Characterizations of Topological Spaces in Terms of Semi-Open Sets, Preopen Sets And Semi-Preopen Sets
And
An Interesting Generalization of Paracompactness

BY

Latha S. R.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE - 641 043
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
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Introduction

INTRODUCTION

This thesis is devoted to the study of characterizations of some important classes of topological spaces and maps in terms of semi-open sets, preopen sets and semi-preopen sets. A subset A of a topological space (X,τ) is called

(1) a semi-open set if $A \subset cl(Int(A))$

(2) a preopen set if $A \subset Int(cl(A))$

(3) a semi-preopen set if $A \subset cl((Int(cl(A)))$.

These sets were introduced by Levine [1963, 7] Mashhour et.al. [1981, 8] and Andrijevic, Mat. Vesnik 38 (1986),24-32 respectively. The class of semi-preopen sets contains both the class of semi-open sets and the class of preopen sets.

Chapter I deals with the characterizations of extremally disconnected spaces, irreducible spaces, semi-irreducible spaces, PS-spaces, partition spaces, S-sets, I-sets, lightly compact spaces, almost regular spaces and β-compact spaces. D. Sivaraj [11] has obtained characterizations of extremally disconnected spaces in terms of semi-open sets, while Takashi Noiri [12] has obtained characterization of such spaces in terms of preopen sets and semi-preopen sets. Julian Dontchev [6] has obtained the characterizations of the remaining spaces in terms of semi-preopen sets. He [6] has also obtained interesting characterizations of almost continuous maps, α-continuous maps, almost weakly continuous maps, almost closed maps, semi-preclosed maps and quasi-irresolute maps in terms of semi-preopen sets.

In chapter 2, we discuss some interesting properties of α-paracompact, α-nearly paracompact subsets and almost-closed mappings. The results are due to Kovacevic [2, 3]. The main results proved are as follows:

(1) If A is an α-Hausdorff α-nearly paracompact subset relative to X\A, then A is closed.

(2) If A is an α-regular (α-almost regular) subset and if U is an open (regularly open) neighbourhood of A and if A is α-paracompact (α-nearly paracompact) relative to X\U, then there exists an open neighbourhood V of A, such that $A \subset V \subset cl(V) \subset U$.

As a consequence of result 2 we get,

(3) Any open (regularly open) α-regular (α-almost regular) subset which is α-paracompact (α-nearly paracompact) relative to its complement is closed.

Regarding almost closed mappings the main results discussed are as follows:

I. Let $f: X \to Y$ be an almost closed mapping of a space X onto Y. Then

(1) If $\{f^{-1}(y): y \in Y\}$ consists of α-Hausdorff subsets which are mutually α-nearly paracompact. Then Y is Hausdorff.

If in addition Y is compact, then f is continuous.

(2) If $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \setminus f^{-1}(y)$, for each y in Y, then f has a closed graph.

(3) If the graph is closed then, the image of an α-regular α-paracompact subset relative to its complement, is closed.

II If f is a closed mapping and if Y is a compact space such that $f^{-1}(y)$ is an α-Hausdorff α-paracompact subset and if every closed subset of X is α-Hausdorff, then X is regular.

Chapter I

CHAPTER - I

CHARACTERIZATIONS OF SOME IMPORTANT CLASSES OF TOPOLOGICAL SPACES IN TERMS OF SEMI-OPEN SETS, PREOPEN SETS AND SEMI - PREOPEN SETS

This chapter deals with some characterizations of extremally disconnected spaces, irreducible spaces, semi-irreducible spaces, PS-spaces, partition spaces, S-sets, I-sets, lightly compact spaces, almost regular spaces and β - compact spaces. In the first section we give the preliminary definitions and results. Second section deals with the characterizations of extremally disconnected spaces. D.Sivaraj [11] has characterized such spaces, in terms of semi-open sets, while Takashi Noiri [12] has generalized such spaces in terms of preopen sets and semi-preopen sets. The characterizations of the remaining spaces are obtained by Julian Dontchev [6] in terms of semi-preopen sets. Section 3 deals with these results. He [6] has also obtained characterizations of mapping in terms of semi-preopen sets. These characterizations are given in section 4.

SECTION 1.1 PRELIMINARIES

By a space X, it is meant the topological space (X, τ). The closure and interior of a subset A of X are denoted by cl(A) and Int(A) respectively. Let A be a subspace of X.

DEFINITION 1.1.1

A subset A of X is **semi-open** if there exists an open set G such that $G \subset A \subset cl(G)$.

Equivalently A is semi-open if $A \subset cl(Int(A))$.

NOTATION

The collection of all semi-open sets is denoted by SO(X). Semi-open sets are termed as **β-sets** (Njastad [9]). The family of all semi-open sets in X is also denoted by τ^{β}.

PROPOSITION 1.1.2

If A is open, then cl(A) is **semi-open.**

PROOF:

$$\begin{aligned} A \text{ is open } &\Rightarrow A = Int(A) \\ &\Rightarrow cl(A) = cl(Int(A)) \subset cl(Int(cl(A))). \end{aligned}$$

Therefore, cl(A) is semi-open.

PROPOSITION 1.1.3

Every open set is a semi-open set.

PROOF:

Let A be an open set.

Then, $A \subset cl(A) = cl(Int(A))$.

Hence A is semi-open.

DEFINITION 1.1.4

The complement of a semi-open set is called a **semi-closed** set.

Equivalently, A is semi-closed iff $Int(cl(A)) \subset A$.

Equivalently, A is semi-closed iff there exists a closed set B such that $Int(B) \subset A \subset B$.

NOTATION

The collection of all semi-closed sets is denoted by $SC(X)$.

REMARK 1.1.5

Closure of a semi-open set is semi-closed.

PROPOSITION 1.1.6

If A is semi-closed, then $Int(cl(A)) = Int(A)$.

PROOF:

A is semi-closed $\Rightarrow Int(cl(A)) \subset A$.

$\Rightarrow Int(Int(cl(A))) \subset Int(A)$

$\Rightarrow Int(cl(A)) \subset Int(A)$.

