

## CHAPTER II

### Vertex corona product of Path and Cycle graph with Origami and Bistar graphs

In this chapter, the b-chromatic number of Vertex corona product of two graphs  $G$  and  $H$  both are discussed. Also the b-chromatic number of vertex corona product of path graph with origami graph, origami graph with path graph, cycle graph with origami graph, origami graph with cycle graph, bistar graph with path graph, path graph with bistar graph, bistar graph with cycle graph, cycle graph with bistar graph are made available.

#### 2.1 Introduction

**b-chromatic number** [Irving et al., 1999]

Let  $G$  be a graph without loops and multiple edges with vertex set  $V(G)$  and edge set  $E(G)$ . The smallest number  $k$  for which  $G$  admits a coloring with  $k$  colors is the chromatic number  $\chi(G)$  of  $G$ . Many graph invariants related to colorings have been defined. Most of them try to minimize the number of colours used to colour the vertices under some constraints. For some invariants, it is meaningful to try to maximize this number; the b-chromatic number is one such example.

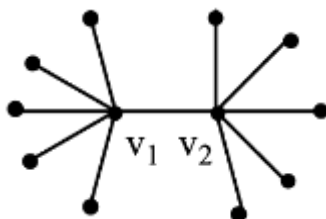
**b-coloring** [Balakrishnan, R 2010, Douglas B. West 2001, Hoang, C. T et al., 2005]

A b-coloring of a graph  $G$  is a proper colouring of the vertices of  $G$  such that there exists a vertex in each color class joined to at least a vertex in each other colour class. Such a vertex is called a dominating vertex. The b-chromatic number of a graph  $G$ , denoted by  $\phi(G)$ , is the maximal integer  $k$  such that  $G$  may have a b-coloring by  $k$  colors. This parameter has been derived by Irving and Manlove in the year 1999.

**Bistar graph** [Jonathan Gross et al., 2004]

Bistar  $(B_{n,n})$  is the graph obtained by joining the center (apex) vertices of two copies of  $K_{1,n}$  by an edge. The vertex set of  $B_{n,n}$  is  $V(B_{n,n}) = \{u, v, u_i, v_i / 1 \leq i \leq n\}$ , where  $u, v$  are apex vertices and  $u_i, v_i$  are pendent vertices. The edge set of  $B_{n,n}$  is  $E(B_{n,n}) = \{uv, uu_i, vv_i / 1 \leq i \leq n\}$ .

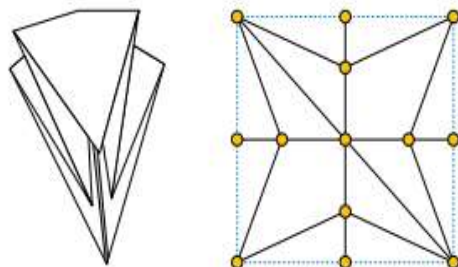
So,  $|V(B_{n,n})| = 2n + 2$  and  $|E(B_{n,n})| = 2n + 1$ .



**Fig 2.1:** Bistar graph

**Origami Graph** [Tiago Januario et al., 1994, Jonathan Gross et al., 2004]

From the point of view of graph drawing, origami folding patterns can be thought of as planar graphs, drawn with straight line edges in the Euclidean plane, with each edge representing a crease that must be folded. For instance, the familiar bird base, a starting point for the classic three-dimensional origami crane, can be thought of as a graph drawing of a planar graph with 13 vertices (Figure 2.2 Origami graph). The NP-completeness of recognizing multi-vertex flat folding patterns, because of the completeness result.



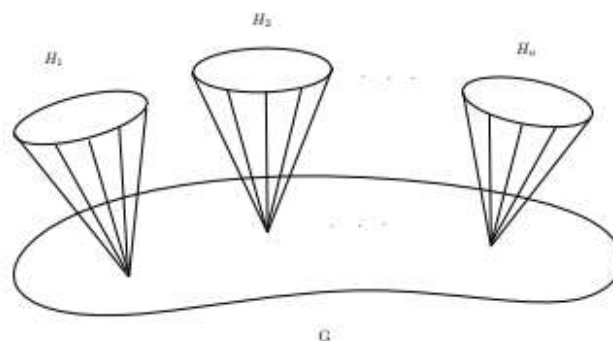
**Fig 2.2:** Origami Graph

**Example:** Figure 2.2 Origami bird base the corresponding folding pattern, interpreted as a graph drawing. The black lines indicate the final creases of the bird base. Temporary creases made while folding the base but later flattened out are not included. Blue dashed lines indicate the boundary of the sheet of paper; these lines are not considered as edges of the graph because they are not creased. Folding patterns that have already been embedded with a given geometry and its proof depends on the specific geometry of the embedding. Here, instead we ask whether an embedding exists.

**Corona Products** [Kalpana, M et al., 2018]

Graph operations play an important role in the study of graph decompositions into isomorphic sub graphs.

Let  $G$  and  $H$  be two simple graphs. If  $|V(G)| = n$  and  $|E(G)| = q$ , we say that  $G$  is an  $(n, q)$ -graph. We also say that  $G$  is of order  $n$ . The corona product  $G \circ H$  of two graphs  $G$  and  $H$  is an important graph operation defined as the graph obtained by taking one copy of  $G$  and  $|V(G)|$  copies of  $H$  and joining the  $i^{\text{th}}$  vertex of  $G$  to every vertex in  $i^{\text{th}}$  copy of  $H$ . If  $G$  is an  $(n, q)$ -graph and  $H$  is an  $(m, q')$ -graph, then  $|V(G \circ H)| = n + nm'$  and  $|E(G \circ H)| = q + nq' + nm'$ . The  $i^{\text{th}}$  copy of  $H$  is denoted by  $H_i, 1 \leq i \leq n$  as shown in Fig. 2.3. It is clear from the definition that corona product of two graphs is not commutative. Obviously,  $G \circ H$  is connected if and only if  $G$  is connected. Also if  $H$  contains at least one edge then  $G \circ H$  is not bipartite graph.



