Some Interesting Results On Fuzzy Metric Spaces

By

S. J. Manjula

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE AND
HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY), COIMBATORE - 641 043
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

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Introduction

INTRODUCTION

In 1965, the concept of fuzzy set was introduced by Zadeh [24]. Since then various authors have introduced several notions like fuzzy topology , fuzzy metric, fuzzy compactness, fuzzy connectedness, fuzzy proximity, fuzzy uniformity,fuzzy convergence etc., and obtained good generalisations of topological results of fuzzy topological spaces.

Many authors like DENG ZI - KE [2], ERCEG [3], KALEVA and SEIKKALA [11], KRAMOSIL AND MICHALEK [13], GEORGE AND VEERAMANI [6] have defined and developed the theory of fuzzy metric spaces. The aim of this thesis is to study fuzzy metric spaces introduced and developed by GEORGE and VEERAMANI [6]. The results obtained in the following two articles are discussed in detail.

1. ON SOME RESULTS IN FUZZY METRIC SPACES [7]
2. SOME THEOREMS IN FUZZY METRIC SPACES [6]

The first chapter is devoted to the study of the article "ON SOME RESULTS IN FUZZY METRIC SPACES". In 1975 KRAMOSIL and MICHALEK [13] defined fuzzy metric space as follows:

The 3 - tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, * is a continuous t - norm, and M is a fuzzy set on $X \times X \times [0,\infty)$ satifying the following conditions.

1. $M(x,y,0) = 0$
2. $M(x,y,t) = 1$ for all $t>0$, if and only if $x=y$

3. $M(x,y,t) = M(y,x,t)$

4. $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$

5. $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous, $x,y,z \in X$ and $t,s > 0$.

In 1994 George and Veeramani modified the definition of fuzzy metric space as follows: The 3 - tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t - norm, and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following condition.

1. $M(x,y,t) > 0$

2. $M(x,y,t) = 1$ for all $t>0$, if and only if $x=y$

3. $M(x,y,t) = M(y,x,t)$

4. $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$

5. $M(x,y,.) : (0,\infty) \to [0,1]$ is continuous, $x,y,z \in X$ and $t,s>0$.

An intersting example of a fuzzy metric space is as follows:

Let $X = N$, the set of natural numbers

For $a,b \in [0,1]$, $a*b = ab$

Define $M: X^2 \times (0,\infty) \to [0,1]$ as

$$M(x,y,t) = x/y \text{ if } x \leq y$$
$$= y/x \text{ if } y \leq x$$

for all $t>0$ and $x,y \in X$

Then $(X,M,*)$ is a fuzzy metric.

Every fuzzy metric M on x induces a topology τ on X given by

$\tau = \{ A...X / x \in A \Leftrightarrow$ there exist $t>0$ and r, $0<r<1$ such that

$$B(x,r,t) \subset A\}$$

where $B(x,r,t) = \{Y \in X / M(x,y,t) > 1-r\}$.

With this definition the authors have proved that every fuzzy metric space is Hausdorff.

With every metric d on set X, the authors have tassociated a fuzzy metric M given by $M(x,y,t) = \frac{t}{t+d(x,y)}$ and this fuzzy metric is called the standard fuzzy metric

It is interesting to note that the topology induced by the metric 'd', is the same as the topology induced by the standard fuzzy metric.

Similar to boundedness in metric spaces, the authors have introduced the concept of F - boundedness in fuzzy metric space as follows:

Let $(X,M,*)$ be a fuzzy metric space. A subset A of X is said to be F - bounded if and only if there exist $t>0$ and $0<r<1$ such that $M(x,y,t) > 1-r$ for all $x,y \in A$.

Some of the results proved here are as follows:

1. Let $(X,M,*)$ be a fuzzy metric space induced by a metric d on X. Then $A \subseteq X$ is F - bounded iff it is bounded
2. Every compact subset A of a fuzzy metric space X is F - bounded.
3. Let $(X,M,*)$ be a fuzzy metric space and τ be the topology induced by the fuzzy metric. Then for a sequence $\{X_n\}$ in X, $X_n \longrightarrow x$ iff $M(X_n,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$.

In 1988 MARIUSZ GRABIEC [16] introduced the concept of cauchy sequence in a fuzzy metric space as follows:o

1. A sequence $\{X_n\}$ in a fuzzy metric space $(X,M,*)$ is a cauchy sequence iff

$$\lim_{n \to \infty} M(X_{n+p}, X_n, t) = 1, \; p>0, \; t>0.$$

It is important to see that if X=R, the set of real numbers and if $S_n = 1 + 1/2 + \ldots\ldots + 1/n$, then the sequence $\{S_n\}$ is a cauchy sequence with respect to the fuzzy metric space, where the fuzzy metric is induced by the Euclidean metric d on R.

If R were complete then we should have an $x \in R$ such that,

$$\lim_{n \to \infty} M(S_n, x, t) = 1$$

ie, $$\lim_{n \to \infty} \frac{t}{t + |S_n - x|} = 1$$

ie, $$\lim_{n \to \infty} |S_n - x| = 0$$

ie, $$\lim_{n \to \infty} S_n = x$$

But this is not true. So R is not complete with respect to the fuzzy metric.

A modified version of cauchy sequence is defined and it is shown that R is complete with respect to this definition. Finally the authors have proved the following Baire's category theorem for fuzzy Metric spaces. "Let X be a complete fuzzy metric space. Then the intersection of a countable number of dense open sets is dense".

Hence any complete fuzzy Metric space is not of first category.

In the second chapter a detailed proof of the ASCOLI - ARZELA Theorem for fuzzy Metric space is given. For this purpose the concepts of Uniform continuity Equicontinuity and Uniform convergence are defined in fuzzy Metric spaces by suitably modifying the classical definition of these concepts.

The important results proved here are as follows:

1. Continuous function on a compact fuzzy metric space is uniformly continuous.
2. Every compact fuzzy metric space is separable.
3. Let $\{f_n\}$ be an equicontinuous sequence of mappings from a fuzzy metric space X to a complete fuzzy metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of X, then $\{f_n\}$ converges at each point of X and the limit function is continuous.
4. Let X be a compact fuzzy metric space and Y be a complete fuzzy metric space. Let A be an equicontinuous family of functions from X to Y. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A such that, cl $\{f_n(x): n = 1,2,3.......\}$ is a compact subset of Y for each $x \in X$. then there exists a continuous function f from X to Y and a sequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.
5. Every fuzzy metric space is metrizable.

Review of Literature

REVIEW OF LITERATURE

The concept of an abstract metric space was introduced by **FRECHET** [5] in 1906. He associated a non-negative real number with each ordered pair of elements of a certain set satisfying certain conditions.

MENGER [17] in 1942 introduced the concept of statistiical metric spaces which is a generalisation of the metric spaces. Instead of associating a number — the distance (p,q) — with every pair of elements p,q, he associated a function $F_{pq}(x)$ and for any positive number x, he defined F_{pq} as the probability that the distance from p to q to be less than x. **WALD** [23] in 1943 constructed a theory of "betweeness" by modifying the triangle inequality of menger and studied the metric spaces.

MENGER [18] in 1951 continued his study of statistical metric spaces adopting wald's version of triangle inequality.

SCHWEIZER and **SKLAR** [21] in 1959 continued the study of statistical metric spaces by menger.

The concept of fuzzy set was introduced by **ZADEH** [24] in 1965. Many topological ideas have been generalised to fuzzy topological spaces.

Separation axioms was generalised by **FORA** [4] **HUTTON** [9] and **REILLY** [9] **MIRA SARKAR** [19].

Uniform structure was generalised by HUTTON [10] **LOWEN** [15]. Proximity structure was generalised by **ARTICO** and **MORESCO** [1], **GHANIM** [8], **KATSARAS** [12]. Many authors like **DENG-ZI-KE** [2], **ERCEG** [3], **KALEVA** and **SEIKKALA** [11], **KRAMOSIL** and **MICHALEK** [13] have introduced the concepts of fuzzy Metric Spaces.

KRAMOSIL and **MICHALEK** [13] generalised the statistical Metric Spaces to the fuzzy situation. It is to be noted that the fuzzy Metric induced by **KRAMOSIL** and **MICHALEK** [13] is not related to the fuzzy Metric introduced by **DENG-ZI-KE** [2], **ERCEG** [3], **KALEVA** and **SEIKKALA** [11]. They consider a Metric as a subset of X x X x $[0,\infty)$ and in making this into a fuzzy set they obtain their fuzzy Metric. Here we discuss briefly some of the interesting articles published by various authors in Metric and fuzzy Metric Spaces.

I. STATISTICAL METRIC SPACES BY SCHWEIZER AND SKLAR [21]

In 1959 the authors continued the study of statistical Metric Spaces by **MENGER** [17]. The author have discussed the articles under the main topics. They are as follows:

1) The axiomatics of statistical Metric Spaces, with particular emphasis on the triangle inequality.
2) The construction and study of particular spaces.
3) A consideration of topological notions in statistical Metric Spaces and a study of the continuity properties of the distances function.

II) METRIC SPACES IN FUZZY SET THEORY BY ERCEG [3] (1979)

Here the author defined Metric in fuzzy set theory and using the definition of uniformity on the fuzzy sets given by **HUTTON** [10].The author defines a uniformity for a Metric Space on a fuzzy set and obtained results on the generation of topologies on fuzzy sets by Pseudo quasi- Metric (p,q metric).(Here it is to be noted that the definition of Metric Space given by the author is entirely different from the definition of **VEERAMANI** and **GEORGE**).