Always $Int(A) \subset Int(cl(A))$.

Hence, $Int(cl(A)) = Int(A)$.

DEFINITION 1.1.7

The **semi-interior** of A is the union of all semi-open sets contained in A and is denoted by $sint(A)$.

DEFINITION 1.1.8

The **semi-closure** of A is the intersection of all semi-closed sets containing A and is denoted by $scl(A)$.

DEFINITION 1.1.9

A is an **α - open** set if $A \subset Int(cl(Int(A)))$

NOTATION

The collection of all α - open sets in X is denoted by τ^{α}. The space (X, τ^{α}) is denoted by X^{*}.

REMARK 1.1.10

Every α - open set is semi-open.

DEFINITION 1.1.11

The complement of an α-open set is called α - **closed.**

DEFINITION 1.1.12

The α - **interior** of A is the union of all α - open sets contained in A and is denoted by α- Int(A).

DEFINITION 1.1.13

The α **-closure** of A is the intersection of all α -closed sets containing A and is denoted by α-cl(A).

DEFINITION 1.1.14

A is said to be **preopen** if $A \subset \text{Int}(\text{cl}(A))$.

NOTATION

The collection of all preopen sets of X is denoted by PO(X).

REMARK 1.1.15

Every open set is preopen.

DEFINITION 1.1.16

The complement of a preopen set is called **preclosed.**

DEFINITION 1.1.17

The **pre-interior** of A is the union of all preopen sets contained in A and is denoted by pint(A).

DEFINITION 1.1.18

The **Preclosure** of A is the intersection of all preclosed sets containing A and is denoted by pcl(A).

PROPOSITION 1.1.19

(a) $\mathrm{scl}(A) = A \cup \mathrm{Int}(\mathrm{cl}(A))$

(b) $\mathrm{pcl}(A) = A \cup \mathrm{cl}(\mathrm{Int}(A))$

(c) $\mathrm{sint}(A) = A \cap \mathrm{cl}(\mathrm{Int}(A))$

(d) $\mathrm{pint}(A) = A \cap \mathrm{Int}(\mathrm{cl}(A))$

PROPOSITION 1.1.20

$$\mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A))) = \mathrm{sint}(\mathrm{scl}(A)).$$

PROOF:

$\mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)) \subset \mathrm{sint}(\mathrm{scl}(A))$.

Let $x \in \mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$.

$\Rightarrow$ $x \in \mathrm{scl}(A)$ and $x \in \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A))$.

If $x \in \mathrm{sint}(\mathrm{scl}(A))$, there is nothing to prove.

If $x \notin \mathrm{sint}(\mathrm{scl}(A)$, then as $\mathrm{sint}(\mathrm{scl}(A)) \subset \mathrm{scl}(A)$,

$x \in \mathrm{scl}(A) - \mathrm{sint}(\mathrm{scl}(A))$.

Therefore, $x \in (\mathrm{scl}(A) - \mathrm{sint}\ (\mathrm{scl}(A))) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$.

$\Rightarrow$ $x \in \mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A))) - \mathrm{sint}(\mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$

$\Rightarrow$ $x \notin \mathrm{sint}(\mathrm{scl}(A)) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$

$\Rightarrow$ $x \notin \mathrm{sint}(\mathrm{scl}(A))$ and $x \notin \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$

which is a contradiction to the fact that $x \in \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$.

Therefore, $x \in \mathrm{sint}(\mathrm{scl}(A))$.

Therefore, $\mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A))) \subset \mathrm{sint}(\mathrm{scl}(A))$. ....(1)

We know that $\mathrm{sint}(A) = A \cap \mathrm{cl}(\mathrm{Int}(A))$.

$\mathrm{Sint}(\mathrm{scl}(A)) = \mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{scl}(A)))$.

But $\mathrm{scl}(A) \subset \mathrm{cl}(A)$.

Therefore, $\mathrm{sint}(\mathrm{scl}(A)) \subset \mathrm{scl}(A) \cap \mathrm{cl}(\mathrm{Int}(\mathrm{cl}(A)))$. ....(2)

Hence from (1) and (2), we get the result.

DEFINITION 1.1.21

A is said to be **semi-preopen** if $A \subset cl(Int(cl(A)))$.

NOTATION

The collection of all semi-preopen subsets of X is denoted by SPO(X).

REMARK 1.1.22

Every preopen set is semi-preopen.

DEFINITION 1.1.23

The complement of a semi-preopen set is called semi-preclosed.

DEFINITION 1.1.24

A is said to be **regular open** if $A = Int(cl(A))$.

NOTATION

The collection of all regular open sets of X is denoted by RO(X).

REMARK 1.1.25

Every regular open set is open.

Every regular open set is preopen.

DEFINITION 1.1.26

The complement of a regular open set is called **regular closed.**

REMARK 1.1.27

Every regular closed set is closed.

REMARK 1.1.28

Every regular closed set is semi-open.

PROPOSITION 1.1.29

A is regular closed iff $A = cl(Int(A))$.

PROOF:

Assume A is regular closed,

iff X-A is regular open,

iff $X-A = Int(cl(X-A))$

iff $A = X - Int(cl(X-A))$

iff $A = X - (X-cl(X-cl(X-A)))$

(since $Int(A) = X-cl(X-A)$)

$= cl(X-cl(X-A))$

iff $A = cl(Int(A))$.

DEFINITION 1.1.30

A set is called **semi-regular** if it is both semi-open and semi-closed.

NOTATION

The collecttion of all semi-regular sets of X is denoted by SR(X). Semi-regular sets are sometimes called regular semi-open.

DEFINITION 1.1.31

The **δ-interior** (resp. the **θ-interior**) of A is the union of all regular open sets of X which are contained in A (resp. the union of all open sets of X whose closures are contained in A) and is denoted by $\delta\text{-}Int(A)$ (resp. $\theta\text{-}Int(A)$).

DEFINITION 1.1.32

A is called **δ-open** (resp. **θ-open**) if $A = \delta\text{-Int}(A)$ (resp. $A = \theta\text{-Int}(A)$).

