Chapter 2

Δ^*-Closed Sets in Topological Spaces

2.1 Introduction

Velicko.N.V (1968) introduced δ-open sets which are stronger than open sets and proved that the collection of δ-open sets denoted by τ_δ-open sets formed a coarser topology on X. Norman Levine (1970) initiated the study of generalised closed sets which were denoted by g-closed sets. By combining the concepts of δ-closed sets and g-closed sets, Julian Dontchev (1996) proposed a class of generalised closed sets called δg-closed sets. As an extension of his work, Sudha.R (2012) established a stronger form of δg-closed sets namely, δg^*-closed sets.

In this chapter a new class of generalised closed sets called Δ^*-closed sets via δg-closed sets is introduced. The dependency of Δ^*-closed sets with various existing closed sets is also analysed. Further preservation of topological properties of Δ^*-closed sets are analysed. Moreover characterization theorems of Δ^*-closed sets on various spaces are also analysed in this chapter.

2.2 Δ^*-closed sets

In this section a new class of generalised closed sets called Δ^*-closed sets is defined and the association between Δ^*-closed sets and various existing closed sets are analysed.

Definition 2.2.1 A subset A of a topological space (X, τ) is said to be a **Δ^*-closed set** if δcl(A) □ U whenever A □ U, U is δg-open in (X, τ).

The class of all Δ^*-closed sets of (X, τ) is denoted by $\Delta^*\mathcal{C}$(X, τ).

Proposition 2.2.2 Every δ-closed set is Δ^*-closed but not conversely.

Proof : Let A be a δ-closed set and U be any δg-open set containing A. Since A is δ-closed, δcl(A) = A. Therefore δcl(A) = A □ U and hence A is Δ^*-closed.

Counter Example 2.2.3 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}}. In this topology the subset {c} is Δ^*-closed but not δ-closed.

Proposition 2.2.4 Every δg^*-closed set is Δ^*-closed but not conversely.

Proof : Let A be a δg^*-closed set and U be any δg-open set containing A in X. Since every δg-open set is g-open and A is δg^*-closed, δcl(A) □ U. Hence A is Δ^*-closed.

Counter Example 2.2.5 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. Then the subset {b} is Δ^*-closed but not δg^*-closed in (X, τ).

Proposition 2.2.6 Every Δ^*-closed set is $\delta g^{\ddagger}$-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any δ-open set containing A in X. Since every δ-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. Hence A is $\delta g^{\ddagger}$-closed.

Counter Example 2.2.7 Let X = {a, b, c}, τ = {ϕ, X, {a}}.Then the subset {a} is $\delta g^{\ddagger}$-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.8 Every Δ^*-closed set is gδ-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any δ-open set containing A in X. Since every δ-open is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, cl(A) □ δcl(A) and therefore cl(A) □ U. Hence A is gδ-closed.

Counter Example 2.2.9 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. Then the subset {a} is gδ-closed but not Δ^*-closed in (X, τ).

Remark 2.210 From the above results we get the following relation.

δ-closedness → δg^*-closedness → Δ^*-closedness → gδ-closedness

That is Δ^*-closedness is properly placed between δg^*-closedness and $g\delta$-closedness.

Proposition 2.2.11 Every Δ^*-closed set is rg-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any regular open set containing A in X. Since every regular open is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, cl(A) □ δcl(A) and so we have cl(A) □ U. Hence A is rg-closed.

Counter Example 2.2.12 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b},{a, c}}. Then the subset {a} is rg-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.13 Every Δ^*-closed set is gpr-closed but not conversely.

Proof : Let A be Δ^*-closed set and U be any regular open set containing A in X. Since every regular open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, pcl(A) □ δcl(A) and so we have pcl(A) □ U. Hence A is gpr- closed.

Counter Example 2.2.14 Let X = {a, b, c}, τ = {ϕ, X, {a}} Then the subset {b} is gpr-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.15 Every Δ^*-closed set is rwg-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any regular open set containing A in X. Since every regular open set is δg-open and A is Δ^*-closed, δcl(A) □ U. As int(A) □ A, we have cl(int(A)) □ cl(A) □ δcl(A) and hence A is rwg-closed.

Counter Example 2.2.16 Let X = {a, b, c}, τ = {ϕ, X, {a, b}}. Then the subset {a} is rwg -closed, but not Δ^*-closed in (X, τ).

Proposition 2.2.17 Every Δ^*-closed set is gspr-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any regular open set containing A in X. Since every regular open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, pcl(A) □ δcl(A) and so we have spcl(A) □ U and hence A is gspr-closed.

Counter Example 2.2.18 Let X = {a, b, c}, τ = {ϕ, X, {a, b}}. Then the subset {b} is gspr-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.19 Every Δ^*-closed set is πg-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, cl(A) □ δcl(A) and so we have cl(A) □ U and hence A is πg-closed.

Counter Example 2.2.20 Let X = {a, b, c}, τ = {ϕ, X, {a}, {b, c}}. Then the subset {c} is πg-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.21 Every Δ^*-closed set is πgp-closed but not conversely.

Proof : Let A be Δ^*-closed and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, pcl(A) □ δcl(A) and therefore we have pcl(A) □ U. Hence A is πg-closed.