III) FIXED POINTS IN FUZZY METRIC SPACES BY GRABIEC [16] (1988)

Using the defintion of fuzzy Metric by **KRAMOSIL** and **MICHALEK** [13] (1988) the author extends two well-known fixed point theorems of **BANACH** and **EDELSTEIN**.

Chapter I

CHAPTER I

INTRODUCTION

This chapter is devoted to the study of fuzzy metric spaces. We discuss here the resutlts obtained by **GEORGE and VEERAMANI** [7]. They have shown that every fuzzy metric induces a Hausdorff topology and every metric induces a fuzzy metric. Moreover a modified definition of cauchy sequence is given with respect to which R is complete. Finally they have proved the **BAIRE'S THEOREM** for fuzzy metric spaces.

Definition : 1.1

A t-norm is a function T: [0,1] x [0,1] --> [0,1] for each x,y and z in [0,1], satisfying the following axioms.

1) $T(x,1) = x$

2) $T(x,y) \leq T(z,y)$ if $x \leq z$

3) $T(x,y) = T(y,x)$

4) $T[x, T(y,z)] = T[T(x,y),z]$

Definition : 1.2

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if ([0,1], $*$) is a topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a,b,c,d \in [0,1]$

Example : 1.3

1) Let $T : [0,1] \times [0,1] \to [0,1]$ be defined by $T(a,b) = \text{Max}\,(a+b-1,0)$. Then T is a continuous t-norm.

Similarly for $a,b \in [0\ 1]$

T_2 : $T(a,b) = ab$

T_3 : $T(a,b) = \text{Min}(a,b)$

T_4 : $T(a,b) = \text{Max}(a,b)$

are all continuous t-norms.

2) $* : [0,1] \times [0,1] \to [0,1]$ defined by $a*b = ab$, for $a,b \in [0,1]$ is a continuous t-norm.

3) $* : [0,1] \times [0,1] \to [0,1]$ defined by $a*b = \text{Min}[a,b]$ for $a,b \in [0,1]$ is a continuous t-norm.

Definition : 1.4

The 3 tuple $(X,M,*)$ is said to be a fuzzy Metric space if x is an arbitrary set, $*$ is a continuous t-norm and Mis a fuzzy set on $X \times X \times (0,\infty) \to [0,1]$ satisfying the following conditions. For $x, y, z \in X$ and $t,S>0$,

1) $M(x,y,t) > 0$

2) $M(x,y,t) = 1$ if and only if $x = y$.

3) $M(x,y,t) = M(y,x,t)$

4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$

5) $M(x,y,.) = (0,\infty) \to [0,1]$ is continuous.

LEMMA : 1.5

The function $M(x,y,.) : (0,\infty) \to [0,1]$ is non-decreasing for all x,y in X.

Proof:

Let $t,s \in (0,\infty)$ be such that $t<s$.

Claim:

$M(x,y,t) \leq M(x,y,s)$

Assume $M(x,y,t) > M(x,y,s)$

Then $M(x,y,t) * M(y,y,s-t) \leq M(x,y,s)$

As $M(y,y,s-t) = 1$,

$M(x,y,t) \leq M(x,y,s) < M(x,y,t)$

i.e: $M(x,y,t) < M(x,y,t)$

a contradiction.

Hence $M(x,y,*)$ is non-decreasing for all x,y in X.

Example 1.6

Let $X = R$ be the set of real numbers.

Define $*$ by, $a * b = ab$, for $a,b \in [0,1]$ and

$M : X \times X \times (0,\infty) \dashrightarrow [0,1]$ as

$M(x,y,t) = [\exp^{(|x-y|/t)}]^{-1}$ for all $x,y \in R$ and $t \in (o,\infty)$.

Then $(X,M,*)$ is a fuzzy Metric Space.

PROOF:

1) For $t \in (0,\infty)$ it is obvious that $M(x,y,t) > 0$.

2) $M(x,y,t) = [\exp^{(|x-y|/t)}]^{-1}$

When $x=y$ we have $|x-y| = 0$

$\therefore M(x,y,t) = (e^0)^{-1} = (1)^{-1} = 1$

$\therefore M(x,y,t) = 1$

Conversely when $M(x,y,t) = 1$

$$[\exp^{(|x-y|/t)}]^{-1} = 1$$

$$[\exp^{(|x-y|/t)}] = 1$$

$$\Rightarrow \frac{|x-y|}{t} = \log 1 = 0$$

$\Rightarrow |x-y| = 0$

$\Rightarrow x = y$

Therefore $M(x,y,t) = 1 \Leftrightarrow x = y$

3) As $|x-y| = |y-x|$ We get $M(x,y,t) = M(y,x,t)$

4) To prove that

$$M(x,y,t) * M(y,z,t) \le M(x,z,t+s)$$

We have

$$M(x,y,t) = (e^{|x-y|/t})^{-1}$$

$$M(y,z,s) = (e^{|y-z|/s})^{-1}$$

$$M(x,z,t+s) = (e^{|x-z|/t+s})^{-1}$$

To Prove

$$(e^{|x-y|/t})^{-1} (e^{|y-z|/s})^{-1} \le (e^{|x-z|/t+s})^{-1}$$

We know that

$$|x-z| \le |x-y| + |y-z|$$

Dividing throughout by t+s

$$\frac{|x-z|}{t+s} \le \frac{|x-y|}{t+s} + \frac{|y-z|}{t+s} \le \frac{|x-y|}{t} + \frac{|y-z|}{s}$$

(i.e) $$\frac{|x-z|}{t+s} \le \frac{|x-y|}{t} + \frac{|y-z|}{s}$$

Taking exponentiation we have,

$$e^{|x-z|/(t+s)} \le e^{|x-y|/t}\, e^{|y-z|/s}$$

$$(e^{|x-z|/t+s})^{-1} \ge (e^{|x-y|/t})^{-1} \quad (e^{|y-z|/s})^{-1}$$

Hence $(e^{|x-y|/t})^{-1} (e^{|y-z|/s})^{-1} \le (e^{|x-z|/t+s})^{-1}$

$\therefore$ $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$

5) Since every exponential function is continuous

$M(x,y,.):(0,\infty) \to [0,1]$ is continuous.

Hence M (x,y,*) is a fuzzy metric space.

$\therefore$ (R,M,*) is a fuzzy Metric space.

Example: 1.7

Let (x,d) be a Metric space. Define $*$ as $a*b=ab$ for $a,b \in [0,1]$ and $M:X \times X \times (0, \infty) \to [0,1]$ as $M(x,y,t)= (e^{d(x,y)/t})^{-1}$ for $x,y \in X$ and $t \in (0,\infty)$. Then $(M,X,*)$ is a fuzzy Metric space.

Remark: 1.8

The above two example hold if we take $a*b=\mathrm{Min}(a,b)$

Example : 1.9

Let $X=N$, the set of natural numbers. Define $a*b=ab$ for $a,b\in[0,1]$ and $M : N \times N \times (0,\infty) \to [0,1]$ as

$M(x,y,t)= x/y$ if $x \leq y$

y/x if $y \leq x$, for all $t>0$, $x,y \in X$

Then $(X,M,*)$ is a fuzzy Metric space.

Remark : 1.10

Let $(X,M,*)$ be a fuzzy Metric space.

Since the function $M(x,y,.)$ is continuous for $x,y \in X$ and given $t>0$, and $0<r<$ whenever $M(x,y,t)>1-r$, there exists a $t_o, 0<t_o<t$ such that $M(x,y,t_o)>1-r$.

Remark 1.11

For any $r_1 > r_2$, we can find a r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find a r_5 such that $r_5 * r_5 \geq r_4$ $[(r_1,r_2,r_3,r_4,r_5 \in (0,1)]$

Definition :1.12 (INDUCED FUZZY METRIC)

Let (x,d) be a metric space. Define $a*b = ab$, for $a,b \in [0,1]$ $M: X \times X \times (0,\infty) \to [0,1]$ as

as $$M(x,y,t) = \frac{kt^n}{t^n+md(x,y)}, \quad k,m,n \in R^+$$

Then $(X,M,*)$ is a fuzzy metric space

When $k=m=n=1$, we get

$$M(x,y,t) = \frac{t}{t+d(x,y)}$$

We call this fuzzy metric induced by a metric d as the standard fuzzy metric.

Definition:1:13

Let $(X,M,*)$ be a fuzzy metric space. Let $x \in X$, $t>0$ and $0<r<1$. The open ball $B(x,r,t)$ with centre at x and radius r is defined as $B(x,r,t) = \{Y \in X;\ M(x,y,t) > 1-r\}$

PROPOSITION : 1.14

Every open ball is an open set.

Proof:

Let $B(x,r,t)$ be an open ball with centre at $x \in X$ and radius r, where $0<r<1$ and $t>0$.

CLAIM I:

$B(x,r,t)$ is an open set.

Let $y \in B(x,r,t)$ Then $M(x,y,t)>1-r$

{(To prove that there exists an open neighbourhood around this point with centre at y and radius r, such that this ball is fully contained in $B(x,r,t)$}.

Let $y \in B(x,r,t)$. Then by definition. $M(x,y,t)>1-r$.

Then there exists a t_0, such that $0<t_0<t$ and $M(x,y,t_0)>1-r$.

