ON B - ALGEBRAS

Thahira Mumtaj, A

(13PMA012)

Thesis submitted to

Avinashilingam Institute for Home Science and Higher Education for Women,

Coimbatore – 641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

March, 2015

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Introduction

INTRODUCTION

In 1966, Imai and Iseki [9, 10] introduced two classes of logical algebras: BCK and BCI algebras. It is known that the class of BCK –algebras is a proper subclass of the class of BCI –algebras. In 2002, Neggers and Kim[19] introduced the notion of B-algebras which is related to several classes of algebras such as BCI/BCK-algebras. The main aim of this thesis is to discuss few interesting articles on B-algebras.

The following articles are chosen for our discussion:

1) **"On B-algebras"**, (2002), by J.Neggers and H.S.Kim.,[19].

2) **"On 0-commutative B-algebras"**, (2005),by H.S.Kim and H.G.Park.,[16].

3**) "On Quadratic B-algebras"**, (2001), by H.K.Park and H.S.Kim.,[21].

4) **"A Note on Normal Subalgebras in B-algebras"**, (2005),by A.Walendziak.,[27].

5) **"On Hom (-,-) as B-algebras"**, (2010), by N.O.Al-Shehrie.,[3].

6) **"The Second Isomorphism Theorem for B-algebras"**, (2014), by C.Endam Joemar and P.Vilela Jocelyn.,[6].

7) **"Derivations of B-algebras"**, (2010), by N.O.Al-Shehrie.,[2].

8)**"A Note on t-Derivations of B-algebras"**, (2014), by R.Soleimani and S.Jahangiri.,[26].

9) **"On (f,g)-Derivations of B-algebras"**,(2014),by L.K.Ardekani and B.Davvaz.,[4].

This thesis is split into four chapters:

In chapter 1, preliminaries on B-algebras due to J. Neggers ,H.S. Kim are presented [19]. Also results on 0-commutative B-algebras due to H.S. Kim ,H.G. park [16], quadratic B - algebras due to H.K.Park, H.S.Kim [21]are discussed.

The interesting results discussed in this chapter are given as follows:

1) If (X; ∗, 0) is a 0-commutative B- algebra, then
   (x ∗ a) ∗ (y ∗ b) = (b ∗ a) ∗ (y ∗ x) for any x, y, a, b ϵ X

2) Every p- semisimple BCI - algebra is a 0 - commutative B - algebra.

3) Let X be a field with | X | ≥ 3. Then every quadratic B - algebra (X; ∗ ,e),e ϵ X, has the form x ∗ y = x - y + e, where x, y ϵ X.

Chapter 2 deals with the study of “B - homomorphisms of B - algebras”.

In this chapter, Some properties of B-homomorphisms ,the second Isomorphism theorem for B-algebras due to C. Endam Joemar and P.Vilela Jocelyn [6] and some properties of Hom (X,Y) as B-algebras due to N.O.Al-Shehrie [3] are investigated.

The following interesting results are discussed:

1) If X is a B-algebra and Y is a 0-commutative B-algebra, then Hom (X,Y) is a 0-commutative B-algebra.
2) Let f : X → Y be a B-homomorphism from X into Y.
   i. If N is a subalgebra of X,then f(N) is a subalgebra of Y. Moreover, if N is commutative, then f(N) is commutative.
   ii. If K is a subalgebra of Y, then $f^{-1}(K)$ is a subalgebra of X containing Ker f.
   iii. If N is a normal subalgebra of X and f is onto, then f(N) is a normal subalgebra of Y.

iv. If K is a normal subalgebra of Y, then $f^{-1}(K)$ is a normal subalgebra of X.

3) If N and K are subalgebras of X with K normal in X, then N / (N∩K) ≅ NK/K.

4) Let H and K be subalgebras of X. Then HK is a subalgebra of X if and only if HK = KH.

Chapter 3 deals with the study of "Derivations and t-Derivations of B- algebras".

In this chapter, some properties of left-right derivations and 0-commutative B-algebras due to N.O.Al-Shehrie [2] and t-derivations of B-algebras due to R.Soleimani and S.Jahangiri [26] are investigated.

The following interesting results are discussed:

1) Let the self map d be a (ℓ,r)-derivation of B-algebra X. Then
   i. d(0) = d(x) ∗ x for all x ϵ X.
   ii. d is 1-1.
   iii. If d is regular, then it is the identity map.
   iv. If there is an element x ϵ X such that d(x) = x, then d is the identity map.
   v. If there is an element x ϵ X such that d(y) ∗ x = 0 or x ∗ d(y) = 0 for all y ϵ X, then d(y) = x for all y ϵ X, i.e. d is constant.

2) Let X be a 0-commutative B-algebra and d_1, d_2 be derivations of X. Then $d_1 \circ d_2 = d_2 \circ d_1$.

3) Let d_t be a self map of an associative 0-commutative B-algebra X. Then d_t *is* a t- derivation of X.

Chapter 4 deals with the study of “(f, g)-Derivations of B-algebras”.

In this chapter, some properties of f-derivations and (f, g)-derivations of B-algebras due to L.K.Ardekani and B.Davvaz [4] are investigated

The following interesting results are discussed:

1) Let d be an (r, l)-f- derivation of B-algebra X. Then, $d(0) = f(x) * d(x)$ and $d(x) = d(x) \wedge f(x)$, for all $x \in X$.

2) Let X be a commutative B-algebra and f, g be endomorphisms. Let d be a self map of X. If d = f, then d is an (f, g)-derivation.

3) If X is a commutative B-algebra , then (Der(X), $\wedge$) is a semi-group where Der(X) denotes the set of all (f, g)-derivations on X.

Review of Literature

REVIEW OF LITERATURE

In 1966, Imai and Iseki [9,10] introduced two classes of logical algebras: BCK and BCI algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.In1983,Hu and Li[7,8] introduced a wide class of abstract algebras:BCH-algebras. They have shown that the class of BCI –algebras is a proper subclass of theclass of BCH algebras. Jun et al.[12] introduced a new notion, called BH –algebras, which is a generalization of BCH/ BCI/BCK-algebras. They also defined the notion of ideals in BH-algebras. In 2005, J.Neggers and H.S.kim [18] introduced the notion of d-algebras, which is another useful generalization of BCK-algebras and then they investigated several relations between d-algebras and BCK-algebras as well as some other interesting relations betweend-algebras and oriented diagraphs.

In 2002, Neggers and Kim[19] introduced the notion of B-algebra, and then Cho and Kim[5] studied some of its properties. Abujabal and Al-Shehrie [1] defined and studied the notion of left derivation of BCI-algebras. Further ,Al- Shehrie[2] has applied the notion of left right derivation in BCI-algebra to B-algebra and obtained some of its properties.

Several other authors have also contributed to the study of the concepts mentioned above. we give here a brief survey of some of the articles published on B-algebras and fuzzy B-algebras.

1 .ON B-ALGEBRAS AND QUASIGROUPS:

J. R.Cho and H.S. Kim (2001) [5]

In this article, the relations between B-algebras and other topics, especially quasigroups are discussed.

2. THE CLASS OF B-ALGEBRAS COINCIDES WITH THE CLASS OF GROUPS:

M. Kondo and Y.B. Jun (2002) [17]

In this article, the authors showed that the class of B-algebras coincides with the class of Groups.

3. ON FUZZY B-ALGEBRAS:

Y.B Jun, E.H. Roh, and H.S. Kim (2002) [13]

In this article, the fuzzification of (normal) B-subalgebras is considered and some related properties are investigated. A characterization of a fuzzy B-algebra is given.

4. B-ALGEBRAS AND GROUPS:

J.Neggers,H.S. Kim (2003) [20]

In this article, the authors gave another proof of the close relationship of B-algebras with groups using the observation that the zero adjoint mapping is surjective. Moreover, the authors find a condition for an algebra defined on the real numbers to be a B-algebra using the analytic method. In addition they note certain other facts about commutative B-algebras.

5. INTUITIONISTIC FUZZY STRUCTURE OF B-ALGEBRAS:

Y.H.Kim and T.E.Jeong (2006) [15]

After the introduction of the concept of fuzzy sets by Zadeh, several researches were conducted on the generalization of the notion of fuzzy sets. The idea of "intuitionistic fuzzy set" was first published by Atanassov, as a generalization of the notion of fuzzy set. In this article, using the Atanassov's idea, the authors establish the intuitionistic fuzzification of the concept of subalgebras in B-algebras, and investigate some of their properties. The authors introduced the notion of equivalence relations on the family of all intuitionistic fuzzy subalgebras of B-algebras and investigated some related properties.

6. INTERVAL VALUED FUZZY B-ALGEBRAS:

A.saeid Borumand (2006) [22]

In this article, the notion of interval-valued fuzzy B-algebras (briefly, i-v fuzzy B-algebras), the level and strong level B-subalgebra is introduced. Some theorems which determine the relationship between these notions and B-subalgebras are studied. The images and inverse images of i-v fuzzy B-subalgebras are defined and how the homomorphic images and inverse images of i-v fuzzy B-subalgebra becomes i-v fuzzy B-algebras are studied.

7. FUZZY CLOSED IDEALS OF B-ALGEBRAS WITH INTERVAL VALUED-MEMBERSHIP FUNCTION:

T. Senapati, M. Bhowmik, M. Pal (2011) [24]

In this article, the notion of a fuzzy closed ideal of a B-algebra is introduced and some related properties are investigated. Also the product of fuzzy B-algebra is investigated.

8. FUZZY B-SUBALGEBRAS OF B-ALGEBRA WITH RESPECT TO T- NORM:

T. Senapati, M.Bhowmik, M. Pal (2012) [23]

In this article, the authors apply the concept of t-norm T to fuzzy structure of B-algebras. The notion of a fuzzy B- subalgebra of B-algebras with respect to t-norm is introduced and several related properties are investigated.

9. FUZZY B-IDEALS ON B-ALGEBRAS:

C. Yamini and S.Kailasavalli (2014) [28]

In this article, the authors introduced the fuzzy B-ideals and investigated how to deal with the homomorphism, Cartesian product of B-ideals and strongest fuzzy relation.

10. FUZZY DOT SUBALGEBRAS AND FUZZY DOT IDEALS OF B-ALGEBRAS:

T. Senapati, M. Bhowmik, M. Pal (2014) [25]

In this article, the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras are introduced and investigated some of their properties. The homomorphic image and inverse image of fuzzy dot subalgebras and fuzzy dot ideals are studied. Also, the notion of fuzzy relations on the family of fuzzy dot subalgebras and fuzzy dot ideals of B-algebras are introduced and investigated some related properties.

11. ON MEDIAL B-ALGEBRAS:

Y.H.Kim (2014) [14]

In this article, the author introduced the notion of medial B-algebras and obtained a fundamental theorem of B-homomorphism for B-algebras.

Chapter – 1

CHAPTER - 1

B-ALGEBRAS AND 0-COMMUTATIVE B-ALGEBRAS

SECTION : 1.1

PRELIMINARIES ON B-ALGEBRAS

Definition : 1.1.1

A **BCI- algebra** is a non empty set X with a constant 0 and a binary operation $*$ denoted by $(X; *, 0)$ satisfying the following axioms:

(BCI 1) $(x * y) * (x * z) * (z * y) = 0$,

(BCI 2) $(x * (x * y)) * y = 0$,

(BCI 3) $x * x = 0$,

(BCI 4) $x * y = 0, y * x = 0 \Longrightarrow x = y$.

Definition : 1.1.2

A BCI –algebra X is said to be a **BCK –algebra** if $0 * x = 0$, for all $x \in X$.

Definition : 1.1.3

For any BCI – algebra X, the set $X_+ = \{x \in X \,/\, 0 * x = 0\}$ is called a **BCK- part of X**.

A BCI- algebra X is said to be **p- semisimple** if $X_+ = \{0\}$.

Theorem : 1.1.4

Let $(X; *, 0)$ be a BCI algebra. Then the following are equivalent.

1) X is p – semisimple

2) $0 * x = 0$ implies $x = 0$,

3) $x * (x * y) = y$,

4) $x *(y * z) = z *(y * x)$,

5) $(x * y) * (z * u) = (x * z) * (y * u)$,

6) $(x * (0 * z) = z * (0 * x)$,

for any x, y, z, u ϵ X.

Proof:

Obvious.