Counter Example 2.2.22 Let X = {a, b, c}, τ = {ϕ, X, {a}} Then the subset {a} is πgp-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.23 Every Δ^*-closed set is πgsp-closed but not conversely.

Proof : Let A be Δ^*-closed set and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, spcl(A) □ δcl(A) and therefore we have spcl(A) □ U. Hence A is πgsp-closed.

Counter Example 2.2.24 Let X = {a, b, c}, τ = {ϕ, X, {a}, {b}, {a, b}}. Then the subset {a} is πgsp-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.25 Every Δ^*-closed set is πgs-closed but not conversely.

Proof : Let A be Δ^*-closed set and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, scl(A) □ δcl(A) and therefore we have scl(A) □ U. Hence A is πgs-closed.

Counter Example 2.2.26 Let X = {a, b, c}, τ = {ϕ, X, {a}}. Then the subset {c} is πgs-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.27 Every Δ^*-closed set is πgα-closed but not conversely.

Proof : Let A be a Δ^*-closed set and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, αcl(A) □ δcl(A) and so we have αcl(A) □ U and hence A is πgα-closed.

Counter Example 2.2.28 Let X = {a, b, c}, τ = {ϕ, X, {a}}. Then the subset {c} is πgα-closed but not Δ^*-closed in (X, τ).

Proposition 2.2.29 Every Δ^*-closed set is πgb-closed but not conversely.

Proof : Let A be Δ^*-closed set and U be any π-open set containing A in X. Since every π-open set is δg-open and A is Δ^*-closed, δcl(A) □ U. For every subset A of X, bcl(A) □ δcl(A) and therefore we have bcl(A) □ U. Hence A is πgb-closed.

Counter Example 2.2.30 Let X = {a, b, c}, τ = {ϕ, X, {a, b}}.Then the subset {a} is πgb-closed but not Δ^*-closed in (X, τ).

Remark 2.2.31 The following counter examples show that Δ^*-closed set is independent from g-closed, δg-closed, αg-closed, *g-closed, $\alpha\hat{g}$ -closed, #gs-closed, $\delta\hat{g}$-closed, gp-closed and g^*p-closed sets.

Counter Example 2.2.32 Let X = {a, b, c}, τ = {ϕ, X, {a}}. In this topology the subset {b} is g-closed, δg-closed, αg-closed, *g-closed, $\alpha\hat{g}$ -closed, #gs-closed, $\delta\hat{g}$ -closed, gp-closed and g^*p-closed but not Δ^*-closed.

Counter Example 2.2.33 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. In this topology the subset {a, b} is Δ^*-closed but not g-closed, δg-closed, αg-closed, *g-closed, $\alpha\hat{g}$-closed, #gs-closed, $\delta\hat{g}$-closed, gp-closed and g^*p-closed.

Remark 2.2.34 The following counter example shows that Δ^*-closed is independent from g^*s-closed and $g^{\#}$s-closed.

Counter Example 2.2.35 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}}. In this topology the subset {a, c} is Δ^*-closed but not g^*s-closed and $g^{\#}$s-closed. Also the subset {b} is g^*s-closed and $g^{\#}$s-closed but not Δ^*-closed.

Remark 2.2.36 The following Counter examples show that Δ^*-closed is independent from $(gs)^*$-closed.

Counter Example 2.2.37 Let X = {a, b, c}, τ = {ϕ, X, {a, b}}. In this topology the subset {b, c} is Δ^*-closed but not $(gs)^*$-closed.

Counter Example 2.2.38 Let X = {a, b, c}, τ = {ϕ, X, {a}, {b, c}}. In this topology the subset {a, c} is $(gs)^*$-closed but not Δ^*-closed.

Remark 2.2.39 The following counter examples show that Δ^*-closed is independent from mildly g-closed.

Counter Example 2.2.40 Let X = {a, b, c}, τ ={ϕ, X, {a, b}}. In this topology the subset {a} is mildly g-closed but not Δ^*-closed.

Counter Example 2.2.41 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. In this topology the subset {a, b} is Δ^*-closed but not mildly g-closed.

Remark 2.2.42 The following Counter examples show that Δ^*-closed is independent from $g^\#$-closed.

Counter Example 2.2.43 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. In this topology the subset {a, c} is Δ^*-closed but not $g^\#$-closed.

Counter Example 2.2.44 Let X = {a, b, c}, τ = {ϕ, X, {a}, {b}, {a, b}}. In this topology the subset {a} is $g^\#$-closed but not Δ^*-closed.

Remark 2.2.45 The following counter example shows that Δ^*-closedness is independent from closedness.