Let $r_0 = M(x,y,t_0) > 1 - r$ ---------------- (1)

Since $r_0 > 1-r$ we can find a s such that $0<s<1$, and $r_0>1-s>1-r$. For r_0 and s with $r_0 > 1-s$ there exists a r_1, $0<r_1<1$ such that $r_0 * r_1 \geq 1-S$.

CLAIM II:

$B(y,1-r_1,t-t_0) \subset B(x,r,t)$

Let $Z \in B(y,1-r_1,t-t_0)$

$$
\begin{aligned}
\text{then } M(y,z,t-t_0) &> 1-(1-r_1)\\
&> 1-1+r_1\\
&> r_1 \quad \text{----------------} \quad (2)
\end{aligned}
$$

$$
\begin{aligned}
\text{consider } M(x,z,t) &\geq M(x,y,t_0) * M(y,z,t-t_0)\\
&\geq r_0 * r_1\\
&\geq 1-S\\
&> 1-r \ \{\because r_0>1-s>1-r)
\end{aligned}
$$

Therefore $Z \in B(x,r,t)$ and hence

$B(y,1-r,t-t_0) \subset B(x,r,t)$

Hence every open ball is an open set.

PROPOSITION : 1.15 (TOPOLOGY INDUCED BY FUZZY METRIC)

Let $(X,M,*)$ be a fuzzy metric space. Define $\tau = \{A \subset X / x \in A$ if and only if there exist $t>0$ and r, $0 < r < 1$ such that $B(x,r,t) \subset A\}$. Then τ is a topology on X

Proof:

1. Obviously $\emptyset$ and X are in τ

2. Let $\{A_\alpha\}$ be an indexed collection of elements of τ.

Let $x \in \cup A\alpha \Rightarrow x \in A\alpha$ for some α.

$\Rightarrow$ There exists a $t > 0$, and r, $0 < r < 1$, such that $B(x,r,t) \subset A_\alpha \subset \cup A_\alpha$. Hence $\cup A_\alpha \in \tau$.

3. Let A_1, A_2......A_n be elements of τ.

Let $y \in \cap A_i$ for $i = 1$ to n.

Then $y \in A_i$ for $i=1$ to n

==>There exists r_i, t_i, for $i=1$ to n such that

$y \in B(x,r_i,t_i) \subset Ai$.

Let $r=\text{Min}\ \{r_1 ......r_n\}$,

$t=\text{Min}\ \{t_1 ......t_n\}$

Then $r<r_i$ for $i=1$ to n

==> $1-r>1-r_i$ for $i=1$ to n

Let $y \in B(x,r,t)$.

since $M(x,y,.)$is monotonically increasing

==> $M(x,y,t_i) \geq M(x,y,t)$

But $M(x,y,t) > 1 - r > 1 - r_i$

Therefore $M(x,y,t_i) \geq M(x,y,t) > 1 - r_i$

==> $y \in B(x,r_i,t_i)$

Therefore $y \in B(x,r,t) \subset B(x,r_i,t_i) \subset A_i$

ie: $y \in B(x,r,t) \subset \bigcap_{i=1}^{n} A_i$

ie: $\bigcap_{i=1}^{n} A_i \in \tau$

Hence τ is a topology on X.

Hence the proof.

Proposition :1.16

The topoloogy in proposition 1.15 is first countable, Since {B(x,1/n,1/n) n={1,2 ......} is a local base for each $x \in X$.

Theorem: 1.17

Every fuzzy metric space is Hausdorff.

proof:

Let $(X,M,*)$ be a fuzzy metric space.

Claim:

$(X,M,*)$ is a Hausdorff space. (i.e) To prove each pair x,y of distinct points of X have disjoint neighbourhoods.

Let x and y be two distinct points of X. Let $t>0$ and $M(x,y,t) = r$ then $0 < r < 1$. For each r_0, such that $r<r_0<1$, there exists a r_1 such that $r_1 * r_1 \geq r_0$ ---- (A)

Then $B(x,1-r,t/2)$ and $B(y,1-r,t/2)$ are the neighbouhoods about the distinct points x and y respectively.

Claim:

$$B(x,1-r_1,t/2) \cap B(y,1-r_1,t/2) = \emptyset$$

Assume the contrary.

Let $Z \in B(x,1-r_1,t/2) \cap B(y,1-r_1,t/2)$

$\Longrightarrow Z \in B(x,1-r_1,t/2) \Longrightarrow M(x,z,t/2) > 1-(1-r_1)$

$> 1-1+r_1$

$> r_1$ ----- (1)

and

$Z \in B(y,1-r_1,t/2) \Longrightarrow M(z,y,t/2) > 1-(1-r_1)$

$> 1-1+r_1$

$> r_1$ ------ (2)

Now $r = M(x,y,t)$

$\geq M(x,z,t/2) * M(z,y,t/2)$

$> r_1 * r_1$ ------ [from (1) and (2)]

$> r$ ------ [from (A)]

Which is a contradiction. .

Therefore every pair x,y of distinct points of X have disjoint neighbourhoods.

Hence (X,M,*) is Hausdorff.

PROPOSITION:1.18

Let (x,d) be a metric space. Let $M(x,y,t) = \frac{t}{t+d\,(x,y)}$ be a fuzzy metric defined on X. Then the topology τ_0 induced by the metric d and the topology τ induced by the fuzzy metric M are the same.

Proof:

Let (x,d) be a metric space.

For n > 0, cosider the ball $B(x,r) = \{y/d(x,y)<r\}$.

Let $y \in B(x,r)$.Then t for t>0, $M(x,y,t) = \frac{t}{t+d(x,y)}$

$$\geq \frac{t}{t+r}$$

Let $r_1 = \frac{t}{t+r}$

Then $0<r_1<1$ and there exists a real number S such that $r_1 > 1-S$. Then $M(x,y,t) = \frac{t}{t+r} = r_1 > 1-S$

Therefore $y \in B\,(x,s,t)$.

To prove the converse part, for t > 0 and 0 < r < 1,

consider $B(x,r,t) = \{y/M(x,y,t)> 1-r\}$ ----- (3)

Let $y \in B(x,r,t)$

Then $\frac{t}{t+d(x,y)} = M(x,y,t) > 1-r$

Therefore $t > t+d(x,y) - tr - d(x,y)\,r$

$d(x,y) < tr + rd\,(x,y)$

$tr > d(x,y) - rd(x,y)$

$tr > d(x,y)\,(1-r)$

Therefore $d(x,y) < \dfrac{tr}{1-r}$

Therefore $y \in B\left[x\,;\ \dfrac{tr}{1-r}\right]$

Therefore $B(x,r,t) \subset B\left[x;\ \dfrac{tr}{1-r}\right]$

Hence the topology τ_0 induced by the metric d and the topology τ induced by the fuzzy metric M are the same.

Definition : 1.19

Let $(X,M,*)$ be a fuzzy Metric space. A subset A of X is said to be F-bounded if and only if there exist $t > 0$ and $0 < r < 1$ such that $M(x,y,t) > 1-r$ for all $x,y \in A$.

Proposition : 1.20

Let $(X,M,*)$ be a fuzzy metric space induced by a metric d on X. Then $A \subseteq x$ is F-bounded if and only if it is bounded.

Proof

Let $(X,M,*)$ be a fuzzy metric induced by a Metric `d' on x. Then $M(x,y,t) = \dfrac{t}{t+d(x,y)}$, $t \in (0,\infty)$

Let $A \subseteq x$. Assume A is F-bounded.

Claim

A is bounded on X with respect to d.

A is F-bounded => there exists $t>0$ and $0<r<1$ such that

$M(x,y,t) > 1-r$ for all $x,y \in A$

$$\Rightarrow \frac{t}{t+d(x,y)} > 1-r$$

$$\Rightarrow t > [t+d(x,y)]\ 1-r$$

$$\Rightarrow d(x,y)(1-r) + t(1-r) < t$$

Therefore $d(x,y) \leq \dfrac{t-t(1-r)}{1-r}$

$$\leq \frac{t-t+tr}{1-r} < \frac{tr}{1-x} = k$$

$$\therefore\ d(x,y) \leq k$$

Hence the subset A of X is bounded with respect to d.

Converse

Assume that A is bounded with respect to the Metric d.

Claim

A is F-bounded.

Since A is bounded there exists a $k > 0$

such that $d(x,y) \leq k$ for every pair x,y of points of A.

Consider $M(x,y,t) = \dfrac{t}{t+d(x,y)}$ for $t \in (0,\infty)$

$$d(x,y) \leq k$$

$$\Rightarrow t+d(x,y) \leq t + k$$

$$\Rightarrow \frac{1}{t+d(x,y)} \geq \frac{1}{t+k}$$

$$M(x,y,t) \geq \frac{t}{t+k} \qquad \text{---------} (1)$$

Consider $\frac{1}{1 + k/t} = 1-r$

$$1 = 1 + \frac{k}{t} - r - \frac{rk}{t}$$

$$\therefore \frac{r(t+k)}{t} = \frac{k}{t}$$

$$r(t+k) = k \Rightarrow r = \frac{k}{t+k}$$

$$1-r = \frac{t}{t+k} \qquad \text{---------}(2)$$

Given $t>0$, choose $r = \frac{k}{t+k}$ Then $0<r<1$

and $M(x,y,t) = \frac{t}{t+d(x,y)} \geq \frac{t}{t+k} \geq 1-r$ [From (1)&(2)]

Therefore $M(x,y,t) > 1-r$ for all $x,y, \in A$.