Definition : 1.1.5

A **B-algebra** is a non-empty set X with constant 0 and binary operation $*$ satisfying the following axioms:

(B1) $x * x = 0$

(B2) $x * 0 = x$

(B3) $(x * y) * z = x * (z * (0 * y))$,

for any x, y, z ϵ X.

It is denoted by $(X; *, 0)$ or simply by X.

Note:

In BCI / BCK / B- algebras, a binary relation "≤" can be defined by, $x \leq y$ if

$x * y = 0$, for x, y ϵ X.

Example : 1.1.6

Let X be the set of all real numbers except for a negative integer –n. Define a binary operation $*$ on X by,

$$x * y = \frac{n(x-y)}{n+y}$$

Then (X; ∗, 0) is a B-algebra.

Example : 1.1.7

Let X = {0, 1, 2, 3, 4, 5} be a set with the following table:

∗	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then (X; ∗, 0) is a B-algebra.

Definition : 1.1.8

Let (X; ∗, 0) be a B-algebra. A non empty subset N of X is called a **B – subalgebra** of X (or a subalgebra of X), if x ∗ y ϵ N, for any x, y ϵ N.

Note:

In example 2.1.7, N_1 ={0,3} is a subalgebra of X, while N_2 = {0,1}is not a subalgebra of X, since $0 * 1 = 2 \notin N_2$. Any subalgebra of a B-algebra is also a B-algebra.

Theorem: 1.1.9

Let (X; ∗, 0) be a B-algebra.

i) If φ ≠ N ⊆ X, then the following are equivalent.

(a) N is a subalgebra of X.

(b) $x * (0 * y), 0 * y \in N$.

ii) Let N be a subalgebra of a B-algebra X and let $x, y \in X$. If $x * y \in N$ then

$y * x \in N$

Proof :

Obvious.

Lemma : 1.1.10

If $(X; *, 0)$ is a B-algebra, then

(BP1) $y * z = y * (0 * (0 * z))$ for any $y, z \in X$

Proof :

Let $(X; *, 0)$ be a B-algebra. Then for any $y, z \in X$,

$y * z = (y * z) * 0$ by (B2)

$= y * (0 * (0 * z))$ by (B3)

Lemma : 1.1.11

If $(X; *, 0)$ is a B-algebra, then

(BP2) $(x * y) * (0 * y) = x$ for any $x, y \in X$

Proof :

Let $(X; *, 0)$ be a B-algebra. From axiom (B3), with $z = 0 * y$, we get for any

$x, y \in X$, $(x * y) * (0 * y) = x * ((0 * y) * (0 * y))$.

Hence axiom (B1) yields,

$(x * y) * (0 * y) = x * 0$

So that from axiom (B2), it follows that,

$(x * y) * (0 * y) = x$.

Lemma : 1.1.12

If $(X; *, 0)$ is a B-algebra, then

(BP3) $x * z = y * z \Longrightarrow x = y$ for any $x, y, z \in X$.

Proof:

Let $(X; *, 0)$ be a B-algebra. If $x * z = y * z$, then

$(x * z) * (0 * z) = (y * z) * (0 * z)$ and thus by the above lemma, it follows that $x = y$.

Proposition : 1.1.13

If $(X; *, 0)$ is a B-algebra, then

(BP4) $x * (y * z) = (x * (0 * z)) * y$, for any $x, y, z \in X$

Proof:

Let $(X; *, 0)$ be a B-algebra.

Using lemma 1.1.10 and (B2),

$(x * (0 * z)) * y = x * (y * (0 * (0 * z)))$ by (B2)

$= x * (y * z)$ by (lemma1.1.10)

Lemma : 1.1.14

Let $(X; *, 0)$ be a B-algebra. Then for any $x, y \in X$

(BP5) $x * y = 0 \Longrightarrow x = y$

(BP6) $0 * x = 0 * y \Longrightarrow x = y$

(BP7) $0 * (0 * x) = x$

Proof:

Let $(X; *, 0)$ be a B-algebra. Let $x, y \in X$

To prove: (BP5)

since $x * y = 0 \Rightarrow x * y = y * y$

By lemma 1.1.12, it follows that $x = y$.

To prove: (BP6)

If $0 * x = 0 * y$, then

$0 = x * x$

$= (x * x)$

$= x * (0 * (0 * x))$

$= x * (0 * (0 * y))$

$= (x * y) * 0$

$= x * y$.

Thus by **(BP5)**, we get $x = y$

To prove : (BP7)

For any $x \in X$, we obtain $0 * x = (0 * x) * 0$

$= 0 * (0 * (0 * x))$ by (B2, B3)

By **(BP6)**, it follows that,

$$x = 0 * (0 * x).$$

Proposition : 1.1.15

Let $(X; *, 0)$ be a B-algebra. Then,

(BP8) $x * y = 0 * (y * x)$

(BP9) $(x * z) * (y * z) = x * y$, for all $x, y \in X$.

Proof:

Obvious.

Theorem : 1.1.16

$(X; *, 0)$ is a B-algebra if and only if it satisfies the following axioms:

i. $x * x = 0$,
ii. $0 * (0 * x) = x$,
iii. $(x * z) * (y * z) = x * y$, for any $x, y, z \in X$.
iv. $0 * (x * y) = y * x$, for any $x, y, z \in X$.

Proof:

Obvious.

Definition : 1.1.17

Let $(X; *, 0)$ be a B-algebra and let $g \in X$. Define $g^n = g^{n-1} * (0 * g)$ $(n \geq 1)$ and $g^0 = 0$.

Note:

$g = g^0 * (0 * g) = 0 * (0 * g) = g$ by (lemma 1.1.14)

Lemma : 1.1.18

Let $(X; *, 0)$ be a B-algebra and let $g \in X$. Then $g^n * g^m = g^{n-m}$ where $n \geq m$.

Proof:

If X is a B-algebra, then by lemma 1.1.14,

$$g^{2} * g = (g' * (0 * g)) * g$$

$$= (g * (0 * g)) * g$$

$$= g * (g * (0 * (0 * g)))$$

$$= g * (g * g)$$

$$= g * 0$$

$$= g$$

Assume that, $g^{n+1} * g = g^{n}$ $(n \geq 1)$

Then, $g^{n+2} * g = (g^{n+1} * (0 * g)) * g$

$$= g^{n+1} * (g * (0 * (0 * g)) \text{ by (B3)}$$

$$= g^{n+1} * 0$$

$$= g^{n+1}$$

Assume $g^{n} * g^{m} = g^{n-m}$, where $n-m \geq 1$

Then, $g^{n} * g^{m+1} = (g^{n} * (g^{m} * (0 * g))$

$$= (g^{n} * g) * g^{m} \text{ by (B3)}$$

$$= g^{n-1} * g^{m}$$

$$= g^{n-(m+1)} \text{ (since } n-m-1 \geq 0)$$

Hence the proof.

Lemma : 1.1.19

Let $(X; *, 0)$ be a B-algebra and let $g \in X$. Then $g^{m} * g^{n} = 0 * g^{n-m}$, where $n \geq m$.

Proof:

If X is a B-algebra, then by applying (B3), (B1) and lemma (1.1.14), we have,

$$g * g^2 = g * (g' * (0 * g))$$

$$= (g * g) * g'$$

$$= 0 * g$$

Assume that $g * g^n = g^{n-1}$ where $(n \geq 1)$

Then, $g * g^{n+1} = g * (g^n * (0 * g))$

$$= (g * g) * g^n \quad \text{by (B3)}$$

$$= 0 * g^n \quad \text{by (B1)}$$

Assume that $g^m * g^n = g^{n-m}$ where $n-m \geq 1$

Then, $g^{m+1} * g^n = (g^m * (0 * g)) * g^n$

$$= g^m * (g^n * g)$$

$$= g^m * g^{n-1}$$

$$= 0 * g^{n-m-1}$$

Hence the lemma.

Theorem : 1.1.20

Let $(X; *, 0)$ be a B-algebra and let $g \in X$. Then

$$g^m * g^n = \begin{cases} g^{m-n} & \text{if } m \geq n \\ 0 * g^{n-m} & \textit{otherwise} \end{cases}$$

Proof:

The proof follows by above two lemmas.

Proposition : 1.1.21

If (X; $*$, 0) is a B-algebra. Then $(a * b) * b = a * b^2$ for any a, b $\in$ X.

Proof:

It follows from (B3) that,

$(a * b) * b = a * (b * (0 * b))$

$= a * b^2$

Proposition : 1.1.22

If (X; $*$, 0) is a B-algebra, then $(0 * b) * (a * b) = 0 * a$ for any a, b $\in$ X.

Proof:

It follows from (B4) and (B1),

$(0 * b) * (a * b) = ((0 * b) * (0 * b)) * a$

$= 0 * a$

Note:

The converse of the above theorem is also true.

Proposition : 1.1.23

If (X; $*$, 0) is a 0 commutative B-algebra, then $(0 * a) * (a * b) = b * a^2$ for any a, b $\in$ X.

Proof:

If X is a 0-commutative B-algebra, then

$$(0 * a) * (a * b) = ((0 * a) * (0 * b)) * a$$

$$= (b * a) * a$$

$$= b * a^2$$

Hence the proof.

SECTION : 1.2

0-COMMUTATIVE B-ALGEBRAS

Definition : 1.2.1

A B-algebra $(X; *, 0)$ is said to be **0-commutative** ((or) simply commutative),

if $x * (0 * y) = y * (0 * x)$ for any $x, y \in X$.

Proposition : 1.2.2

If $(X; *, 0)$ is a 0-commutative B-algebra, then $(0 * x) * (0 * y) = y * x$, for any

$x, y \in X$.

Proof:

Since, X is a 0-commutative B-algebra, for any $x, y \in X$, by **(BP1)**,

$(0 * x) * (0 * y) = y * (0 * (0 * x))$

$= y * x$.

Theorem : 1.2.3

If $(X; *, 0)$ is a 0-commutative B-algebra, then $a * (a * b) = b$ for any $a, b \in X$

Proof:

If X is a commutative, then by **(B4)**,

$a * (a * b) = (a * (0 * b))) * a$

$= (b * (0 * a)) * a$

$= b * (a * a)$

$= b$.

Note:

The converse of the above theorem is also true.

Corollary : 1.2.4

If $(X; *, 0)$ is a 0-commutative B-algebra, then the left cancellation laws holds.

That is, $a * b = a * b' \Longrightarrow b = b'$.

Proof:

Let $(X; *, 0)$ is a 0-commutative B-algebra, by the above theorem,

$b = a * (a * b)$

$= a * (a * b')$

$= b'$.

Proposition : 1.2.5

If $(X; *, 0)$ is a 0-commutative B- algebra, then $(0 * a) * (a * b) = b * a^2$ for any $a, b \in X$.

Proof:

$(0 * a) * (a * b) = ((0 * a) * (a * b)) * a$

$= (b * a) * a$

$= b * a^2$

Hence the proof.

Theorem :1.2.6

If $(X; *, 0)$ is a 0-commutative B-algebra, then $(x * a) * (y * b) = (b * a) * (y*x)$ for any $x, y, a, b \in X$.

Proof:

Let $(X; *, 0)$ is a 0-commutative B-algebra, then for any $x, y, a, b \in X$,

$(x * a) * (y * b) = x * [(y * b) * (0 * a)]$

$= x * [y * \{(0 * a) * (0 * b)\}]$

$= x * (y * (b * a))$

$= [x * (0 * (b * a))] * y$

$= [(b * a) * (0 * x)] * y$

$= (b * a) * [y * (0 * (0 * x))]$

$= (b * a) * (y * x)$

Corollary : 1.2.7

If $(X; *, 0)$ is a 0-commutative B-algebra, then $(x * z) * (y * z) = x * y$, for any $x, y, z \in X$.

Proof:

Since, $(X; *, 0)$ is a 0-commutative B-algebra, for any $x, y, a \in X$,

$$(x * a) * (y * b) = (b * a) * (y * x) \qquad (1)$$

Put $x = a$ in (1)

Then,

$(a * a) * (y * b) = (b * a) * (y * a)$

$\Rightarrow 0 * (y * b) = (b * a) * (y * a)$

$\Rightarrow b * y = (b * a) * (y * a)$

Corollary : 1.2.8

If $(X; *, 0)$ is a 0-commutative B-algebra, then $(x * y) * (z * x) = x * y$, for any $x, y, z \in X$.

Proof:

By theorem 1.2.6, $x * y = (x * y) * (z * z)$

$= (z * y) * (z * x)$ for any $x, y, z \in X$.