NOTATION

The collection of all δ-open (resp. θ-open) sets form a topology on X and is denoted by τ_δ (resp. τ_θ).

DEFINITION 1.1.33

A point $x \in X$ is called **δ-adherent** point of A, if $\text{Int}(\text{cl}(G)) \cap A \neq \phi$ for every open set G containing x.

DEFINITION 1.1.34

A point $x \in X$ is called **θ-adherent** point of A, if $A \cap \text{cl}(G) \neq \phi$ for every open set G containing x.

DEFINITION 1.1.35

The set of all δ-adherent (resp. θ-adherent) points of A is called the **δ-closure** (resp. **θ-closure) of A.**

NOTATION

The δ-closure (resp. θ-closure) of A is denoted by $\delta\text{-cl}(A)$ (resp. $\theta\text{-cl}(A)$).

PROPOSITION 1.1.36

$$\text{cl}(A) \subset \delta\text{-cl}(A) \subset \theta\text{-cl}(A).$$

PROOF :

Let $x \in \text{cl}(A)$.

Then for every open set U containing x,

$U \cap A \neq \phi$

$\Rightarrow$ $\mathrm{Int}(\mathrm{cl}(U)) \cap A \neq \phi$ (Hence $x \in \delta\text{-cl}(A)$)

$\Rightarrow$ $\mathrm{cl}(U) \cap A \neq \phi$ (Hence $x \in \theta\text{-cl}(A)$).

DEFINITION 1.1.37

A set is called a **CO-set** iff its closure is open.

DEFINITION 1.1.38

A set is called a **residual set** iff its interior is empty.

DEFINITION 1.1.39

A set is called an **NDB set** iff its boundary is nowhere dense.

DEFINITION 1.1.40

A space X is **resolvable** iff it has two disjoint dense subsets. In the opposite case it is called irresolvable.

DEFINITION 1.1.41

A non-void space X is **irreducible** if it satisfies the following equivalent conditions:

(a) Every two non-void open subset of X intersect.

(b) X is not the union of a finite family of two closed proper subsets.

(c) Every non-void open subset of X is dense.

(d) Every open subset of X is connected.

NOTATION

An irreducible space is called sometimes hyperconnected.

DEFINITION 1.1.42

A space X is **semi-irreducible** (or FCC) if it satifies the following equivalent conditions:

(a) Every disjoint family of non-void open subsets of X is finite.

(b) X is the union of a finite number of irreducible spaces.

DEFINITION 1.1.43

A space X is a **PS-space** iff each preopen subset of X is semi-open.

DEFINITION 1.1.44

A space X is a **partition space** iff every open subspace is closed.

Equivalently, a space X is a partition space iff every subspace of X is preopen.

NOTATION

A partition space is called locally indiscrete.

DEFINITION 1.1.45

A is called an **S-set** (resp. **I-set**) in X iff from $A \subset \bigcup_{i \in I} cl(U_i)$, where each U_i is semi-open it follows that $A \subset \bigcup_{i \in J} cl(U_i)$ (resp. $A \subset \bigcup_{i \in J} Int(cl(U_i))$) for some finite $J \subset I$ or equivalently iff from $A \subset \bigcup_{i \in I} A_i$, where each A_i is regular closed, it follows that $A \subset \bigcup_{i \in J} A_i$ (resp. $A \subset \bigcup_{i \in J} Int(A_i)$) for some finite $J \subset I$.

DEFINITION 1.1.46

A **finite dense subsystem** (resp. **finite interior dense subsystem**) of a cover of a space X is a finite subcollection, whose closures (resp. interiors of the closures) cover X.

DEFINITION 1.1.47

A space X is **lightly compact** if it satisfies the following equivalent conditions:

(a) Every locally finite family of open sets is finite.

(b) Every countable open cover has a finite dense subsystem.

DEFINITION 1.1.48

A space X is said to be β **-compact** if every cover of X by β -open sets has a finite subcover.

DEFINITION 1.1.49

A space X is **almost regular** iff each regular closed subset A and a point $x \notin A$ have open disjoint neighbourhoods or equivalently iff for every regular closed subsets $A \subset X$, we have $\theta\text{-cl}(A) = \text{cl}(A)$.

DEFINITION 1.1.50

A space X is called **extremally disconnected** (briefly E.D.), if cl(U) is open in X for every open set U of X, or equivalently if every two disjoint open sets of X have disjoint closures.

REMARK 1.1.51

Extremal disconnectedness of a topological space cannot be characterized by the statement that every two disjoint preopen sets have disjoint closures as is seen from the following example.

EXAMPLE 1.1.52

Let $X = \{a,b,c\}$ and $\tau = \{\phi, X, \{a,b\}\}$.

Consider the sets $\{a\}$, $\{b\}$.

Clearly they are disjoint.

$\text{cl}\{a\} = X$ and $\text{cl}\{b\} = X$.

$\text{Int}(\text{cl}\{a\}) = X$ and $\text{Int}(\text{cl}\{b\}) = X$.

$\{a\} \subset X = \text{Int}(\text{cl}\{a\})$ and

$\{b\} \subset X = \text{Int}(\text{cl}\{b\})$.

Therefore, $\{a\}$, $\{b\} \in PO(X)$.

$\{a\} \cap \{b\} = \phi$.

$cl\{a\} \cap cl\{b\} = X \cap X \neq \phi$.

DEFINITION 1.1.53

Subsets A and B of the space X are said to be **completely separated** if there is a continuous function $g: X \to [0,1]$ such that $g(A) = 0$ and $g(B) = 1$.

SECTION 1.2 CHARACTERIZATIONS OF EXTREMALLY DISCONNECTED SPACES.

In this section we discuss the characterizations due to Sivaraj [11] and Noiri [12]. The following results are needed to obtain the characterizations of extremally disconnected spaces [11] in terms of semi-open sets.

THEOREM 1.2.1 [11]

A space X is extremally disconnected iff τ^{β} is a topology on X iff $\tau^{\alpha} = \tau^{\beta}$.

THEOREM 1.2.2

[T.Noiri, Acta Math. Acad. Sci. Hungur. 35(1980), 103].