Hence it is **F** - bounded.

Hence the proof.

THEOREM: 1.21

Every compact subset A of a fuzzy metric space X is F-bounded.

Proof:

Let A be a compact subset of a fuzzy metric space.

Claim:

A is F - bounded

Fix $t > 0$ and $0 < r < 1$ consider the open cover $\{B(x,r,t) : x \in A\}$ of A. Since A is compact, there exists a finite subcollection $x_1, x_2 \ldots\ldots x_n \in A$ such that $A \subseteq \bigcup_{i=1}^{n} B(x_i, r_i, t)$. Let $x, y, \in A$.

$$\left.\begin{array}{l} \text{Then } x \in B(x_i,r,t) \Longrightarrow M(x,x_i,t) > 1-r \text{ and} \\ \qquad y \in B(x_j,r,t) \Longrightarrow M(y,x_j,t) > 1-r \end{array}\right] \qquad ----(1)$$

$$\text{Let } \alpha = \text{Min } [M(x_i,x_j,t); 1 \le i,j \le n] \qquad ----(2)$$

Then $\alpha > 0$

consider $M(x,y,3t) \ge M(x,x_i,t) * M(x_i,y,2t)$

$M(x_i,y,2t) \ge M(x_i,x_j,t) * M(x_j,y,t)$

Therefore $M(x,y,3t) \ge M(x,x_i,t) * M(x_i,x_j,t) * M(x_j,y,t)$

$> (1-r) * (1-r) * \alpha$ [from (1) & (2)]

Since $0 < r < 1$ and $\alpha > 0$ there exists a s such that $0 < s < 1$, and $(1-r) * (1-r) * \alpha > 1-S$

$\Rightarrow M(x,y,3t) > 1-S$ for all $x, y \in A$

Let $3t = t'$ then

$M(x,y,t') > 1-S$ for all $x, y \in A$.

Hence A is F-bounded.

PREPOSITION : 1.22

In a fuzzy Metric Space every compact set A is closed and bounded.

PROOF:

(i) Let $(X,M,*)$ be a fuzzy Metric Space. From Theorem 1.21 Every compact subset A of a fuzzy Metric Space is F-bounded.

From proposition 1.20 A is bounded. Hence in a fuzzy. Metric space every compact set is bounded.

(ii) To prove Every compact set A is closed.

To prove A is closed it is enough to show that A contains all its limit points. Let $y \in x-A$ and $x \in A$. Since the fuzzy Metric Space is Hausdorff there exist disjoint neighbourhoods U_{xy} and U_x containing y and x respectively. The collection $\{U_x | x \in A\}$ is an open cover for A. Since A is compact it has finite subcover.

i.e. there exists elements $x_1, x_2, \ldots\ldots x_n \in A$ such that

$$A \subset \bigcup_{i=1}^{n} U_{xi}$$

As $U_{xi} \cap U_{yxi} = \emptyset$ for i = 1 to n

$$\left(\bigcap_{i=1}^{n} U_{yxi}\right) \cap A = \emptyset$$

i.e. $Y \in \bigcap_{i=1}^{n} U_{yxi}$ and $\left(\bigcap_{i=1}^{n} U_{yxi}\right) \cap A = \emptyset$

$\therefore$ y is not a limit point of A.

(i.e) A contains all its limit points.

Therefore A is closed.

Hence in a fuzzy metric space every compact set A is closed and bounded.

Theorem : 1.23

Let $(X,M,*)$ be a fuzzy metric space and τ be the topology induced by the fuzzy Metric. Then for a sequence $\{xn\}$ in X $x_n \longrightarrow x$ if and only if $M(xn,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$.

PROOF:

Let $(X,M,*)$ be a fuzzy Metric Space and τ be the topology induced by the fuzzy Metric. Let $\{xn\}$ be a sequence in X converging to x.

CLAIM:

$M(x_n,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$

Let $t>0$. Since $x_n \longrightarrow x$ then for $0<r<1$, there exist $n_o \in N$. such that $x_n \in B(x,r,t)$ for all $n \geq no$. From the definition it follows that $M(xn,x,t) > 1-r$ for $n \geq n_o$.

Since `r'is arbitrary $\Rightarrow 1-M(x_n,x,t) < r$

$\Rightarrow M\ (x_n,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$.

CONVERSE

Assume that $M(xn,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$.

CLAIM

x_n converges to x

For each $t>0$, $M(xn,x,t) \longrightarrow 1$ as $n \longrightarrow \infty$.

Therefore for $0<r<1$, there exists $n_o \in N$ such that $1-M(xn,x,t) < r$ for all $n \geq n_o$.

It follows that $M(xn,x,t) > 1-r$ for all $n \geq n_o$.

Thus $x_n \in B(x,r,t)$ for all $n \geq n_o$. and hence $x_n \longrightarrow x$.

Hence the proof.

DEFINITION : 1.24

A sequence $\{X_n\}$ in a Fuzzy Metric Space $(X,M,*)$ is a Cauchy sequence if and only if $\lim_{n \to \infty} M(x_n+p,x_n,t) = 1$ for $p>0$, $t>0$.

DEFINITION : 1.25

A fuzzy Metric Space in which every cauchy sequence is convergent is called a complete fuzzy Metric Space.

PREPOSITION : 1.26

It is important to note that with respect to the above definition, R fails to be complete. For example, consider $Sn = 1 + 1/2 + 1/3 + \ldots\ldots + 1/n$ in $(R,M,.)$

where $M(x,y,t) = \dfrac{t}{t+d(x,y)}$, where d is the Euclidean Metric on R. Then $M(S_n+p, S_n,t) = \dfrac{t}{t+|S_{n+p}-S_n|}$

$S_{n+p} = 1 + 1/2 + 1/3 + \ldots + 1/n + 1/n+1 + 1/n+2 + \ldots + 1/n+p$

$S_n \quad = 1 + 1/2 + 1/3 + \ldots + 1/n$

$S_{n+p}-S_n = (1/n+1) + (1/n+2) + \ldots + (1/n+p)$

Therefore $|S_{n+p} - S_n| = |1/n+1 + 1/n+2 + \ldots + 1/n+p|$

$= (1/n+1) + (1/n+2) + \ldots + (1/n + p)$

Therefore $M(S_{n+p},S_n,t) = \dfrac{t}{t+(1/n+1) + (1/n+2) + \ldots + (1/n + P)}$

Therefore $\lim\limits_{n\to\infty} M(S_{n+p},S_n,t) = 1$.

Therefore We have $\{S_n\}$ is a cauchy sequence in the fuzzy Metric space R. If R were fuzzy complete, then there exists $x \in R$ such that $M(S_n,x,t) \to 1$ as $n \to \infty$.

From this it follows that,

$$\frac{t}{t+|S_n-x|} \to 1 \text{ as } n \to \infty.$$

Further $|S_n-x| \to 0$ as $n \to \infty$.

Therefore $S_n \to x$ in R which is not true. Hence to make

R a complete fuzzy Metric space the authors have redefined cauchy sequence as follows:

DEFINITION : 1.27

A sequence $\{x_n\}$ in a fuzzy Metric Space $(X,M,*)$ is a cauchy sequence if and only if for each $\epsilon > 0$, $t > 0$, there exists $n_o \in N$ such that $M(x_n,x_m,t) > 1 - \epsilon$ for all $n,m \geq n_o$.

DEFINITION : 1.28

Let $(X,M,*)$ be a fuzzy Metric Space. A closed ball with centre $x \in X$ and radius r, $0<r<1$, is defined as $\{y \in X : M(x,y,t) \geq 1-r\}$ where $t>0$, and is denoted as $B[x,r,t]$.

THEOREM : 1.29

Every closed ball is a closed set.

PROOF:

Let $(X,M,*)$ be a fuzzy Metric Space. Let $y \in \overline{B[x,r,t]}$

CLAIM:

$y \in B[x,r,t]$.

Since X is first countable, there exists a sequence $\{y_n\}$ in $B[x,r,t]$ such that $y_n \longrightarrow y$.

Therefore $\lim\limits_{n \to \infty} M(y_n,y,t) = 1$. For a given $\epsilon>0$ and $t>0$,

$$M(x,y,t+\epsilon) \geq M(x,y_n,t) * M(y_n,y,\epsilon) \qquad \text{-----------(1)}$$

$$\lim_{n \to \infty} M(x,y_n,t) = M(x,y,t) \geq 1-r \qquad \text{-----------(A)}$$

$$\lim_{n \to \infty} M(y_n,y,\epsilon) = M(y,y,\epsilon) = 1 \qquad \text{-----------(B)}$$

Applying (A) and (B) in (1) we have

$$M(x,y,t+\epsilon) \geq M(x,y,t) * M(y,y,\epsilon)$$

$$M(x,y,t+\epsilon) \geq \lim_{n\to\infty} M(x,y_n,t) * \lim_{n\to\infty} M(y_n,y,\epsilon)$$

$$\geq (1-r) * 1 = 1-r.$$

[If $M(x,y_n,t]$ is bounded, $\{y_n\}$ has a subsequence, which we again denote by $\{y_n\}$ for which $\lim_{n\to\infty} M(x,y_n,t)$ exists].