Corollary : 1.2.9

If $(X; *, 0)$ is a 0-commutative B-algebra, then $(x * a) * y = (0 * a) * (y * x)$, for any $x, y, a \in X$.

Proof:

Since $(X; *, 0)$ is a 0-commutative B-algebra, then

$$(x * a) * (y * 0) = (0 * a) * (y * x) \qquad (1)$$

put b = 0 in (1),

$$(x * a) * (y * 0) = (0 * a) * (y * x)$$

$$\Longrightarrow (x * a) * y = (0 * a) * (y * x)$$

Corollary : 1.2.10

If $(X; *, 0)$ is a 0-commutative B-algebra, then $x * (y * b) = b * (y * x)$, for any

$x, y, b \in X$.

Proof:

Put $a = 0$ in $(x * a) * (y * b) = (b * a) * (y * x)$

$$x * (y * b) = (x * 0) * (y * b)$$

$$= (b * 0) * (y * x)$$

$$= b * (y * x)$$

Theorem :1.2.11

If (X; $*$, 0) is a 0-commutative B-algebra, then $(x * y) * z = (x * z) * y$, for any $x, y, z \in X$.

Proof:

$$(x * y) * z = x * [z * (0 * y))]$$

$$= x * [y * (0 * z))]$$

$$= (x * z) * y.$$

Proposition : 1.2.12

Let (X; $*$, 0) is a 0-commutative B-algebra. Then (X: $\leq$) is a partially ordered set, where $x \leq y$ if and only if $x * y = 0$.

Proof:

Obvious.

Theorem : 1.2.13

If (X; $*$, 0) is a 0-commutative B-algebra, then

$[(x * y) * (x * z)] * (z * y) = 0$, for any $x, y, z \in X$.

Proof:

By applying theorem 1.2.6,

$$[(x * y) * (x * z)] * (z * y) = [(z * y) * (x * x)] * (z * y)$$

$$= [(z * y) * 0] * (z * y)$$

$$= (z * y) * (z * y)$$

= 0.

Hence the proof.

Theorem: 1.2.14

If (X; ∗, 0) is a 0-commutative B-algebra, then[x ∗ (x ∗ y)] ∗ y = 0, for any x, y, z ϵ X.

Proof:

By applying theorem 1.2.11,

[x ∗ (x ∗ y)] ∗ y = (x ∗ y) ∗ (x ∗ y)

= 0.

Theorem: 1.2.15

Every 0-commutative B-algebra is a BCI –algebra.

Proof:

It follows from proposition 1.2.12 and theorems 1.2.13 and 1.2.14.

Note:

The converse of above theorem need not be true in general.

Example: 1.2.16

Let X = {0, 1, 2, 3} be a set with the following table:

∗	0	1	2	3
0	0	0	3	3
1	1	0	3	2
2	2	3	0	1
3	3	3	0	0

Then it is a BCI –algebra, but not a 0-commutative B-algebra, since

$3 * (0 * 2) = 0$

$\neq 2 * (0 * 3).$

Theorem :1.2.17

Every 0-commutative B-algebra is a p- semisimple BCI –algebra.

Proof:

It follows from theorem 1.1.4 &1.2.15.

Theorem : 1.2.18

Every p- semisimple BCI –algebra is a 0-commutative B-algebra.

Proof:

It is enough to show, **(B3)**

For any x, y, z $\in$ X, by theorem 1.1.14,

$x * (z * (0 * y)) = (x * 0) * (z * (0 * y))$

$= (x * z) * (0 * (0 * y))$

$= (x * z) * y.$

Hence the proof.

SECTION : 1.3

QUADRATIC B - ALGEBRAS

Definition : 1.3.1

Let X be a field with $|X| \geq 3$. An algebra $(X; *)$ is said to be **quadratic** if $x * y$ is defined by $x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, ,where $a_1,...,a_6 \in X$ are fixed.

Definition : 1.3.2

A quadratic algebra $(X; *)$ is said to be a **quadratic B – algebra**, if for some fixed $e \in X$ it satisfies the following conditions :

(QB1) $x * x = e$

(QB2) $x * e = x$

(QB3) $(x * y) * z = x * (z * (e * y))$, for any $x, y, z \in X$.

Theorem : 1.3.3

Let X be a field with $|X| \geq 3$. Then every quadratic B- algebra $(X; *,e)$, $e \in X$, has the form $x * y = x - y + e$, where $x, y \in X$.

Proof:

Let $x * y = Ax^2 + Bxy + cy^2 + Dx + Ey + F$ (1)

for some fixed $A,B,C,D,E,F \in X$. Then,

$e = x * x = (A+ B + C) x^2 + (D + E) x + F$ (2)

Let x=0 in (2), then F = e

hence, $x * y = Ax^2 + Bxy + cy^2 + Dx + Ey + e$ (3)

if y = x in (3),then

$e = x * x = (A+B+C)x^2 +(D+E)x + e$, for any $x \in X$ and hence $A+B+C = 0 = D+E$

that is, $E = -D$ and $B = -A - C$

Hence (3)becomes,

$$x * y = (x-y)(Ax - (y + D)) + e \quad (4)$$

Let $y = e$ in (4).then by (QB2),

$$x = x * e = (x - e)(A x - (e + D -1)(x - e) = 0$$

Since, X is a field either $x - e = 0$ or $Ax – ce + D - 1=0$

Since $|X| \geq 3$, we have $Ax - Ce + D - 1 = 0$, for any $x \in X$.

This means that $A = 0$, $1 - D + Ce = 0$.

Hence (4) becomes

$$x * y = (x - y) + C(x - y)(e - y) + e \quad (5)$$

If we replace e by x, and x by y respectively in (5), then

$$e * x = (e - x) + C(e - x)(e - x) + e.$$

It follows that

$$e * (e * x) = e * [(e - x) + C(e - x)^2 + e]$$

$$= x – C (e - x)^2 + C(e - x)\{1 + C(e - x)\}^2$$

$$= x + C^3 (e - x)^4 + 2C^2(e - x)^3 .$$

Since $x = e * (e * x)$, we obtain

$$C^2 (e - x)^3 \{-C x + 2 + C e\} = 0.$$

Since X is a field with $|X| \geq 3$, we obtain $C = 0$

Thus, every quadratic B-algebra $(X; *, e)$ has the form $x * y = x - y + e$, where

$x, y \in X$.

Example : 1.3.4

Let R be the set of all real numbers. Define $x * y = x - y + \sqrt{2}$. Then $(R;*, \sqrt{2})$ is a quadratic B-algebra.

Example : 1.3.5

Let $K = GF(p^n)$ be a Galois field. Define $x * y = x - y + e$, $e \in K$. Then $(K; *, e)$ is a quadratic B-algebra.

Proposition : 1.3.6

Let X be a field with $|X| \geq 3$. If $(X; *, e)$ is a quadratic B-algebra, then $(x * y) * (x * z) = z * y$ for any $x, y, z \in X$.

Proof:

Obvious.

Theorem: 1.3.7

Let X be a field with $|X| \geq 3$. Then every quadratic B-algebra on X is a BCI-algebra.

Proof:

Obvious.

Chapter – 2

CHAPTER - 2

B-HOMOMORPHISMS OF B - ALGEBRAS

SECTION : 2.1

NORMAL B-ALGEBRAS AND HOM (-,-) AS B - ALGEBRAS

Definition : 2.1.1

A B-algebra X is said to be **associative**, if $(x * y) * z = x *(y * z)$, for all

$x, y, z \in X$.

Definition : 2.1.2

A nonempty subset N of A is said to be **normal** (or a normal subalgebra) of A, if $(x * a) * (y * b) \in N$ for any $x * y, a * b \in N$.

Theorem : 2.1.3 [27]

Let N be a subalgebra of a B-algebra X. Then the following statements are equivalent:

(i) N is a normal subalgebra.

(ii) If $x \in X$ and $y \in N$, then $x * (x * y) \in N$.

Proof:

To prove: (i) ⇒ (ii)

Let $x \in X$ and $y \in N$. Then $x * x = 0 \in N$ and $0 * y \in N$.

Since N is normal,

$(x * 0) * (x * y) \in N$. Thus $x * (x * y) \in N$.

To prove: (ii) ⇒ (i)

Let $x * y, a * b \in N$. By theorem 1.1.9 (ii), $b * a \in N$.

By (BP9), we have

$(0 * a) * (0 * b) = (0 * a) * [(0 * a) * (b * a)]$ and using (ii) we get,

$(0 * a) * (0 * b) \in N$.

Applying (B3) twice we obtain

$$x * (x * [(0 * a) * (0 * b)]) = x * [(x * b) * (0 * a)]$$

$$= (x * a) * (x * b).$$

From this, combining (ii) with (1) we get

$(x * a) * (x * b) \in N$.

We have $[(x * a) * (x * b)] * (y * x) \in N$, because N is a subalgebra.

Using (B3) and (BP9) we get,

$$[(x * a) * (x * b)] * (y * x) = (x * a) * [(y * x) * (0 * (x * b))]$$

$$= (x * a) * [(y * x) * (b * x)]$$

$$= (x * a) * (y * b).$$

Therefore $(x * a) * (y * b) \in N$. consequently, N is normal.

Corollary : 2.1.4

In 0-commutative B-algebras,the concepts of subalgebras and normal sub-algebras coincide.

Proof:

Obvious.

Lemma : 2.1.5

Let X be a B-algebra,

(i) If {Nα : α ∈ Λ} is any nonempty collection of subalgebras of X, then $\bigcap_{\sim\in\Lambda} N\sim$ is a subalgebras of X.

(ii) If {Nα : α ∈ Λ} is any non-empty collection of subalgebras of X,then $\bigcap_{\sim\in\Lambda} N\sim$ is a normal subalgebras of X.

Proof:

Let X be a B-algebra.

(i) Follows from the fact that a non –empty intersection of a system of subalgebras is a subalgebra.

(ii) Let {Nα : α ∈ Λ}be any non-empty collection of normal subalgebras of X.

By (i), Nα is a subalgebra of X. Suppose ,x ∗ y ,a ∗ b ϵ $\bigcap_{\sim\in\Lambda} N\sim$.

Then x ∗ y, a ∗ b ∈ Nα for all α ∈ Λ.

Since each Nα is normal, (x ∗ a) ∗ (y ∗ b) ∈ Nα for all α ∈ Λ.

Therefore, (x ∗ a) ∗ (y ∗ b) ∈ $\bigcap_{\sim\in\Lambda} N\sim$ and so, $\bigcap_{\sim\in\Lambda} N\sim$ is a normal subalgebra of X.

Definition : 2.1.6

A mapping f : X→Y between B-algebras X and Y is called a **B-homomorphism** if f(x ∗ y) = f(x) ∗ f(y),for all x, y ϵ X.

Example : 2.1.7

Let X = {0, 1, 2, 3} be a set with the following table:

∗	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X; *, 0)$ is a B-algebra. If we define $f(0) = 0$, $f(1) = 3$, $f(2) = 3$and $f(3) = 0$, then

$f: X \to Y$ is a B-homomorphism.

Definition : 2.1.8

(i) A B-homomorphism $f: X \to X$ is called a **B- monomorphism** , **B- epimorphism** or **B-isomorphism** if f is one-one ,onto or a bijection respectively.

(ii) A B-isomorphism $f: X \to X$ is called a **B- automorphism**.

(iii) A mapping f of a B-algebra X into itself is called a B-endomorphism of X if

$f(x * y) = f(x) * f(y)$ for all $x, y \in X$ and $f(0) = 0$.

(iv) The trivial B-homomorphism "0" is defined as $0(x) = 0$ for $x \in X$.

(v) The subset $\{x \in X / f(x) = 0$ in Y is called the **kernel** of the B-homomorphism

$f : X \to Y$ and it is denoted by Ker f.

Note:

Any kernel of a B-homomorphism $f : X \to Y$ is a subalgebra of X.

Notation:

The set of all B-homomorphisms of a B-algebra X into a B-algebra Y is denoted by Hom(X,Y).

The following example shows that (Hom (X,Y); $*$,0) may not be a B-algebra in general, where $*$ is defined as follows:

$(f * g)\, x = f(x) * g(x)$ for f, g ϵ Hom(X,Y),for x ϵ X.

Example : 2.1.9

Let X= {0,1,2,3,4,5} be a B-algebra with cayley table as follows:

∗	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Define a map f : X→X by f(x) = 0,for all x ϵ X, and a map g: X→X by g(x) = 0,for all x ϵ X.

