Chapter 7

Δ^*-Locally Closed sets in Topological Spaces

7.1 Introduction

A subset of a topological space is said to be a locally closed set (Bourbaki, 1966) if it is the intersection of an open set and a closed set. Ganster and Reilly (1989), defined LC-continuity and LC-irresolute map by using locally closed sets. Further the concepts of generalized locally closed (briefly, glc) sets, GLC-continuous maps and GLC-irresolute maps were introduced by Balachandran et.al.,(1996). Since then several topologists contributed their study to the development of generalizations of locally closed sets and locally continuous maps in topological spaces. In this chapter, three types of locally closed sets namely, Δ^*lc-sets, Δ^*lc*-sets and Δ^*lc**-sets are introduced and their interrelations and properties are studied. Also the nature of these sets in three different spaces called Δ^*-door space, Δ^*-submaximal space and Δ^{**}-submaximal space are discussed. Furthermore Δ^*-locally continuous and Δ^*-locally irresolute maps are defined. Their properties using ${}_{\Delta^*}T_{\delta}$-space and some important results under composition of mappings are also analyzed in this chapter.

7.2 Δ^*-Locally Closed Sets

Definition 7.2.1 Let A be a subset of (X, τ). Then A is called a

a) Δ^*-locally closed set, i.e., **Δ^*lc-set** if there exists a Δ^*-open set U and a Δ^*-closed set F of (X, τ) such that A = U ∩ F.

b) Δ^*lc*-set if there exists a Δ^*-open set U and a δ-closed set F of (X, τ) such that A = U ∩ F.

c) Δ^*lc^{**}-set if there exists a δ-open set U and a Δ^*-closed set F of (X, τ) such that A = U ∩ F.

The collection of all Δ^*lc (resp., Δ^*lc^*-sets, Δ^*lc^{**}) sets of (X, τ) is denoted by $\Delta^*LC(X, \tau)$ (respectively $\Delta^*LC^*(X, \tau)$, $\Delta^*LC^{**}(X, \tau)$).

Proposition 7.2.2 For a topological space (X, τ), the following inclusions are true.

a) $\delta LC(X, \tau) \square \Delta^*LC(X, \tau)$

b) $\delta LC(X, \tau) \square \Delta^*LC^{**}(X, \tau) \square \Delta^*LC(X, \tau)$

c) $\delta LC(X, \tau) \square \Delta^*LC^*(X, \tau) \square \Delta^*LC(X, \tau)$

Proof : It follows from the fact that every δ-closed set is Δ^*-closed set in (X, τ).(Proposition 2.2.2).

Remark 7.2.3 The reverse implications of the above results are not true as seen from the following example.

Counter Example 7.2.4 Let {a, b, c} and τ = {ϕ , X, {a}}. Then we have

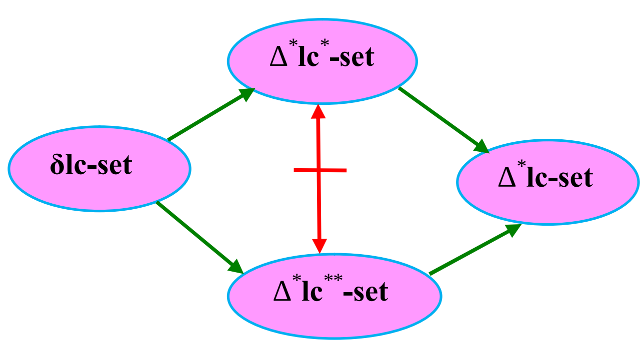
$\delta LC(X, \tau)$ = {ϕ , X};

$\Delta^*LC(X, \tau)$ = { ϕ , X, {a}, {b, c}};

$\Delta^*LC^*(X, \tau)$ = {ϕ , X, {a}};

$\Delta^*LC^{**}(X, \tau)$ = { ϕ , X, {b, c}}.

Remark 7.2.5 From the above Proposition 7.2.2 and Counter example 7.2.4 it can be seen that $\Delta^*LC^*(X, \tau)$ and $\Delta^*LC^{**}(X, \tau)$ are independent.



Proposition 7.2.6 If $\Delta^*C(X, \tau)$ □ $\delta GO(X, \tau)$ then $\Delta^*LC^{**}(X, \tau) = \delta LC(X, \tau)$.

Proof : By Proposition 7.2.2(b), $\delta LC(X, \tau)$□$\Delta^*LC^{**}(X, \tau)$ ------------(1).