In particular for $n \in N$, take $\epsilon = 1/n$. then we have

$$M(x,y,t+\epsilon) = M(x,y,t+1/n) \geq 1-r.$$

Hence $M(x,y,t) = \lim_{n\to\infty} M(x,y,t+1/n) \geq 1-r$

Therefore $M(x,y,t) \geq 1-r$

Thus $y \in B\,[x,r,t]$.

Therefore Every closed ball is a closed set.

Therefore $B[x,r,t]$ is a closed set.

THEOREM : 1.30

BAIRE'S THEOREM:

STATEMENT

Let X be a complete fuzzy Metric Space. Then the intersection of a countable number of dense open sets is dense.

PROOF

Let X be a complete fuzzy Metric Space. Let Bo be a non-empty open set. Let D_1, $D_2 \ldots$ be a countable dense open sets in X. Since B_o is a non-empty open set and D_1 is dense in X, $B_o \cap D_1 \neq \emptyset$. Therefore there exists a x_1 such that $x_1 \in B_o \cap D_1$. Since $B_o \cap D_1$ is open, there exist $0<r_1<1$, $t_1>0$ such that $B(x_1,r_1,t) \subset B_o \cap D_1$. Choose $r_1' < r_1$ and $t_1' = \text{Min}\{t_1,1\}$ such that $B\,[x_1,r_1',t_1'] \subset B_o \cap D_1$.

Let $B_1=B(x_1,r_1',t_1')$. Then B_1 is a non-empty open set.

Since D_2 is dense in X, $B_1 \cap D_2 \neq \emptyset$.

Let $x_2 \in B_1 \cap D_2$. Again by the same arguement as the first case, $B_1 \cap D_2$ is an open set, and since $x_2 \in B_1 \cap D_2$ there exists a neighbourhood around x_2 of radius $0<r_2<1/2$ and $t_2>0$ such that $B(x_2,r_2,t_2) \subset B_1 \cap D_2$.

Choose $r_2'<r_2$ and $t_2'= \mathrm{Min}\{t_2,1/2\}$.

Such that $B[x_2,r_2',t_2'] \subset B_1 \cap D_2$.

Let $B_2 = B(x_2,r_2',t_2')$.

Similarly proceeding by induction we can find a xn, such that $x_n \in B_{n-1} \cap D_n$. Since $B_{n-1} \cap D_n$, being the intersection of two open sets it is open, there exists $0<r_n<1/n$ and $t_n>0$ such that, $B(x_n,r_n,t_n) \subset B_{n-1} \cap D_n$.

Choose $r_n'<r_n$ and $t_n'= \mathrm{Min}\{t_n,1\}$

such that $B[x_n,r_n',t_n'] \subset B_{n-1} \cap D_n$ --------(1)

Let $B_n= B(x_n,r_n',t_n')$

Now we claim that $\{x_n\}$ is a cauchy sequence.

For a given $t>0$, $\epsilon>0$, choose n_o such that $1/n_o<t$ and $1/n_o<\epsilon$.

Then for $n \geq n_o$, $m \geq n$ we have
$$M(x_n,x_m,t) \geq M(x_n,x_m,1/n) \geq 1-(1/n) \geq 1-\epsilon$$

Hence {xn} is a cauchy sequence.

Since X is complete, $x_n \longrightarrow x$ in X. But we know that $x_k \in B[x_n,r_n',t_n']$ for all $k \geq n$. Since by theorem 1.29 we have "Every closed ball is a closed set" and hence $B[x_n,r_n',t_n']$ is a closed set. Let $x \in B[x_n,r_n',t_n']$

From (1) $B[x_n,r_n',t_n'] \subset B_{n-1} \cap D_n$

ie: $x \in B\,[x_n, r_n', t_n'] \subset B_{n-1} \cap D_n$ for all n

Therefore $B_o \cap \left(\bigcap_{n=1}^{\infty} D_n \right) \neq \emptyset$

Hence $\bigcap_{n=1}^{\infty} D_n$ is dense in X. Hence the intersection of a countable number of dense open sets is dense.

Therefore $\bigcap_{n=1}^{\infty} D_n$ is dense in X.

Hence the proof.

Chapter II

CHAPTER II

INTRODUCTION

The concepts uniform continuity, equicontinuity, uniform convergence introduced by **GEORGE AND VEERAMANI** [6] are discussed in this chapter. They have shown that a sequence of functions is uniformly continuous (equi continuous) with respect to a metric d iff it is uniformly continuous (equi continuous) with respect to the induced standard fuzzy metric. Further they have proved **ASCOLI-ARZELA THEOREM** and **NAGATA SIMRNOV THEOREM** and established Every fuzzy Metric space is metrizable.

THEOREM : 2.1

Every fuzzy metric space is normal

Proof:

Let $(X,M,*)$ be a fuzzy metric space.

Let F and G be two disjoint closed sets in X.

Claim:

There exist disjoint open sets U and V containing F and G respectively.

Let $x \in F$ then $x \notin G$

This => $x \in G^c$ (complement of G).

As complement of G is open there exists $r_x, t_x > 0$, $0 < r_x < 1$ such that $B(x,r_x,t_x) \cap G = \emptyset$ for all $x \in F$

Let $y \in G$ then $y \in F^c$ therefore there exists r_y, $0 < r_y < 1$ and $t_y > 0$ such that $B(y,r_y,t_y) \cap F = \emptyset$ for all $y \in G$.

Let $S = \text{Min}\{r_x,t_x,r_y,t_y\}$. Then there exists a $S_o \in (0,s)$ such that $(1-S_o) * (1-S_o) > 1-S$

Define $U = \bigcup_{x\in F} B(x,S_0,s/2)$ and $V = \bigcup_{y\in G} B(y,S_0,s/2)$

Each $B(x,S_0,s/2)$ being an open set and as arbitrary union of opens sets is open, $U = \bigcup_{x\in F} B(x,S_0,s/2)$ is open. Similarly, $V = \bigcup_{y\in G} B(y,S_0,s/2)$ is open. Now clearly U and V are open sets such that $F \subset U$ and $G \subset V$.

Claim

$U \cap V$ is empty.

Assume the contrary ie., $U \cap V \neq \emptyset$. Let $Z \in U \cap V$ then we have $Z \in U$ and $Z \in V$. This => there exist $x \in F$ and $y \in G$ such that $Z \in B(x,S_0,s/2)$ and $Z \in B(y,S_0,s/2)$ ------(1)

Therefore $M(x,y,s) \geq M(x,z,s/2) * M(y,z,s/2)$

From (1) $z \in B(x,S_0,s/2)$

=> $M(x,z,s/2) > 1-S_0$ ------(2)

$Z \in B(y,S_0,s/2)$

=> $M(y,z,s/2) > 1-S_0$ -------(3)

$$\begin{aligned} \text{Therefore } M(x,y,s) &\geq M(x,z,s/2)*M(y,z,s/2) \quad \text{[From (2)\&(3)]} \\ &\geq (1-S_0) * (1-S_0) \\ &> 1 - S \end{aligned}$$

=> $M(x,y,s) > 1 - S$

Hence $y \in B(x,s,s)$

But $B(x,s,s) \subset B(x,r_x,t_x)$ as $S < t_x, r_x$.

=> $y \in B(x,r_x,t_x)$ --------(4)

Since $y \in G$ --------(5)

From (4) and (5) we have $y \in B(x,r_x,t_x) \cap G$. Thus $B(x,r_x,t_x) \cap G$ is non-empty which is a contradiction. Therefore $U \cap V$ is empty. Hence X is normal.

Uniform continuity in fuzzy metric spaces is defined as follows:

DEFINITION: 2.2

A mapping f from a fuzzy metric space X to a fuzzy metric space Y is said to be uniformly continuous, if for given $r,t > 0$, $0<r<1$, we can find $r_o,t_o>0$, $0<r_o<1$, such that $M(x,y,t_o) > 1 - r_o$ implies $M\{f(x),f(y),t)\} > 1 - r$ for all $x,y \in X$.

Here a generalisation of a theorem in metric spaces which states that a continuous function on a compact space is uniformly continuous is given as follows:

THEOREM : 2.3

Continuous function on a compact fuzzy metric space is uniformly continuous:

Proof:

Let f be a continuous map form a compact fuzzy metric space X to a fuzzy metric space Y. Let $S,t>0$, $0<S<1$ be given. Then we can find $r\in(0,1)$ such that, $(1-r)*(1-r)>1-S$ Since 'f' is continuous, for each $x \in X$, we can find $r_x,t_x>0$, $0<r_x<1$ such that $M(x,y,t_x)>1-r_x$ implies $M\{f(x),f(y),t/2\} > 1 - r$. For $r_x\in(0,1)$, we can find $S_x\in(0,r_x)$ such that $(1-S_x) * (1 - S_x) > 1-r_x$. ----------(A) Since X is compact, every open covering $\{B(x,S_x,t_x/2)/x\in X\}$ of X contians a finite subcollection that also covers X. Hence there exist $x_1,x_2.......x_k$ in X such that

$$X = \bigcup_{i=1}^{k} B(x_i,S_{xi},t_{xi}/2).$$ Take $S_o = \text{Min } S_{xi}$ and

t_o = Min $t_{xi}/2$ for i = 1,2.....k.

For any x,y $\in$ X, if $M(x,y,t_o) > 1 - S_o$, then

$M(x,y,t_{xi}/2) > 1 - S_{xi}$ --------(1)

Since x $\in$ X there exists a x_i such that,

$M(x,x_i,t_{xi}/2) > 1 - S_{xi}$ --------(2)

Therefore $M\{f(x),f(x_i),t/2)\} > 1 - r$.