If a space X is extremally disconnected, then $scl(A) = cl(A)$ for every semi-open subset A of X.

THEOREM 1.2.3

[S.G.Crossley, and S.K.Hildebrand, Texas J.Sci.22(1971),99].

If A is a subset of a space X, then $sint(A) = X-scl(X-A)$ and $scl(A) = X-sint(X-A)$.

THEOREM 1.2.4 [11]

If A is a semi-open subset of a space X, then $cl(A) = \delta\text{-}cl(A)$.

PROOF :

We know that $cl(A) \subset \delta\text{-}cl(A)$.

Therefore, it is enough to prove that,

$$\delta\text{-}cl(A) \subset cl(A).$$

Let $x \in \delta\text{-}cl(A)$.

Suppose $x \notin cl(A)$.

Therefore, there exists an open set G containing x such that $G \cap A = \phi$

$\Rightarrow$ $G \cap Int(A) = \phi$ (since $Int(A) \subset A$).

CLAIM : $Int(cl(G)) \cap A = \phi$.

Suppose not, that is, $Int(cl(G)) \cap A \neq \phi$

As A is semi-open, $A \subset cl(Int(A))$.

Therefore, $Int(cl(G)) \cap cl(Int(A)) \neq \phi$.

There exists $y \in Int(cl(G)) \cap cl(Int(A))$.

$\Rightarrow$ There exists a neighbourhood W of y, such that $W \subset cl(G)$.

Since $y \in cl(Int(A))$, $W \cap Int(A) \neq \phi$.

$\Rightarrow$ There exists $z \in W \cap Int(A)$.

As $z \in cl(G)$ also, $W \cap Int(A) \cap G \neq \phi$.

$\Rightarrow$ $Int(A) \cap G \neq \phi$ which is a contradiction.

Therefore $Int(cl(G)) \cap A = \phi$.

$\Rightarrow$ $x \notin \delta\text{-}cl(A)$.

Hence $\delta\text{-}cl(A) \subset cl(A)$.

THEOREM 1.2.5 [11]

The following are equivalent for any space X.

(i) X is extremally disconnected.

(ii) The closure of every semi-open set in X is open.

(iii) The semi-closure of every semi-open set in X is open.

(iv) The δ - closure of every semi-open set in X is open.

(v) Every two disjoint semi-open sets in X are completely separated.

(vi) Every two disjoint semi-open sets in X have disjoint closures.

(vii) cl(A) = scl(A) for every semi-open set A in X.

(viii) The semi-closure of every semi-open set in X is closed.

(ix) Int(A) = sint(A) for every semi-closed set A in X.

(x) The semi-interior of every semi-closed set in X is open.

PROOF : (i) $\Rightarrow$ (ii)

Let A be a semi-open set in X.

There exists an open set G such that, $G \subset A \subset cl(G)$

$\Rightarrow$ $cl(G) \subset cl(A) \subset cl(G)$

$\Rightarrow$ cl(A) = cl(G) = open (since X is extremally disconnected).

Hence closure of every semi-open set in X is open .

(ii) $\Rightarrow$ (i) follows from the fact that every open set is semi-open.

(i) $\Rightarrow$ (iii)

Let A be a semi-open set in X.

By Theorem 1.2.2, scl(A) = cl(A).

Since X is extremally disconnected, by (i) $\Rightarrow$ (ii) we get, cl(A) is open. Hence scl(A) is open.

(iii) $\Rightarrow$ (i)

Let G be an open set in X.

To prove cl(G) = scl(G) .

Suppose $x \notin scl(G)$.

Then, there is a semi-open set V containing x such that $V \cap G = \phi$, which implies that $scl(V) \cap G = \phi$ (since if $y \in scl(V) \cap G$, then $V \cap G \neq \phi$ as G is semi-open).

Since $scl(V) \cap G$ is an open set containing x, $x \notin cl(G)$.

Therefore, $cl(G) \subset scl(G)$.

Always, $scl(G) \subset cl(G)$.

Therefore, $cl(G) = scl(G)$.

By (iii), cl(G) is open in X.

Hence X is extremally disconnected.

(iii) $\Rightarrow$ (iv)

Let A be a semi-open set in X.

From (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii), we get cl(A) is open.

Therefore by Theorem 1.2.4, δ-cl(A) is open.

(iv) $\Rightarrow$ (i)

Let A be an open set in X.

Then A is semi-open and hence by Theorem 1.2.4, $cl(A) = \delta\text{-}cl(A)$.

Therefore by (iv), we get cl(A), is open.

Hence X is extremally disconnected.

(iv) $\Rightarrow$ (iii)

Since (iv) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii), we get, (iv) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (v)

Let A and B be two disjoint semi-open sets in X.

By Theorem 1.2.1, A and B are open in X^*.

Since X^* is extremally disconnected [by proposition 7 of Njastad[9]] A and B are completely separated in X^* [Willard, General Topology, Addison - Wesley Publishing Company. Inc., 1970].

Hence A and B are completely separated in X (by proposition 8 of Njastad [9]).

(v) $\Rightarrow$ (vi) clear.

(vi) $\Rightarrow$ (ii)

Let A be a semi-open set in X.

Hence cl(A) is semi-closed.

Since A and X-cl(A) are disjoint semi-open sets, by (vi) we have,

$cl(A) \cap cl(X - cl(A)) = \phi$.

That is, $cl(A) \cap (X - Int(cl(A))) = \phi$.

$\Rightarrow cl(A) \subset Int(cl(A))$.

Hence $Int(cl(A)) = cl(A)$.

$\Rightarrow$ cl(A) is open in X.

(i) $\Rightarrow$ (vii) follows from Theorem 1.2.2.

(vii) $\Rightarrow$ (viii) and (ix) $\Rightarrow$ (x) are obvious.

(viii) $\Rightarrow$ (vii)

For any subset A of X, $A \subset scl(A) \subset cl(A)$.

Therefore $cl(A) = cl(scl(A))$.

If A is semi-open by (viii), scl(A) is closed.

Therefore $scl(A) = cl(scl(A))$.