Let A be a Δ^*lC^{**}-set. Then there exists a δ-open set U and Δ^*-closed set F of X such that $A = U \cap F$. By the assumption F is δg-open. By Proposition 2.3.5, F is δ-closed set of X. Therefore A is δlc-set and $\Delta^*LC^{**}(X, \tau)$ □ $\delta LC(X, \tau)$ -----------------(2).

Hence from (1) and (2), $\Delta^*LC^{**}(X, \tau) = \delta LC(X, \tau)$.

Definition 7.2.7 A topological space (X, τ) is said to be a Δ^***-door space** if each subset of (X, τ) is either Δ^*-open or Δ^*-closed in (X, τ).

Example 7.2.8 Let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}\}$. Then (X, τ) is a Δ^*-door space.

Remark 7.2.9 If (X, τ) is a Δ^*-door space then $\Delta^*LC(X, \tau) = P(X)$.

Proposition 7.2.10 Let (X,τ) be a $T_{\Delta^*\delta}$-space. Then the following results hold:

a) $\Delta^*LC(X, \tau) = \delta LC(X, \tau) = \Delta^*LC^*(X, \tau) = \Delta^*LC^{**}(X, \tau)$

b) $\Delta^*LC(X, \tau) \subseteq GLC(X, \tau)$

c) $\Delta^*LC(X, \tau) \subseteq GLSC(X, \tau)$

Proof : (a) Since (X,τ) is a $T_{\Delta^*\delta}$-space, every Δ^*-open set is δ-open and every Δ^*-closed set δ-closed. Hence we have $\Delta^*LC(X, \tau) \subseteq \delta LC(X, \tau)$. By Proposition 7.2.2(a). $\delta LC(X, \tau) \subseteq \Delta^*LC(X, \tau)$. Hence $\Delta^*LC(X, \tau) = \delta LC(X, \tau)$.

(b) and **(c)** follow from **(a)**, since for any space (X, τ), $\delta LC(X, \tau) \subseteq LC(X, \tau) \subseteq GLC(X, \tau)$ and $\delta LC(X, \tau) \subseteq LC(X, \tau) \subseteq GLSC(X, \tau)$.

Proposition 7.2.11 For a subset A of (X, τ), if A □ $\Delta^*LC(X, \tau)$ then $A = U \cap \Delta^*cl(A)$ for some Δ^*-open set U in (X, τ).

Proof : Let A □ $\Delta^*LC(X, \tau)$. Then there exists a Δ^*-open set U and a Δ^*-closed set F of X such that A = U ∩ F. Since A □ U and A □ $\Delta^*cl(A)$, A □ U ∩ $\Delta^*cl(A)$-----------(1). Conversely by the definition of Δ^*-closure, $\Delta^*cl(A)$ □ F and hence U ∩ $\Delta^*cl(A)$ □ U ∩ F = A----------- (2). Therefore from (1) and (2), A = U ∩ $\Delta^*cl(A)$.

Proposition 7.2.12 For a subset A of (X, τ), if A □ $\Delta^*LC^{**}(X, \tau)$ then A = U ∩ $\Delta^*cl(A)$ for some δ-open set U in (X, τ).

Proof : Let A □ $\Delta^*LC^{**}(X, \tau)$. Then by the definition, A = U ∩ F where U is a δ-open set and F is a Δ^*-closed set containing A. Since F is a Δ^*-closed set, $\Delta^*cl(A)$ □ F which implies that U ∩ $\Delta^*cl(A)$ □ U ∩ F = A. Since A □ U and A □ $\Delta^*cl(A)$, we have A □ U ∩ $\Delta^*cl(A)$. Therefore A = U ∩ $\Delta^*cl(A)$ where U is a δ-open set in (X, τ).

Definition 7.2.13 A subset A of (X, τ) is called a Δ^***-dense set** if $\Delta^*cl(A) = X$.

Example 7.2.14 Let X = {a, b, c} and τ = {ϕ , X, {a}, {a, b}}. Then the Δ^*-dense sets are X and {a, b}.

Proposition 7.2.15 In a topological space (X, τ), every Δ^*-dense set is δ-dense set but not conversely.

Proof : Let A be a Δ^*-dense set in (X, τ). Then Δ^*cl(A) = X. Since Δ^*cl(A) □ δcl(A), δcl(A) = X. Hence A is δ-dense.

