CHAPTER – IV

CHAPTER IV

FUZZY COMPLETELY IRRESOLUTE AND FUZZY WEAKLY COMPLETELY IRRESOLUTE FUNCTIONS

Definition: 4.1

Let f : (X,τ) → (X,τ_1) be a function between two fuzzy topological spaces. Then f is called :

i. **fuzzy almost open** if the image of every fuzzy regular open subset of X is fuzzy open in Y.

ii. **a fuzzy R-map** iff $f^{-1}(\alpha)$ is a fuzzy regular open subset of X for each fuzzy regular open subset α of Y.

iii. **fuzzy completely continuous** iff $f^{-1}(\alpha)$ is a fuzzy regular open subset of X for each open subset α of Y.

iv. **a fuzzy strongly continuous** function iff $f^{-1}(\alpha)$ is a fuzzy clopen in X for every fuzzy subset α of Y.

Definition: 4.2

Two non-empty fuzzy subsets α and β are **fuzzy semi-separated** iff there exists two fuzzy semi-open subsets μ and γ such that α ⊆ β , β ⊆ γ , $\alpha \not{q} \gamma$ and $\beta \not{q} \mu$. A fuzzy subset which cannot be expressed as the union of two fuzzy semi-separated subsets is said to be a **fuzzy semi-connected set**.

Definition: 4.3

A fuzzy point x_α is said to be **a fuzzy δ - cluster point** of a fuzzy set A iff every fuzzy regularly open q-nbd U of x_α is q-coincident with A. The set of all fuzzy δ - cluster point of A is called the fuzzy δ - closure of A and is denoted by $[A]_\delta$.

A fuzzy set A is **fuzzy δ-closed** iff $A = [A]_\delta$ and the complement of a fuzzy δ-closed set is called **fuzzy δ-open.**

Definition: 4.4

A function f: X→Y from a fuzzy topological space (X,τ) to another fuzzy topological space (Y,τ_1) is said to be **fuzzy completely irresolute function** iff $f^{-1}(\alpha)$ is a fuzzy regular open subset of X for every semi-open subset α in Y.

Equivalently we may say that f is a **fuzzy completely irresolute** function iff $f^{-1}(\beta)$ is a fuzzy regular closed subset of X for every semi-closed subset β in Y.

Example: 4.5

Let μ_1 , μ_2 and μ_3 be fuzzy subsets of I = [0, 1] defined as follows:

$$\mu_1(x) = \begin{cases} 0 & , \ 0 \le x \le \frac{1}{2}, \\ 2x-1 & , \ \frac{1}{2} \le x \le 1, \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & , \ 0 \le x \le \frac{1}{4}, \\ -4x+2 & , \ \frac{1}{4} \le x \le \frac{1}{2}, \\ 0 & , \ \frac{1}{2} \le x \le 1, \end{cases}$$

$$\mu_3(x) = \begin{cases} 0 & , 0 \le x \le \frac{1}{4}, \\ \frac{1}{3}(4x-1) & , \frac{1}{4} \le x \le 1. \end{cases}$$

Then $\tau = \{0, \mu_1, \mu_2, \mu_1 \cup \mu_2, 1\}$ and $\tau_1 = \{0, \mu_2, 1\}$ are fuzzy topologies on I. Let us consider the function f : $(I, \tau) \to (I, \tau_1)$,defined by f (x) = x $\forall$ x ϵ I.

Now $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_2) = \mu_2$. Since every fuzzy open subset is fuzzy semi-open, μ_2 is fuzzy semi-open in (I, τ_1). And both μ_1 and μ_2 are fuzzy regular open subsets of (I, τ).

Thus f is a fuzzy completely irresolute function.

Remark: 4.6

Since every fuzzy open subset is fuzzy semi-open, it can be seen that every fuzzy completely irresolute function is fuzzy completely continuous. But the converse is not true which is shown by the following example.

Example: 4.7

Taking fuzzy topology τ from example 4.5 and τ' as

$\{0, \mu_1, \mu_1', \mu_1 \wedge \mu_1', \mu_1 \vee \mu_1', 1\}$ on I. Let us consider a function f: $(I, \tau') \to (I, \tau)$ defined by $f(x) = \frac{1}{2}x$ $\forall$ x ϵ I.