Hence $cl(A) = scl(A)$.

(x) $\Rightarrow$ (ix)

For any subset A of X, $Int(A) \subset sint(A) \subset A$.

Therefore, $Int(A) = Int(sint(A))$.

If A is semi-closed, by (X) sint(A) is open and so $Int(A) = sint(A)$.

(vii) $\Rightarrow$ (vi)

Let A and B be disjoint semi-open sets in X.

Then scl(A) and scl(B) are semi-open sets.

CLAIM : $scl(A) \cap scl(B) = \phi$.

Suppose $scl(A) \cap scl(B) \neq \phi$.

Then there exists $y \in scl(A)$ and $y \in scl(B)$.

Since $y \in scl(B)$ and as $scl(A)$ is semi-open, $scl(A) \cap B \neq \phi$.

Therefore, there is an $z \in scl(A)$ and $z \in B$.

Therefore, as B is semi-open, $B \cap A \neq \phi$, a contradiction.

Therefore, $scl(A) \cap scl(B) = \phi$.

Using (vii), we get, $cl(A) \cap cl(B) = \phi$.

(vii)$\Longleftrightarrow$ (ix) follows immediately from Theorem 1.2.3.

Before discussing the characterizations of extremally disconnected spaces in terms of preopen sets and semi-preopen sets sets [12], let us give some important properties needed to obtain those characterizations.

THEOREM 1.2.6 [4]

If A is preopen, then $\alpha\text{-}cl(A) = \theta\text{-}cl(A)$.

Theorem 1.2.7 [4]

A topological space (X, τ) is extremally disconnected iff $SO(X) \subset PO(X)$.

THEOREM 1.2.8 [4]

$\tau^{\alpha} = SO(X) \cap PO(X)$.

THEOREM 1.2.9 [4]

If a space (X, τ) is extremally disconnected, then $scl(A) = \theta - cl(A)$ for each $A \in PO(X)$.

PROOF :

Since (X, τ) is extremally disconnected, it follows from Theorems 1.2.7 and 1.2.8 that $SO(X) = \tau^{\alpha}$.

Therefore, $scl(A) = \alpha\text{-}cl(A)$ for each $A \subset X$.

Hence from Theorem 1.2.6 we get $scl(A) = \theta\text{-}cl(A)$ for each $A \in PO(X)$.

THEOREM 1.2.10 [4]

If a topological space (X, τ) is extremally disconnected, then $scl(A) = \theta\text{-}cl(A)$ for each $A \in SO(X)$.

PROOF :

Follows from Theorems 1.2.7 and 1.2.9.

THEOREM 1.2.11 [4]

If A is a subset of a space (X, τ), then $Int(cl(A)) \subset scl(A)$.

THEOREM 1.2.12 [4]

Let A be a subset of a space (X, τ). Then $A \in PO(X)$ iff $scl(A) = Int(cl(A))$.

PROOF :

Let $A \in PO(X)$.

Then $scl(A) \subset scl(Int(cl(A)))$.

Since $Int(cl(A)) \in SC(X)$, $scl(A) \subset Int(cl(A))$.

By Theorem 1.2.11, it follows that $scl(A) = Int(cl(A))$.

The converse is obvious.

THEOREM 1.2.13 [12]

If $A \in PO(X)$, then $cl(A) = \delta\text{-}cl(A) = \theta\text{-}cl(A)$.

PROOF :

By proposition 1.1.36 $cl(A) \subset \delta\text{-}cl(A) \subset \theta\text{-}cl(A)$.

Therefore, it is enough to show that $\theta\text{-}cl(A) \subset cl(A)$.

Take any $x \in X\text{-}cl(A)$.

Then $U \cap A = \phi$, for some open set U containing x.

CLAIM : If U is an open set with $U \cap A = \phi$, then $U \cap cl(A) = \phi$.

Suppose $U \cap cl(A) \neq \phi$.

Then, there exists $y \in U \cap cl(A)$.

$y \in U$ and $y \in cl(A) \Rightarrow U \cap A \neq \phi$ which is a contradiction.

Hence the claim.

Therefore, $U \cap Int(cl(A)) = \phi$.

By the above claim we get that $cl(U) \cap Int(cl(A)) = \phi$.

$cl(U) \cap A = \phi$. (since $A \in PO(X)$).

Therefore, $x \in X-(\theta\text{-}cl(A))$.

Thus $X-cl(A) \subset X-(\theta\text{-}cl(A))$.

That is, $\theta\text{-}cl(A) \subset cl(A)$.

Thus $cl(A) = \delta\text{-}cl(A) = \theta\text{-}cl(A)$.

THEOREM 1.2.14 [12]

If $A \in SPO(X)$, then $cl(A) = \delta\text{-}cl(A)$.

PROOF :

Since $cl(A) \subset \delta\text{-}cl(A)$, it is enough to prove $\delta\text{-}cl(A) \subset cl(A)$.

Assume that $x \in X-cl(A)$, then $U \cap A = \phi$ for some open set U containing x.

Since U is open and $U \cap A = \phi$ as in the proof of the above theorem, we get, $cl(U) \cap Int(cl(A)) = \phi$.

$Int(cl(U)) \cap cl(Int(cl(A))) = \phi$.

$Int(cl(U)) \cap A = \phi$, since A is semi-preopen.

Therefore $x \in X-(\delta\text{-}cl(A))$.

Thus $X - cl(A) \subset X - (\delta\text{-}cl(A))$.

(ie) $\delta\text{-}cl(A) \subset cl(A)$.

Now let us discuss the characterizations due to Noiri [12].

THEOREM 1.2.15 [12]

The following are equivalent for a space X.

(a) X is extremally disconnected.

(b) The closure of every semi-preopen set of X is open.

(c) The δ- closure of every semi-preopen set of X is open.

(d) The δ-closure of every preopen set of X is open.

(e) The θ-closure of every preopen set of X is open.

(f) The closure of every preopen set of X is open.

PROOF :

(a) $\Rightarrow$ (b) follows from (i) $\Rightarrow$ (ii) of Theorem 1.2.5.

(b) $\Rightarrow$ (c).