Counter Example 7.2.16 Let X = {a, b, c} and τ = {ϕ , X, {a}, {a, b}}. Then the subset {c} is δ-dense in (X, τ) but not Δ^*-dense set in (X, τ) since Δ^*cl{c} = {c} ≠ X whereas Δ^*cl{a, b} = X.

Proposition 7.2.17 In a topological space (X, τ), every πg-dense set is Δ^*-dense set but not conversely.

Proof : Let A be a πg-dense set in (X, τ). Then πgcl(A) = X. Since πgcl(A) □ Δ^*cl(A), we have Δ^*cl(A) = X. Hence A is Δ^*-dense.

Counter Example 7.2.18 Let X = {a, b, c} and τ = {ϕ , X, {a}}. Then the subset {a, b} is Δ^*-dense but not πg-dense since πgcl[{a, b}] = {a,} ≠ X.

Definition 7.2.19 A topological space (X, τ) is called a **Δ^*-submaximal** (resp.,Δ^{**}-submaximal) space if every Δ^*-dense (resp., δ-dense) subset is Δ^*-open in (X, τ).

Proposition 7.2.20 Every Δ^*-submaximal space is a πg-submaximal space but not conversely.

Proof : Let (X, τ) be a Δ^*-submaximal space and A is πg-dense subset of (X, τ). Since every πg-dense subset is Δ^*-dense (Proposition 7.2.19), A is Δ^*-dense and A is Δ^*-open. We know that every Δ^*-open set is πg-open set (Proposition 2.5.8). Therefore A is πg-open. Hence (X, τ) is a πg-submaximal space.

Counter Example 7.2.21 Let X = {a, b, c} and τ = {ϕ , X, {a}}. Then (X, τ) is πg-submaximal space but not a Δ^*-submaximal space since the subset {a, b} □ (X, τ) is Δ^*-dense but not Δ^*-open in (X, τ).

Proposition 7.2.22 Every Δ^{**}-submaximal space is a Δ^*-submaximal space but not conversely.

Proof : Let (X, τ) be a Δ^{**}-submaximal space and A be a Δ^*-dense subset of (X, τ). By Proposition 7.2.17, A is δ-dense in (X, τ). By the assumption, A is Δ^*-open and hence (X, τ) is a Δ^*-submaximal space.

Counter Example 7.2.23 Let X = {a, b, c} and τ = {ϕ , X, {a, b}}. Then (X, τ) is a Δ^*-submaximal as the Δ^*-dense subsets X and {a, b} are Δ^*-open. But (X, τ) is not a Δ^{**}-submaximal since the subset {b, c} in (X, τ) is δ-dense but not Δ^*-open in (X, τ).

Remark 7.2.24 From the Proposition 7.2.24 and Proposition 7.2.22 we have

Δ^{**}-submaximal $\longrightarrow$ Δ^{*}-submaximal $\longrightarrow$ πg-submaximal

Proposition 7.2.25 Let f : (X, τ) $\longrightarrow$ (Y, σ) be a Δ^*-irresolute map. Then the following statements are true.

a) If B $\in$ Δ^*LC(Y, σ) then f^{-1}(B) $\in$ Δ^*LC(X, τ).

b) If B $\in$ Δ^*LC(Y, σ) then f^{-1}(B) $\in$ gδLC(X, τ).

c) If B $\in$ gδLC(Y, σ) and (Y, σ) is a $g\delta T_{\Delta^*}$-space then f^{-1}(B) $\in$ Δ^*LC(X, τ).

Proof : Let f : (X, τ) $\longrightarrow$ (Y, σ) be a Δ^*-irresolute map.

a) Let B $\in$ Δ^*LC(Y, σ). Then there exists a Δ^*- open set G and Δ^*-closed set H such that B = G ∩ H which implies that f^{-1} (B) = f^{-1}(G) ∩ f^{-1}(H). Since f is Δ^*-irresolute, f^{-1}(G) and f^{-1}(H) are Δ^*-open and Δ^*-closed respectively. Hence f^{-1} (B) $\in$ Δ^*LC(X, τ).

b) Let B $\in$ Δ^*LC(Y, σ). Then there exists a Δ^*- open set G and Δ^*-closed set H such that B = G ∩ H which implies that f^{-1}(B) = f^{-1}(G) ∩ f^{-1}(H). Since f is Δ^*-irresolute, f^{-1}(G) and f^{-1}(H) are Δ^*-open and Δ^*-closed respectively. Since every Δ^*-closed is gδ-closed (Proposition 2.2.8), f^{-1}(G) and f^{-1}(H) are gδ-open and gδ-closed respectively. Therefore f^{-1}(B) $\in$ gδLC(X, τ).

c) Let B $\in$ gδLC(Y, σ). Then there exists a gδ-open set G and a gδ-closed set H in (Y, σ) such that B = G ∩ H. Since (Y, σ) is a

$g\delta T_{\Delta^*}$-space, G is Δ^*-open and Δ^*-closed also. Then B □ $\Delta^*LC(Y, \sigma)$. Hence by the result (a), $f^{-1}(B)$ □ $\Delta^*LC(X, \tau)$.