**Fig 2.3:** The corona product of two graphs

## Vertex Corona

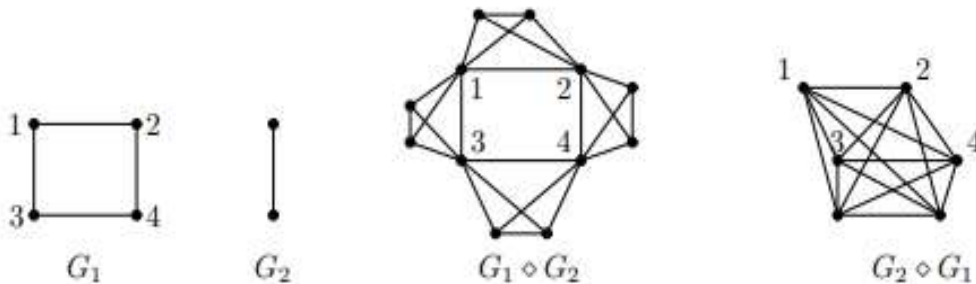
The Vertex Corona product of  $G$  and  $H$  is the graph  $G \circ H$  obtained by taking one copy of  $G$ , called the centre,  $|V(G)|$  copies of  $H$ , called the outer graph, and making the  $i^{\text{th}}$  vertex  $G$  adjacent to every vertex of the  $i^{\text{th}}$  copy of  $H$ , where  $1 \leq i \leq |V(G)|$ . This graph product was introduced by [Frucht et al., 1970].

## Edge Corona

Let  $G_1$  and  $G_2$  be two graphs on disjoint sets of  $n_1$  and  $n_2$  vertices,  $m_1$  and  $m_2$  edges, respectively. The edge corona  $G_1 \diamond G_2$  of  $G_1$  and  $G_2$  is defined as the graph obtained by taking one copy of  $G_1$  and  $m_1$  copies of  $G_2$  and then joining two end-vertices of the  $i^{\text{th}}$  edge of  $G_1$  to every vertex in the  $i^{\text{th}}$  copy of  $G_2$ .

Note that, the edge corona  $G_1 \diamond G_2$  of  $G_1$  and  $G_2$  has  $n_1 + m_1 n_2$  vertices and  $m_1 + 2m_1 n_2 + m_1 m_2$  edges

**Example:** Let  $G_1$  be the cycle of order 4 and  $G_2$  be the complete graph  $K_2$  of order 2. The two edge coronas  $G_1 \diamond G_2$  and  $G_2 \diamond G_1$  are depicted in Figure 2.4



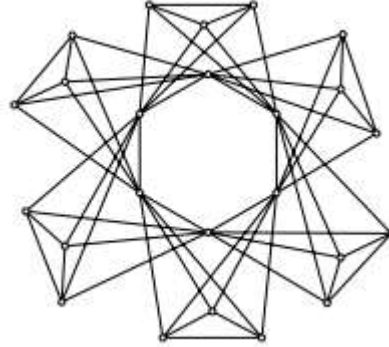
**Fig 2.4:** Edge corona graphs

## Neighbourhood corona

Let  $G_1$  and  $G_2$  be two graphs on  $n_1$  and  $n_2$  vertices,  $m_1$  and  $m_2$  edges respectively. Then the neighbourhood corona,  $G_1 * G_2$  is the graph obtained by taking  $n_1$  copies of  $G_2$  and for each  $i$ , making all vertices in the  $i^{\text{th}}$  copy of  $G_2$  adjacent with the neighbours of  $v_i$ ,  $i = 1, 2, \dots, n$ .

The neighbourhood corona  $G_1 * G_2$  of  $G_1$  and  $G_2$  has  $n_1 + n_1 n_2$  vertices and  $m_1(2n_2 + 1) + n_1 m_2$  edges and when  $G_2 = K_1$ ,  $G_1 * G_2$  is the splitting graph. Note that in general this operation is not commutative.

**Example:** The following figure illustrates Neighbourhood corona.



**Fig 2.5:**  $C_6 * K_2$

The corona product is not commutative. For example,  $K_3 \circ P_2 \neq P_2 \circ K_3$  since the number of vertices differs. Also the corona product is not associative. It is easy to prove that  $\chi(G \circ H) < \Delta(G \circ H) + 2$  for all  $G$  and  $H$  for the corona product. We are interested in proving the tight bound of the [Behzad et al., 1965] conjecture for the corona product of certain classes of graphs.

## 2.2: The b-chromatic number of the corona graph of Path and Cycle with Origami graph

**Theorem 2.2.1:** For any cycle graph  $C_n$  and an origami graph  $O_n$ , the b-chromatic number of  $C_n \circ O_n$  is

$$\varphi(C_n \circ O_n) = \begin{cases} n + 3 & \text{for } n = 4 \\ n + 2 & \text{for } n = 5 \\ n + 1 & \text{for } n = 6 \\ n & \text{for } n \geq 7 \end{cases}$$

**Proof:**

Let  $V(C_n) = \{a_i : 1 \leq i \leq n\}$  and  $V(O_n) = \{x_i, y_i, z_i : 1 \leq i \leq n\}$

$$\text{Let } V(C_n \circ O_n) = \{a_i : 1 \leq i \leq n\} \cup \{x^i_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{y^i_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{z^i_j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

We prove the results by the following four cases.