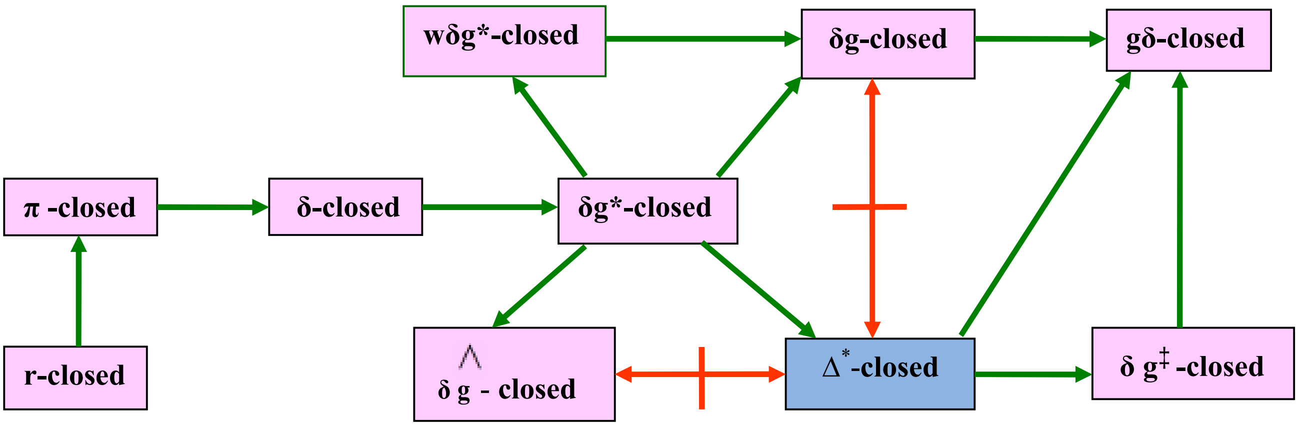
Counter Example 2.2.46 Let X = {a, b, c, d}, τ = {ϕ, X, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}}. In this topology the subset {a, d} is Δ^*-closed but not a closed set and the subset {b} is closed but not a Δ^*-closed set.

Remark 2.2.47 The following Counter examples show that Δ^*-closed set is independent from $w\delta g^*$-closed.

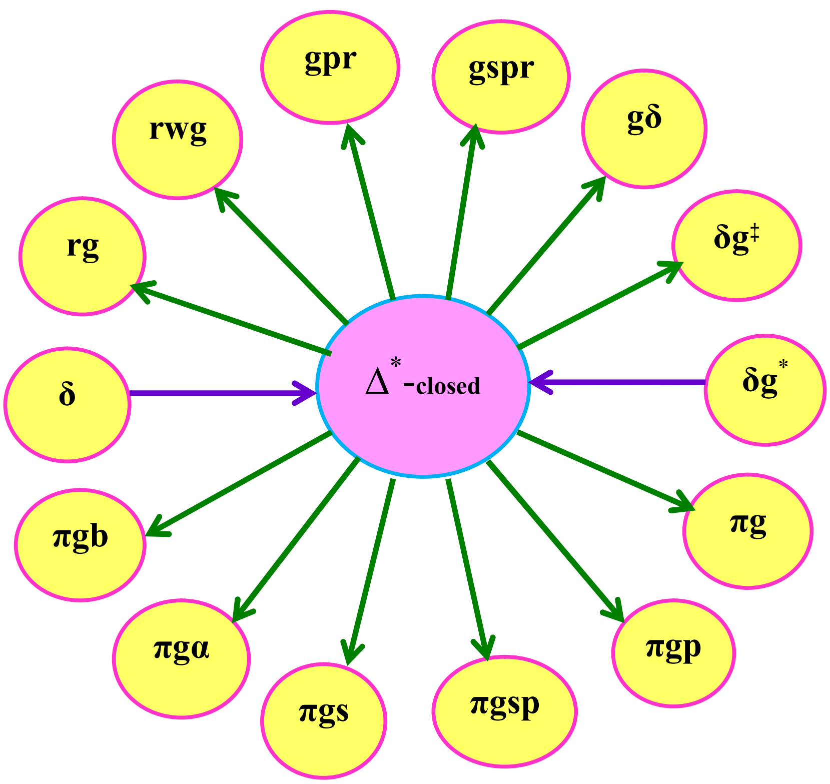
Counter Example 2.2.48 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}, {a, c}}. In this topology the subset {b} is Δ^*-closed but not $w\delta g^*$-closed.

Counter Example 2.2.49 Let X = {a, b, c}, τ = {ϕ, X, {a}}. In this topology the subset {b} is $w\delta g^*$-closed but not Δ^*-closed.