Then f, g ϵ Hom(X,Y) but $f * g \notin$ Hom(X,Y), for

$(f * g)(3 * 1) = (f * g)(4)$

$= f(4) * g(4)$

$= 4$ and

$(f * g)(3) * (f * g)(1) = (f(3) * g(3)) * (f(1) * g(1))$

$= 3 * 2$

$= 5.$

Therefore, $((f * g)(3*1)) \neq (f * g)(3) * (f * g)(1)$.

Hence Hom(X,Y) is not a B-algebra.

Theorem : 2.1.10

If X is a B-algebra and Y is an associative B-algebra, then Hom(X ,Y) is an associative B-algebra.

Proof:

Let f, g ϵ Hom(X,Y) and x ϵ X. Then,

$(f * g)(x * y) = f(x * y) * g(x * y)$

$= (f(x) * f(y)) * (g(x) * g(y))$

$= (f(x) * (f(y)) * (g(x)) * g(y))$

$= (f(x) * (0 * g(y))) * (f(y) * g(x)$ by (BP4)

$= ((f(x) * 0) * g(y)) * (f(y) * g(x))$

$= (f(x) * g(y)) * (f(y) * g(x))$ by (B2)

$= (f(x) * (g(y) * f(y))) * g(x)$

$= f(x) * (g(x) * (0 * (g(y) * f(y))))$ by (B3)

$= f(x) * g(x) * (f(y) * g(y)))$ by (BP8)

$= f(x) * g(x)) * (f(y) * g(y))$

$= (f * g(x) * (f * g) (y)$

Then f $*$g ϵ Hom (X, Y), for all f, g ϵ Hom(X,Y).

Since Y is a B-algebra, Hom(X,Y) is a B-algebra for all f, g, h ϵ Hom(X,Y) .

Now let f, g, h ϵ Hom(X,Y) and x ϵ X. Then

$(f * g) * h) (x) = (f(x) * g(x) * h(x)$

$= f(x) * (g(x) * h(x))$

$= (f * (g * h)) (x)$ because Y is an associative B-algebra.

Hence Hom(X,Y) is an associative B-algebra.

Theorem : 2.1.11

If X is a B-algebra and Y is a 0-commutative B-algebra, then Hom(X,Y) is a 0-commutative B-algebra.

Proof:

Let f, g $\in$ Hom(X,Y) and x $\in$ X. Then

$(f * g)(x * y) = f(x * y) * g(x * y)$

$= (f(x) * (f(y) * (g(x) * g(y))$

$= (g(y) * f(y)) * (g(x) * f(x))$ by theorem 1.2.6.

$= (0 * (f(y) * g(y))) * (0 * (f(x) * g(x)))$

$= (f(x) * g(x)) * (f(y) * g(y))$

$= (f * g)(x) * (f * g)(y)$

Therefore, $f * g \in$ Hom(X, Y) for all f, g $\in$ (Hom X, Y) .

Since Y is a B-algebra, Hom(X,Y) is a B-algebra for all f, g, h $\in$ Hom(X,Y).

Now let f, g $\in$ Hom(X,Y) and x $\in$ X. Then

$((f * 0) * g)(x) = (f(x) * 0) * g(x)$

$= g(x) * (0 * f(x))$

$= ((g * 0) * f)(x)$ because Y is a 0-commutative B-algebra.

Hence the proof.

Definition : 2.1.12

Let M and Θ be subsets of X and Hom (X,Y) respectively. We define orthogonal subsets $M^{\perp}$ and $\Theta^{\perp}$ of M and Θ, respectively, by

$M^{\perp} = \{f \in \text{Hom}(X,Y) \,/\, f(x) = 0, \text{for all } x, y \in X\}$ and

$\Theta^{\perp} = \{x \in X / f(x) = 0$, for all $f \in Hom(X,Y)\}$.

Theorem : 2.1.13

Let X be a B-algebra, Y be an associative B-algebra, $M \subseteq X$ and $\Theta \subseteq Hom(X,Y)$. Then $M^{\perp}$ and $\Theta^{\perp}$ are normal subalgebras of Hom(X,Y) and X, respectively.

Proof:

Let $f * g, h * k \in M^{\perp}$. Then $(f * g)(x) = 0$, for all $x \in M$ and $(h * k)(x) = 0$, for all $x \in M$. By theorem 2.1.10, we have that Hom(X,Y) is an associative B-algebra. Thus

$$((f * h) * (g * k))(x) = (((f * h) * g) * k)(x)$$

$$= ((f * (g * (0 * h))) * k)(x) \text{ by (B3)}$$

$$= ((f * ((g * (0 * h)) * k)(x)$$

$$= ((f * g) * (h * k))(x)$$

$$= (f * g)(x) * (h * k)(x) = 0 \text{ for all } x \in M.$$

Thus, $(f * h) * (g * k) \in M^{\perp}$ and so $M^{\perp}$ is a normal subalgebra of Hom(X,Y).

Now, let $x * y, a * b \in \Theta^{\perp}$, hence $f(x * y) = 0$ and $f(a * b) = 0$, for all $f \in Hom(X,Y)$.

Since, Y is an associative B-algebra, in a similar way we can prove that

$f((x * a) * (y * b)) = 0$,for all $f \in Hom(X,Y)$ and then $(x * a) * (y * b) \in \Theta^{\perp}$, for all

$f \in Hom(X,Y)$.

Therefore, $\Theta^{\perp}$ is a normal subalgebra of X.

Theorem : 2.1.14

Let X be a B-algebra, Y be a 0-commutative B-algebra, $M \subseteq X$ and $\Theta \subseteq Hom(X,Y)$. Then $M^{\perp}$ and $\Theta^{\perp}$ are normal subalgebras of Hom(X,Y) and X, respectively.

Proof:

Let $f * g, h * k \in M^{\perp}$.

Then $(f * g)(x) = 0$, for all $x \in M$ and $(h * k)(x) = 0$, for all $x \in M$.

From theorem 2.1.11, we know that Hom(X,Y) is a 0-commutative B-algebra.

Hence

$((f * h) * (g * k))(x) = ((k * h) * (g * f)(x)$

$= ((0 * (h * k)) * (0 * (f * g)))(x)$

$= (0(x) * (h * k)(x)) * (0(x) * (f * g)(x) = 0$ for all $x \in M$.

Thus $(f * h) * (g * k) \in M^{\perp}$ is a normal subalgebra of Hom(X,Y).

Now, let $x * y, a * b \in \Theta^{\perp}$. Then $f(x * y) = 0$ and $f(a * b) = 0$ for all $f \in$ Hom(X,Y)

Since Y is a 0-commutative B-algebra, in similar way we can prove that

$f((x * a) * (y * b)) = 0$, for all $f \in$ Hom(X,Y) and then $(x * a) * (y * b) \in \Theta^{\perp}$ for all $f \in$ Hom(X,Y). Therefore, $\Theta^{\perp}$ is normal subalgebra of X.

SECTION : 2.2

THE SECOND ISOMORPHISM THEOREM FOR B-ALGEBRAS

Definition : 2.2.1

Let H, K be subalgebras of X. Define the subset HK of X to be the set

HK = $\{x \in X \,/\, x = h * (0 * k)$ for some $h \in H, k \in K\}$.

Example : 2.2.2

Let X = {0, 1, 2, 3, 4, 5} be a set with the following table:

∗	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then (X; ∗, 0) is a B-algebra .

Let H = {0, 3} and K = {0, 4}.

Clearly, H and K are subalgebras of X.

However, HK = {0, 2, 3, 4} is not a subalgebra of X since, $4*3=1 \notin HK$.

Note:

In general, HK need not be a subalgebra. In the succeeding results, a necessary and sufficient condition for HK to be a subalgebra will be proved.

Lemma: 2.2.3

Let H and K be subalgebras of X. Then

i) H ⊆ HK, KH and K ⊆ HK, KH,

ii) HH = H,

iii) H ⊆ K implies HK = KH = K.

Proof :

Let H and K be subalgebras of X.

If $h \in H$, then $h = h * 0 = h * (0 * 0) \in HK$ by (B1) and (B2).

Also, $h = 0 * (0 * h) \in KH$ by (BP1). Thus, H ⊆ HK, KH.

Similarly, K ⊆ HK, KH.

(ii) and (iii) can be proved easily.

The following theorem gives the necessary and sufficient condition for HK to be a subalgebra.

Theorem : 2.2.4

Let H and K be subalgebras of X. Then HK is a subalgebra of X if and only if HK = KH.

Proof :

Suppose HK is a subalgebra of X. Let $x \in KH$. Then $x = k * (0 * h)$ for some $k \in K$, $h \in H$. By Lemma 2.2.3(i), $k, h \in HK$.

Since HK is a subalgebra, $0 * h \in HK$ by Theorem 1.1.9and so $x = k * (0 * h) \in HK$.

Thus, KH ⊆ HK. (1)

On the other hand, let $x \in HK$.

Then $0 * x \in HK$.

Hence, $0 * x = h * (0 * k)$ for some $h \in H$, $k \in K$.

Since H and K are subalgebras, $0 * h \in H$ and $0 * k \in K$.

Thus, by (BP7) and (BP8), we have,

$$x = 0 * (0 * x)$$
$$= 0 * (h * (0 * k))$$
$$= (0 * k) * h$$
$$= (0 * k) * (0 * (0 * h)) \in KH.$$

Hence, $HK \subseteq KH$. (2)

Therefore, by (1) and (2) HK = KH.

Conversely, suppose that HK = KH and let $x, y \in HK$. Then

$x = h_1 * (0 * k_1)$, $y = h_2 * (0 * k_2)$ for some $h_1, h_2 \in H$, $k_1, k_2 \in K$.

Now, since $(0 * k_2) * h2 = (0 * k_2) * (0 * (0 * h_2)) \in KH = HK$,

$(0 * k_2) * h_2 = h_3 * (0 * k_3)$ for some $h_3 \in H$, $k_3 \in K$.

Similarly, $k_1 * (0 * h_3) = h_4 * (0 * k_4)$ for some $h_4 \in H$, $k_4 \in K$.

Thus, by (BP7), (BP8), and (B3), we have

$$x * y = (h_1 *(0*k_1))*(h_2 *(0*k_2))$$
$$= (h_1 * (0 * k_1)) * [0 * ((0 * k_2) * h_2)]$$
$$= (h_1 * (0 * k_1)) * [0 * (h_3 * (0 * k_3))]$$
$$= (h_1 * (0 * k_1)) * ((0 * k_3) * h_3)$$
$$= h_1 * [((0 * k_3) * h_3) * (0 * (0 * k_1))]$$
$$= h_1 * [((0 * k_3) * h_3) * k_1]$$

$= h_1 * [(0 * k_3) * (k_1 * (0 * h_3))]$

$= h_1 * [(0 * k_3) * (h_4 * (0 * k_4))]$

$= h_1 * [((0 * k_3) * k_4) * h_4]$

$= h_1 * [((0 * k_3) * k_4) * (0 * (0 * h_4))]$

$= (h_1 * (0 * h_4)) * ((0 * k_3) * k_4)$

$= (h_1 * (0 * h_4)) * [0 * (k_4 * (0 * k_3))] \in HK.$

Therefore, HK is a subalgebra of X.

Corollary : 2.2.5

If H and K are subalgebras of a commutative B-algebra X, then HK is a subalgebra of X.

Proof :

Since X is commutative, HK = KH.

By Theorem 2.2.4, HK is a subalgebra of X.

Lemma : 2.2.6

If N and K are subalgebras of X with K normal in X, then $N \cap K$ is a normal subalgebra of N.

Proof :

Since $N \cap K \subseteq N$, $N \cap K$ is a subalgebra of N by Lemma 2.1.5 (i).

Let $x \in N$ and $y \in N \cap K$. Since K is normal of X, $y \in K$, $x \in N \subseteq X$, we have $x * (x * y) \in K$ by Theorem 2.1.3.

Since N is a subalgebra and $x, y \in N$, we have $x * (x * y) \in N$.

Thus, $x * (x * y) \in N \cap K$.

By Theorem 2.1.3, $N \cap K$ is normal of N.

Lemma : 2.2.7

If N and K are subalgebras of X with K normal in X, then NK is a subalgebra of X.

Proof :

Let $x, y \in NK$. Then $x = n_1 * (0 * k_1)$ and $y = n_2 * (0 * k_2)$ for some $n_1, n_2 \in$ N and $k_1, k_2 \in K$. Since N and K are subalgebras of X, $n_1 * n_2 \in N$ and $k_1 * k_2 \in K$.