Proposition 7.2.26 The following results are true for any two subsets A and B of (X, τ).

a) If A,B □ $\Delta^*LC^*(X, \tau)$, then $A \cap B$ □ $\Delta^*LC^*(X, \tau)$.

b) If A □ $\Delta^*LC(X, \tau)$ and B is Δ^*-open then $A \cap B$ □ $\Delta^*LC(X, \tau)$.

c) If A □ $\Delta^*LC^*(X, \tau)$ and B is Δ^*-open then $A \cap B$ □ $\Delta^*LC^*(X, \tau)$.

d) If A □ $\Delta^*LC^{**}(X, \tau)$ and B is Δ^*-open then $A \cap B$ □ $\Delta^*LC(X, \tau)$.

Proof : a) Follows from the fact that the intersection of two Δ^*-open sets is Δ^*-open and the intersection of two δ-open sets is δ-open.

b) and **c)** Follows from the fact that the intersection of two Δ^*-open sets is Δ^*-open.

d) Follows from the fact that the intersection of δ-open and Δ^*-open sets is Δ^*-open. (by Theorem 2.5.21).

Theorem 7.2.27 Suppose that $X = \bigcup\{Z_i :$ i □ □ and □ is finite$\}$ and let A be a subset of X. If $A \cap Z_i$ □ $\Delta^*LC^{**}(X, \tau)$ for each i, then A□ $\Delta^*LC^{**}(X, \tau)$.

Proof : $A = A \cap X \Longrightarrow A = A \cap (\bigcup\{Z_i :$ i □ □ $\}) = \bigcup\{A \cap Z_i :$ i □ □$\}$. Since $A \cap Z_i$ □ $\Delta^*LC^{**}(X, \tau)$, we have $A \cap Z_i = U_i \cap F_i$ where U_i is δ-open and F_i is

Δ^*-closed. Hence $A = \bigcup\{ U_i \cap F_i : i$ □ □ $\} = \bigcup\{U_i\} \cap \{\bigcup F_i\} = U \cap F$ where U is δ-open and F is Δ^*-closed. Hence $A \in \Delta^*LC^{**}(X, \tau)$.

7.3 Δ^*-Locally Continuous and Δ^*-Locally Irresolute Functions

In this section Δ^*LC-continuous maps, Δ^*LC^*-continuous maps Δ^*LC^{**}-continuous maps, Δ^*LC-irresolute maps, Δ^*LC^*-irresolute maps and Δ^*LC^{**}-irresolute maps are defined and their properties are analysed.

Definition 7.3.1 Let $f : (X, \tau) \to (Y, \sigma)$ be a map. Then f is called

a) Δ^***LC-continuous** if $f^{-1}(V) \in \Delta^*LC(X, \tau)$ for each $V \in \sigma$.

b) Δ^***LC*****-continuous** if $f^{-1}(V) \in \Delta^*LC^*(X, \tau)$ for each $V \in \sigma$.

c) Δ^***LC******-continuous** if $f^{-1}(V) \in \Delta^*LC^{**}(X, \tau)$ for each $V \in \sigma$.

Proposition 7.3.2 Let $f : (X, \tau) \to (Y, \sigma)$ be a map. Then the following statements are true.

a) If f is δLC-continuous then it is Δ^*LC-continuous, Δ^*LC^*-continuous and Δ^*LC^{**}continuous.

b) If f is Δ^*LC^*-continuous or Δ^*LC^{**}-continuous then it is Δ^*LC-continuous.

Proof : a) Follows from the Proposition 7.2.2 and by the fact that every δlc-set is Δ^*lc-set, Δ^*lc^*-set and Δ^*lc^{**}-set.

b) Since every Δ^*lc^*-set is Δ^*lc-set and every Δ^*lc^{**}-set is Δ^*lc-set, the proof follows.

The converse of the above proposition is not true which can be seen from the following example.