Now

$$\begin{aligned} M(y,x_i,t_{xi}) &\geq M(x,y,t_{xi}/2) * M(x,x_i,t_{xi}/2) \\ &\geq (1 - S_{xi}) * (1 - S_{xi}) \qquad \text{[from (1) \& (2)]} \\ &> (1 - r_x) \qquad \text{[from (A)]} \end{aligned}$$

Therefore $M(y,x_i,t_{xi}) > (1 - r_x)$

Therefore $M\{f(y),f(xi),t/2) > 1 - r$.

$$\begin{aligned} \text{Now, } M\{f(x),f(y),t\} &\geq M\{(f(x),f(x_i),t/2) * M\{f(y),f(x_i),t/2) \\ &\geq (1 - r) * (1 - r) \\ &> 1 - S. \end{aligned}$$

Therefore $M(f(x),f(y),t) > 1-s$.

$\therefore M\{f(x),f(y),t\} > 1 - S$

Hence $M(x,y,t_o) > 1 - S_o$ implies

$M\{f(x),f(y),t\} > 1 - S$ for all x,y $\in$ X

Hence f is uniformly continuous.

Proposition: 2.4

Let f be a uniformly continuous mapping of the fuzzy Metric space X into the fuzzy Metric space Y. If $\{x_n\}$ is a cauchy sequence in X, then $\{f(x_n)\}$ is also a cauchy sequence in Y.

Proof:

Let $(X,M,*)$ be a fuzzy Metric space and $\{x_n\}$ a cauchy sequence in X and f be a uniformly continuous map from a fuzzy Metric space X to a fuzzy metric space Y.

Claim:

$\{f(x_n)\}$ is a cauchy sequence in Y.

Since f is uniformly continuous for given $r,t > 0$, $0<r<1$, we can find $r_o,t_o>0$, $0<r_o<1$, such that $M(x,y,t_o)>1-r_o$ implies $M\{f(x),f(y),t\}>1-r$ for all $x,y \in X$. -------(1)

Since $\{x_n\}$ is a cauchy sequence in $(X,M,*)$

there exists $n_o \in N$ such that

$\Rightarrow M(x_n,x_m,t_o)>1 - r_o$ for $n, m > N_o$

From (1) we get $M\{f(x_n),f(x_m),t\} > 1 - r$ for $n,m > N_o$

$\Rightarrow f(x_n)$ is a cauchy sequence in Y.

Hence the proof.

PROPOSITION: 2.5

Every compact fuzzy metric space is separable

Proof:

Let $(X,M,*)$ be a compact fuzzy metric space. Let $t>0$ and $r>0$, where $0<r<1$. The collection $\{B(x,r,t)/x \in X\}$ is an open cover of X. Since X is compact it has a finite subcover. Therefore there exist $x_1,x_2.....x_n$ in X such that $X = \bigcup_{i=1}^{n} B(x_i,r,t)$. In particular, for each $n \in N$, we can find a finite set A_n such that $X = \bigcup_{a \in A_n} B(a,1/n,1/n)$.

Let $A = \bigcup_{n=1}^{\infty} A_n$.

Then A is countable and $X \subset A$ Let $x \in A$. Then for each n, there exists $a_n \in A_n$ such that $x \in B(a_n, 1/n, 1/n)$. Thus a_n converges to x.But $a_n \in A$ for all n, and hence $x \in A$ Therefore A is dense in X and X is separable.

Hence Every compact fuzzy Metric space is separable.

The concepts equicontinuity and uniform convergence play an important role in the study of Ascoli-Arzela theorem for Fuzzy Metric Spaces. These concepts are defined here and proved that uniform convergence and equicontinuity in Metric Spaces (X,d) are equivalent to uniform convergence and equicontinuity in the fuzzy Metric Spaces (X,M,*) induced by the Metric d respectively.

DEFINITION : 2.6

Let X be any non-empty set and (Y,M,*) be a fuzzy Metric Space. Then a sequence {fn} of functions from X to Y is said to converge uniformly to a function f from X to Y, if given $r,t > 0$, $0 < r < 1$, there exists $n_o \in N$ such that $M\{f_n(x),f(x),t\}>1-r$ for all $n \geq n_o$ and for all $x \in X$.

DEFINITION 2.7

A family of mapping F from a fuzzy Metric Space X to a fuzzy Metric Space Y is said to be equicontinuous if for given $r,t>o$, $o<r<1$, we can find r_o ,$t_o>o$, $o<r_o<1$, such that $M(x,y,t_o)>1-r_o$ implies $M\{f(x),f(y),t\} > 1-r$ for all $f \in F$.

PROPOSITION: 2.8

Let X be any non-empty set and (Y,d) be a Metric Space. Let (Y,M,*) be the induced fuzzy Metric Space. Then a Sequence of functions $\{f_n\}$ from X to Y converges uniformly to a function `f' from X to Y, with respect to the Metric `d' if and only if $\{f_n\}$ converges uniformly to `f' with respect to the fuzzy Metric M.

PROOF:

Let $(X,M,*)$ be a fuzzy Metric Space. (Y,d) be a Metric Space and $(Y,M,*)$ be the induced fuzzy Metric Space.

Assume that $\{f_n\}$ converges uniformly to `f' with respect to the fuzzy Metric M. Then for given $r,t>0$, $0<r<1$, there exists $K \in N$ such that $M\{f_n(x),f(x),t\}>1-r$ for all $n \geq k$. Let $\epsilon > 0$ be given. Take $r = \dfrac{\epsilon}{t+\epsilon}$

$$\text{Then} \qquad M\{f_n(x),f(x),t\} > 1 \frac{\epsilon}{t+\epsilon}$$

$$\text{But} \qquad M\{f_n(x),f(x),t\} = \frac{t}{t+d\{f_n(x),f(x)\}} > 1- \frac{\epsilon}{t+\epsilon}$$

$$\Rightarrow \qquad \frac{t}{t+d\{f_n(x),f(x)\}} > \frac{t}{t+\epsilon}$$

$$\Rightarrow \qquad t+\epsilon > t + d\{f_n(x),f(x)\}$$

Therefore $d\{f_n(x),f(x)\} < \epsilon$ for all $n \geq k$.

Therefore $\{f_n\}$ converges uniformly to `f' with respect to the Metric `d' on Y.

CONVERSE:

Assume that $\{f_n\}$ converges uniformly to `f' with respect to the Metric `d' on Y. Then for given $\epsilon>0$, and for all $n \geq K$, $d\{f_n(x), f(x)\} < \epsilon$ for all $n \geq K$.

$$\Rightarrow \qquad t+d\{f_n(x),f(x)\} < t+\epsilon.$$

$$\Rightarrow \quad \frac{t}{t+d\{f_n(x),\ f(x)\}} > \frac{t}{t+\epsilon}$$

$$\Rightarrow \quad \frac{t}{t+d\{f_n(x),f(x)\}} > \frac{t+\epsilon-\epsilon}{t+\epsilon}$$

$$\Rightarrow \quad \frac{t}{t+d\{f_n(x),f(x)\}} > 1 - \frac{\epsilon}{t+\epsilon}$$

$$\Rightarrow \quad M\{f_n(x),f(x),t\} = \frac{t}{t+d\{f_n(x),f(x)\}} > 1 - \frac{\epsilon}{t+\epsilon}$$

$$\Rightarrow M\{f_n(x),f(x),t\} > 1-r \text{ for all } n \geq K \text{ where } r= \frac{\epsilon}{t+\epsilon}$$

Therefore given $r,t>0$, $0<r<1$, if $\epsilon = \frac{rt}{1-r}$ there exists a $K \in N$ such that $M\{f_n(x),f(x),t\}> 1-r$ for all $n \geq K$. Hence $\{f_n\}$ converges uniformly to `f' with respect to the fuzzy Metric M.

Hence the proof.

PROPOSITION: 2.9

Let (X,d) and (Y,d') be the given Metric spaces. Let (X,M,*) and (Y,M,*) be the corresponding induced fuzzy Metric Spaces. Then a family F of functions from X to Y is equicontinuous with respect to the Metric if and only if F is equicontinuous with respect to the fuzzy Metric.

PROOF:

Let (X,d) and (Y,d') be the Metric Spaces and let (X,M,*) and (Y,M,*) be the corresponding induced fuzzy Metric spaces.

Assume that F is equicontinuous with respect to the Metric. Then for given $\epsilon>0$, there exists a $\delta>0$ such that $d(X,Y)<\delta$ implies $d'\{f(x),f(y)\}<\epsilon$ for all $f \in F$.

