Consider f is a J-open function. By hypothesis, f is bijective and J-continuous. Hence f is a J-Homeomorphism. In the other way, consider f is a J-Homeomorphism. By the definition of J-Homeomorphism, f is a J-open function.

Proposition 7.7.21. If a function f : (Y,ζ) → (Z,σ) is a bijective function and a J-continuous function. Then f: (Y,ζ) → (Z,σ) is a J-Homeomorphism if and only if f is a J-closed function.

Proof Consider f is a J-Homeomorphism. Then f is a J-open function. By **Proposition 7.4.4.**, f is a J-closed function. In the other way, consider f is a J-closed function. By **Proposition 7.4.4.**, f is a J-open function. By hypothesis, f is a J-continuous function and bijective function. Hence f is a J-Homeomorphism.

Remark 7.7.22. The above discussion is portrayed in the following diagram.



Observation 7.7.23. The composition of two J-Homeomorphisms is not a J-Homeomorphism, since composition of two J-continuous functions is not a J-continuous function.

Theorem 7.7.24. If a J-Homeomorphism f : (Y,ζ) → (Z,σ) is a Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTC-spaces.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism. Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a J-continuous function, f^{-1}(U) is J-closed in (Y,ζ).Since (Y,ζ) is a JTC-space, f^{-1}(U) is closed in (Y,ζ).Hence f is a continuous function ----- (2).Moreover, let U be any open subset in (Y,ζ). Since f is an open function, the image f(U) is a J-open set in (Z,σ). Since (Z,σ) is a JTC-space, the image f(U) is an open set in (Z,σ) ----- (3).From (1),(2) and (3),we get f is a Homeomorphism.

Corollary 7.7.25. If a J-Homeomorphism f : (Y,ζ) → (Z,σ) is a Homeomorphism when (Y,ζ) is a JTδ-space and (Z,σ) is a JTC-space.

Theorem 7.7.26. If a J-Homeomorphism f : (Y,ζ) → (Z,σ) is a Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTδ-spaces.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism. Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a J-continuous function, f^{-1}(U) is J-closed in (Y,ζ).Since (Y,ζ) is a JTδ-space, therefore f^{-1}(U) is δ-closed in (Y,ζ) which gives f^{-1}(U) is closed in (Y,ζ).Hence f is a continuous function ----- (2).Moreover, let U be any open set in (Y,ζ).If f is a J-open function the image f(U) is a J-open set in (Z,σ).Since (Z,σ) is a JTδ-space which implies f is a δ-open set, therefore the image f(U) is an open set in (Z,σ) ----- (3).From (1),(2) and (3),we get f is Homeomorphism.

Theorem 7.7.27. If a function f : (Y,ζ) → (Z,σ) is a J-Homeomorphism,then f is a g-Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTg-spaces.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism. Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a J-continuous function, f^{-1}(U) is J-closed in (Y,ζ).Since (Y,ζ) is a JTg-space, therefore f^{-1}(U) is g-closed in (Y,ζ).Hence f is a g-continuous function ----- (2).Moreover, let U be any open set in (Y,ζ).Since f is a J-open function the image f(U) is a J-open set in (Z,σ).Since (Z,σ) is a JTg-space which implies f is an g-open set in (Z,σ) ----- (3).From (1),(2) and (3),we get f is a g-Homeomorphism.

Theorem 7.7.28. If a J-Homeomorphism f: (Y,ζ) → (Z,σ),then f is a δg*-Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTδg*-spaces.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism.Then f is a bijective function ----- (1). Let U be any closed set in (Z,σ).Since f is a J-continuous function, f^{-1}(U) is J-closed in (Y,ζ).Since (Y,ζ) is a JTδg*-space, therefore f^{-1}(U) is δg*-closed in (Y,ζ).Hence f is a δg*-continuous function ----- (2).Moreover, let U be any open set in (Y,ζ).Since f is a J-open function the image f(U) is a J-open set in (Z,σ).Since (Z,σ) is a JTδg*-space which implies f is a δg*-open set in (Z,σ) ----- (3).From (1),(2) and (3),we get f is a δg*-Homeomorphism.

Theorem 7.7.29. If f : (Y,ζ) → (Z,σ) and g : (Z,σ) → (P,μ) are J-Homeomorphisms then g ∘ f : (Y,ζ) → (P, μ) is a J-Homeomorphism when (Z,σ) is a JTC-space.

Proof Consider f : (Y,ζ) → (Z,σ) and g : (Z,σ) → (P, μ) are J-Homeomorphisms. Then f is a bijective function -----(1). Let U be an open set in (Y,ζ).Since f is a J-open function, f(U) is a J-open set in (Z,σ). Since (Z,σ) is a JTC-space, f(U) is open in (Z,σ). Also g: (Z,σ) → (P, μ) is a J-open function, g(f(U)) is J-open in (P, μ). Hence g ∘ f is a J-open function -----(2).Let U be a closed set in (P,μ). Then g is a bijective function -----(3).

From (1) and (3), $g \circ f$ is bijective -----(4). Since g: (Z,σ) → (P,μ) is J-continuous and (Z,σ) is a JTC-space, the inverse image $g^{-1}(U)$ is J-closed implies that $g^{-1}(U)$ is closed in (Z,σ) . Since f: (Y,ζ) → (Z,σ) is J-continuous , $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is a J-closed set in (Y,ζ).That is $g \circ f$ is J-continuous -----(5). Hence from (2),(4) and (5), $g \circ f$ is a J-Homeomorphism.

Corollary 7.7.30. If f : (Y,ζ) → (Z,σ) and g : (Z,σ) → (P,μ) are J-Homeomorphisms then $g \circ f$: (Y,ζ) → (P, μ) is a J-Homeomorphism when (Z,σ) is a JTδ-space.