Let A be a semi-preopen set.

Then cl(A) = δ-cl(A) by Theorem 1.2.14.

By (b), cl(A) is open.

Therefore, the δ-closure of every semi-preopen set of X is open.

(c) $\Rightarrow$ (d) follows from the fact that every preopen set is a semi-preopen set.

(d) $\Rightarrow$ (e) and (e) $\Rightarrow$ (f) follows from theorem 1.2.13.

(f) $\Rightarrow$ (a).

Since every open set is preopen, we get by (f), that X is extremally disconnected.

THEOREM 1.2.16 [12]

The following are equivalent for a space X.

(a) X is extremally disconnected.

(b) scl(A) = θ-cl(A) for every $A \in PO(X) \cup SO(X)$.

(c) scl(A) = cl(A) for every $A \in SPO(X)$.

(d) scl(A) = δ-cl(A) for every $A \in SPO(X)$.

PROOF :

(a) $\Rightarrow$ (b) follows from Theorems 1.2.9 and 1.2.10.

(b) $\Rightarrow$ (a)

First, let A be any preopen set of X.

By Theorem 1.2.12, $Int(cl(A)) = scl(A)$.

By (b), $scl(A) = \theta\text{-}cl(A)$.

By Theorem 1.2.13 $\theta\text{-}cl(A) = cl(A)$.

Hence $Int(cl(A)) = cl(A)$, which in turn$\Rightarrow$that cl(A) is open.

Hence by Theorem 1.2.15, X is extremally disconnected.

(a) $\Rightarrow$ (c)

Let A be a semi-preopen set.

By Theorem 1.2.11, $Int(cl(A)) \subset scl(A)$.

Therefore, $Int(cl(A)) \subset scl(A) \subset cl(A)$.

Since X is extremally disconnected, by Theorem 1.2.15 cl(A) is open.

Therefore, $scl(A) = cl(A)$.

(c) $\Rightarrow$ (d) follows immediately from Theorem 1.2.14.

(d) $\Rightarrow$ (a)

Let U and V be any two disjoint open sets.

By the claim in (vii) $\Rightarrow$ (vi) of Theorem 1.2.5 we get,

$scl(U) \cap scl(V) = \phi$.

Therefore, by (d) and by theorem 1.2.14,

$scl(U) = \delta\text{-}cl(U) = cl(U)$.

Similarly, $scl(V) = cl(V)$.

Therefore, $cl(U) \cap cl(V) = \phi$.

Therefore, X is extremally disconnected.

THEOREM 1.2.17 [12]

The following are equivalent for a space X.

(a) X is extremally disconnected.

(b) If $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then $cl(A) \cap cl(B) = \phi$.

(c) If $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then,

$\delta\text{-}cl(A) \cap \delta\text{-}cl(B) = \phi$.

(d) If $A \in PO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then

$\theta\text{-}cl(A) \cap \delta\text{-}cl(B) = \phi$.

(e) If $A \in PO(X)$, $B \in SO(X)$ and $A \cap B = \phi$, then

$cl(A) \cap cl(B) = \phi$.

PROOF :

(a) $\Longrightarrow$ (b)

Let $A \in SPO(X)$, $B \in SO(X)$ and $A \cap B = \phi$.

Then $A \cap Int(B) = \phi$.

Therefore, $cl(A) \cap Int(B) = \phi$.

By (a), using Theorem 1.2.15, we get, cl(A) is open.

Therefore, $cl(A) \cap cl(Int(B)) = \phi$.

Since $B \in SO(X)$, $B \subset cl(Int(B)) \Longrightarrow cl(B) \subset cl(Int(B))$.

Therefore, $cl(A) \cap cl(B) = \phi$.

(b) $\Longrightarrow$ (c)

If $A \in SPO(X)$, then $cl(A) = \delta\text{-}cl(A)$.

Since every semi-open set is semi-preopen, $cl(B) = \delta\text{-}cl(B)$.

Therefore, $(\delta\text{-}cl(A)) \cap (\delta\text{-}cl(B)) = \phi$.

(c) $\Longrightarrow$ (d)

If $A \in PO(X)$, then by Theorem 1.2.13,

$cl(A) = \delta\text{-}cl(A) = \theta\text{-}cl(A)$.

Therefore, $(\delta\text{-}cl(A)) \cap (\delta\text{-}cl(B)) = \phi$.

$\Longrightarrow$ $(\theta\text{-}cl(A)) \cap (\delta\text{-}cl(B)) = \phi$.

(d) $\Rightarrow$ (e)

$A \in PO(X) \Rightarrow cl(A) = \theta\text{-}cl(A)$ (by Theorem 1.2.13)

$B \in SO(X) \Rightarrow B \in SPO(X)$

$\Rightarrow cl(B) = \delta\text{-}cl(B)$ (by Theorem 1.2.14).

Therefore, by (d), $cl(A) \cap cl(B) = \phi$.

(e) $\Rightarrow$ (a)

Let A and B be disjoint open sets.

Then A is preopen and B is semi-preopen.

Therefore, by (e) $cl(A) \cap cl(B) = \phi$.

Therefore, X is extremally disconnected.

THEOREM 1.2.18 [12]

The following are equivalent for a space X.

(a) X is extremally disconnected.

(b) If $A \in SO(X)$ and $B \in SPO(X)$, then $cl(A) \cap cl(B) = cl(A \cap B)$.

(c) If $A \in SO(X)$ and $B \in SPO(X)$, then $A \cap B \in SPO(X)$.

PROOF :

CLAIM : If U is open in X, then $U \cap cl(S) \subset cl(U \cap S)$, for every subset S of X.

Let $x \in U \cap cl(S)$.

Therefore, $x \in U$ and $x \in cl(S)$.

Let V be any neighbourhood of x.

Then $V \cap S \neq \phi$.

Also, as $V \cap U$ is open and $x \in V \cap U$, $(V \cap U) \cap S \neq \phi$.

That is, $V \cap (U \cap S) \neq \phi$, for every neighbourhood V of x.

Therefore, $x \in cl(U \cap S)$.