By Theorem2.4, $n_2 * (n_2 * (k_1 * k_2)) \in K$. Therefore, by (B1), (B2), (B3), and (BP7), we have

$$x * y = (n_1 * (0 * k_1)) * (n_2 * (0 * k_2))$$

$$= n_1 * [(n_2 * (0 * k_2)) * (0 * (0 * k_1))]$$

$$= n_1 * [(n_2 * (0 * k_2)) * k_1]$$

$$= [n_1 * ((0 * n2) * (0 * n_2))] * [(n_2 * (0 * k_2)) * k_1]$$

$$= ((n_1 * n_2) * (0 * n_2)) * [(n_2 * (0 * k_2)) * k_1]$$

$$= (n_1 * n_2) * \{[(n_2 * (0 * k_2)) * k_1] * (0 * (0 * n_2))\}$$

$$= (n_1 * n_2) * \{[(n_2 *(0 * k_2)) * k_1] * n_2\}$$

$$= (n_1 * n_2) * \{[n_2 * (k_1 * (0 * (0 * k_2)))] * n_2\}$$

$$= (n_1 * n_2) * [(n_2 *(k_1 * k_2)) * n_2]$$

$$= (n_1 * n_2) * \{0 * [n_2 * (n_2 * (k_1 * k_2))]\} \in NK.$$

Therefore, NK is a subalgebra of X.

Lemma : 2.2.8

If N and K are normal subalgebras of X, then NK = KN is a normal subalgebra of X.

Proof :

By Lemma 2.2.7 and Theorem 2.2.4, NK = KN is a subalgebra of X. Let $x \in X$ and $y \in NK$. Then $y = n * (0 * k)$ for some $n \in N$ and $k \in K$. Therefore, by (B1), (B2), (B3), (BP7), (BP8), and (BP4), we have

$$
\begin{aligned}
x * (x * y) &= x * [x * (n * (0 * k))] \\
&= x * ((x * k) * n) \\
&= x * [((0 * (0 * x)) * k) * (0 * (0 * n))] \\
&= x * [(0 * (k * x)) * (0 * (0 * n))] \\
&= (x * (0 * n)) * (0 * (k * x)) \\
&= [(x * (0 * n)) * ((0 * x) * (0 * x))] * (0 * (k * x)) \\
&= [((x * (0 * n)) * x) * (0 * x)] * (0 * (k * x)) \\
&= [(x * (0 * n)) * x] * [(0 * (k * x)) * (0 * (0 * x))] \\
&= [(x * (0 * n)) * x] * [(0 * (k * x)) * x] \\
&= [x * (x * (0 * (0 * n)))] * [0 * (x * (0 * (k * x)))] \\
&= (x * (x * n)) * [0 * (x * (x * k))] \in NK.
\end{aligned}
$$

By Theorem 2.1.3, NK is normal in X.

Lemma : 2.2.9

If $f : X \to Y$ and $g: Y \to Z$ are B-homomorphisms, then $f \circ g: X \to Z$ is also a B-homomorphism.

Proof:

Obvious.

Corollary : 2.2.10

The composition of B-monomorphisms is a B-monomorphism, the composition of B-epimorphisms is a B-epimorphism, the composition of B-isomorphisms is a B-isomorphism, and the composition of B-automorphisms is a B-automorphism.

Proof:

Obvious.

Note:

(i) The identity mapping $id_X : X \rightarrow X$ of a B-algebra X is an automorphism of X.

(ii) If $f: X \rightarrow Y$ is a B-homomorphism from X into Y, then $f(0_X)=0_Y$ and $f(0_X * x) = 0_Y * f(x)$ for all $x \in X$.

Lemma : 2.2.11

Let $f: X \rightarrow Y$ be a B-homomorphism from X into Y.

(i) If N is a subalgebra of X, then f(N) is a subalgebra of Y . Moreover, if N is commutative, then f (N) is commutative.

(ii) If K is a subalgebra of Y, then $f^{-1}(K)$ is a subalgebra of X containing Ker f.

(iii) If N is a normal subalgebra of X and f is onto, then f(N) is a normal subalgebra of Y

(iv) If K is a normal subalgebra of Y, then $f^{-1}(K)$ is a normal subalgebra of X.

Proof :

Let $f: X \rightarrow Y$ be a B-homomorphism.

(i) Suppose N is commutative. If $x, y \in f(N)$, then there exist $a, b \in N$ such that $f(a) = x$ and $f(b) = y$. Since N is commutative,

$x * (0_Y * y) = f(a) * (f(0_X) * f(b))$

$= f(a * (0_X * b))$

$= f(b * (0_X * a))$

$= f(b) * (f(0_X) * f(a))$

$= y * (0_Y * x).$

Therefore, f(N) is commutative.

(ii) The proof is obvious.

(iii)Let N be a normal subalgebra of X. By (i), f(N) is a subalgebra of Y . Let $x \in Y$ and $y \in f(N)$. Then $y = f(n)$ for some $n \in N$. Since f is onto, there exists $a \in X$ such that $f(a) = x$.

Since N is normal in X, $a * (a * n) \in N$.

Thus, $x * (x * y) = f(a) * (f(a) * f(n))$

$= f(a * (a * n)) \in f(N).$

Therefore, f(N) is normal in y.

(iv)Let K be a normal subalgebra of Y. By (ii), $f^{-1}(K)$ is a subalgebra of X.

Let $x \in X$ and $y \in f^{-1}(K)$. Then $f(x) \in Y$ and $f(y) \in K$.

Since K is normal subalgebra of Y,

$f(x * (x * y)) = f(x) * (f(x) * f(y)) \in K.$

Thus, $x * (x * y) \in f^{-1}(K)$ and so $f^{-1}(K)$ is normal in X.

Definition : 2.2.12

If N is a normal subalgebra of X, then $(X/N; *, [0]_N)$ is a B-algebra, where $X/N = \{[x]_N : x \in X\}$ and $*$ is defined by $[x]_N * [y]_N = [x * y]_N$. For $x \in X$, $[x]_N$ is the equivalence class containing x, that is, $[x]_N = \{y \in X : x \sim N\ y\}$, where $x \sim N\ y$ if and only if $x * y \in N$ for any x, $y \in X$. The algebra X/N is called **the quotient B-algebra** of X by N.

Theorem : 2.2.13 (first Isomorphism theorem for B-algebras)

Let f: $X \to Y$ be a B-homomorphism from X into Y. Then X/Ker f ≅ Im f. In particular, if f is surjective, then X/ Ker f ≅ Y .

Proof:

Obvious

Theorem : 2.2.14 (Third Isomorphism theorem for B-algebras)

Let N and K be subalgebras of X and let $K \subseteq N$. Then $X/N \cong (X/K) / (N/K)$

Proof:

Obvious.

Lemma: 2.2.15

If K and N are Normal subalgebras of X such that $K \subseteq N$, then N/K is a normal subalgebra of X/K.

Proof:

Since K and N are normal subalgebras of $X \ni K \subseteq N$, N/K is well defined. Also X/K and N/K are B-algebras. since $K \subseteq N$, $N/K \subseteq X/K$ and so N/K is a subalgebra of X/K. Let $[x]_k * [y]_K$,$[a]_K * [b]_K \in N/K$. Since $[x * y]_K = [x]_K * [y]_K \in N/K$, $x * y \in N$. Similarly, $a * b \in N$.

Since N is normal, $(x * a) * (y * b) \in N$ and so

$([x]_K * [a]_K) * ([y]_K * [b]_K) = [x * a]_K * [y * b]_K$

$= [(x * a) * (y * b)]_K \in N/K.$

Therefore, N/K is a normal subalgebra of X/K.

Note:

This means that (X / K) / (N / K) in theorem 2.2.14 indeed is well defined.

Theorem : 2.2.16 (Second Isomorphism theorem for B-algebras)

If N and K are subalgebras of X with K normal in X, then

$$N / (N \cap K) \cong \frac{NK}{K}.$$

Proof:

Let N and K be subalgebras of X with K normal in X.

By lemma 2.2.6, $N \cap K$ is a normal subalgebra of N. Thus, $N \cap K$ is well defined.

By lemma 2.2.7 and theorem 2.2.4, NK= KN is a subalgebra of X.

Thus, $N / (N \cap K)$ is well- defined.

By Lemma 2.2.7 and Theorem 2.2.4, NK = KN is a subalgebra of X.

By Lemma 2.2.3(i), K is normal in NK. Hence, NK/K is well-defined.

Define f: $N \rightarrow NK/K$ by $f(n) = [n]_k$ by f(n) = [n]K for all $n \in N$.

Since $N \subseteq NK$, $f(n) = [n]_K \in NK / K$ for all $n \in N$. Let $m, n \in N$ such that m = n.

Then $f(m) = [m]_K = [n]_K = f(n)$. Thus, f is well-defined.

Let $[x]_K \in NK / K$ .Then $x = n * (0 * k)$ for some $n \in N$, $k \in K$.

It follows that f is onto. since

$[x]_K = [n * (0 * k)]_K$

$= [n]_K * [0 * k]_K$

$= [n]_K$

$= f(n).$

Also, f is a B-homomorphism since,

$f(a * b) = [a * b]_K$

$= [a]_K * [b]_K$

$= f(a) * f(b).$

Hence, by Theorem 2.2.14, $N / \text{Ker } f \cong NK / K$.

But $\text{Ker } f = \{n \in N: f(n) = K\}$

$= \{n \in N: [n]_K = [0]_k \}$

$= \{n \in N : n \in K\}$

$= N \cap K$.

Therefore, $N / (N \cap K) \cong NK / K$.

Chapter – 3

CHAPTER - 3

DERIVATIONS AND t - DERIVATIONS OF B - ALGEBRAS

SECTION 3.1:

DERIVATIONS OF B-ALGEBRAS

Definition : 3.1.1

A self map d of a B-algebra X is said to be **regular**, if $d(0) = 0$.

If $d(0) \neq 0$ then d is called an **irregular** map.

Notation :

For a B-algebra, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition : 3.1.2

Let X be a B-algebra .

By a **(ℓ, r) - derivation** of X, we mean a self map d of X satisfying the identity

$d(x * y) = (d(x) * y) \wedge (x * d(y))$,for all $x, y \in X$.

If X satisfies the identity $d(x * y) = (x * d(y)) \wedge (d(x) * y)$, for all $x, y \in X$, then we say that d is a **(r, ℓ) - derivation** of X.

Moreover ,if d is both a (ℓ, r) and a (r, ℓ) derivation ,we say that d is a **derivation** of X.

Example : 3.1.3

Let X = {0, 1, 2, 3} be a 0-commutative B-algebra with cayley table as follows:

$*$	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Define a map d: X→X by :

$$d(x) = \begin{cases} 3 \ if \ x = 0 \\ 2 \ if \ x = 1 \\ 1 \ if \ x = 2 \\ 0 \ if \ x = 3 \end{cases}$$

Then d is a derivation of X.

Example : 3.1.4

Let $\mathbb{Z}$ be the set of all integers " - " the minus operation on $\mathbb{Z}$. Then ($\mathbb{Z}$, -, 0) is a B-algebra. Let d(x) = x-1 for all x$\in \mathbb{Z}$. Then

$$(d(x) - y) \wedge (x - d(y)) = ((x-1) - y) \wedge (x - (y - 1))$$

$$= (x - y - 1) \wedge (x - y + 1)$$

$$= (x - y + 1) - 2$$

$$= x - y - 1$$

$$= d(x - y) \text{ for all } x, y \in z$$

and so d is a (r, ℓ)-derivation of X.

But $(1 - d(0)) \wedge (d(1) - 0) = (1 - (-1) \wedge (0-0)$

$= 2 * 0$

$= 0-(0-2)$

$= 2$

$\neq 0$

$= d(1)$

$= d(1-0)$

and thus d is not a (r, ℓ)-derivation of X.

Proposition : 3.1.5

Let d be a (ℓ, r) - derivation of B-algebra X. Then

i. $d(0) = d(x) * x$ for all, $x \in X$
ii. d is 1-1.
iii. If d is regular, then it is the identity map.
iv. If there is an element $x \in X$ such that $d(x) = x$, then d is the identity map.
v. If there is an element $x \in X$ such that $d(y) * x = 0$ or $x * d(y) = 0$ for all $y \in X$, then $d(y) = x$ for all $y \in X$, i.e. d is constant.