Counter Example 7.3.3 a) Let X = {a, b, c} = Y and τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a ,b}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*LC-continuous, Δ^*LC*-continuous and Δ^*LC**-continuous but not δLC-continuous since for the open set {a, b} □ (Y, σ), f^{-1}{a, b} = {a, b} □ δLC(X, τ).

b) Let X = {a, b, c} = Y and τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {b, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*LC-continuous but it is neither Δ^*LC*-continuous nor Δ^*LC**-continuous, since for the open set {b, c} □ (Y, σ), f^{-1}{b, c} = {b, c} □ Δ^*LC*(X, τ) and for the open set {a} □ (Y, σ), f^{-1}{a} = {a} □ Δ^*LC**(X, τ).

Definition 7.3.4 Let f : (X, τ) → (Y, σ) be a map. Then f is called

a) Δ^***LC-irresolute** if f^{-1}(V) □ Δ^*LC(X, τ) for each V □ Δ^*LC(Y, σ).

b) Δ^***LC*-irresolute** if f^{-1} (V) □ Δ^*LC*(X, τ) for each V □ Δ^*LC*(Y, σ).

c) Δ^***LC**-irresolute** if f^{-1}(V) □ Δ^*LC**(X, τ) for each V □ Δ^*LC**(Y, σ).

Proposition 7.3.5 Let f : (X, τ) → (Y, σ) be a Δ^*-irresolute map. Then f is Δ^*LC-irresolute but not conversely.

Proof : It follows by the definitions of Δ^*-irresolute map, Δ^*LC-irresolute map and by the Proposition 7.2.29 (a).

Counter Example 7.3.6 Let X = {a, b, c} = Y and τ = {ϕ , X, {a}, {b}, {a, b}, {a, c}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = a, f(b) = c, f(c) = b. Then f is Δ^*LC-irresolute but not Δ^*-irresolute since for the Δ^*-open set {b} □ (Y, σ), f^{-1}{b} = {c} is not Δ^*-open in (X, τ).

Proposition 7.3.7 If f : (X, τ) → (Y, σ) is Δ^*LC-continuous (Δ^*LC^*-continuous (or) Δ^*LC^{**}-continuous) and (X, τ) is a ${}_{\Delta^*}T_{\delta}$-space then f is δLC-continuous.

Proof : Let f : (X, τ) → (Y, σ) be Δ^*LC-continuous map and V be an open set in (Y, σ). Since f is Δ^*LC- continuous (Δ^*LC^*-continuous (or) Δ^*LC^{**}-continuous), f^{-1} (V) is Δ^*lc-set (Δ^*lc^*-set, Δ^*lc^{**}-set) in (X, τ). Since (X, τ) is ${}_{\Delta^*}T_{\delta}$-space, by the Proposition 7.2.10 (a), we have f^{-1}(V) is δlc-set in (X, τ). Hence f is δLC-continuous.

Remark 7.3.8 The Δ^*LC-irresolute maps and Δ^*LC-continuous maps are independent as seen from the following examples.

Counter example 7.3.9 Let X = {a, b, c} = Y and τ = {ϕ , X, {a}, {b, c}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = b, f(c) = a. Then f is Δ^*LC-continuous but not Δ^*LC-irresolute since for the set {a} ∈ Δ^*LC(Y, σ), f^{-1}{a} = {c} is not Δ^*-open in Δ^*LC(X, τ).

Counter example 7.3.10 Let X = {a, b, c} = Y and τ = {ϕ , X, {a}, {b, c}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*LC-irresolute but not Δ^*LC-continuous since for the open set {a, b} ∈ (Y, σ), f^{-1}{a, b} = {a, b} is not in Δ^*LC(X, τ).

Proposition 7.3.11 Any map defined on a δ-door space is Δ^*LC-irresolute.

Proof : Let f : (X, τ) → (Y, σ) be a map where (X, τ) is a δ-door space and (Y, σ) is any space. Let A ∈ Δ^*LC(Y, σ). Then by the assumption on

(X, τ), $f^{-1}(A)$ is either δ-open or δ-closed. Since every δ-closed is Δ^*-closed, $f^{-1}(A) \in \Delta^*LC(X, \tau)$. Hence f is Δ^*LC-irresolute.

Proposition 7.3.12 Let f : (X, τ) → (Y, σ) be a map. If f is Δ^*LC-continuous and Contra δ-continuous where (Y, σ) is a Δ^*T_δ-space then f is a Δ^*LC-irresolute map.