Proof Proof is clear.

Definition 7.7.31. A bijective function f : (Y,ζ) → (Z,σ) is said to be a **J$\mathcal{C}$-Homeomorphism** if both the functions f and f^{-1} are J-irresolute.

The family of all J$\mathcal{C}$-Homeomorphisms of a topological space (Y,ζ) onto itself is denoted by J$\mathcal{C}\hbar$(Y,ζ).

Example 7.7.32. Let f : (Y,ζ)→ (Z,σ) be the function defined by f(p) = p, f(q) = r, f(r) = q. Consider Y = Z = {p,q,r} with ζ = {Y, ϕ, {p}} and σ = {Z,ϕ,{p},{p,q}}.Then f is a bijective function, f and f^{-1} is both J-irresolute as JO(Y,ζ)= P(Y) – {q,r} = JO(Z,σ).

J$\mathcal{C}$-Homeomorphism is independent with J-Homeomorphism. It can be seen from the following Counter Example.

Counter Example 7.7.33. From the **Counter Example 7.7.10.,** a function f is a J-Homeomorphism but not a J$\mathcal{C}$-Homeomorphism. Because for the J-open set {p,r} in (Y,ζ),the inverse image $(f^{-1})^{-1}$ ({p,r}) = {q,r} is not a J-open set in (Z,σ).

Theorem 7.7.34. The composition of two J$\mathcal{C}$-Homeomorphisms is a J$\mathcal{C}$-Homeomorphism.

Proof By **Proposition 6.2.41.**, Composition of two J-irresolute functions is J-irresolute and hence the proof follows.

Theorem 7.7.35. If a J-Homeomorphism f : (Y,ζ) → (Z,σ) is a J$\mathcal{C}$-Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTC-spaces.

Proof Given f : (Y,ζ) →(Z,σ) is a J-Homeomorphism. Then f is a bijective function ----- (1). Let U be any J-closed set in (Z,σ) which gives U is a closed set in (Z,σ)(by assumption, (Z,σ) is a JTC-space).Since f is a J-continuous function,f^{-1}(U) is J-closed in (Y,ζ).Hence f is a J-irresolute function ----- (2).Moreover, let U be any J-open set in (Y,ζ) which gives U is an open set in (Y,ζ) (by assumption, (Y,ζ) is a JTC-space). Since f is a J-open function, the image f(U) is a J-open set in (Z,σ).That is $(f^{-1})^{-1}$(U) is a J-open set in (Z,σ).Therefore f^{-1} is a J-irresolute function in (Z,σ) ----- (3).From (1),(2) and (3),we get f is J$\mathcal{C}$-Homeomorphism.

Corollary 7.7.36. If a J-Homeomorphism f : (Y,ζ) → (Z,σ) is a J$\mathcal{C}$-Homeomorphism when both spaces (Y,ζ) and (Z,σ) are JTδ-spaces.

Theorem 7.7.37. The set J$\mathcal{C}\hbar$(Y, ζ) is a group under the composition of functions.

Proof Let us define a binary operation $*$: J$\mathcal{C}\hbar$(Y, ζ) × J$\mathcal{C}\hbar$(Y, ζ) → J$\mathcal{C}\hbar$(Y, ζ) by (f $*$ g) = (g ∘ f) for every f, g∈ J$\mathcal{C}\hbar$(Y,ζ) and ∘ is the usual operation of composition of functions. Then by the **Theorem 7.7.34.**, (g∘ f)∈ J$\mathcal{C}\hbar$(Y,ζ) ---(1). We know that the composition of functions is associative ---(2) and the identity function I : (Y,ζ) → (Y,ζ) belongs to J$\mathcal{C}\hbar$(Y,ζ) serves as the identity element ---(3). If f ∈J$\mathcal{C}\hbar$(Y,ζ) then f is bijective and hence f^{-1} exists and f^{-1}∈ J$\mathcal{C}\hbar$(Y,ζ) such that (f ∘ f^{-1}) = f^{-1}∘ f = I. So the inverse exists for each element of J$\mathcal{C}\hbar$(Y,ζ) ---(4). Therefore from (1),(2),(3) and (4),J$\mathcal{C}\hbar$(Y,ζ) forms a group under the operation of composition of functions.

Theorem 7.7.38. A function f : (Y,ζ) → (Z,σ) is a J$\mathcal{C}$-Homeomorphism. Then the function f induces an isomorphism from the group J$\mathcal{C}\hbar$(Y,ζ) onto the group J$\mathcal{C}\hbar$(Z,σ).

Proof Using the function f, let us define a function θ_f : J$\mathcal{C}\hbar$(Y,ζ) → J$\mathcal{C}\hbar$(Z,σ) by $\theta_f(\hbar)$ = f ∘$\hbar$∘ f^{-1} for every $\hbar$∈J$\mathcal{C}\hbar$(Y,ζ). Then θ_f is a bijective function. Further for every $\hbar_1$,

$\hbar_2 \in JC\hbar(Y,\zeta)$, $\theta_f(\hbar_1, \hbar_2) = f \circ(\hbar_1 \circ \hbar_2)\circ f^{-1} = (f \circ \hbar_1 \circ f^{-1}) \circ (f \circ \hbar_2 \circ f^{-1}) = \theta_f(\hbar_1) \circ \theta_f(\hbar_2)$. Therefore θ_f is a homomorphism. Hence θ_f is an isomorphism induced by f.

Theorem 7.7.39. The $J\mathcal{C}$-Homeomorphism is an equivalence relation in the collection of all topological spaces.

Proof Since the composition of function is an equivalence relation. The reflexivity and symmetric relations are immediate and the transitivity follows from the **Theorem 7.7.34.**