Now both μ_1, μ_1' are fuzzy regular open in (I, τ'), since cl μ_1 = int $\mu_1 = \mu_1$ and cl μ_1' = int $\mu_1' = \mu_1'$. But both μ_1' and μ_2' are fuzzy semi-open subsets of (I,τ) (Since cl $\mu_2 = \mu_1'$ and cl $\mu_1 = \mu_2'$). Also μ_3 is fuzzy semi-open in (I, τ), since $\mu_1 \le \mu_3 \le$ cl μ_1.

Now $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = 0$, $f^{-1}(\mu_2) = f^{-1}(\mu_1 \cup \mu_2) = \mu_1'$, since μ_1' is fuzzy regular open in (I, τ') f is fuzzy completely continuous.

Again $f^{-1}(\mu_1') = 1$, $f^{-1}(\mu_2') = \mu_1$.

But $f^{-1}(\mu_3)(x) = \mu_3(\frac{1}{2}x) = \begin{cases} 0 & , \ 0 \leq x \leq \frac{1}{2} \\ \frac{1}{3}(2x-1) & , \ \frac{1}{2} \leq x \leq 1, \end{cases} \quad x \in I.$

Which is not fuzzy open in (I, τ'). So $f^{-1}(\mu_3)$ is not fuzzy regular open in (I, τ'). Hence f is not fuzzy completely irresolute.

Remark: 4.8

From the definitions, it follows that every fuzzy completely irresolute function is fuzzy irresolute but the converse may fail to hold, which is shown by the example.

Example: 4.9

Taking the topology τ from example 4.5, let us consider a function $f : (I, \tau) \to (I, \tau_1)$ defined by $f(x)=x \ \forall \ x \in I$. Now $f^{-1}(0) = 0$, $f^{-1}(1) = 1$, $f^{-1}(\mu_1) = \mu_1$, $f^{-1}(\mu_2) = \mu_2$ and $f^{-1}(\mu_1 \cup \mu_2) = \mu_1 \cup \mu_2$, since $cl\ \mu_1 = \mu_2'$, $cl\ \mu_2 = \mu_1'$, and $\mu_1 \leq \mu_3 \leq cl\ \mu_1$, μ_2 and μ_3 are fuzzy semi-open subsets of (I, τ).

Also $f^{-1}(\mu_1') = \mu_1'$, $f^{-1}(\mu_2') = \mu_2'$ and $f^{-1}(\mu_3) = \mu_3$. Thus f is fuzzy irresolute but not fuzzy completely irresolute.

Theorem: 4.10

Let $f : (X, \tau) \to (Y, \tau_1)$ be a function from a fuzzy topological space (X, τ) to another fuzzy topological space (Y, τ_1). Consider the following statements:

a. f is fuzzy completely irresolute function.

b. For each fuzzy point $x_p \in X$ and each fuzzy semi-open subset α of Y containing $f(x_p)$ there exists a fuzzy regular open subset β of X containing x_p such that $f(\beta) \subseteq \alpha$.

c. For each fuzzy point $x_p \in X$ and each fuzzy semi-open subset α of Y containing $f(x_p)$ there exists a fuzzy δ- open subset β of X containing x_p such that $f(\beta) \subseteq \alpha$.

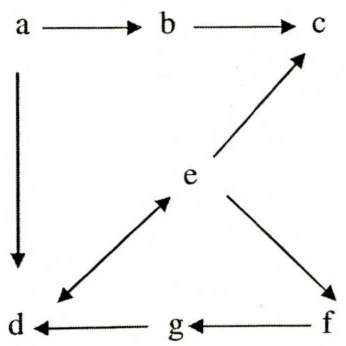
d. The inverse image of every fuzzy semi-open subset of Y is a fuzzy δ- open subset in X.

e. The inverse image of every fuzzy semi-closed subset of Y is a fuzzy δ- closed subset in X.

f. $f(cl_\delta(\gamma)) \subset scl\, f(\gamma)$ for each subset γ of X.

g. $cl_\delta(f^{-1}(\beta)) \subset f^{-1}(scl(\beta))$ for each fuzzy subset β of Y.