Proof : Let f : (X, τ) → (Y, σ) be Δ^*LC-continuous and Contra δ-continuous map. Let (Y, σ) be a Δ^*T_δ-space. Let $G \in \Delta^*LC(Y, \sigma)$. Then there exists a Δ^*-open set U and a Δ^*-closed set F of (Y, σ) such that G = U ∩ F. Since (Y, σ) is a Δ^*T_δ-space, U is δ-open and F is δ-closed. As every δ-open set is open, U is open. Since f is Δ^*LC-continuous, $f^{-1}(U)$ is Δ^*lc-set. Since f is Contra δ-continuous, $f^{-1}(F)$ is δ-open and hence Δ^*-open. Also $f^{-1}(G) = f^{-1}(U) \cap f^{-1}(F)$. By Proposition 7.2.30 (b), $f^{-1}(G)$ is a Δ^*lc-set in (X, τ). Therefore f is Δ^*LC-irresolute.

Proposition 7.3.13 Let f : (X, τ) → (Y, σ) be a map. If f is Δ^*LC-continuous and contra Δ^*-irresolute where (Y, σ) is a Δ^*T_δ -space then f is Δ^*LC-irresolute.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*LC-continuous and contra Δ^*-irresolute and let (Y, σ) is a Δ^*T_δ -space. Let $G \in \Delta^*LC(Y, \sigma)$. Then there exists a Δ^*-open set U and a Δ^*-closed set F in (Y, σ) such that G = U ∩ F. Since (Y, σ) is a Δ^*T_δ -space, U is δ-open and F is δ-closed in (Y, σ). Also $f^{-1}(G) = f^{-1}(U) \cap f^{-1}(F)$. Since f is Δ^*LC-continuous, $f^{-1}(U)$ is a Δ^*lc-set. Since every δ-closed set is Δ^*-closed, F is

Δ^*-closed in (Y, σ). Since f is contra Δ^*-irresolute, f^{-1}(F) is Δ^*-open in (X, τ). Therefore by Proposition 7.2.30(b), f^{-1}(G) is a Δ^*lc-set in (X, τ). Hence f is Δ^*LC-irresolute.

Proposition 7.3.14 Let f : (X, τ) → (Y, σ) be a map. If f is Δ^*LC^*-continuous and contra δ-continuous where (Y, σ) is a $\Delta^* T_\delta$ -space then f is Δ^*LC^*-irresolute map.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*LC^*-continuous and contra δ-continuous in which (Y, σ) is a $\Delta^* T_\delta$ -space. Let G$\in\Delta^*LC^*$(Y, σ). Then there exists a Δ^*-open set U and a δ-closed set F in (Y, σ) such that G = U ∩ F. Since (Y, σ) is a $\Delta^* T_\delta$ -space, U is δ-open in (Y, σ). Since f is contra δ-continuous, f^{-1}(F) is δ-open in (X, τ). Since every δ-open set is Δ^*-open, f^{-1}(F) is Δ^*-open in (X, τ). Therefore by Proposition 7.2.30 (c), f^{-1}(G) is a Δ^*lc^*-set in (X, τ). Hence f is Δ^*lc^*-irresolute.

Proposition 7.3.15 Let (X, τ) be a $\Delta^* T_\delta$ -space. If (X, τ) is a Δ^*-submaximal space then every map having (X, τ) as its domain is Δ^*LC-irresolute.

Proof : Let (X, τ) be a $\Delta^* T_\delta$ -space and a Δ^*-submaximal space. Let f : (X, τ) → (Y, σ) be any map. Then by Proposition 7.2.27, P(X) = Δ^*LC(X, τ). If U is Δ^*lc-set of (Y, σ) then f^{-1}(U) $\in$ P(X) = Δ^*LC(X, τ) and hence f is Δ^*LC-irresolute.

Proposition 7.3.16 Let f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) be any two maps. Then

a) (g ∘ f) : (X, τ) → (Z, η) is Δ^*LC-irresolute (resp. Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute) if f is Δ^*LC-irresolute (resp. Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute) and g is also Δ^*LC-irresolute.(resp. Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute).

b) (g ∘ f) : (X, τ) → (Z, η) is Δ^*LC- continuous if f is Δ^*LC-irresolute and g is Δ^*LC-continuous.