**Case 1:** For  $n=4$

Assign the following  $n+3$  colors as b-chromatic number for  $(C_n \circ O_n)$

- For  $a_i : 1 \leq i \leq 4$ , assign the color  $c_i$
- For  $x^i_j : 1 \leq i \leq 4, 1 \leq j \leq 3$ , assign the color  $c_{i+n}$
- For  $x^i_j : 1 \leq i \leq 4, j = 4$ , if  $i = 3$ , assign the color  $c_{n-1}$ , else assign the color  $c_n$
- For  $y^i_j : 1 \leq i \leq 4, 1 \leq j \leq 4, 1 \leq k \leq n+3$ , assign the color  $c_k$
- For  $z^i_j : 1 \leq i \leq 4, 1 \leq j \leq 4, 1 \leq k \leq n+3$ , assign the color  $c_k$

The above coloring is assigned by the proper coloring procedure.

For the graph  $(C_n \circ O_n)$ ,  $n = 4$  the b-coloring procedure depends the vertex set of  $O_n$ . But the origami graph  $O_n$  has  $n + 3$  vertices of degree  $n + 2$  then we can assign  $n + 3$  colors to get b-coloring.

**Case 2:** For  $n = 5$

Assign the following  $n + 2$  colors as b-chromatic number for  $C_n \circ O_n$ .

- For  $a_i : 1 \leq i \leq 5$ , assign the color  $c_i$
- For  $x^i_j : 1 \leq i \leq 5, 1 \leq j \leq 2$ , assign the color  $c_{n+i-1}$
- For  $x^i_j : 1 \leq i \leq 5, 3 \leq j \leq n, 1 \leq k \leq n+2$ , assign the color  $c_k$
- For  $y^i_j : 1 \leq i \leq 5, 1 \leq j \leq 5, 1 \leq k \leq n+2$ , assign the color  $c_k$
- For  $z^i_j : 1 \leq i \leq 5, 1 \leq j \leq 5, 1 \leq k \leq n+2$ , assign the color  $c_k$

The above coloring is assigned, by the proper coloring procedure.

For the graph  $C_n \circ O_n$ ,  $n = 5$ , the b-coloring procedure depends the vertex set of  $O_5$ . But the graph  $O_5$  has 7 vertices with degree 6, and then we can assign 7 colors to get the b-coloring.

**Case 3:** For  $n = 6$

Assign the following  $n + 1$  colors as b-chromatic number for  $(C_n \circ O_n)$ .

For  $a_i : 1 \leq i \leq n$  assign the color  $c_i$

- For  $x^i_j : 1 \leq i \leq n, 1 \leq j \leq n-1, 1 \leq k \leq n+1$ , assign the color  $c_k$
- For  $x^i_j : 1 \leq i \leq n, j = n, 1 \leq k \leq n+1$ , assign the color  $c_k$
- For  $y^i_j : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n+1$ , assign the color  $c_k$
- For  $z^i_j : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n+1$ , assign the color  $c_k$

The above coloring is assigned, by the proper coloring procedure.

For the graph  $C_n \circ O_n, n = 6$ , the b-coloring procedure depends the vertex set of  $O_n$ . But the graph  $O_n$  has  $n + 1$  vertices with degree  $n$ , and then we can assign  $n + 1$  colors to get the b-coloring.