Then the following implications hold true:



Proof:

(a)⇒(b):

Let f be a fuzzy completely irresolute function, x_p a fuzzy point in X and α a fuzzy semi-open subset in Y such that $f(x_p) \in \alpha$.

Then $x_p \in f^{-1}(f(x_p)) \subseteq f^{-1}(\alpha)$.

Let $\beta = f^{-1}(\alpha)$ which is fuzzy regular open in X (since f is fuzzy completely irresolute) containing x_p. .Now $f(\beta) = ff^{-1}(\alpha) \subseteq \alpha$.

Hence condition (b) is satisfied.

(b)⇒(c):

Since every fuzzy regular open subset is fuzzy δ-open the result follows immediately.

(a)⇒(d):

By the same argument as in (b).

(d)⇒(e):

This is obvious, being a complement of each other.

(e)⇒(c):

Let α be a fuzzy semi-open subset of Y containing $f(x_p)$.Then by (e), $f^{-1}(1-\alpha) = 1-f^{-1}(\alpha)$ is fuzzy δ-closed in X. Therefore $f^{-1}(\alpha)$ is fuzzy δ-open in X.

Also $x_p \in f^{-1}(\alpha) = \beta$ (say).

Thus $f(\beta) = f f^{-1}(\alpha) \subseteq \alpha$.

(e)⇒(f):

Let $f(\gamma)$ be a fuzzy semi-closed subset of Y for any fuzzy subset γ of X. Then $f(\gamma) = scl\ f(\gamma)$.

By (e), $f^{-1}(scl(f(\gamma)))$ is fuzzy δ - closed in X.

Let $x_p \notin f^{-1}(\text{scl}(f(\gamma)))$.

Then for some fuzzy open q-nbd, η of x_p, int cl $\eta \, \not q \, f^{-1}(\text{scl}(f(\gamma)))$ and hence

int cl $\eta \, \not q \, \gamma$. Thus $x_p \notin cl_\delta(\gamma)$.

It follows that $cl_\delta(\gamma) \subset f^{-1}(\text{scl (f }(\gamma)))$

That is, f $(cl_\delta(\gamma)) \subset \text{scl}(f(\gamma))$.

(f)⇒(g):

Let β be a fuzzy subset of Y. By (f), we have

f $(cl_\delta\ (f^{-1}(\beta))) \subset$ scl (β) and hence cl$(f^{-1}(\beta)) \subset f^{-1}(\text{scl}(\beta))$.

(g)⇒(d):

Let $x_p \in X$ and β be any fuzzy semi-open subset containing $f(x_p)$.

Then $\gamma = 1 - \beta$ is fuzzy semi – closed in Y.

By (g), $cl_\delta\ (f^{-1}(\gamma)) \subset f^{-1}(\text{scl}(\gamma)) = f^{-1}(\gamma)$ which implies that

$cl_\delta\ (f^{-1}(\gamma)) = f^{-1}(\gamma)$ is fuzzy δ - closed subset of X containing x_p.

This satisfies condition (e) but (e) ⇔ (d).

Hence the result.

The composition of fuzzy completely irresolute functions with other fuzzy functions are given in the following theorem.

Theorem: 4.11

1. If f : X ⟶ Y is fuzzy completely irresolute and g : Y ⟶ Z is fuzzy irresolute, then g ∘ f : X ⟶ Z is a fuzzy completely irresolute function.

2. If f : X ⟶ Y is fuzzy strongly continuous and g : Y ⟶ Z is fuzzy completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy completely irresolute function.
3. If f : X ⟶ Y is fuzzy completely continuous and g : Y ⟶ Z is fuzzy completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy completely irresolute.
4. If f : X ⟶ Y is fuzzy R - map and g : Y ⟶ Z is fuzzy completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy completely irresolute.
5. If f : X ⟶ Y is fuzzy almost open surjection and g : Y ⟶ Z is fuzzy function such that g ∘ f : X ⟶ Z is a fuzzy completely irresolute then g is fuzzy irresolute.

Definition: 4.12

A function f: X ⟶ Y from a fuzzy topological space (X, τ) to another fuzzy topological space (Y, τ_1) is said to be **fuzzy weakly completely irresolute** iff $f^{-1}(\alpha)$ is a fuzzy regular open subset of X for each fuzzy semi θ - open subset α in Y or iff $f^{-1}(\beta)$ is a fuzzy regular closed subset of X for each fuzzy semi θ - closed subset β in Y.