Proof:

i. Let $x * X$. Then $x * x = 0$ and so

$d(0) = d(x * x) \wedge (x * d(x))$

$= (x * d(x)) * [(x * d(x)) * d(x) * x]$

$= [(x * d(x)) * (0 * d(x) * x))] *(x * d(x)]$

$= [(x * d(x)) *(x * d(x))] * (x * d(x))$

$= 0 * (x * d(x))$

= d(x) x

ii. Let x, y ϵ X such that d(x) = d(y).

Then by (i), we have d(0) = d(x) x.

Also, by (i), d(0) = d(y) y.

Thus, d(x) x = d(y) y

Therefore, d(x) x = d(y) y.

By cancellation law, x = y.

That is d is 1-1.

iii. suppose that d is regular, and x ϵ X, thus

d(0) = 0.

0 = d(x) x by (i)

⟹ d(x) = x for all x ϵ X.

That is d is the identity map

iv. Suppose d(x) = x for some x ϵ X. Then

d(x) x = 0

⟹ d(0) = 0 by (i)

Using (iii), we have that d is the identity map.

v. follows directly from x y = 0 ⟹ x = y.

Proposition : 3.1.6

Let X = {0, 1, 2, 3} be a 0-commutative B-algebra with cayley table as follows

∗	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Define a map d: X→X by

$$d(x) = \begin{cases} 2 \; if \; x = 0 \\ 3 \; if \; x = 1 \\ 0 \; if \; x = 2 \\ 1 \; if \; x = 3 \end{cases}$$

then d is a derivation of X.

Proposition : 3.1.7

Let (X; ∗, 0) be a 0-commutative B-algebra and d is a (ℓ, r) - derivation of X. Then

(i) d(x ∗ y) = d(x) ∗ y.

(ii) d(x) ∗ d(y) = x ∗ y, for all x, y ϵ X.

Proof:

(i) Let x, y ϵ X, then

d(x ∗ y) = (d(x) ∗ y) ∧ (x ∗ d(y))

= (x ∗ d(y)) ∗ [(x ∗ d(y)) ∗ (d(x) ∗ y)]

= d(x) ∗ y, since x ∗ (x ∗ y) = y

(ii) Let x, y ϵ X, then from proposition 3.1.5 (i) we have, d(0) = d(x) ∗ x

Also, $d(0) = d(y) \square y$

Thus, $d(x) \square x = d(y) \square y$

That is, $(d(y) \square y) \square (d(x) \square x) = 0$

$(x \square y) \square (d(x) \square d(y)) = 0$, by theorem 1.2.6

$\Rightarrow d(x) \square d(y) = x \square y$

Proposition : 3.1.8

Let $(X; \square, 0)$ be a 0-commutative B-algebra and d is a (ℓ, r)-derivation of X. Then

(i) $d(x \square y) = x \square d(y)$

(ii) $d(x) \square d(y) = x \square y$

Proof:

Obvious.

Definition : 3.1.9

Let X be a B-algebra and d_1, d_2 be two self maps of X. we define

$d_1 \circ d_2 : X \rightarrow X$ as $d_1 \circ d_2(x) = d_1(d_2(x))$, for all $x \in X$.

Proposition : 3.1.10

Let $(X; \square, 0)$ be a 0-commutative B-algebra and d is a (ℓ, r)-derivation of X.

Then $d_1 \circ d_2$ is also a (ℓ, r)-derivation of X**.**

Proof:

Let X be a 0-commutative B-algebra and d_1, d_2 are (ℓ, r)-derivation of X and let

$x, y \in X$. then $(d_1 \circ d_2)(x \square y) = d_1 [d_2(x) \square y) \wedge (x \square d_2(y))]$

$= d_1 [d_2(x) \square y)$ by theorem 1.2.3

Using proposition 3.1.7 (i), we get

$$(d_1 \circ d_2)(x \ast y) = d_1(d_2(x)) \ast y$$

$$= (x \ast d_1(d_2(y))) \ast [(x \ast d_1(d_2(y))) \ast (d_1(d_2(x)) \ast y)]$$

$$= (d_1 \circ d_2)(x) \ast y) \wedge (x \ast (d_1 \circ d_2)(y))$$

which implies $(d_1 \circ d_2)$ is a (ℓ, r)-derivation.

Proposition : 3.1.11

Let X be a 0-commutative B-algebra and let d_1, d_2 is a (r, ℓ)-derivation of X. Then $d_1 \circ d_2$ is also a (r, ℓ)-derivation of X.

Proof:

Obvious.

Theorem : 3.1.12

Let X be a 0-commutative B-algebra and let d_1, d_2 be derivation of X. Then $d_1 \circ d_2$ is also a derivation of X.

Proof:

The proof follows by above two propositions.

Theorem : 3.1.13

Let X be a 0-commutative B-algebra and d_1, d_2 is a derivations of X. Then $d_1 \circ d_2 = d_2 \circ d_1$.

Proof:

Let X be a 0-commutative B-algebra and d_1, d_2 is a derivations of X. Since d_2 is a (ℓ, r)-derivation of X, then from proposition 3.1.7 (i), we have,

$(d_1 \circ d_2)\ (x \square y) = d_1 (d_2 (x \square y))$

$= d_1 (d_2(x) \square y)$ for all x. y ϵX.

but d_1 is (r, ℓ)-derivation of X, so from proposition 3.1.9 (i), we obtain

$d_1 (d_2(x) \square y) = d_2 (x) \square d_1(y)$

Thus, we have for all x, y ϵ X, we have

$(d_1 \circ d_2)\ (x \square y) = d_2 (x) \square d_1 (y)$ (1)

Also since d_1 is (r, ℓ) - derivation of X, we have

$(d_2 \circ d_1) (x \square y) = d_2(x \square d_1(y))$, for all x, y ϵ X

But d_2 is (ℓ, r) - derivation of X, so

$d_2(x \square d_1(y)) = d_2(x) \square d_1(y)$, for all x, y ϵ X.

Thus, for all x, y ϵ X, we have

$(d_2 \circ d_1) (x \square y) = d_2(x) \square d_1(y)$ (2)

By (1) and (2),

$d_1 \circ d_2 (x \square y) = (d_2 \circ d_1) (x \square y)$ for all x, y ϵ X

putting y = 0,we get

$d_1 \circ d_2(x) = (d_2 \circ d_1) (x)$ for all x, y ϵ X

$\Longrightarrow d_1 \circ d_2 = d_2 \circ d_1$.

Hence the proof.

SECTION : 3.2

t-DERIVATIONS OF B-ALGEBRAS

Definition : 3.2.1

Let X be a B-algebra. For any $t \in X$, we define a self map $d_t: X \rightarrow X$ by

$d_t(x) = x * t$ for all $x \in X$.

Definition : 3.2.2

The self map d_t of a B-algebra X is said to be ***t*-regular** if (0) = 0.

Lemma : 3.2.3

Let d_t be a self map of a B-algebra X. Then the following hold:

(i) d_t is one-one.

(ii) $d_t(x) * d_t(y) = x * y$ for all $x, y \in X$.

Proof:

It is sufficient to prove (ii). By applying,

(BP9) $(x * y) * (z * y) = x * z$, we obtain

$$d_t(x) * d_t(y) = (x * t) * (y * t)$$

$$= x * y.$$

Lemma : 3.2.4

Let d_t be a self map of a 0-commutative B-algebra X. Then the following hold:

(i) $d_t(x * y) = d_t(x) * y$ for all $x, y \in X$.

(ii) If d_t is t-regular, then it is an identity.

Proof:

(i) Since d_t is a self map of a B-algebra X, then by theorem 1.2.11,

$d_t(x * y) = (x * y) * t$

$= (x * t) * y$

$= d_t(x) * y.$

(ii) Let d_t be t-regular and $x \in X$.

Then 0 = (0) and by (i),

$0 = d_t(x) * x.$

$\Longrightarrow d_t(x) = x$ for all $x \in X$.

Therefore d_t is an identity.

This completes the proof.

Definition : 3.2.5

Let X be a B-algebra.

Then for any $t \in X$, the self map $d_t : X \to X$ is called a **left- right t - derivation** (or briefly (l,r)-t-derivation) of X if it satisfies the identity $d_t(x * y) = (d_t(x) * y) \wedge (d_t(y))$ for all $x, y \in X$.

Similarly, if d_t satisfies the identity $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$, then it is called **right-left t-derivation** (or briefly (r, l)-t-derivation) of X.

Moreover, if d_t is both a (l, r)- and a (r, l)-t-derivation of X, then d_t is a **t-derivation** of X.

Example : 3.2.6

Let X be a B-algebra of all real numbers except for a negative integer $-n$, with a binary operation $*$ on X by $x * y = \frac{n(x-y)}{n+y}$. For any $t \in X$, define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$. First, we show that X is a 0-commutative B-algebra.

For any $x, y \in X$,

$$x * (0 * y) = x * \frac{n(0-y)}{n+y}$$

$$= x * \frac{-ny}{n+y}$$

$$= \frac{nx+xy+ny}{n}$$

Also, $y * (0 * x) = y * \frac{n(0-x)}{n+x}$

$$= y * \frac{-nx}{n+x}$$

$$= \frac{ny+yx+nx}{n}$$

Hence X is a 0-commutative B-algebra.

Next for all $x, y, t \in X$,

$$(x * y) * t = n\,\frac{n(x-y)-t(n+y)}{(n+y)(n+t)} \quad \text{and}$$

$$(x * t) * y = n\,\frac{n(x-y)-t(n+y)}{(n+y)(n+t)}$$

Since X is a 0-commutative B-algebra, for all $x, y, t \in X$,

$$(d_t(x) * y) \wedge (x * d_t(y)) = (x * (y * t)) * ((x * (y * t)) * ((x * t) * y))$$

$$= (x * y) * t$$

$= d_t(x * y)$

So d_t is a (l, r)-t-derivation of X. But d_t is not a (r, l)-t-derivation of X.

Example : 3.2.7

Let X = {0, 1, 2} be a B-algebra with the following table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

For any $t \in X$, define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t$ for all $x \in X$. Then, d_t is a (l, r)-t-derivation of X, which is not a (r, l)-t-derivation of X.

If we set x = 0, y = 2 and t = 1, then

$d_t(x * y) = (x * y) * t$

$= 0$

$\neq 2$

$= ((x * t) * y) * (((x * t) * y) * (x * (y * t)))$

$= (x * d_t(y)) \Lambda (d_t(x) * y)$.

But if for any $t \in X$, define a self map $d_t : X \rightarrow X$ by $d_t(x) = x * t = x$ then X is a t-derivation of X, which is t-regular.

Theorem : 3.2.8

Let d_t be a self map of a B-algebra X. Then,

(i) If d_t is a (l,r)-t-derivation and t-regular of X, then $d_t(x) = d_t(x) \Lambda x$ for all $x \in X$.

(ii) If d_t is a (r,l)-t-derivation of X, then $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if d_t is t-regular.

Proof:

(i) If d_t is a (l, r)-t-derivation and t-regular of X, then by **(B2)**

$$d_t(x) = d_t(x \square 0)$$

$$= (d_t(x) \square 0) \wedge (x \square d_t(0))$$

$$= d_t(x) \wedge (x \square 0)$$

$$= d_t(x) \wedge x.$$

(ii) Let d_t be a (r,l)-t-derivation of X. If d_t is t-regular, then by **(B2)**

$$d_t(x) = d_t(x \square 0)$$

$$= (x \square d_t(0)) \wedge (d_t(x) \square 0)$$

$$= (x \square 0) \wedge d_t(x)$$

$$= x \wedge d_t(x).$$

Conversely, suppose that

$d_t(x) = x \wedge d_t(x)$ for all, $x, y \in X$, then

$$d_t(0) = 0 \wedge d_t(0)$$

$$= d_t(0) \square (d_t(0) \square 0)$$

$$= d_t(0) \square d_t(0)$$

$$= 0.$$

So d_t is t-regular.

Theorem : 3.2.9

Let d_t be a self map of an associative 0-commutative B-algebra X. Then d_t is a t-derivation of X.