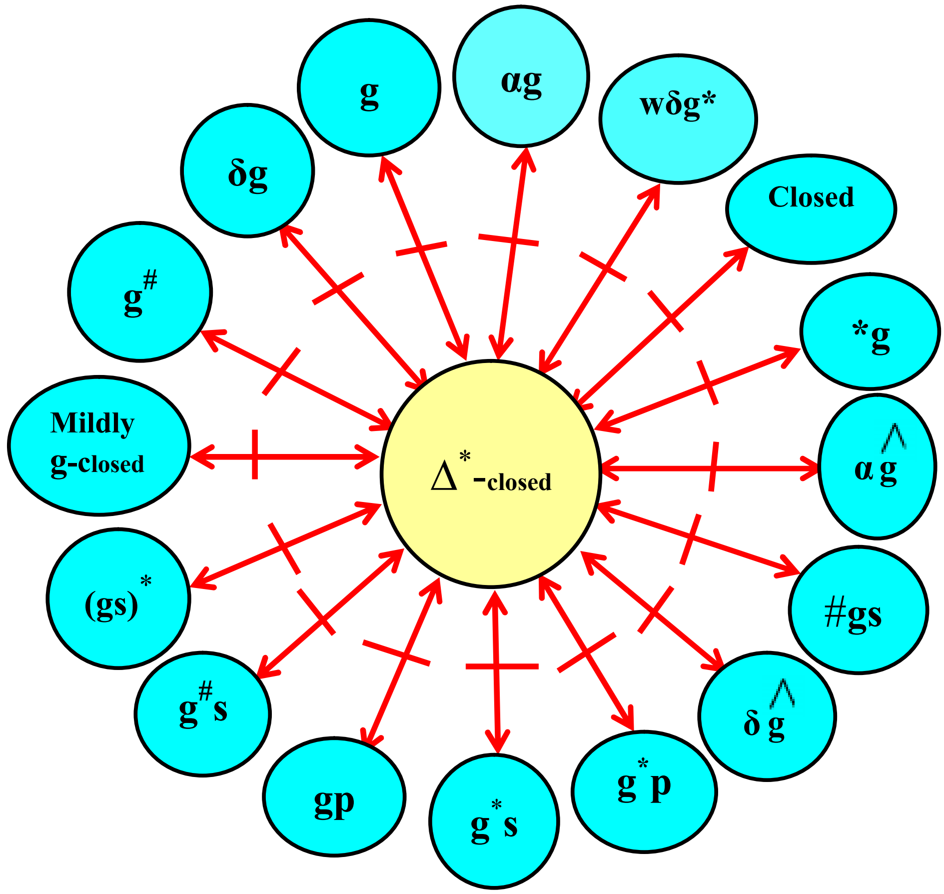
Remark 2.2.50 From the above relations we get the following implications between various g-closed sets using δ-closure.



Remark 2.2.51 The following diagram shows the dependency of Δ^*-closed set with various existing closed sets.



Remark 2.2.52 The following diagram shows the independency of Δ^*-closed with other closed sets.



2.3 Properties of Δ^*-closed sets in Topological Spaces

Theorem 2.3.1 The finite union of Δ^*-closed sets is Δ^*-closed .

Proof : Let $\{A_i / i = 1, 2, 3.......n\}$ be a finite class of Δ^*-closed sets of X. Let $A = \bigcup_{i=1}^{n} A_i$.

Let V be a δg open set containing A. This implies $A_i \subseteq V$ for every i. By assumption δcl(A_i) □ V for every i. This implies $\bigcup_{i=1}^{n}$ δcl(A_i) □ V. Then δcl($\bigcup_{i=1}^{n} A_i$) □ V. Thus δcl(A) □ V. Hence finite union of Δ^*-closed sets is Δ^*-closed.

Remark 2.3.2 The following example shows that the difference of any two Δ^*-closed sets in X need not be a Δ^*-closed set.

Example 2.3.3 Let X = {a, b, c}, τ = {ϕ, X, {a, b}}. Then the subset {c} and {b, c} are Δ^*-closed sets but their difference {b} is not Δ^*-closed.

Theorem 2.3.4 Let A be a Δ^*-closed set of X. Then δcl(A)−A does not contain a non empty δg-closed set.

Proof : Let A be a Δ^*-closed. Let F be a δg-closed set contained in δcl(A)−A. Now F^c is a δg-open set in X such that A □ F^c. Since A is Δ^*-closed set of X, δcl(A) □ F^c. Thus F □ $(\delta cl(A))^c$. Also F □ δcl(A) − A. Therefore F □ $(\delta cl(A))^c \cap \delta cl(A)$ = ϕ. Hence F = ϕ.

Proposition 2.3.5 If A is δg-open as well as Δ^*-closed set of (X, τ) then A is a δ-closed set of (X, τ).

Proof : Since A is δg-open and Δ^*-closed, δcl(A) □ A which implies A = δcl(A). Hence A is δ-closed.

Proposition 2.3.6 If A is Δ^*-closed and δg-open then (A ∩ F) is δ-closed where F is δ-closed in (X, τ) .

Proof : Since A is Δ^*-closed and δg-open, A is δ-closed by Proposition 2.3.5. Since F is given to be δ-closed, (A ∩ F) is δ-closed.

Theorem 2.3.7 The intersection of a Δ^*-closed set and a δ-closed set is always Δ^*-closed.

Proof : Let A be a Δ^*-closed set and F be a δ-closed set. Let V = A ∩ F. Let U be a δg-open set such that V □ U. Then A ∩ F □ U which implies A □ (U □ F^c). Here F^c is δ-open. So F^c is δg-open. Hence (U □ F^c) is δg-open and by assumption A □ (U □ F^c) which implies δcl(A) □ (U □ F^c) which implies δcl(A) ∩ F □ U.

Now δcl(V) = δcl(A ∩ F) □ δcl(A) ∩ δcl(F) = δcl(A) ∩ F □ U. Therefore δcl(V) □ U. Hence A ∩ F is Δ^*-closed.

Theorem 2.3.8 If A is a Δ^*-closed set in the space X and A □ B □ δcl(A) then B is also Δ^*-closed.

