Chapter VIII

CHAPTER VIII

RANK AND PERIMETER PRESERVERS FOR MATRICES OVER SEMIRINGS

In this chapter, S denotes a chain semiring. In this chapter, the set of linear operators preserving the rank and the perimeter of every rank-1 matrix over any chain semiring are characterized.

Definition: 8.1

Let T is a linear operator on $M_{m,n}(S)$. Then

(i) T **preserve rank 1** if $r(T(A)) = 1$ whenever $r(A) = 1$for all $A \in M_{m,n}(S)$.

(II) T **preserve perimeter k** of rank-1 matrices if $P(T(A)) = k$ whenever $P(A) = k$ for all $A \in M_{m,n}(S)$ with $r(A) = 1$.

Theorem: 8.2

If T is a (P, Q, B)-operator on $M_{m,n}(S)$, then T preserves both rank and perimeter of rank-1 matrices.

Proof:

If T is a (P, Q, B)-operator on $M_{m,n}(S)$, there exist $m \times m$ and $n \times n$ permutation matrices P and Q, respectively such that $T(A) = (A \circ B)Q$, or $m = n$ and $T(A) = P(A^t \circ B)Q$ for all A in $M_{m,n}(S)$, where $B \in M_{m,n}(S)$ is a matrix with all nonzero entries and $r(B) = 1$. Then we can write $B = cd^t$, where none of entries c or d is zero. Let A be a rank-1 matrix in $M_{m,n}(S)$ with a factorization $A = ab^t$. For the case $T(A) = P(A \circ B)Q$, we have the following:

$$T(A) = P(ab^t \circ cd^t)Q = P(a \circ c)(b \circ d)^t Q = (P(a \circ c))(Q^t(b \circ d))^t \qquad (*)$$

Thus (∗) implies that

$$r(T(A)) = r\left((P(a \circ c)) \quad (Q^t(b \circ d))^t\right) = 1 \text{ and}$$

$$P(T(A)) = |P(a \circ c)| + |Q^t(b \circ d)| = |a \circ c| + |b \circ d| = |a| + |b| = P(A)$$

For the case m = n and $T(A) = (A^t \circ B)Q$, we can show that r(T(A)) = 1and P(T(A)) = P(A) by the similar method as above.

Lemma: 8.3

Let T be a linear operator on $M_{m,n}(S)$. If T preserves rank and perimeter 2 of every rank-1 matrix, then the following statements hold:

(i) T maps a cell into a cell with nonzero scalar multiplication;

(ii) T maps a line matrix into a line matrix.

Proof:

(i) Follows from the property that T preserves perimeter 2. (ii) If not, there exist two distinct cells E and F in same row (or column) such that T(E) and T(F) lie in two different row and different columns. Then we have r(E+F) = 1, while r(T(E+F)) = r(T(E)+T(F)) = 2. This contradicts to the fact that T preserves rank 1.

The following is an example of a linear operator that preserves rank and perimeter 2 of rank 1 matrices, but it does not preserve perimeter 2n ($n \geq 2$) and is not a (P, Q, B)-operator.

Example: 8.4

Let T be a linear operator on $M_{m,n}(S)$ with $n \geq 2$ defined by

$$T(A) = \left(\sum_{i,j=1}^{n} a_{ij}\right) E_{kk} = \max\{a_{ij} \mid i, j = 1, \ldots\ldots\ldots n\} E_{kk}$$

For all $A = [a_{ij}] \in M_{n,n}(S)$, where k is a fixed integer in {1, 2,.........n}. Then it is easy to verify that T is a linear operator and preserves rank and perimeter 2 of each rank-1 matrix. But T does not preserve perimeter 2n; for, if $J \in M_{n,n}(S)$ is a matrix whose entries are all 1, then J has rank 1 and perimeter 2n, but $T(J) = E_{kk}$ has rank 1 and perimeter 2. Hence T is not a (P, Q, B)-operator by Theorem 8.2.

Lemma: 8.5

Let T be a linear operator on $M_{m,n}(S)$. Suppose that T preserves rank and perimeters 2 and $p(\geq 3)$ of rank-1 matrices. Then

(i) T maps two distinct cells in a row (column) into two distinct cells in a row or in a column with nonzero scalar multiplication;

(ii) If T maps a row matrix into a row (or column if m =n) matrix, then T maps every row matrix into a row (or column if m =n) matrix and if T maps a column matrix in to column (or row if m = n) matrix then T maps every column matrix into a column (or row if m = n) matrix.