Therefore, $U \cap cl(S) \subset cl(U \cap S)$.

Hence the claim.

(a) $\Rightarrow$ (b)

Let $A \in SO(X)$ and $B \in SPO(X)$.

Therefore, by Theorem 1.2.15, cl(B) is open in X and we obtain,

$cl(A) \cap cl(B) \subset cl(Int(A)) \cap cl(B)$.

$\subset cl(Int(A) \cap cl(B))$ (by the above claim).

Let $x \in cl(A) \cap cl(B)$.

Then $x \in cl(Int(A) \cap cl(B))$.

Let U be any neighbourhood of x.

Then $U \cap Int(A) \cap cl(B) \neq \phi$.

$\Rightarrow$ $U \cap Int(A) \cap B \neq \phi$

$\Rightarrow$ $U \cap A \cap B \neq \phi$.

Hence $x \in cl(A \cap B)$.

Therefore, $cl(A) \cap cl(B) \subset cl(A \cap B)$.

Therefore, $cl(A) \cap cl(B) = cl(A \cap B)$.

(b) $\Rightarrow$ (c)

Let $A \in SO(X)$ and $B \in SPO(X)$.

Then we have,

$A \cap B \subset cl(Int(A)) \cap cl(Int(cl(B)))$ (since $A \in SO(X)$ and $B \in SPO(X)$).

$= cl(Int(A) \cap Int(cl(B)))$ (by (b)).

$= cl(Int(A \cap cl(B)))$ (since $Int(A \cap B) = Int(A) \cap Int(B)$).

$\subset cl(Int(cl(A) \cap cl(B)))$ (since $A \subset cl(A)$).

$= cl(Int(cl(A \cap B)))$.

Therefore, $A \cap B \subset cl(Int(cl(A \cap B)))$.

Therefore, $A \cap B \in SPO(X)$.

(c) $\Rightarrow$ (a)

To show that X is extremally disconnected, it is enough to show that $cl(A) \cap cl(B) = cl(A \cap B)$ for all open sets A and B.

Let A and B be open sets of X.

Then $cl(A)$ and $cl(B)$ are semi-open.

Also $cl(B)$ is semi-preopen. Hence by (c), $cl(A) \cap cl(B)$ is semi-preopen.

Therefore, $cl(A) \cap cl(B) \subset cl(Int(cl(cl(A) \cap cl(B))))$.

$\subset cl(Int(cl(cl(A)) \cap cl(cl(B))))$

(since $cl(A \cap B) \subset cl(A) \cap cl(B)$).

$\subset cl(Int(cl(A) \cap cl(B)))$.

$= cl(Int(cl(A)) \cap Int(cl(B)))$

(since $Int\ (A \cap B) = Int(A) \cap Int(B)$).

$\subset cl(cl(A) \cap Int(cl(B)))$.

$\subset cl(cl(A \cap Int(cl(B))))$ (by the claim).

$= cl(A \cap Int(cl(B)))$.

$\subset cl(A \cap cl(B))$.

$\subset cl(A \cap B)$ (by the claim).

Therefore, $cl(A) \cap cl(B) \subset cl(A \cap B)$.

We know that, $cl(A \cap B) \subset cl(A) \cap cl(B)$.

Therefore, $cl(A) \cap cl(B) = cl(A \cap B)$.

Hence, X is extremally disconnected.

SECTION 1.3 CHARACTERIZATIONS OF IRREDUCIBLE SPACES, SEMI-IRREDUCIBLE SPACES, PS-SPACES, PARTITION SPACES, S-SETS, I-SETS, LIGHTLY COMPACT SPACES, ALMOST REGULAR SPACES AND β- COMPACT SPACES.

In this section we discuss the results due to Julian Dontchev [6]. He [6] has obtained a set of equivalent conditions for semi-preopen sets and proved some properties of such sets.

THEOREM 1.3.1 [6]

For a subset A of a space X the following conditions are equivalent:

(1) There exists a preopen set U such that $U \subset A \subset cl(U)$.

(2) $A \subset cl(Int(cl(A))) \Longleftrightarrow cl(A) = cl(Int(cl(A)))$.

(3) $cl(A)$ is regular closed.

(4) $A \subset sint(scl(A))$.

(5) $cl(A)$ is semi-open.

(6) A is dense in a regular closed subspace of X.

(7) A is dense in a semi-open subspace of X.

(8) $scl(A)$ is semi-open.

(9) $pcl(A) \subset cl(Int(cl(pcl(A))))$.

PROOF :

(1) $\Rightarrow$ (2)

By (1), $U \subset A \subset cl(U)$.

$\Rightarrow cl(U) \subset cl(A) \subset cl(U)$

$\Rightarrow cl(U) = cl(A)$.

Since U is a preopen set, $U \subset Int(cl(U)) = Int(cl(A))$.

Therefore, $cl(U) \subset cl(Int(cl(A)))$.

Therefore, $A \subset cl(Int(cl(A)))$.

Hence $cl(A) \subset cl(Int(cl(A)))$.

Since $Int(cl(A)) \subset cl(A)$, we get, $cl(A) = cl(Int(cl(A)))$.

(2) $\Rightarrow$ (3)

By Proposition 1.1.29, A is regular closed iff, $A = cl(Int(A))$.

By (2), $cl(A) = cl(Int(cl(A)))$.

$\Rightarrow cl(A)$ is regular closed.

$(3) \Rightarrow (1)$

Let $U = \text{pint}(A)$.

Then U is preopen and $U \subset A$.

$$\text{Thus } \text{Int}(\text{cl}(A)) = \text{cl}(A) \cap \text{Int}(\text{cl}(A))$$
$$\subset A \cap \text{cl}(\text{Int}(\text{cl}(A))).$$

But $A \cap \text{Int}(\text{cl}(A)) = \text{pint}(A) = U$.

Therefore, $\text{Int}(\text{cl}(A)) \subset U$.

$\Rightarrow$ $\text{cl}(\text{Int}(\text{cl}(A))) \subset \text{cl}(U)$.

By (3), $\text{cl}(A)$ is regular closed.