Proof : Let U be δg-open set of X such that B □ U. Then A □ U. Since A is Δ^*-closed δcl(A) □ U. Since B □ δcl(A), δcl(B) □ δcl(δcl(A)) = δcl(A). Hence δcl(B) □ U. Therefore B is also a Δ^*-closed set.

Theorem 2.3.9 Let A be a Δ^*-closed set of X. Then A is δ-closed if and only if δcl(A) − A is δg-closed.

Proof : (Necessity) : Let A be a δ-closed subset of X. Then δcl(A) = A and therefore δcl(A) – A = ϕ which is δg-closed.

(**Suffficiency) :** Let δcl(A)−A be a δg-closed set. Since A is Δ^*-closed, by Theorem 2.3.4, δcl(A)−A does not contain a non empty δg-closed set which implies δcl(A)–A = ϕ. That is δcl(A) = A. Hence A is δ-closed.

Characterization Theorems :

Theorem 2.3.10 Let A be a subset of a semi regular space (X, τ). Then

a) A is Δ^*-closed if and only if A is δg^*-closed.

b) If in addition, (X, τ) is a $T_{1/2}$-space then A is Δ^*-closed if and only if A is δg-closed.

Proof : Let A be a subset of a semi regular space (X, τ).

(a) ⟹ (b) : We know that by Proposition 2.2.4, every δg^*-closed is Δ^*-closed.

Conversely let A be a Δ^*-closed set. Let A □ U where U is g-open. In a semi regular space, every g-open is δg-open.(By Theorem 3.4 of Julian Dontchev (1996)).

Since A is Δ^*-closed, δcl(A) □ U which implies that A is δg^*-closed.

(b) ⟹ (a) : By the result (a), A is Δ^*-closed if and only if A is δg^*-closed. Then A is δg^*-closed if and only if A is δg-closed (By Theorem 2.2.54 of Sudha (2014)) and hence A is Δ^*-closed if and only if it is δg-closed .

Theorem 2.3.11 In an almost weakly Hausdorff space (X, τ), the g-closed subsets of (X, τ_S) are Δ^*-closed.

Proof : Let A ⊆ X be a g-closed subset of (X, τ). Let x □ δcl A). If {x} is δ-open, x □ δcl(A) implies every δ-neighborhood of {x} intersects A. Since {x} is δ-open, {x} ∩ A ≠ ϕ. This implies x □ A. If not, then X \ {x} is δ-open, since X is almost weakly Hausdorff. Assume that x □ A. Since A is g-closed in (X, τ_S) then δcl(A) □ X \{x}. That is x □ δcl(A). By contradiction x □ A. Thus δcl(A) = A or equivalently A is δ-closed and hence by Proposition 2.2.2, A is Δ^*-closed in (X, τ).

Definition 2.3.12 A topological space (X, τ) is called a **R_1-space** (Davis, 1961) if every two different points with distinct closures have disjoint neighbourhoods.

Remark 2.3.13 In general the concepts of Δ^*-closed and g^*-closed are not equivalent. But for a compact subset A of an R_1-topological space, they coincide. This can be seen in the following theorem.

Theorem 2.3.14 For a compact subset A of an R_1-topological space (X, τ) the following conditions are equivalent.

a) A is a g^*-closed set.

b) A is a Δ^*-closed set.

Proof : (a) ⟹ (b) : Let A be a g^*-closed set. Let $A \subseteq U$ where U is δg-open. In R_1-spaces, the concepts of closure and δ-closure coincide for compact sets. (By Theorem 3.6, Jankovic (1985)). Hence U is g-open and $\delta cl(A) = cl(A) \subseteq U$. Therefore A is a Δ^*-closed set.

(b) ⟹ (a) : Let A be a Δ^*-closed set. Assume $A \subseteq U$ where U is g-open. By Theorem 3.6, Jankovic (1985), U is δg-open and $cl(A) = \delta cl(A) \subseteq U$. Therefore A is a g^*-closed set.

Corollary 2.3.15 In Hausdorff spaces, a finite set is g^*-closed if and only if it is Δ^*-closed.

Definition 2.3.16 A **partition space** (Nieminen, 1977) is a topological space where every open set is closed.

Concerning partition spaces, the following characterization via Δ^*-closed sets is obtained.

Theorem 2.3.17 If (X, τ_S) is a $T_{1/2}$-space, then the following conditions are equivalent.

a) (X, τ_S) is a Partition space.

b) Every subset of (X, τ) is Δ^*-closed.

Proof : a) ⟹ b) Let $A \subseteq U$ where U is δg-open and A is an arbitrary subset of X. Hence in (X, τ_S), U is g-open. Since (X, τ_s) is a Partition space, U is open. By criteria, (X, τ_S) is a Partition space and hence U is clopen. Thus $\delta cl(A) = U$. Hence $\delta cl(A) \subseteq \delta cl(U) = U$ implying that A is Δ^*-closed .

b) ⟹ a) Let A be open in (X, τ_S). Then A is δ-open in (X, τ). Then A is δg-open in (X, τ). Then by (b), $\delta cl(A) \subseteq A$ or equivalently, A is δ-closed and hence closed in (X, τ_S). Therefore (X, τ_S) is a Partition space.