Given $r,t>0$, $0<r<1$, let $\epsilon = \frac{rt}{1-r}$ and $\frac{t}{t+d(x,y)} > 1 - \frac{\delta}{t+\delta}$

implies $\frac{t}{t+d\{f(x),f(y)\}} > 1 - r$ for all $f \in F$

We have $d(X,Y) < \delta$

$$t+d(X,Y) < t+\delta$$

$$\frac{1}{t+d(X,Y)} > \frac{1}{t+\delta}$$

$$d'\{f(x),f(Y)\} < \epsilon = \frac{rt}{1-r}$$

$$t+d'\{f(x),f(y)\} < t + \frac{rt}{1-r}$$

$$t+d'\{f(x),f(y)\} < \frac{t}{1-r}$$

$$\Rightarrow \frac{1}{t+d'\{f(x),f(y)\}} > \frac{1 - r}{t}$$

$$\Rightarrow \frac{t}{t+d'\{f(x),f(y)\}} > 1 - r$$

Taking $t_o = t$ and $r_o = \frac{\delta}{t_o+\delta}$

then $M(X,Y,t_o) > 1-r_o$

i.e. $\frac{t_o}{t_o+d(x,y)} > 1-r_o$

$$\Rightarrow \frac{t}{t+d(x,y)} > 1 - \frac{\delta}{t+\delta}$$

$$\Rightarrow \frac{t}{t+d(x,y)} > \frac{t}{t+\delta}$$

$$\Rightarrow M\ \{f(x),f(y),t\} > 1-\ r$$

Therefore $M(X,Y,t_o) > 1-r_o$ implies $M\{f(x),f(y),t\} > 1-r$ for all $f \in F$. Hence F is equicontinuous with respect to the fuzzy Metric.

The Converse part can also be proved in the same way.

Hence the proof.

THEOREM : 2.10

Let $\{f_n\}$ be an equicontinuous sequence of mapping from a fuzzy Metric space X to a complete fuzzy Metric space Y. If $\{f_n\}$ converges for each point of a dense subset D of x, then $\{f_n\}$ converges at each point of X and the limit function is continuous.

PROOF

Let $t,S>0$, be two real numbers, then we can find r in $(0,1)$, such that $(1-r) * (1-r) *(1-r) > 1-S$. Let $F = \{f_n\}$ be an equicontinuous sequence of mapping from a fuzzy Metric space X to a complete fuzzy Metric space Y. From the definition of equicontinuity, we have for the given $r,t>0$, we can find $r_1,t_1>0$, $0<r_1<1$, such that for $x,y \in X$, $M\ (x,y,t_1) > 1-r_1$ implies $M\ \{f_n(x),f_n(y),t/3) > 1-r$ for all $f_n \in F$ ---(1).
Since D is dense in X, $D \cap B(x,r_1,t_1) \neq \emptyset$.
Let $y \in B(x,r_1,t_1) \cap D$. Then $y \in D$ and from our hypothesis $\{f_n(y)\}$ converges. Therefore $\{f_n(y)\}$ is a cauchy sequence in Y. Therefore for given $r,t>0$, there exists a $n_o \in N$ such

that for all $m,n \geq n_o$,

$M\{f_n(y), f_m(y), t/3) > 1 - r$ -------(2)

Since $F = \{f_n\}$ is an equicontinuous family, then for given $r,t>0$, we can find $r_1, t_1 > 0$, $0<r_1<1$ such that $x,y \in X$,

$M(x,y,t_1) > 1-r$, implies $M\{f_n(x), f_n(y), t/3\} > 1-r$ -----(3)

for all $f_n \in F$. Therefore for any $x \in X$ and $n,m \geq$

$$n_o M\{f_n(x), f_m(x), t\} \geq M\{f_n(x), f_n(y), t/3) * M\{f_n(Y), f_m(y), t/3\}$$

$$* M\{f_m(x), f_m(y), t/3\}$$

$$\geq (1-r)*(1-r)*(1-r) \quad [\text{from}(1),(2),\text{and}(3)]$$

$$> 1 - S$$

Hence $\{f_n(x)\}$ is a cauchy sequence in Y.

Since the space Y is complete, the cauchy sequence $\{f_n(x)\}$ in Y converges.

Let $f(x) = \lim f_n(x)$

Claim:2

The limit function f is continuous.

Let $s_o, t_o > 0$ be given. Then we can find a $r_o \in (0,1)$ such that $(1-r_o) * (1-r_o) * (1-r_o) > (1-s_o)$.

Since F is equicontinuous for given $r_o, t_o > 0$, $0<r_o<1$, there exists $r_2, t_2 > 0$, $0<r_2<1$, such that,

$M(x,y,t_2) > 1-r_2$ implies $M\{f_n(x), f_n(y), t_o/3\} > 1-r_o$ for all $f_n \in F$.

Since $f_n(x)$ converges to $f(x)$, for given $r_o, t_o > 0$, $0<r_o<1$, we can find a $K \in N$ such that

$M\{f_n(x), f(x), t_o/3\} > 1-r_o$ for all $n \geq K$ ------(2).

Similarly since $f_n(y)$ converges to $f(y)$, we can find a

$j \in N$ Such that, $M\ (f_n(y), f(y), t_o/3\} > 1 - r_o$ ------(3)

for all $n \geq j$.

Hence for all $n \geq \text{Max}\ (k,j)$

$$\begin{aligned} M\ \{f(x), f(y), t_o)\} &\geq M\ \{f(x), f_n(x), t_o/3\} * \\ &\quad M\ \{f_n(x), f_n(y), t_o/3\} * \\ &\quad M\ \{f_n(y), f(y), t_o/3\} \\ &\geq (1-r_o)*(1-r_o)*(1-r_o)\ \{\text{from } (1),(2)\&\ (3)\} \\ &> 1 - S_o \end{aligned}$$

Hence the limit function f is continuous.

Hence the proof.

THEOREM:2.11

Ascoli - Arzela Theorem:

STATEMENT:

Let X be a compact fuzzy Metric space and Y be a complete fuzzy metric space. Let A be an equicontinuous family of function from X to Y. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in A such that $cl\{f_n(x): n=1,2,3....\}$ is a compact subset of Y for each $x \in X$. Then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.

Proof:

Let $(X,M,*)$ be a compact fuzzy metric space and hence by proposition: 2.5 X is separable.

Let $D = \{x_1, x_2......\}$ be a countable dense subset of X.

For each i, $cl\{f_n(x_i): n=1,2...\}$ is a compact subset of Y.

As every fuzzy Metric space is first countable, and every compact subset of Y is sequentially compact. Therefore we get a subsequence $\{g_n\}$ of $\{f_n\}$ such that $\{g_n(x_i)\}$ converges for each i=1,2... By theorem: 2.10 we get a function f from X to Y such that $g_n(x)$ converges to f(x) for all $x \in X$.

Claim

g_n converges uniformly to f on X.

Let $t>0$ and $s \in (0,1)$, Then there exists a $r \in (0,1)$ such that $(1-r)*(1-r)*(1-r)>1-s$. Since A is equicontinuous there exists $r_1, t_1>0$, $0<r_1<1$, such that $M(x,y,t_1)>1-r_1$ implies $M(g_n(x),g_n(y),t/3)>1-r$ for all n.

Since X is compact, by theorem: 2.3 f is uniformly continuous.Therefore for given $r,t>0$, $0<r1$, there exists $t_2>0$, $r_2 \in (0,1)$ such that $M(x,y,t_2) > 1-r_2$ implies $M(f(x),f(y),t/3)>1-r$.

Let $t_o=\min\{t_1,t_2\}$ and $r_o=\min\{r_1,r_2\}$. Since $(X,M,*)$ is a compact fuzzy metric space, for given $r_o, t_o, >0$, $0<r_o<1$,

kThere exists $x_1, x_2 \ldots\ldots x_k$ in X

such that $X \subseteq \cup B(x_i,r_o,t_o)$ i=1 for some finite K.

If $x \in X$, there exist i, $1 \le i \le k$, such that $M(x,x_i,t_o)>1-r_o$.

But $t_o=\mathrm{Min}\{t_1,t_2\}$ and $r_o=\min\{r_1,r_2\}$ and hence

$M(g_n(x),g_n(xi),t/3) > 1-r$ (By the equicontinuity of A) --(1)

and $M(f(x_i),f(x),t/3) > 1-r$ -------(2)

(By the uniform continuity of f)

Since $g_n(x_j)$ converges to $f(x_j)$, for $t>o$, $r \in (0,1)$, there exist $n_o \in N$, such that $M(g_n(x_j),f(x_j),t/3) > 1-r$ ------(3)

for all $n \geq n_o$, and for all $j=1,2,.....k$.

$$\begin{aligned} \text{Now, } M(g_n(x),f(x),t/3) &\geq M(g_n(x),g_n(x_i),t/3) * \\ &\quad M(g_n(x_i),f(x_i),t/3) * \\ &\quad M(f(x_i),f(x),t/3) \\ &\geq (1-r) * (1-r) * (1-r) \\ &> 1-s \text{ for all } x \in X. \end{aligned}$$

Hence g_n converges to 'f' uniformly on X.

Therefore, f is a continuous function from X to Y.

Hence the proof

DEFINITION:2.12

A collection ß of subsets of X is said to be countably locally finite if ß can be written as the countable union of collection $ß_n$, each of which is locally finite.

DEFINITION: 2.13

Let a be a collection of subsets of the space X. A collection ß of subsets of X is said to be a refinement of `a', if for each element B of ß, there is an element A of a containing ß.

PROPOSITION: 2.14

Let (X,M,*) be a fuzzy metric space. If A is an open covering of X, then there is a open covering B of X such that B is countably locally finite refinement of A.

Proof:

Since A is an open covering of x, by well ordering theorem, there exists a order '<' for othe collection A, such that every non-empty subset A has a smallest element.

For $U \in A$, and $\cap \in N$, define,

$S_n(U) = \{x \in X / B(x,1/n,1/n) \subset U\}$.