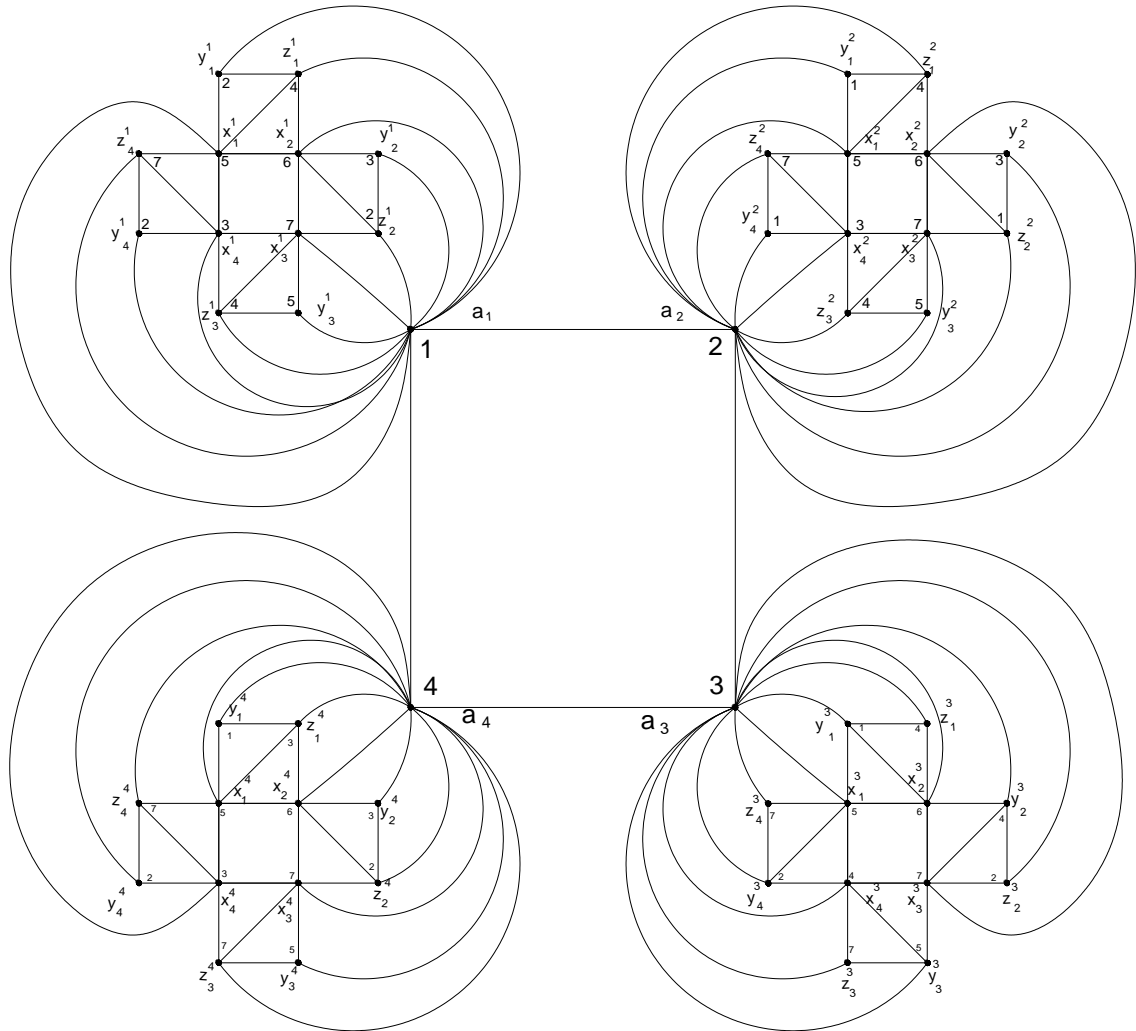
**Case 4:** For  $n \geq 7$

Assign the following  $n$  colors as b-chromatic number for  $C_n \circ O_n$ .

- For  $a_i : 1 \leq i \leq n$  assign the color  $c_i$
- For  $x^i_j : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n$ , assign the color  $c_k$
- For  $y^i_j : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n$ , assign the color  $c_k$
- For  $z^i_j : 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq n$ , assign the color  $c_k$

The above coloring is assigned, by the proper coloring procedure. For the graph  $C_n \circ O_n, n \geq 7$ , the b-coloring procedure depends the vertex set of  $O_n$ . But the graph  $O_n$  has  $n$  vertices with degree  $n - 1$ , and then we can assign  $n$  colors to get the b-coloring.

Hence the proof.



**Fig 2.6:**  $\varphi(C_4 \circ O_4) = 7$

### 2.2.2: The b-chromatic number of corona graph of $\varphi[O_n \circ C_n]$

#### Algorithm:

Input:  $O_n \circ C_n, n \geq 4$ .

$$V \leftarrow \left\{ x_1, x_2, x_3, \dots, x_n, y_1, y_2, y_3, \dots, y_n, z_1, z_2, z_3, \dots, z_n, \right. \\ \left. a^1_1, a^1_2, \dots, a^n_n, b^1_1, b^1_2, \dots, b^n_n, d^1_1, d^1_2, \dots, d^n_n \right\}$$

for  $i = 1$  to  $n$

$z_i \leftarrow i$ ;

end for

for  $i = 1$  to  $n$

$x_i \leftarrow n + 1$ ;

end for

for  $i = 1$  to  $n - 2$

$y_i \leftarrow i + 1$ ;

end for

for  $i = n - 1$  to  $n$

$y_i \leftarrow 2n - 2$ ;

end for

for  $i = 1$  to  $n, j = 1$  to  $n, k = 1$  to  $2n$

$a^i_j \leftarrow k$ ;

end for

for  $i = 1$  to  $n, j = 1$  to  $n, k = 1$  to  $2n$

$b^i_j \leftarrow k$ ;

end for

for  $i = 1$  to  $n$ ,  $j=1$  to  $n$ ,  $k=1$  to  $2n$

$d^i_j \leftarrow k$ ; end for

end procedure

**output:** vertex colored  $O_n \circ C_n$ .

**Theorem 2.2.2:** For any positive integer  $n$ ,  $n \geq 4$ , the b-chromatic number of corona graph of origami graph with cycle graph is  $\chi_b(O_n \circ C_n) = 2n$ , for  $n \geq 4$ .

**Proof:**

$$\text{Let } V(O_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\}$$

$$\text{and } V(C_n) = \{a_i : 1 \leq i \leq n\}$$

By the definition of corona product, each vertex of  $O_n$  is adjacent to every vertex of number of copies of  $C_n$ .

i.e.,  $\{x_i, y_i, z_i : 1 \leq i \leq n\} \in V(O_n)$  is adjacent to every vertex of

$$\{a^i_j, b^i_j, d^i_j : 1 \leq i \leq n, 1 \leq j \leq n\} \in V(C_n).$$

Then the vertex set of the corona product  $O_n \circ C_n$  is

$$V(O_n \circ C_n) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\} \cup \{z_i : 1 \leq i \leq n\} \cup$$

$$\{a^i_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{b^i_j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{d^i_j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Assign the coloring as per the algorithm.

By this coloring procedure, we get that  $\chi_b(O_n \circ C_n) = 2n$

To prove the lower bound, let us assume that, the b-chromatic number of corona product of origami graph  $O_n$  with cycle graph  $C_n$  is greater than  $2n$ .

That is the b-chromatic number of  $\chi_b(O_n \circ C_n) = 2n + 1$

Assigning  $2n + 1$  colors, the graph  $O_n \circ C_n$  should have  $2n + 1$  vertices with  $2n$  degree along with distinct colors. But the graph  $O_n \circ C_n$  having only  $2n$  vertices with maximum degree  $2n - 1$ , which is the contradiction.