2.4 Δ^*-Closure operator

In this section the notion of Δ^*-closure operator is introduced and its properties are analysed.

Definition 2.4.1 The Δ^*-closure operator of a topological space (X, τ) is denoted by Δ^***cl(A)** and defined as follows.

$\Delta^*cl(A) = \cap \{ F \subseteq X : A \subseteq F$ and F is Δ^*-closed in $(X, \tau) \}$.

Proposition 2.4.2 If A is a Δ^*-closed set in (X, τ) then $\Delta^*cl(A) = A$.

Proof : Let A be Δ^*-closed in (X, τ). Then by the definition of $\Delta^*cl(A)$, the smallest F in the intersection of Δ^*-closed sets containing A is A itself. Hence $\Delta^*cl(A) = A$.

Remark 2.4.2(a) : Since the intersection of two Δ^*-closed sets need not be Δ^*-closed, $\Delta^*cl(A)$ may not be a Δ^*-closed set.

Example 2.4.2(b) : Let X = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}}. Then Δ^*-closed sets in (X, τ) are ϕ , X, {b}, {c}, {a, b}, {a, c}, {b, c}.

Here $\Delta^*cl(\{a\}) = \cap \{X, \{a, b\}, \{a, c\} \} = \{a\} \neq \Delta^*$-closed in (X, τ).

Remark 2.4.3 For a subset A of (X, τ), $A \subseteq \Delta^*cl(A) \subseteq \delta g^*cl(A) \subseteq \delta cl(A)$.

Proposition 2.4.4 Let A and B be subsets of (X, τ). Then the following statements are true.

a) $\Delta^*cl(\phi) = \phi$ and $\Delta^*cl(X) = X$.

b) If $A \square B$, then $\Delta^*cl(A) \square \Delta^*cl(B)$.

c) $A \square \Delta^*cl(A)$.

d) $\Delta^*cl(A) \square \Delta^*cl(B) = \Delta^*cl(A \square B)$.

e) $\Delta^*cl(A \cap B) \square \Delta^*cl(A) \cap \Delta^*cl(B)$

f) $\Delta^*cl(\Delta^*cl(A)) = \Delta^*cl(A)$

Proof : (a), (b) and (c) follow from the Definition 2.4.1.

d) Since $A \square A \square B$ and $B \square A \square B$, by (b), $\Delta^*cl(A) \square \Delta^*cl(A \square B)$ and $\Delta^*cl(B) \square \Delta^*cl(A \square B)$. Hence $\Delta^*cl(A) \square \Delta^*cl(B)) \square \Delta^*cl(A \square B)$.

To prove the reverse inequality, let $x \square \Delta^*cl(A) \square \Delta^*cl(B)$. Then $x \square \Delta^*cl(A)$ and $x \square \Delta^*cl(B)$. Therefore there exist Δ^*-closed sets U and V in X such that $A \square U$ and $B \square V$ where $x \square U$ and $x \square V$. Hence we have $A \square B \square U \square V$ and $x \square U \square V$.

By Proposition 2.3.1, $(U \square V)$ is Δ^*-closed and hence $x \square \Delta^*cl(A \square B)$.

e) Since $A \cap B \square A$ and $A \cap B \square B$ by (b) $\Delta^*cl(A \cap B) \square \Delta^*cl(A)$ and $\Delta^*cl(A \cap B) \square \Delta^*cl(B)$. Hence $\Delta^*cl(A \cap B) \square \Delta^*cl(A) \cap \Delta^*cl(B)$.

f) Follows from the definition of $\Delta^*cl(A)$.

Definition 2.4.5 Let U be any subset of (X, τ). Using Δ^*-closure operator, a new class of sets denoted by $\Delta^*\tau^\#$ is defined as follows.

$\Delta^*\tau^\# = \{ U : \Delta^*cl(X - U) = X - U \}$.

Proposition 2.4.6 For any topology τ, we have $\tau_\delta \square \Delta^*\tau^\#$

Proof : Obvious from Remark 2.4.3 and Definition 2.4.5.

Remark 2.4.7 : $\tau_\delta \square \delta g^*\tau^\# \square \Delta^*\tau^\#$

Proof : It follows from the implication, δ-closed $\longrightarrow$ δg^*-closed $\longrightarrow$ Δ^*-closed.

2.5 Δ^*-Open sets

In this section the concept of Δ^*-open sets in topological spaces is introduced and their properties are analysed.

Definition 2.5.1 A subset A of a topological space(X, τ) is called Δ^*-open if its complement A^c is Δ^*-closed in (X, τ). The collection of all Δ^*-open sets in (X, τ) is denoted by $\Delta^*O(X, \tau)$.

Theorem 2.5.2 If a subset A of a topological space (X, τ) is δ-open, then it is Δ^*-open in (X, τ).

Proof : Let A be a δ-open set in a topological space (X, τ). Then A^c is δ-closed in (X, τ). By Proposition 2.2.2, A^c is Δ^*-closed in (X, τ). Hence A is Δ^*open in (X, τ).

Remark 2.5.3 The converse of the above theorem need not be true as seen in the following example.

Counter example 2.5.4 Let X = {a, b, c}, τ = {ϕ, X, {a}, {a, b}}. Then the subset {a} is Δ^*-open but not δ-open in (X, τ).