Definition: 8.6

For a linear operator T on $M_{m,n}(S)$ preserving rank and perimeter 2 of rank-1 matrices, we define the corresponding mapping $T': \Delta_{m,n} \rightarrow \Delta_{m,n}$ by $T'(i, j) = (k, l)$ whenever $T(E_{ij}) = b_{ij}E_{kk}$ for some nonzero scalar $b_{ij} \in S$.

Lemma: 8.7

Let T be a linear operator preserving both ranks and perimeters 2 and k $(k \geq 4, k \neq n+1)$ of rank-1 matrices. Then T' is a bijection of $\Delta_{m,n}$.

Proof:

By Lemma 8.3, we have that for any $E_{ij} \in E_{mn}$ there exist $E_{rl} \in E_{mn}$ and nonzero $b_{ij} \in S$ such that $T(E_{ij}) = b_{ij}E_{rl}$. Without loss of generality, we may assume that T maps the i^{th} row of a matrix into the r^{th} row with nonzero scalar multiplication. Suppose that T' (i, j) = T' (p, q) for some distinct pairs (i, j), $(p, q) \in \Delta_{m,n}$. By the definition of T', we have $T(E_{ij}) = b_{ij}E_{rl}$ and $T(E_{pq}) = b_{pq}E_{rl}$ for some nonzero scalars b_{ij}, $b_{pq} \in S$. Lemma 8.5 implies that $i \neq p$ and $j \neq q$. Furthermore T maps the i^{th} row and the p^{th} row of a matrix into the r^{th} row.

Case1:

$4 \leq k \leq n$: claim: we can choose a $2 \times (k-2)$ submatrix A from i^{th} and p^{th} row, but T(A) is a $1 \times k$ submatrix in the r^{th} row. If the claim is true, then P(A) = k, while P(T(A) = k+1, a contradiction.

Proof of the claim:

By Lemma 8.5, T maps distinct cells in each row (or column) to distinct cells with nonzero scalar multiplication. Now, choose E_{ij}, E_{pj} but do not choose E_{iq}, E_{pq}. Since there is a cell E_{ph_1} $(h_1 \neq j,q)$ in the p^{th} row such that $T'(p, h_1) = T'(i, q)$ but $T'(i, h_1) \neq T'(p, j)$, we can choose a $2 \times (4-2)$ submatrix $E_{ij} + E_{ih_1} + E_{pj} + E_{ph_1}$ whose image under T is 1×4 submatrix in the r^{th} row. Therefore the claim is satisfied for k = 4. Assume that for k = s with $4 \leq s \leq n-1$, the claim is true. Then there is a $2 \times (s-2)$ submatrix.

$$X = E_{ij} + \sum_{t=1}^{s-3} E_{ih_t} + E_{pj} + \sum_{t=1}^{s-3} E_{ph_t} .$$

Such that T(X) is an $1 \times s$ submatrix in the i^{th} row, where $\{j, q, h_1, \ldots\ldots\ldots\ldots h_{s-3}\}$ is the set of distinct indices. Now, we can choose a cell $E_{ph_{s-2}}$ ($h_{s-2} \neq j, q, h_1, \ldots\ldots\ldots, h_{s-3}$) such that $T'(i, h_{s-2}) \neq T'(p, j), T'(p, q), T'(p, h_1), \ldots\ldots\ldots\ldots T'(p, h_{s-3})$. Then we have a $2 \times ((s+1)-2)$ submatrix $A = E_{ij} + \sum_{t=1}^{s-2} E_{ih_t} + E_{pj} + \sum_{t=1}^{s-2} E_{ph_t}$ such that T(A) is an $1 \times (s+1)$ submatrix in the r^{th} row. Thus the claim is satisfied for $k = s+1$. By the mathematical induction, the claim true.

Case 2:

$k = n+\alpha \geq n+2$. Consider a matrix

$$Y = \sum_{s=1}^{n} E_{is} + \sum_{t=1}^{n} E_{pt} + \sum_{h=1}^{\alpha-2} \sum_{g=1}^{n} E_{hg}$$

With rank 1 and perimeter k. Then T maps the ith and pth row Y into the rth row with nonzero scalar multiplication by Lemma 8.5. Thus the perimeter of T(Y) is less than k, a contradiction.

Hence $T'(i, j) \neq T'(p, q)$ for any two distinct pairs $(i, j), (p, q) \in \Delta_{m,n}$. Therefore T' is bijection on $\Delta_{m,n}$.