Proof : a) Let V □ $\Delta^*LC(Z, \eta)$ (resp., V □ $\Delta^*LC^*(Z, \eta)$, V □ $\Delta^*LC^{**}(Z, \eta)$). Since g is Δ^*LC-irresolute (resp., Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute), $g^{-1}(V)$ □ $\Delta^*LC(Y, \sigma)$ (resp., $g^{-1}(V)$ □ $\Delta^*LC^*(Y, \sigma)$, $g^{-1}(V)$ □ $\Delta^*LC^{**}(Y, \sigma)$). Since f is Δ^*LC-irresolute (resp., Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute), $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1}$ □ $\Delta^*LC(X, \tau)$ (resp., $(g \circ f)^{-1}$ □ $\Delta^*LC^*(X, \tau)$, $(g \circ f)^{-1}$ □ $\Delta^*LC^{**}(X, \tau)$). Therefore $(g \circ f)$ is Δ^*LC-irresolute (resp., Δ^*LC^*-irresolute, Δ^*LC^{**}-irresolute).

b) Let V be any open set in (Z, η). Since g is Δ^*LC-continuous, g □ $\Delta^*LC(Y, \sigma)$. Since f is Δ^*LC-irresolute, $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1}$ □ $\Delta^*LC(X, \tau)$. Therefore $(g \circ f)$ is Δ^*LC-continuous.

Remark 7.3.17 The composition of two Δ^*LC- continuous maps need not be a Δ^*LC- continuous as seen from the following example.

Counter example 7.3.18 Let f : (X, τ) → (Y, σ) be a map defined by f(a) = c, f(b) = b, f(c) = a and g : (Y, σ) → (Z, η) be a map defined by g(a) = c, g(b) = b, g(c) = a where τ = { ϕ , X, {a}, {b, c}}, σ = { ϕ , Y, {a, b}} and η = { ϕ , z, {a}, {a, b}, {a, c}}. Let $(g \circ f)$: (X, τ) → (Z, η) be the composition map defined by $(g \circ f)(a) = a$, $(g \circ f)(b) = b$ and $(g \circ f)(c) = c$. Then both f and g are Δ^*LC- continuous but the composition map $(g \circ f)$ is not Δ^*LC- continuous, since for the subset {a, b} □ (Z, η), $(g \circ f)^{-1}\{a, b\} = \{a, b\}$ □ $\Delta^*LC(X, \tau)$.

Proposition 7.3.19 For any two maps f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η), the following statements are true.

a) $(g \circ f)$: (X, τ) → (Z, η) is Δ^*LC- continuous if f is Δ^*LC-irresolute and g is δLC-continuous (resp., Δ^*LC^*-continuous, Δ^*LC^{**}-continuous).

b) $(g \circ f)$: (X, τ) → (Z, η) is Δ^*LC- continuous (resp. Δ^*LC^*-continuous) if f is Δ^*LC-irresolute (resp. Δ^*LC^*-irresolute) and g is Δ^*-continuous.

c) $(g \circ f) : (X, \tau) \to (Z, \eta)$ is Δ^*LC- continuous (resp., Δ^*LC^*-continuous, Δ^*LC^{**}-continuous) if f is Δ^*LC- continuous and g is δ-continuous.

Proof : a) Let V be any open set in (Z, η). Since g is δLC-continuous (resp., Δ^*LC^*-continuous, Δ^*LC^{**}-continuous), $g^{-1}(V) \square \delta LC(Y, \sigma)$. By Proposition 7.2.2, $g^{-1}(V) \square \Delta^*LC(Y, \sigma)$. Since f is Δ^*LC-irresolute, $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1} \square \Delta^*LC(X, \tau)$. Hence $(g \circ f)$ is Δ^*LC- continuous.

b) Let V be any open set in (Z, η). Since g is Δ^*- continuous, $g^{-1}(V)$ is Δ^*-open in (Y, σ) and hence $g^{-1}(V)$ is Δ^*lc-set (resp., Δ^*lc^*-set) in (Y, σ). Since f is Δ^*LC-irresolute (resp., Δ^*LC^*-irresolute), $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1} \square \Delta^*LC(X, \tau)$. Hence $(g \circ f)$ is Δ^*LC- continuous.(resp., Δ^*LC^*-continuous).

c) Let V be any open set in (Z, η). Since g is δ-continuous, $g^{-1}(V)$ is δ-open in (Y, σ). Since every δ-open is Δ^*-open, $g^{-1}(V)$ is Δ^*-open in (Y, σ) and hence $g^{-1}(V)$ is Δ^*lc-set(resp., Δ^*lc^*-set, Δ^*lc^{**}-set) in (Y, σ). Since f is Δ^*LC-continuous.(resp., Δ^*LC^*-continuous, Δ^*LC^{**}-continuous), $f^{-1}[g^{-1}(V)] = (g \circ f)^{-1} \square \Delta^*LC(X, \tau)$ (resp., $\square \Delta^*LC^*(X, \tau)$, $\square \Delta^*LC^{**}(X, \tau)$). Hence $(g \circ f)$ is Δ^*LC- continuous. (resp., Δ^*LC^*-continuous, Δ^*LC^{**}-continuous).