Therefore assigning  $2n + 1$  colors is impossible.

$$\therefore \chi(O_n \circ C_n) \leq 2n$$

$$\text{Hence } \chi(O_n \circ C_n) = 2n.$$

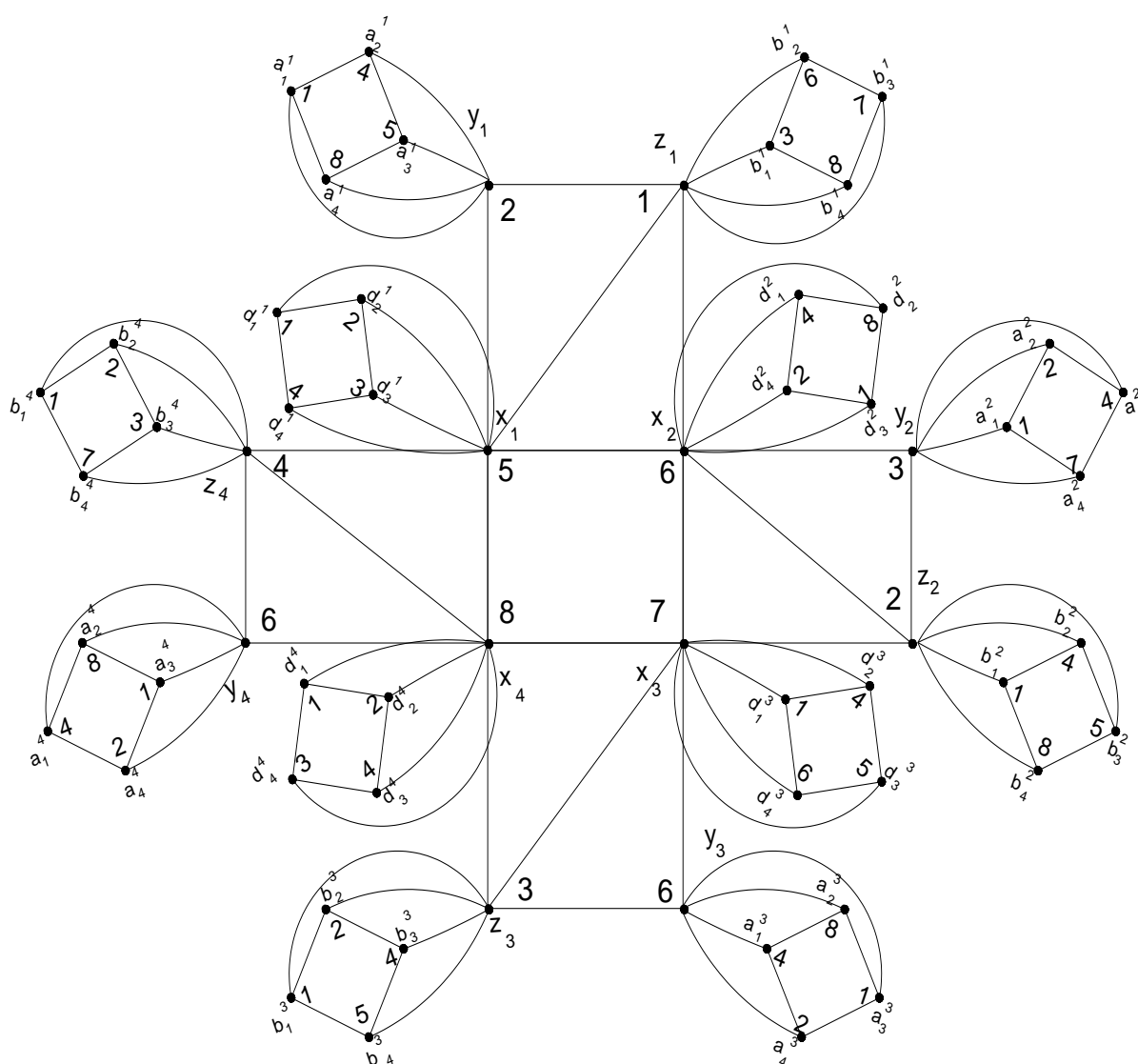


Fig 2.7:  $\chi(C_4 \circ O_4) = 8$

**Theorem 2.2.3:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number of the corona graph of origami graph with path graph is  $\varphi(o_n \circ p_n) = n + 4$  for  $n \geq 4$ .

**Proof:**

Let  $V(O_n) = \{a_i, b_i, d_i : 1 \leq i \leq n\}$  and  $V(P_n) = \{p_i : 1 \leq i \leq n\}$

Let  $V(O_n \circ P_n) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\} \cup \{p_i : 1 \leq i \leq n\}$

Consider  $n \geq 4$ , assign the following  $n + 4$  colors as  $b$ -chromatic number for  $O_n \circ P_n$ .

- For  $p_i : 1 \leq i \leq n$  assign the colour  $c_i : 1 \leq i \leq n + 4$  by the proper coloring procedure.
- For  $d_i : 1 \leq i \leq n$  assign the color  $c_i$
- For  $a_i : 1 \leq i \leq 4$  assign the color  $c_{\lfloor \frac{n}{2} \rfloor + i - 1}$  and for the remaining vertices repeat the colours in the same order
- And for  $b_i : 1 \leq i \leq n$  assign the color  $c_i : n \leq i \leq n + 4$  by the proper coloring procedure.

An easy check shows that this colouring is a  $b$ -coloring.

The colouring procedure depends the degree of the origami graph which is  $n + 3$ .

Then we can assign maximum number of  $n + 4$  colors to get a  $b$ -coloring.