Remark: 4.13

Every fuzzy completely irresolute function is fuzzy weakly completely irresolute, since every fuzzy semi-θ-open subset is fuzzy semi-open. But the converse is not true, as shown in the following example.

Example: 4.14

In example 4.5 and 4.7 we have shown that f : (I, τ) ⟶ (I, τ) is not a fuzzy completely irresolute function. Now we will show that f is a fuzzy weakly completely irresolute function.

Taking x = ¾ and p = 4/5, we have μ_1' (x) = ½ and p+μ_1' (x) >1.

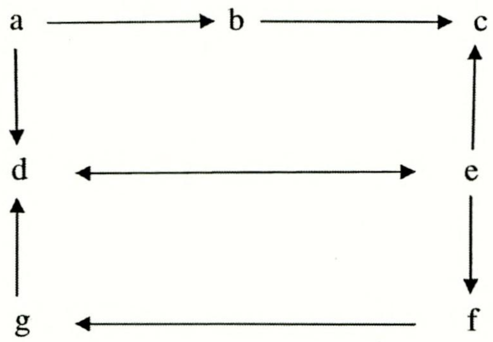
Thus x_p is a fuzzy point which is quasi-coincident with a fuzzy semi-open subset μ_1'. It is obvious that scl $\mu_1' \leq 1$. Thus by definition, 1 is the only fuzzy semi-θ-open subset in (I,τ). Since$f^{-1}(1)$ =1, which is fuzzy regular open in (I,τ), f is fuzzy weakly completely irresolute.

Theorem: 4.15

For any function f: (X,τ) ⟶ (Y, τ_1). Consider the following statements:

a. f is fuzzy weakly completely irresolute.
b. For each fuzzy point x_p ∈X and each fuzzy semi-θ-open subset α in Y containing f (x_p), there exists a fuzzy regular open subset β of X containing x_p such that f(β)⊆ α.
c. For each fuzzy point x_p ∈X and each fuzzy semi-θ-open subset α in Y containing f (x_p), there exists a fuzzy δ - open subset β of X containing x_p such that f(β) ⊆ α.
d. The inverse image of every fuzzy semi-θ-open subset of Y in fuzzy δ-open subset in X.
e. The inverse image of every fuzzy semi-θ-closed subset of Y in fuzzy δ-closed subset in X.
f. f($cl_\delta(\alpha)$) ⊂ $cl_{S-\theta}$ f(α) for each subset α of X.
g. $cl_\delta(f^{-1}(\beta)) \subset f^{-1}$ ($cl_{S-\theta}(\beta)$) for each fuzzy subset β of Y.

Then the following implications hold true.



Proof:

It is similar to the proof of theorem 4.10.

The composition of fuzzy weakly completely irresolute functions with other fuzzy functions are given in the following theorem.

Theorem: 4.16

1. If f : X ⟶ Y is fuzzy completely irresolute and g : Y ⟶ Z is fuzzy weakly completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy weakly completely irresolute.
2. If f : X ⟶ Y is fuzzy completely continuous and g : Y ⟶ Z is fuzzy weakly completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy weakly completely irresolute.
3. If f : X ⟶ Y is fuzzy R - map and g : Y ⟶ Z is fuzzy weakly completely irresolute, then g ∘ f : X ⟶ Z is a fuzzy weakly completely irresolute.
4. If f : X ⟶ Y is fuzzy completely irresolute and g : Y ⟶ Z is fuzzy semi - irresolute, then g ∘ f : X ⟶ Z is a fuzzy weakly completely irresolute.
5. If f : X ⟶ Y is fuzzy almost open surjection and g : Y ⟶ Z is fuzzy function such that g ∘ f : X ⟶ Z is a fuzzy weakly completely irresolute, then g is fuzzy semi - irresolute.

6. If $f : X \longrightarrow Y$ is fuzzy weakly completely irresolute and $g : Y \longrightarrow Z$ is fuzzy strongly irresolute, then $g \circ f : X \longrightarrow Z$ is a fuzzy completely irresolute.