Proposition 2.5.5 Every clopen set in (X, τ) is Δ^*-open.

Proof : Let A be a clopen set. Then cl(A) = A and int(A) = A. Hence int(cl(A)) = A. Thus A is regular open and therefore A is δ-open which implies that A is Δ^*-open.

Remark 2.5.6 A Δ^*-open set is need not be clopen as seen from the following examples.

Counter example 2.5.7 Let X = {a, b, c} and τ = {ϕ, X, {a}}. Then {a} is Δ^*-open but not clopen .

The following proposition can be proved similar to 2.5.2.

Proposition 2.5.8 Every Δ^*-open set is gδ-open, rg-open, gpr-open, $\delta g^\ddagger$ -open and πg-open but not conversely.

Lemma 2.5.9 For a subset of A of (X, τ), δcl(X − A) = X − δInt(A). (Velicko, 1968).

Theorem 2.5.10 A subset A of a topological space (X, τ) is Δ^*-open if and only if G □ δInt(A) whenever A □ G and G is δg-closed.

Proof : Assume that A is Δ^*-open. Then A^c is Δ^*-closed. Let G be a δg-closed set in (X, τ) contained in A. Then G^c is a δg-open set in (X, τ) containing A^c. Since A^c is Δ^*-closed, δcl(A^c) □ G^c, equivalently G □ δInt(A).

Conversely assume that G □ δInt(A),whenever G □ A and G is δg-closed in (X, τ). Let A^c be contained in F where F is δg-open. Then F^c □ A. By criteria, F^c □ δInt(A). This implies δcl(A^c) □ F. Thus A^c is Δ^*-closed and hence A is Δ^*-open.

Proposition 2.5.11 If δInt(A) □ B □ A and A is Δ^*-open in (X, τ), then B is Δ^*-open in (X, τ).

Proof : Follows from Lemma 2.5.9 and Theorem 2.3.8

Theorem 2.5.12 If A and B are Δ^*-open sets in (X, τ) then A ∩ B is Δ^*-open in (X, τ).

Proof : Let A and B be Δ^*-open sets in X. Then X − A and X − B are Δ^*-closed sets and (X − A) □ (X − B) = X − (A ∩ B) is Δ^*-closed. Hence (A ∩ B) is Δ^*-open.

Theorem 2.5.13 A subset A of (X, τ) is Δ^*-open if and only if the only δg-open set containing δInt(A) □ A^c is X.

Proof : (Necessary) : Let A be a Δ^*-open set and G be a δg-open set such that δInt(A) □ A^c □ G. So δInt(A) □ G and A^c □ G. Since A^c is Δ^*-closed, δcl(A^c) □ G. Hence δInt(A) □ δcl(A^c) □ G. Because δcl(A^c) = $(\delta Int(A))^c$, we have δInt(A) □ $(\delta Int(A))^c$ □ G which means that G = X.

(Sufficiency) : Suppose that F is δg-closed and F □ A. Then δInt(A) □ A^c □ δInt(A) □ F^c. As δ-open implies δg-open, we get δInt(A) is δg-open. Also F^c is δg-open. Hence δInt(A) □ F^c is δg-open. It follows by hypothesis that δInt(A) □ F^c = X.

Hence (δInt(A) □ F^{c}) ∩ F = X ∩ F = F. Therefore δInt(A) ∩ F = F. Therefore F □ δInt(A) which implies that A is Δ^*-open.

We have an alternative characterization of Δ^*-closed sets in Theorem 2.5.16

Proposition 2.5.14 For each a □ (X, τ) either {a} is δg-closed or {a} is Δ^*-open in (X, τ). That is for any space (X,τ), X = δGC(X,τ) □ Δ^*O(X, τ).

Proof : Suppose that {a} is not δg-closed in (X, τ). Then $\{a\}^{c}$ is not δg-open and the only δg-open set containing $\{a\}^{c}$ is the space (X, τ) itself. That is $\{a\}^{c}$ □ X. Therefore δcl($\{a\}^{c}$) □ X which implies $\{a\}^{c}$ is Δ^*-closed and hence {a} is Δ^*-open (X, τ).

Definition 2.5.15 The intersection of all δg-open subsets of (X, τ) containing A is called the δg-kernel of A and is denoted by **δg-ker(A).**

i.e., δg-ker(A) = ∩ { U / U is δg-open in (X, τ) and A □ U}.

Theorem 2.5.16 For a subset A of (X, τ), the following properties are equivalent.

a) A is Δ^*-closed.

b) δcl(A) □ δg-ker(A).

c) i) δcl(A) ∩ B □ A where B □ δGC(X,τ).

ii)δcl(A) ∩ V □ δg-ker(A) where V □ Δ^*O(X, τ).

Proof: (a) ⟹ (b) : Let x □ δg-ker(A). Then there exists a set U □ δGO(X, τ) such that x □ U and A □ U. Since A is Δ^*-closed, δcl(A) □ U and therefore x □ δcl(A). Hence δcl(A) □ δg-ker(A).

(b) ⟹ (c) : i) First we claim that δg-ker(A) ∩ B □ A, for a δg-closed set B.