Hence the proof

**Theorem 2.2.4:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number of the corona graph of Path graph with origami graph is  $\varphi(P_n \circ O_n) = \begin{cases} n + 1, & \text{for } n \leq 6 \\ n, & \text{for } n \geq 7 \end{cases}$

**Proof:**

Let  $V(P_n) = \{p_i : 1 \leq i \leq n\}$  and  $V(O_n) = \{a_i, b_i, d_i : 1 \leq i \leq n\}$

$V(P_n \circ O_n) = \{a_i : 1 \leq i \leq n\} \cup \{b_i : 1 \leq i \leq n\} \cup \{d_i : 1 \leq i \leq n\} \cup \{p_i : 1 \leq i \leq n\}$

We prove the results by the following two cases

**Case 1:** For  $n \leq 6$ .

Assign the following  $n + 1$  colors as b-chromatic number for  $(P_n \circ O_n)$

- For  $p_i : 1 \leq i \leq n$ , assign the color  $c_i$ .
- For  $d_i : 1 \leq i \leq 2$ , assign the color  $c_{i+4}$  and for remaining vertices assign the colors  $c_i : 2 \leq i \leq n + 1$  by the proper coloring procedure.
- For  $a_i : 1 \leq i \leq n$ , assign the color  $c_i : n \leq i \leq n + 1$ , where color class  $c_i$  is not equal to colour class of path.
- And for  $b_i : 1 \leq i \leq 2$ , assign the color  $c_{i+3}$ , and for remaining vertices assign the colors  $c_i : 1 \leq i \leq n + 1$ , by the proper coloring procedure.

For the graph  $P_n \circ O_n$ ,  $n \leq 6$ , the b-coloring procedure depends the vertex set of  $(O_n)$ .  $O_n$  has  $n + 1$  vertices of degree  $n$ , then we can assign  $n + 1$  color to get  $b$ -coloring.

**Case 2:** For  $n \geq 7$

Assign the following  $n$  colors as b-chromatic number for  $(P_n \circ O_n)$

- For  $p_i : 1 \leq i \leq n$ , assign the color  $c_i$ .
- For  $d_i : 1 \leq i \leq n$ , assign the color  $c_i : 1 \leq i \leq n$ , with the proper coloring procedure.
- For  $a_i : 1 \leq i \leq n$ , assign the color  $c_i : 1 \leq i \leq n$ , with the proper coloring procedure.
- And for  $b_i : 1 \leq i \leq n$ , assign the color  $c_i : 1 \leq i \leq n$ , with the proper coloring procedure
- For the graph  $P_n \circ O_n$ ,  $n \geq 7$ , the b-coloring procedure depends the vertex set of  $(P_n)$ .
- That is  $p_i : 1 \leq i \leq n$ . This vertex set has  $n$  vertices of degree  $3n + 1$ . Hence we can assign  $n$  colors to get  $b$ -coloring.

Hence the proof

### 2.3: The $b$ -chromatic number of the Corona graph of Path and Cycle with Bistar graph

**Theorem 2.3.1:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number for the corona product of bistar graph with path graph is  $\varphi(B_{n,n} \circ P_n) = n + 2$  for  $n \geq 1$ .

**Proof:**

Let  $V(B_{n,n}) = \{U, V, U_i, V_i : 1 \leq i \leq n\}$  and  $V(P_n) = \{p_i : 1 \leq i \leq n\}$

$$V(B_{n,n} \circ P_n) = \{u\} \cup \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup$$

Let  $\{u^j_i : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{v^j_i : 1 \leq i \leq n, 1 \leq j \leq n\} \cup$

$$\{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$$

Consider  $n \geq 1$ , assign the following  $n+2$  colours as  $b$ -chromatic number for  $B_{n,n} \circ P_n$

- For  $U$ , assign the color  $c_1$
- For  $V$ , assign the color  $c_2$
- For  $u_i : 1 \leq i \leq n$ , assign the color  $c_{n+2}$
- For  $V_i : 1 \leq i \leq n$ , assign the color  $c_{n+2}$
- For  $u_i^j$  assign the  $l : 1 \leq l \leq n+2$  colors except the colors of vertices  $u$  &  $\{u_i : 1 \leq i \leq n\}$
- For  $V_i^j$  assign the  $k : 1 \leq k \leq n+2$  colors except the colors of vertices  $v$  &  $\{v_i : 1 \leq i \leq n\}$
- For  $x_i : 1 \leq i \leq n$ , assign the color  $c_{i+1}$
- For  $y_i : 1 \leq i \leq n$ , assign the color  $c_{i+2}$

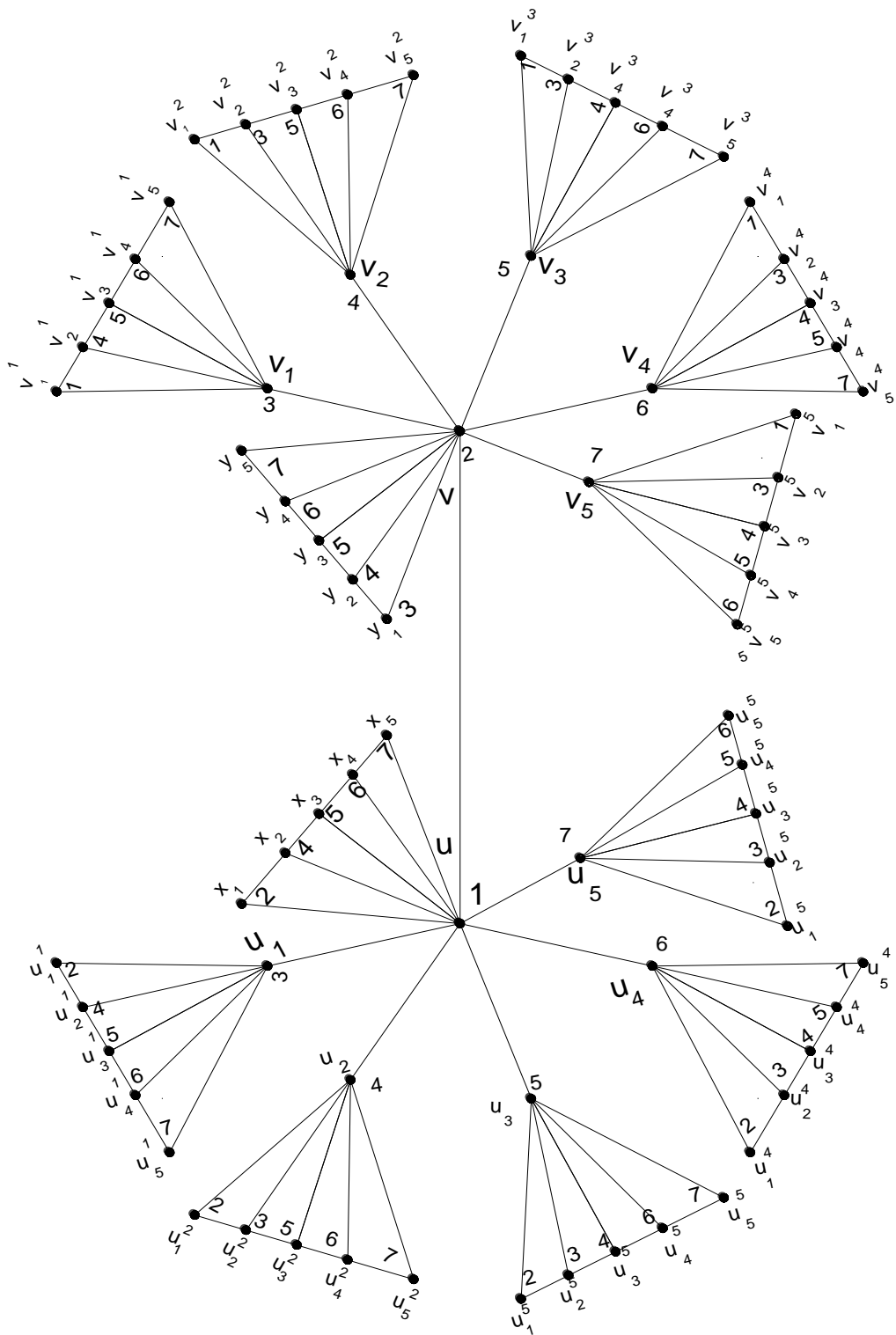
An easy check shows that this coloring is a  $b$ -Coloring.