The condition of $k \neq n+1$ Lemma 8.6 must be necessary. The following example shows that T is a linear operator preserving both ranks and perimeters 2 and n+1 of rank-1 matrices, but T' is not a bijection of $\Delta_{m,n}$.

Example: 8.8

Consider a linear operator T on $M_{2,3}(S)$ defined by

$$T = \left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+e & b+f & c+d \\ 0 & 0 & 0 \end{bmatrix}.$$

Then we can easily shows that T preserves both ranks and perimeters 2 and 4 of rank-1 matrices. But T' is not a bijection of $\Delta_{2,3}$.

Notation: 8.9

Let $\mathcal{R}_i$ = {E_{ij}| 1 $\leq$ j $\leq$ n}, C_j = {E_{ij}| 1 $\leq$ i $\leq$ m}, $\mathbb{R}$ = {$\mathcal{R}_i$| 1 $\leq$ i $\leq$ m} and $\mathbb{C}$ = {C_j| 1 $\leq$ j $\leq$ n}. For a linear operator T on $M_{m,n}(S)$, define $T^*(A) = [T(A)]^*$ for all A in $M_{m,n}$ (S). Let $T^*(\mathcal{R}_i)$ = { $T^*(E_{ij})$| 1 $\leq$ j $\leq$ n} for each i = 1,..........,m and $T^*(C_j)$ = {$T^*(E_{ij})$| 1 $\leq$ i $\leq$ m} for each j = 1,..........n.

Theorem: 8.10

Let T be a linear operator on $M_{m,n}(S)$, where S is chain semiring. Then the following are equivalent:

(i) T is a (P, Q, B)-operator;

(ii) T preserves both rank and perimeter of rank-1 matrices;

(iii) T preserves both rank and perimeters 2 and k (k $\geq$ 4, k $\neq$ n+1) of rank-1 matrices.

Proof:

(i) $\Rightarrow$ (ii): Clear by Proposition 8.2 (ii) $\Rightarrow$ (iii): It is obvious.

(iii) $\Rightarrow$ (i): Assume (iii). Then the corresponding mapping T : $\Delta_{m,n} \rightarrow \Delta_{m,n}$ is a bijection by Lemma 8.7. Furthermore, there are cases:

(a) T^* maps $\mathbb{R}$ onto $\mathbb{R}$ and maps $\mathbb{C}$ onto $\mathbb{C}$, or

(b) T^* maps $\mathbb{R}$ onto $\mathbb{C}$ and maps $\mathbb{C}$ onto R.

Case (a): We note that $T^*(\mathcal{R}_i) = \mathcal{R}_{\sigma(i)}$ and $T^*(C_j) = C_{\tau(j)}$ for all i = 1, …., m and j = 1, …., n, where σ and τ are permutations of {1, …., m} and {1, …., n}, respectively. Let P and Q be the permutation matrices corresponding to σ and

T, respectively. Then for any $E_{ij} \in \mathbb{E}_{m,n}$, we can write $T(E_{ij}) = b_{ij}E_{\sigma(i)\,\tau(j)}$ for some nonzero scalar $b_{ij} \in S$. Now we claim that $B = [b_{ij}] \in M_{m,n}(S)$ has rank 1. For, consider an $m \times n$ matrix J, all of whose entries are 1.

Then we have

$$T(J) = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} E_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} T(E_{ij})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} b_{ij}E_{\sigma(i)\,\mathcal{T}(j)} = PBQ.$$

Since J has rank 1, it follows that $r(T(J)) = 1$, and hence $r(B) = r(PBQ) = r(T(J)) = 1$. Therefore for any $A = [a_{ij}] \in M_{m,n}(S)$, we have

$$T(A) = T\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}E_{ij}\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}T(E_{ij})$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ij}E_{\sigma(i)\,\mathcal{T}(j)} = P(A \circ B)Q.$$

Thus T is a (P, G, B)-operator.

Case (b): We note that m = n and $T^*(\mathcal{R}_i) = \mathcal{C}_{\sigma(i)}$ and $T^*(\mathcal{C}_j) = \mathcal{R}_{\mathcal{T}(j)}$ for all i, j, where σ and $\mathcal{T}$ are some permutations of {1, …., m}. By an argument similar to case (a), we obtain that T(A) is of the form $T(A) = P(A^t \circ B)Q$, and thus T is a (P, Q, B)-operator.