Proof:

Since X is an associative 0-commutative B-algebra, we have

$d_t (x * y) = (x * y) * t$

$= (x * (y * t)) * 0$ (by B1)

$= ((x * (y * t)) * ((x * (y * t)) * (x * (y * t)))$

$= ((x * (y * t)) * ((x * (y * t)) * ((x * (y * t))$

$= (x * (y * t)) * ((x * (y * t)) * ((x * (y * t))$

$= ((x * t) * y) \Lambda (x * (y * t))$

$= (d_t (x) * y) \Lambda (x * (d_t(y))$

Again,

$d_t(x * y) = (x * y) * t$

$= ((x * t) * y) * 0$

$= ((x * t) * y) * (((x * t) * y) * ((x * t) * y))$

$= ((x * t) * y) * (((x * t) * y) * ((x * y) * t))$ [by (14)]

$= ((x * t) * y) * (((x * t) * y) * (x * (y * t)))$ [by (16)]

$= (x * (y * t)) \Lambda ((x * t) * y)$

$= (x * d_t(y)) \Lambda (d_t(x) * y)$

Lemma : 3.2.10

Let d_t be a (r, l)-t-derivation of a 0-commutative B-algebra X. Then

$d_t(x * y) = x * d_t(y)$ for all, $x, y \in X$.

Proof:

Since d_t is a (r, l)-t-derivation of X,

$$d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$$

$$= (d_t(x) * y) * ((d_t(x) * y) * (x * d_t(y)))$$

$$= x * d_t(y).$$

Definition : 3.2.11

Let X be a B-algebra and d_t, d_t' be two self maps of X. Then we define

$d_t \circ d_t' : X \to X$ by $(d_t \circ d_t')(x) = d_t(d_t'(x))$, for all $x \in X$.

Theorem : 3.2.12

Let X be a 0-commutative B-algebra and d_t, d_t' are (r,l)-t-derivations of X. Then $d_t \circ d_t'$ is a t-derivation of X.

Proof:

Since d_t, d_t' are two self maps of X, by Lemma 3.2.4(i) and by theorem 1.2.3, for all $x, y \in X$,

$$(d_t \circ d_t')(x * y) = d_t(d_t'(x * y))$$

$$= d_t(d_t'(x) * y)$$

$$= d_t(d_t'(x)) * y$$

$$= (x * d_t(d_t'(y))) * (x * d_t(d_t'(y))) * (d_t(d_t'(x)) * y)$$

$$= (d_t(d_t'(x)) * y) \wedge (x * d_t(d_t'(y)))$$

$$= ((d_t \circ d_t')(x) * y) \wedge (x * (d_t \circ d_t'(y)).$$

Next, since d_t ,d'_t are (r,l)-t-derivations of X, by lemma 3.2.10 and by theorem 1.2.3,for all x, y ϵ X, we have

$((d_t \circ d_t') (x * y) = d_t (d_t' (x * y))$

$= d_t (x * d_t' (y))$

$= x * d_t (d_t' (y))$

$= d_t (d_t'(x)) * y) * d_t (d_t'(x) * y) * (x * d_t (d_t' (y)))$

$= (x * d_t (d_t' (y))) \Lambda (d_t (d_t'(x)) * y)$

$= (x * ((d_t \circ d_t')(y)) \Lambda ((d_t \circ d_t')(x) * y)$

Theorem : 3.2.13

Let X be a 0-commutative B-algebra and let d_t be a (r, l)-t-derivation and d_t' be self map of X. Then $d_t \circ d_t' = d_t' \circ d_t$

Proof:

Suppose d_t is a (r, l)-t-0derivation and d_t' is a self map of X. By Lemmas 3.2.4(i) and 3.2.10, for all x, y ϵ X,

$(d_t \circ d_t')(x * y) = d_t (d_t'(x * y))$

$= d_t (d_t' (x) * y))$

$= d_t' (x) * d_t(y).$

Again, by lemmas 3.2.10 and 3.2.4 (i), for all x, y ϵ X,

$(d_t' \circ d_t) (x * y) = d_t '(d_t (x * y))$

$= d_t '(x * d_t(y))$

$= d_t '(x) * d_t(y).$

Therefore, $(d_t \circ d_t ') (x * y) = (d_t' \circ d_t) (x * y).$

By putting $y = 0$, for all $x \in X$, we get

$(d_t \circ d_t{}')(x) = (d_t{}' \circ d_t)(x)$.

Hence, $d_t \circ d_t{}' = d_t{}' \circ d_t$.

This completes the proof.

Definition : 3.2.14

Let X be a B-algebra and let d_t and $d_t{}'$ be two self maps of X.

Then we define $d_t \square d_t{}' : X \to X$ by $(d_t \square d_t{}')(x) = d_t(x) \square d_t{}'(x)$ for all $x \in X$.

Theorem : 3.2.15

Let d_t, $d_t{}'$ be two (r, l)-t-derivations of a 0-commutative B-algebra X. Then $d_t \square d_t{}' = d_t{}' \square d_t$.

Proof:

Since d_t is a (r,l) -t-derivations of X, for all $x, y \in X$ by Lemmas 3.2.4 (i) and 3.2.10,

$(d_t \circ d_t{}')(x \square y) = (d_t (d_t{}')(x \square y))$

$= d_t (d_t{}')(x) \square y$

$= d_t{}'(x) \square d_t(y)$.

Again, since d_t is a (r, l)- t-derivation of X, then by Lemmas 3.2.10 and 3.2.4 (i),

$(d_t \circ d_t{}')(x \square y) = (d_t (d_t{}')(x \square y))$

$= d_t(x) \square d_t{}'(y))$

$= d_t(x) \square d_t{}'(y)$.

Therefore,

$d_t{}'(x) \square d_t(y) = d_t(x) \square d_t{}'(y)$.

By putting y = x, for all $x \in X$, we get,

$d_t'(x) \circ d_t(x) = d_t(x) \circ d_t'(x)$.

Hence, $d_t \circ d_t' = d_t' \circ d_t$. This proves the theorem.

Chapter – 4

CHAPTER - 4

(f, g) - DERIVATIONS OF B-ALGEBRAS

SECTION : 4.1

f - DERIVATIONS OF B-ALGEBRAS

Notation :

If (X; , 0) is a 0-commutative B-algebra, then y (y x) = x for all x, y ϵ X that means x ^ y = x.

Definition : 4.1.1

Let X be a B-algebra.

A **left-right f-derivation** (briefly, (l, r) - f-derivation) of X is a self map d of X satisfying the identity d(x y) = (d(x) f(y)) (f(x) d(y)), for all x, y X, where f is an endomorphism of X.

If d satisfies the identity d(x y) = (f(x) d(y)) (d(x) f(y)), for all x, y X, then we say d is a **right-left f-derivation** (briefly, (r,l)-f-derivation) of X.

Moreover, if d is both an (l, r)-f-derivation and (r, l)-f-derivation, we say d is an **f-derivation**.

Example : 4.1.2

Let X = {0, 1, 2} and the binary operation is defined as follows:

	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then, (X, □, 0) is a B-algebra. Define the map d, f : X → X by

$$d(x) = f(x) = 0 \text{ if } x = \begin{cases} 0 \text{ } if \text{ } x = 0 \\ 2 \text{ } if \text{ } x = 1 \\ 1 \text{ } if \text{ } x = 2 \end{cases}$$

Then, f is an endomorphism.

Then d is both (l, r)- and (r, l)- f-derivation of X. So d is an f-derivation.

Now, we define d’ = 0.

Then, d' is not an (l, r)-f-derivation, since d' (1 □ 2) = 0

but (d' (1) □ f(2)) □ (f(1) □ d' (2)) = (0 □ 1) □ (2 □ 0) = 2.

Also, d' is not an (r, l)-f-derivation,

since d' (1 □ 2) = 0

but (f(1) □ d' (2)) □ (d' (1) □ f(2)) = (2 □ 0) □ (0 □ 1) = 2.

Theorem : 4.1.3

Let d be an (l, r) – f - derivation of B-algebra X. Then, d (0) = d(x) □ f(x), for all x □ X.

Proof:

For all x □ X, we have,

d(0) = d(x □ x)

= (d(x) □ f(x)) □ (f(x) □ d(x))

= (f(x) □ d(x)) □ ((f(x) □ d(x)) □ (d(x) □ f(x)))

= ((f(x) □ d(x)) □ (0 □ (d(x) □ f(x)))) □ (f(x) □ d(x))

= ((f(x) □ d(x)) □ (f(x) □ d(x))) □ (f(x) □ d(x))

= 0 □ (f(x) □ d(x))

= d(x) □ f(x).

Theorem : 4.1.4

Let d be an (r, l) -f- derivation of B-algebra X. Then, d(0) = f(x) □ d(x) and d(x) = d(x) □ f(x), for all x □ X**.**

Proof:

For all x □ X, we have,

d(0) = d(x □ x)

= (f(x) □ d(x)) □ (d(x) □ f(x))

= (d(x) □ f(x)) □ ((d(x) □ f(x)) □ (f(x) □ d(x)))

= ((d(x) □ f(x)) □ (0 □ (f(x) □ d(x)))) □ (d(x) □ f(x))

= ((d(x) □ f(x)) □ (d(x) □ f(x))) □ (d(x) □ f(x))

= 0 □ (d(x) □ f(x))

= f(x) □ d(x).

Also, we have for all x □ X,

d(x) □ 0 = d(x)

= d(x □ 0)

= (f(x) □ d(0)) □ (d(x) □ f(0))

= d(x) □ (d(x) □ (f(x) □ d(0)))

= d(x) □ (d(x) □ (f(x) □ (f(x) □ d(x)))).

By left cancellation law, $0 = d(x) * (f(x) * (f(x) * d(x))$

We get $d(x) * f(x) = d(x)$.

Corollary : 4.1.5

Let d be an (l, r)-f-derivation ((r, l)-f-derivation) of B-algebra X. Then,

(i) d is injective if and only if f be injective.

(ii) If d is regular, then d = f.

(iii) If there is an element $x_0 \in X$ such that $d(x_0) = f(x_0)$, then d = f.

Proof:

Let d be an (l, r)-f-derivation.

(i) Suppose that d is injective and $f(x) = f(y)$, $x, y \in X$.

Then, $d(0) = d(x) * f(x)$ and $d(0) = d(y) * f(y)$, by Theorem 4.1.3.

So, $d(x) * f(x) = d(y) * f(y)$. Thus, $d(x) = d(y)$, by right cancellation law

Therefore, x = y, since d is injective.

Conversely, suppose that f is injective and $d(x) = d(y)$, $x, y \in X$

Then, $d(0) = d(x) * f(x)$ and $d(0) = d(y) * f(y)$, by Theorem 4.1.3

$\Longrightarrow d(x) * f(x) = d(y) * f(y)$.

Thus $f(x) = f(y)$, by left cancellation law.

Therefore x = y, since f is injective.

ii) Suppose that d is regular and $x \in X$. Then $0 = d(0) = d(x) * f(x)$, by Theorem 4.1.3.

Hence, $d(x) = f(x)$.

iii) Suppose that there is an element $x_0 \in X$ such that $d(x_0) = f(x_0)$. Then, $d(x_0) * f(x_0) = 0$. So, $d(0) = 0$, by Theorem 4.1.3.

Part (ii) implies that $d = f$.

Similarly, when d is an (r, l)-f-derivation, the proof follows by Theorem 4.1.4.

SECTION : 4.2

(f, g) - DERIVATIONS OF B – ALGEBRAS

Definition : 4.2.1

Let X be a B-algebra.

A **left-right (f, g)-derivation** (briefly, (l, r)-(f, g)-derivation) of X is a self map d of X satisfying the identity $d(x * y) = (d(x) * f(y)) * (g(x) * d(y))$, for all $x, y \in X$, where f, g are endomorphisms of X.

If d satisfies the identity $d(x * y) = (f(x) * d(y)) * (d(x) * g(y))$, for all $x, y \in X$, then we say d is a **right-left (f, g)- derivation** (briefly, (r, l)-(f, g)-derivation) of X.

Moreover, if d is both an (l, r) – and (r, l)-(f, g)- derivation, then d is a **(f, g)-derivation**.

Note :

If the function g is equal to the function f, then the (f, g) - derivation is f-derivation defined in Definition 4.1.1. Also, if we choose the functions f and g as the identity functions, then the (f, g)-derivation that we define coincides with the derivation defined in Definition 3.1.2.

Example : 4.2.2

Let $(X; *, 0)$, d and f are as Example 4.1.2. Define g = I, where I is an identity function. Then d is an (f, g)-derivation.

But d is not an (l, r)-(g, f)-derivation, since $d(1 * 2) = 1$

but $(d(1) * g(2)) * (f(1) * d(2)) = (2 * 2) * (2 * 1) = 0$.