The coloring procedure depends the number of vertices of Bistar graph ( $B_{n,n}$ ) that is

$B_{n,n}$  has degree  $n+1$ . Hence we can assign maximum of  $n+2$  colors to get a

$b$ -coloring.

Hence the proof.



**Fig 2.8 :**  $\phi(B_{5,5}oP_5)=7$

**Theorem 2.3.2:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number of corona graph of bistar graph with cycle graph is  $\varphi(B_{n,n} \circ C_n) = n + 2$  for  $n \geq 1$ .

**Proof:**

Let  $V(B_{n,n}) = \{U, V, U_i, V_i : 1 \leq i \leq n\}$  and  $V(C_n) = \{p_i : 1 \leq i \leq n\}$

Let

$$V(B_{n,n} \circ P_n) = \{u\} \cup \{v\} \cup \{u_i : 1 \leq i \leq n\} \cup \{V_i : 1 \leq i \leq n\}$$

$$\{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{V_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Consider  $n \geq 1$  assign the following  $n + 2$  colours as  $b$ -chromatic number for  $B_{n,n} \circ C_n$

- For  $U$ , assign the color  $c_1$
- For  $V$ , assign the color  $c_2$
- For  $u_i : 1 \leq i \leq n$ , assign the color  $c_{n+2}$
- For  $V_i : 1 \leq i \leq n$ , assign the color  $c_{n+2}$
- For  $u_i^j$  assign the colour  $c_k : 1 \leq k \leq n+2$ , except the colors of vertices  $u$  &  $\{u_i : 1 \leq i \leq n\}$
- For  $V_i^j$  assign the colour  $m : 1 \leq m \leq n+2$ , except the colors of vertices  $v$  &  $\{v_i : 1 \leq i \leq n\}$

An easy check shows that this coloring is a  $b$ -coloring.

The colouring procedure depends the number of vertices of Bistar graph ( $B_{n,n}$ ) the bistar graph  $B_{n,n}$  has degree  $n+1$ . Hence we can assign maximum of  $n+2$  colours to get a  $b$ -coloring.

Hence the proof.

**Theorem 2.3.3:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number of corona

$$\text{graph of path graph with bistar graph is } \varphi(P_n \circ B_{n,n}) = \begin{cases} n+2 & \text{for } n=1 \\ n+3 & \text{for } n \geq 2 \end{cases}$$

**Proof:**

$$\text{Let } V(p_n) = \{p_i : 1 \leq i \leq n\} \text{ and } V(B_{n,n}) = \{U, V, U_i, V_i : 1 \leq i \leq n\}$$

Let

$$V(P_n \circ B_{n,n}) = \{p_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{V_i : 1 \leq i \leq n\} \cup$$

$$\{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{V_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Consider the following cases

**Case 1:** For  $n=1$ ,

Assign the following  $n+2$  colours as  $b$ -chromatic number for  $P_1 \circ B_{1,1}$

- For  $p_1$ , assign the color  $c_1$
- For  $u_1$ , assign the color  $c_2$
- For  $v_1$ , assign the color  $c_3$
- For  $u_1^1$ , assign the color  $c_3$
- For  $v_1^1$ , assign the color  $c_2$

An easy check shows that this coloring is a  $b$ -coloring.

**Case2:** For any positive integer  $n, n \geq 2$

Assign the following  $n+3$  colors as  $b$ -chromatic for  $P_n \circ B_{n,n}, n \geq 2$

- For  $p_i : 1 \leq i \leq n$ , assign the color  $c_i$
- For  $u_i : 1 \leq i \leq n$ , assign the color  $c_{n+2}$
- For  $V_i : 1 \leq i \leq n-1$ , assign the color  $c_{n+2}$
- For  $v_n$  assign the colors  $c_{n+3}$
- For  $u_i^j$  and  $V_i^j : 1 \leq i \leq n, 1 \leq j \leq n$  assign the  $d : 1 \leq d \leq n+3$  as by the proper colouring way.