Definition 7.3.20 Let X be a set such that $X = A \square B$ for some $A \square X$ and $B \square X$. Let $f : A \to Y$ and $h : B \to Y$ be any two functions. Then f and h are said to be **compatible** if $f(x) = h(x)$ for every $x \square A \cap B$.

If $f : A \to Y$ and $h : B \to Y$ are compatible where $X = A \square B$ then the function $(f \nabla h) : X \to Y$ defined by $(f \nabla h)(x) = f(x), \forall x \in A$ and $(f \nabla h)(x) = h(x), \forall x \in B$ is called the **combination of f and h**.

Theorem 7.3.21 Let $X = A \square B$ where A and B are Δ^*-closed sets of (X, τ) and $f : (X, \tau_A) \to (Y, \sigma)$ and $h : (B, \tau_B) \to (Y, \sigma)$ are compatible functions. If f and h are $\Delta^{*}LC^{**}$-continuous (resp., $\Delta^{*}LC^{**}$-irresolute) then $(f \nabla h): X \to Y$ is $\Delta^{*}LC^{**}$-continuous (resp., $\Delta^{*}LC^{**}$-irresolute).

Proof : Let V $\in$ σ (resp.,V $\in$ $\Delta^{*}LC^{**}(Y, \sigma)$). Then $(f \nabla h)^{-1}(V) \cap B = h^{-1}(V)$ hold. By the assumption $(f \nabla h)^{-1}(V) \cap A \in \Delta^{**}LC^{**}(A, \tau_B)$ and $(f \nabla h)^{-1}(V) \cap B \in \Delta^{**}LC^{**}(B, \tau_B)$. By Proposition 7.2.31, $(f \nabla h)^{-1}(V) \in \Delta^{**}LC^{**}(X, \tau)$ and hence $(f \nabla h): X \to Y$ is $\Delta^{*}LC^{**}$-continuous (resp., $\Delta^{*}LC^{**}$-irresolute).

The following example shows that the **Pasting Lemma** for $\Delta^{*}LC^{*}$-continuous functions need not be true.

Example 7.3.22 Let X = {a, b, c}, τ = {ϕ, {a}, {a,b}}, σ = {ϕ, Y, {a}}. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined by f(a) = a, f(b) = b and f(c) = a. Then $\Delta^{*}LC^{*}(X, \tau)$ = P(X)−{a, c}. Let A = {a, c} and B = { b, c}. Then {A, B} forms a Δ^{*}-closed cover for X and $f|A : (A, \tau_A) \to (Y, \sigma)$ and $g|B : (B, \tau_B) \to (Y, \sigma)$ are both $\Delta^{*}LC^{*}$-continuous but f =(f | A)□ (f | A) is not $\Delta^{*}LC^{*}$-continuous.

The following theorem shows that the **Pasting Lemma** holds good for $\Delta^{*}LC^{**}$-continuous (resp., $\Delta^{*}LC^{**}$-irresolute) functions.

Theorem 7.3.23 If f: (X,τ) ⟶ (Y,σ) is $\Delta^{*}LC^{**}$-continuous and a subset B is Δ^{*}-closed in (X,τ), then the restriction of f to B, say $f|B : (B, \tau_B) \to (Y, \sigma)$ is $\Delta^{*}LC^{**}$-continuous.

Proof : Let V be an open set of (Y,σ). Then $f^{-1}(V) = G \cap F$ for some δ-open set G and Δ^{*}-closed set F of (X,τ). We have $(f|B)^{-1}(V) = (G \cap B) \cap (F \cap B)$ where (G∩B) is a δ-open set in (B, τ_B) and (F∩B) is a Δ^{*}-closed set in (B, τ_B) by Theorem 2.3.7. Hence $(f|B)^{-1}(V)$ □ $\Delta^{*}LC^{**}(B, \tau_B)$. This implies that $f|B : (B, \tau_B) \to (Y, \sigma)$ is $\Delta^{*}LC^{**}$-continuous.