Let $R_n(U) = S_n(U) - \bigcup\limits_{V \subset U} V$

CLAIM

If $V, W \in A$ with $V \subset W$ and if $x \in R_n(V)$, and $y \in R_n(w)$, then $M(x,y,1/n) \leq 1 - 1/n$.

PROOF:

Let $x \in R_n(v)$, $y \in R_n(w)$ and $V \subset W$

As $B(x,r,t) = \{Y \in X : M(x,y,t) > 1-r\}$

it is enough to prove that $Y \notin B(x,1/n,1/n)$

$$R_n(w) = Sn(w) - \bigcup\limits_{V \subset W} V$$

$\therefore$ $Y \in R_n(w) \Rightarrow Y \in S_n(w)$ and $Y \notin V$ __________________(1)

$$R_n(v) = S_n(v) - \bigcup\limits_{U \subset V} U$$

$\therefore$ $x \in R_n(v) \Rightarrow x \in S_n(v)$ and $x \notin U$ __________________ (2)

But $S_n(v) = \{x \in X / B(x,1/n,1/n) \subset V\}$

$\therefore$ $B(x,1/n,1/n) \subset V$ __________________ (3)

As $Y \notin V$ we have $Y \notin B(x,1/n,1/n)$

$\Rightarrow M(x,y,1/n) \leq 1-1/n$

Hence the proof of claim 1.

For $n \in N$, there exist $S \in (o,1/n)$ such that

$(1-s)*(1-s)*(1-s) > 1-1/n$

Let $E_n(U) = \bigcup \{B(x,s,1/3n) : x \in R_n(U)\}$. Since each $B(x,s,1/3n)$ is an open set, $E_n(U)$ is an open set.

CLAIM:2

$E_n(U)$'s are disjoint open sets.

Given $V,W \in A$ with $V \subset W$, and if $x \in E_n(v), Y \in E_n(w)$ we prove $M(x,y,1/3n) \leq 1-s$.

Assume the contrary, then $M(x,y,1/3n) > 1-s$ ______________(A)

Since $x \in E_n(v) \Rightarrow x \in \cup \{B(z,s,1/3n) : z \in R_n(v)\}$

then there exists a $x_o \in R_n(v)$ such that

$$x \in B(x_o,s,1/3n)$$

$$\Rightarrow M(x_o,x,1/3n) > 1-s \qquad \text{______________(B)}$$

Given $y \in E_n(w) \Rightarrow y \in \cup\{B(z,s,1/3n) : z \in R_n(w)\}$ then there exists a $y_o \in R_n(w)$ such that $y \in B(y_o,s,1/3n)$

$$\Rightarrow M(y_o,y,1/3n) > 1-s \qquad \text{______________(C)}$$

Since $V \subset W$,

$$\begin{aligned} 1-1/n &\geq M(x_o,y_o,1/n) \\ &\geq M(x,x_o,1/3n) * M(x,y,1/3n) * M(y,y_o,1/3n) \quad [\text{from A,B \& C}] \\ &\geq (1-s)*(1-s)*(1-s) \\ &> 1-1/n \quad (\text{since 's' is real}) \end{aligned}$$

Therefore $1-1/n > 1-1/n$

Which is a contradition and hence we have

$M(x,y,1/3n) \leq 1-s$ --------(I)

Hence $E_n(U)$'s are disjoint

Define $E_n = \{E_n(U) : U \in A\}$

But we have $E_n(U) = \cup \{B(x,s,1/3n) / x \in R_n(U)\}$

For $y \in E_n(U)$, there exists $X \in R_n(U)$

such that $y \in B(x,s,1/3n)$.

As $S < 1/n$, $y \in B(x,s,1/3n) \subset B(x,1/n,1/n) \subset U$

Therefore $y \in U$

Hence $E_n(U) \subset U$ for all $U \in A$.

Therefore E_n refines A; that is for each elements in E_n, There exists an element U of A, Such that $E_n(U) \subset U$.

Hence E_n refines A.

Claim

E_n is locally finite.

Since $S \in (0,1)$, we can find a $r_o \in (0,1)$ such that

$(1-r_o) * (1-r_o) * (1-r_o) > 1-s$

Now we claim that for each $x \in X$, the ball $B(x,r_o,1/6n)$ intersects atmost one element of E_n. Suppose $B(x,r_o,1/6n)$ intersect $E_n(U)$ and $E_n(V)$ with $U<V$, then there exist a $y \in E_n(U)$ and $Z \in E_n(V)$ such that

$y \in E_n(U) \Rightarrow y \in B(x,r_o,1/6n)$, $x \in R_n(V)$

$\Rightarrow M(x,y,1/6n) > 1-r_o$ ---------------(1)

$Z \in E_n(V) \Rightarrow Z \in B(x,r_o,1/6n)$, $x \in R_n(W)$

$\Rightarrow M(x,z,1/6n) > 1-r_o$ ---------------(2)

Since $U<V$, we have $M(y,z,1/3n) > 1-s$

That is $M(y,z,1/3n) \geq M(x,y,1/6n) * M(x,z,1/6n)$

$\geq (1-r_o) * (1-r_o)$

$> 1-s$ ---------------(3)

Which is a contradiction to (I)

Hence E_n is locally finite.

Now consider the collection $B = \cup \{E_n / n \in N\}$

Let $x \in X$. A being a cover of x, we can find a $U \in A$ such that U is the first element of A that contains X.

Since U is open, we can choose n so that $B(x,1/n,1/n) \subset U$.

Hence $x \in S_n(U)$, but U being the first element of A that contains x, $x \in R_n(U)$ and hence $x \in E_n$.

Thus we got a collection B of sets satisfying the required condition.

Hence B is countably locally finite refinement of A.

Hence the proof.

THEOREM: 2.15

Every fuzzy Metric Space has a basis that is countably locally finite.

PROOF:

Let $(X,M,*)$ be a fuzzy Metric Space.

CLAIM: 1

It has a basis that is countably locally finite.

For a given $m \in N$, define $A_m = \{B(x,1/m,1/m): x \in X\}$. Then A_m covers x for each m. By proposition: 2.14 there exists an open covering D_m of X which is countably locally finite refinement of A_m. Let $D = \cup D_m$, $m \in M$, then D is countably locally finite.

CLAIM: 2

D is a basis for X.

Let $x \in X$. Given $r,t>0$, $0<r<1$, there exist a $m \in M$ such that $(1-1/m)*(1-1/m)>1-r$ and $1/m < t/2$. If B is the element of D_m which contains x, then there exist a $x_o \in X$, such that $B \subset B(x_o,1/m,1/m)$. (Since D_m refines A_m).

For any y in B,
$$\begin{aligned} M(x,y,t)>M(x,y,2/m) &\geq M(x,x_o,1/m)*M(y,x_o,1/m) \\ &\geq (1-1/m) * (1-1/m) \\ &> 1-r. \end{aligned}$$

Thus $y \in B(x,r,t)$ and hence $B \subset B(x,r,t)$. Hence every fuzzy Metric Space has a basis that is countably locally finite. Hence the proof.

REMARK: 2.16

Since the topology induced by a Metric and the corresponding fuzzy Metric are same by the above theorem we get that every Metric space has a basis that is countably locally finite.

THEOREM: 2.17

THE NAGATA - SMIRNOV METRIZATION THEOREM :

Let X be a regular space with a basis $\mathcal{B}$ that is countably locally finite. Then X is Metrizable.

THEOREM: 2.18

Every fuzzy Metric space is Metrizable.

PROOF:

Every fuzzy Metric space is regular by theorem 2.1 and since every fuzzy Metric space has a basis that is countably locally finite, the result follows from the Nagata-Smirnov theorem (theorem: 2.17).

PROPOSITON: 2.19

Let X be a compact Metric space and Y be a complete Metric space. Let A be an equicontinuous family of functions from X to Y and $\{f_n\}_{n=1}^{\infty}$ be a sequence in A such that $cl\{f_n(x):n=1,2,\}$ is a compact subset of Y for each $x \in X$. Then there exists a continuous function f from X to Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.

PROOF:

Let (Y,d) be the given Metric space and let (Y,M,*) be the induced fuzzy Metric space. We have $\{x_n\}$ is a cauchy sequence in (Y,d) iff $\{x_n\}$ is a cauchy sequence in (Y,M,*). x_n converges to x in (Y,d) iff x_n converges to x in (Y,M,*). Hence (Y,M,*) is complete iff (Y,d) is complete.

By applying proposition: 2.8 and propositon: 2.9 and by using **ASCOLI-ARZELA THEOREM** we get that there exists a continuous function f from a compact Metric space X to a complete Metric space Y and a subsequence $\{g_n\}$ of $\{f_n\}$ such that g_n converges uniformly to f on X.

Summary and Conclusion

SUMMARY AND CONCLUSION

In this thesis we have made an attempt to study generalisation of some of the results on Metric spaces to fuzzy metric spaces introduced by George and Veeramani [6] and [7].

In chapter I the concept of fuzzy Metric Space and its properties are studied. The important result discussed is the BAIRE'S THEOREM for fuzzy Metric Spaces.

In Chapter II we have discussed uniform continuity, equicontinuity and uniform convergence for fuzzy Metric space. Two interesting results discussed here are the ASCOLI-ARZELA Theorem and NAGATA-SIMNRNOV Theorem for fuzzy Metric Spaces.

It is interesting to note here that even though the Metric is fuzzy the topology it induces is not fuzzy. For future study many results on Metric Spaces can be attempted for fuzzy Metric Spaces.

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