An easy check shows that this coloring is a  $b$ -coloring.

The coloring procedure depends the number of vertices of bistar graph  $(B_{n,n})$ . Here the  $B_{n,n}$  has degree  $n+2$ . Hence we can assign maximum of  $n+3$  colours to get a  $b$ -coloring.

Hence the proof.

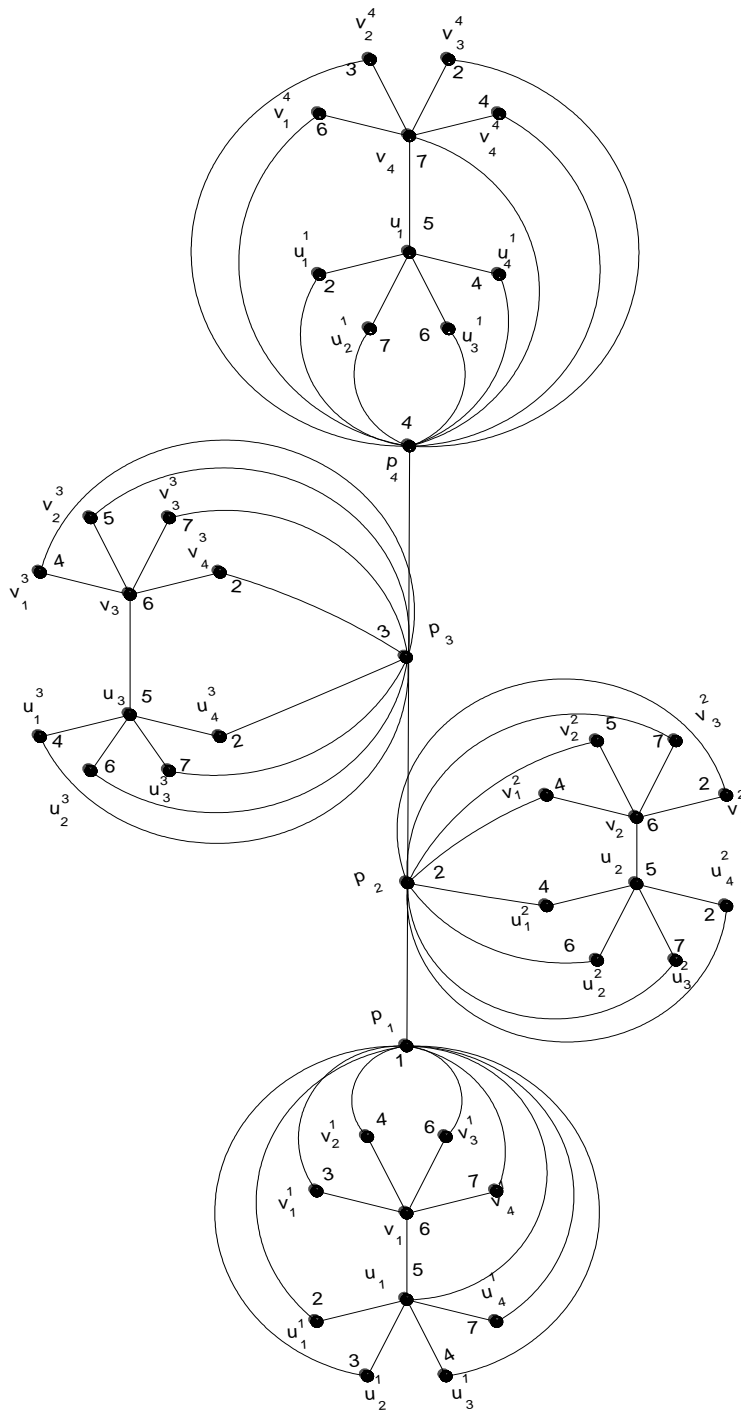


Fig 2.9 :  $\varphi(P_4 \circ B_{4,4}) = 7$

**Theorem 2.3.4:** For any positive integer  $n, n \geq 1$  the  $b$ -chromatic number of the corona graph of cycle graph with bistar graph is  $\phi(c_n \circ B_{n,n}) = n + 3$  for  $n \geq 1$ .

**Proof:**

Let  $V(c_n) = \{r_i : 1 \leq i \leq n\}$  and  $V(B_{n,n}) = \{U, V, U_i, V_i : 1 \leq i \leq n\}$

Let

$$v(c_n \circ B_{n,n}) = \{r_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup$$

$$\{u_i^j : 1 \leq i \leq n, 1 \leq j \leq n\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq n\}$$

Consider  $n \geq 1$ , assign the following  $n + 3$  colours as  $b$ -chromatic number for  $c_n \circ B_{n,n}$ .

- For  $r_i : 1 \leq i \leq n$ , assign the color  $c_i$
- For  $u_i : 1 \leq i \leq n$ , assign the color  $c_{n+1}$
- For  $v_i : 1 \leq i \leq n - 1$ , assign the color  $c_{n+2}$
- And for  $v_n$ , assign the color  $c_{n+3}$
- For  $u_i^j$ , assign the colours  $c_i : 1 \leq i \leq n + 3$  except the colors of vertices  $u_i$  &  $r_i : 1 \leq i \leq n$
- For  $v_i^j$ , assign  $c_i : 1 \leq i \leq n + 3$  the colors except the colors of vertices  $v_i$  &  $r_i : 1 \leq i \leq n$

An easy check shows that this coloring is a  $b$ -coloring.

The coloring procedure depends the number of vertices of bistar graph  $(B_{n,n})$ .

The  $B_{n,n}$  has degree  $n + 2$ .

Hence we can assign maximum of  $n + 3$  colors to get a  $b$ -coloring.

Hence the proof.