Let x □ δg-ker(A) ∩ B and assume that x □ A. Since the set X − {x} □ δGO(X, τ) and A □ X −{x}, δg-ker(A) □ X − {x}. Then we have x □ X − {x} which is a contradiction.

Thus δg-ker(A) ∩ B □ A for every δg-closed set B. Hence by using (b), δcl(A) ∩ B □ δg-ker(A) ∩ B □ A.

ii) Proof follows similar to the above proof.

(c) ⟹ (b) : By Theorem 2.3.1 and (c),

δcl(A) = δcl(A) ∩ X = δcl(A) ∩ [δGC(X, τ) □ Δ^*O(X, τ)]

= [δcl(A) ∩ δGC(X, τ)] □ [δcl(A) ∩ Δ^*O(X, τ)]

□ A □ δg-ker(A) = δg-ker(A).

That is δcl(A) □ δg-kernel(A)holds.

(b) ⟹ (a) : Let U □ δGO(X, τ) such that A □ U. Then we have that δg-ker(A) □ U and so by (b) δcl(A) □ U. Therefore A is Δ^*-closed.

Corollary 2.5.17 Let P = { A − δcl(A) ∩ V □ δg-ker(A) where V is a Δ^*-open set}. Then

a) If $\bigcap_{i\ \varepsilon}$ A_i □ P and A_i is Δ^*-closed in (X, τ) for each i, then ∩ A_i is Δ^*-closed.

b) If P = P(X) and A_i is Δ^*-closed in X for each i □ ε then $\bigcap_{i\ \varepsilon}$ A_i is Δ^*-closed in (X, τ).

c) If δcl(A_i) ∩ V □ A, where V is a Δ^*-open set and A_i is Δ^*-closed in (X,τ) for each i □ ε, then $\bigcap_{i\ \varepsilon}$ A_i is Δ^*-closed in (X, τ).

Proof : a) By Theorem 2.5.16, δcl(A_i) ∩ B □ A_i for a δg-closed set B and each i □ ε. Then we have δcl($\bigcap_{i\ \varepsilon}$ A_i) ∩ B □ $\bigcap_{i\ \varepsilon}$ A_i. Using assumption and Theorem 2.5.16(c), $\bigcap_{i\ \varepsilon}$ A_i is Δ^*-closed.

The proof of (b) and (c) follow from (a).

Theorem 2.5.18 For each x □ X, x □ Δ^*cl(A) if and only if U ∩ A ≠ ϕ for every Δ^*-open set U in (X, τ) containing x.

Proof : (Necessary) : Let x □ Δ^*cl(A). Suppose that there exists a Δ^*-open set U in (X, τ) containing x such that U ∩ A = ϕ. Hence X − U is Δ^*-closed in (X, τ) containing A which implies that $\Delta^*cl(A)$ □ X − U. Hence x □ $\Delta^*cl(A)$ which is a contradiction. Hence U ∩ A ≠ ϕ.

(Sufficiency) : Let us assume that U ∩ A ≠ ϕ for every Δ^*-open set U in (X, τ) containing x. Suppose that x □ $\Delta^*cl(A)$. By the definition of $\Delta^*cl(A)$, these exists a Δ^*-closed set U in (X, τ) containing A such that x □ U. Hence X − U is Δ^*-open set in (X, τ) containing x. Since A □ U, we have (X − U) ∩ A = ϕ which is a contradiction. Hence x □ $\Delta^*cl(A)$.

Theorem 2.5.19 Every Δ^*-closed set is δ-closed in (X, τ) if and only if $\Delta^*\tau^\# = \tau_\delta$.

Proof : Let every Δ^*-closed set in in (X, τ) is δ-closed Then A □ $\Delta^*\tau^\#$ implies A □ τ_δ. Therefore $\Delta^{□}\tau^\#$ □ τ_δ. The other way τ_δ □ $\Delta^{□}\tau^\#$ is always true. Therefore $\Delta^{□}\tau^\# = \tau_\delta$. Conversely, let $\Delta^{□}\tau^\# = \tau_\delta$. Let A be a Δ^*-closed set. Therefore $\Delta^*cl(A) = A$. So (X−A) □ $\Delta^{□}\tau^\#$ which implies (X−A) □ τ_δ. That is A is δ-closed.

In a semi regular space closedness coincides with δ-closedness. That is $\tau_\delta = \tau$. The following corollary is hence derived.

Corollary : Every Δ^*-closed set is δ-closed in semi regular space if and only if $\Delta^*\tau^\# = \tau$.

Theorem 2.5.20 If a subset A is Δ^*-closed in X then δcl(A) − A is Δ^*-open.

Proof : Suppose that A is Δ^*-closed in X . Let F □ δcl(A) − A and F be a δg-closed set. Since A is Δ^*-closed, δcl(A) − A does not contain a non empty δg-closed set (by Theorem 2.3.4). Thus F = ϕ. Therefore F □ δInt[δcl(A) − A]. Hence δcl(A) − A is Δ^*-open.

Theorem 2.5.21 The intersection of a Δ^*-open set and a δ-open set is always Δ^*-open.

Proof : It follows from Theorem 2.3.7