Also, d is not an (r, l)-(g, f)-derivation, since d(1 ∗ 2) = 1

but (g(1) ∗ d(2)) ∗ (d(1) ∗ f(2)) = (1 ∗ 1) ∗ (2 ∗ 1) = 0.

Example : 4.2.3

Let X = {0, 1, 2, 3} and binary operation ∗ is defined as:

∗	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then, (X; ∗, 0) is a B- algebra. Define maps d, f, g: X → X by

d(x) = f(x) = $\begin{cases} 0 \; if \; x = 0 \\ 2 \; if \; x = 1 \\ 1 \; if \; x = 2 \\ 3 \; if \; x = 3 \end{cases}$ and g(x) = $\begin{cases} 0 \; if \; x = 0 \\ 3 \; if \; x = 1,2,3 \end{cases}$

Then f and g are B- endomorphisms. Then d is an (f, g)-derivation.

Theorem : 4.2.4

Let d be a self map of a B-algebra X. Then, the following hold:

(i) If d is a regular (l, r)-(f, g)-derivation of X, then d(x) = d(x) ∗ g(x), for all x ∈ X.

(ii) If d is an (r, l)-(f, g)-derivation of X, then d(x) = f(x) ∗ d(x), for all x ∈ X if and only if d is a regular.

Proof:

i) Suppose that d is a regular (l, r)-(f, g)-derivation of X and x ∈ X. Then,

d(x) = d(x ∗ 0)

$= (d(x) * f(0)) * (g(x) * d(0))$

$= d(x) * g(x)$.

(ii) Suppose that d is an (r, l)-(f, g)-derivation of X. If $d(x) = f(x) * d(x)$, for all $x \in X$, then

$d(0) = f(0) * d(0)$

$= d(0) * (d(0) * 0)$

$= d(0) * d(0)$

$= 0$.

Conversely, suppose that $d(0) = 0$. Then,

$d(x) = d(x * 0)$

$= (f(x) * d(0)) * (d(x) * g(0))$

$= f(x) * d(x)$, for all $x \in X$.

Theorem : 4.2.5

Let X be a commutative B-algebra. Then, for all $x, y \in X$,

(i) If d is an (l, r)-(f, g)-derivation of X, then $d(x * y) = d(x) * f(y)$. Moreover, $d(0) = d(x) * f(x)$.

(ii) If d is an (r, l)-(f, g)-derivation of X, then $d(x * y) = f(x) * d(y)$. Moreover $d(0) = f(x) * d(x)$.

(iii) If d is an (l, r)-(f, g)-derivation ((r, l)-(f, g)-derivation), then Corollary 4.1.5 is valid.

Proof:

Obvious.

Theorem : 4.2.6

Let X be a commutative B-algebra and f, g be endomorphisms. Let d be a self map of X. If d = f, then d is an (f, g)-derivation.

Proof:

Suppose that $x, y \in X$. Then,

$$
\begin{aligned}
d(x * y) &= f(x * y) \\
&= f(x) * f(y) \\
&= d(x) * f(y) \\
&= (d(x) * f(y)) \wedge (g(x) * d(y)).
\end{aligned}
$$

So, d is an (r, l)-(f, g)-derivation. Also, we have

$$
\begin{aligned}
d(x * y) &= f(x * y) \\
&= f(x) * f(y) \\
&= f(x) * d(y) \\
&= (f(x) * d(y)) \wedge (d(x) * g(y)).
\end{aligned}
$$

Hence, d is an (l, r) - (f, g)-derivation. Therefore, d is an (f, g)-derivation.

Theorem : 4.2.7

Let $(X; *, 0)$ be a commutative B-algebra X. For all $x \in X$,

i)If d be an (l, r)-(f, g)-derivation, then

$$
d(x^m * x^n) = \begin{cases} d(0) * (0 * f(x)^{m-n} if\ m \geq n \\ d(0) * f(x)^{n-m}, if\ m < n \end{cases}
$$

ii) If d be an (r, l)-(f, g)-derivation, then

$$
d(x^m * x^n) = \begin{cases} f(x)^{m-n-1} * (0 * d(x) if\ m \geq n \\ (0 * d(x)) * f(x)^{n-m-1} if\ m < n \end{cases}
$$

Proof:

Clearly, $(x * y^n) * y = x * y^{n+1}$, for all $x, y \in X$ and $n \geq 1$. So by induction we get,

$d(x^n) = d(0) * (0 * f(x))^n$, where d is an (l, r) - (f, g)-derivation.

Also, we have

$d\,(x^n) = f(x)^{\,n-1} * (0 * d(x))$, where d is an (r, l) - (f, g) - derivation.

The results follow by theorem 1.1.20.

Theorem : 4.2.8

Let X be a commutative B-algebra and f, g be endomorphisms such that $f \circ f = f$. Also, let d and d' be (l, r) - (f, g)-derivations ((r, l) - (f, g) - derivations) of X. Then, $d \circ d'$ is also an (l, r) - (f, g)-derivation ((r, l)-(f, g)- derivation) of X.

Proof:

Let d and d' are the (l, r)-(f, g)-derivations of X. Then, by Theorem 4.2.5, for all $x, y \in X$, we have

$(d \circ d')(x * y) = d\,(d'(x) * f(y))$

$= d\,(d'(x)) * f(f(y))$

$= d \circ d'(x) * f(y)$

$= (d \circ d'(x) * f(y)) \wedge (g(x) * d \circ d'(y))$.

Thus, $d \circ d'$ is a (l, r) - (f, g) - derivation of X.

Now, suppose that d, d' are (r, l)-(f, g)-derivations of X. Similarly, we can prove $d \circ d'$ is a (r, l) - (f,g) - derivation of X.

Theorem : 4.2.9

Let X be a commutative B-algebra, d and d' be (f, g) - derivations of X such that $f \circ d = d \circ f$, $d' \circ f = f \circ d'$. Then, $d \circ d' = d' \circ d$.

Proof:

Since d' is an (l, r)-(f, g)-derivation and d is an (r, l)-(f, g)-derivation of X, for all $x, y \in X$,

$(d \circ d')(x \ast y) = d((d'(x) \ast f(y)) \wedge (g(x) \ast d'(y)))$

$= d(d'(x) \ast f(y))$

$= f \circ d'(x) \ast d \circ f(y)$ (1)

Also, since d' is an (l, r)-(f, g)-derivation and d is an (r, l)-(f, g)-derivation of X, for all $x, y \in X$,

we have $(d' \circ d)(x \ast y) = d'((f(x) \ast d(y)) \wedge (d(x) \ast g(y)))$

$= d'(f(x) \ast d(y))$

$= d' \circ f(x) \ast f \circ d(y)$

$= f \circ d'(x) \ast d \circ f(y)$ (2)

By (1) and (2), we have $d, d'(x \ast y) = d' \circ d(x \ast y)$, for all $x, y \in X$.

By putting $y = 0$, we get $d \circ d'(x) = d' \circ d(x)$, for all $x \in X$.

Definition : 4.2.10

Let X be a B-algebra and d, d_0 be two self maps of X. We define $d \circ d' : X \rightarrow X$ as follows: $(d \circ d')(x) = d(x) \ast d'(x)$, for all $x \in X$.

Theorem : 4.2.11

Let X be a commutative B-algebra and d, d_0 be (f, g)- derivations of X. Then,

(i) $(f \circ d') \bullet (d \circ f) = (d \circ f) \bullet (f \circ d')$.

(ii) $(d \circ d') \bullet (f \circ f) = (f \circ f) \bullet (d \circ d')$.

Proof:

(i) Since d is an (r, l)-(f, g)-derivation and d' is an (l, r)-(f, g)-derivation of X, then for all x, y □ X,

(d ∘ d')(x □ y) = d((d' (x) □ f(y)) □ (g(x) □ d' (y)))

= d(d' (x) □ f(y))

= (f(d' (x)) □ d(f(y)))∧ (d(d' (x)) □ g(f(y)))

= (f ∘ d' (x)) □ (d ∘ f(y)).

Also, d is a (l, r)-(f, g)-derivation and d' is a (r, l) - (f, g)-derivation of X.

Hence, for all x, y □ X,

(d ∘ d') (x □ y) = d ((f(x) □ d' (y)) ∧ (d' (x) □ g(y)))

= d (f(x) □ d' (y))

= (d (f(x)) □ f(d' (y))) ∧ (g(f(x)) □ d(d' (y)))

= (d ∘ f(x)) □ (f ∘ d' (y)).

Now, we obtain

(f ∘ d' (x)) □ (d ∘ f(y)) = (d ∘ f(x)) □ (f ∘ d' (y)), for all x, y □ X.

By putting x = y, we have

(f ∘ d' (x)) □ (d ∘ f(x)) = (d ∘ f(x)) □ (f ∘ d' (x)).

So, (f ∘ d’ • d ∘ f) (x) = (d ∘ f • f ∘ d') (x), for all x □ X.

ii) The proof is similar to the proof of (i)

Note:

Let Der (X) denotes the set of all (f, g) - derivations on X. Let d, d' □ Der(X).

Define the binary operation ∧ as follows: (d ∧ d') (x) = d(x) ∧ d' (x), for all x □ X.

Theorem: 4.2.12

If X is a commutative B-algebra, then (Der(X), $\wedge$) is a semi- group, where Der (X) denotes the set of all (f, g) - derivations on X.

Proof:

Suppose that d, d' are (l, r)-(f, g)-derivations of X. We prove d $\wedge$ d' is also an (l, r) - (f, g)-derivation. For all x, y $\in$ X,

(d $\wedge$ d') (x $*$ y) = d(x $*$ y) $\wedge$ d' (x $*$ y)

= d(x $*$ y)

= d(x) $*$ f(y)

= (d(x) $\wedge$ d' (x)) $*$ f(y)

= (d $\wedge$ d') (x) $*$ f(y)

= ((d $\wedge$ d') (x) $*$ f(y)) $\wedge$ (g(x) $*$ (d $\wedge$ d') (y)).

So, d $\wedge$ d' is a (l, r) - (f, g) - derivation of X .

Therefore d $\wedge$ d'$\in$ Der(X)

Let d, d', d'' $\in$ Der (X). We prove d $\wedge$ (d' $\wedge$ d') = (d $\wedge$ d') $\wedge$ d' (associative property).

If x, y $\in$ X, then

(d $\wedge$ (d' $\wedge$ d')) (x $*$ y) = d(x $*$ y) $\wedge$ (d' $\wedge$ d') (x $*$ y)

= d(x $*$ y).

Also, we have

((d $\wedge$ d') $\wedge$ d') (x $*$ y) = (d $\wedge$ d') (x $*$ y)

= d(x $*$ y) $\wedge$ d' (x $*$ y)

$$= d(x \square y).$$

This shows that

$$(d \wedge (d' \wedge d'))(x \square y) = ((d \wedge d') \wedge d')(x \square y), \text{ for all } x, y \square X.$$

By putting $y = 0$, we obtain

$$d \wedge (d' \wedge d') = (d \wedge d') \wedge d'.$$

Therefore, (Der (X), $\wedge$) is a semigroup.

Summary and Conclusion

SUMMARY AND CONCLUSION

Y.Imai and K.Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [9,10].It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In 2002, Neggers and Kim[19] introduced the notion of B-algebra, and then Cho and Kim[5] studied some of its properties. Abujabal and Al-Shehrie [1] defined and studied the notion of left derivation of BCI-algebras. Further, Al- Shehrie [2] has applied the notion of left-right derivation in BCI-algebra to B-algebra and obtained some of its properties.

In the **first chapter**, preliminaries on B-algebras due to J.Neggers, H.S. Kim[19] are presented .Also some results on 0-commutative B-algebras due to H.S. Kim, H.G. park[16] and quadratic B-algebras due to H.K.Park ,H.S.Kim [21] are discussed.

In **chapter two**, some properties of B-homomorphisms of B-algebras, the second Isomorphism theorem for B-algebras due to C. Endam Joemar and P.Vilela Jocelyn[6] and some properties of Hom (X,Y) as B-algebras due to N.O.Al-Shehrie[3] are discussed.

In **chapter three**, some properties of left-right derivations and 0 - commutative B-algebras due to N.O.Al-Shehrie [2] and t-derivations of B-algebras due to R.Soleimani and S.Jahangiri[26] are studied.

In the **fourth chapter**, some properties of f - derivations and (f, g)-derivations of B-algebras due to L.K.Ardekani and B.Davvaz [4] are discussed.

A deep study of B-algebras can be extended to different types of algebras and fuzzy algebras. So, it provides a lot of scope for further research.

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