$\Rightarrow$ $\text{cl}(A) = \text{cl}(\text{Int}(\text{cl}(A)))$.

Therefore, $U \subset A \subset \text{cl}(A) = \text{cl}(\text{Int}(\text{cl}(A))) \subset \text{cl}(U)$.

$\Rightarrow$ $U \subset A \subset \text{cl}(U)$.

$(2) \Rightarrow (4)$

By the definition of semi-closure of A, we have, $A \subset \text{scl}(A)$.

By (2), $A \subset \text{cl}(\text{Int}(\text{cl}(A)))$.

Hence $A \subset \text{scl}(A) \cap \text{cl}(\text{Int}(\text{cl}(A)))$.

By proposition 1.1.20, $\text{scl}(A) \cap \text{cl}(\text{Int}(\text{cl}(A))) = \text{sint}(\text{scl}(A))$.

Therefore, $A \subset \text{sint}(\text{scl}(A))$.

Therefore, $A \subset \text{sint}(\text{scl}(A))$.

$(4) \Rightarrow (2)$

By proposition 1.1.20 , $\text{sint}(\text{scl}(A)) = \text{scl}(A) \cap \text{cl}(\text{Int}(\text{cl}(A)))$.

By (4), $A \subset \text{sint}(\text{scl}(A)) = \text{scl}(A) \cap \text{cl}(\text{Int}(\text{cl}(A))) \subset \text{cl}(\text{Int}(\text{cl}(A)))$.

$\Rightarrow$ $A \subset \text{cl}(\text{Int}(\text{cl}(A)))$.

$(3) \Rightarrow (5)$

It follows from the remark 1.1.28.

(5) $\Rightarrow$ (3)

By (5), cl(A) is semi-open.

That is, $cl(A) \subset cl(Int(cl(A)))$.

Therefore, $cl(A) = cl(Int(cl(A)))$.

Hence cl(A) is regular closed.

(3) $\Rightarrow$ (6)

Since by (3), cl(A) is regular closed, A is dense in the regular closed subspace cl(A) of X.

(6) $\Rightarrow$ (7)

The result follows from the remark 1.1.28.

(7) $\Rightarrow$ (3)

Let B be a semi-open subspace of X in which A is dense.

By the definition of semi-open, we have, $B \subset cl(Int(B))$.

Also, $cl(Int(B)) \subset cl(B)$.

Thus, $cl(B) = cl(Int(B))$.

A is dense in a semi-open subspace B.

$\Rightarrow$ $cl(A) = B$.

Therefore $cl(A) = cl(B) = cl(Int(B)) = cl(Int(cl(A)))$

Hence cl(A) is regular closed.

(5) $\Rightarrow$ (8)

CLAIM : $Int(cl(A)) \subset scl(A) \subset cl(Int(cl(A)))$

By 1.1.19 , $scl(A) = A \cup Int(cl(A))$

we get, $Int\ (cl(A)) \subset scl(A)$.

From the definition of scl(A),

$scl(A) \subset cl(A) = cl(Int(cl(A)))$ [since cl(A) is semi-open by (5)].

Hence $Int(cl(A)) \subset scl(A) \subset cl(Int(cl(A)))$

Therefore, scl(A) is semi-open.

(8) $\Rightarrow$ (2)

Since $scl(A)$ is semi-open, $scl(A) \subset cl(Int(scl(A)))$.

But $scl(A) \subset cl(A)$ $\Rightarrow$ $scl(A) \subset cl(A)$

$\Rightarrow$ $cl(Int(scl(A))) \subset cl(Int(cl(A)))$.

From the definition of $scl(A)$, we have, $A \subset scl(A)$. Combining the above two inclusions, we get,

$A \subset scl(A) \subset cl(Int(scl(A))) \subset cl(Int(cl(A)))$.

$\Rightarrow$ $A \subset cl(Int(cl(A)))$.

(2) $\Rightarrow$ (9)

By (2), $cl(A) = cl(Int(cl(A)))$.

From the definition of $pcl(A)$, $A \subset pcl(A) \subset cl(A)$.

Hence $pcl(A) \subset cl(A) = cl(Int(cl(A)))$

$= cl(Int(cl(pcl(A))))$

Therefore, $pcl(A) \subset cl(Int(cl(pcl(A))))$.

(9) $\Rightarrow$ (2)

By the definition of $pcl(A)$, $A \subset pcl(A) \subset cl(A)$.

By (9), $pcl(A) \subset cl(Int(cl(pcl(A))))$

Combining both, $A \subset pcl(A) \subset cl(Int(cl(pcl(A)))) \subset cl(Int(cl(A)))$

Hence, $A \subset cl(Int(cl(A)))$.

THEOREM 1.3.2 [6]

(1) Semi-open sets = semi-preopen and an NDB set.

(2) Semi-regular sets = semi-preopen and semi-closed.

(3) Regular closed sets = semi-preopen and α- closed

= semi-preopen and closed.

PROOF:

(1) Assume that A is semi-open.

To prove, A is semi-preopen and an NDB set.

A is semi-open $\Rightarrow$ there exists an open set U such that $U \subset A \subset cl(U)$.

U is open $\Rightarrow$ U is preopen.

Therefore there exists a preopen set U such that $U \subset A \subset cl(U)$.

Therefore A is semi-preopen.

Now to prove A is an NDB set.

To prove Fr(A) is nowhere dense.

$$\begin{aligned} Fr(A) &= cl(A) \cap cl(X-A) \\ &\subset cl(U) \cap cl(X-A) \quad [\text{Since } A \subset cl(U)] \\ &\subset cl(U) \cap cl(X-U) \\ &= Fr(U) \quad [\text{since } U \subset A \Rightarrow (X-A) \subset (X-U) \\ &\qquad\qquad \Rightarrow cl(X-A) \subset cl(X-U)] \end{aligned}$$

Hence $Fr(A) \subset Fr(U)$.

$$\begin{aligned} \text{By definition } Fr(U) &= cl(U) \setminus Int(U) \\ &= cl(U) \setminus U \quad [\text{since U is open}] \\ &= \phi. \end{aligned}$$

Consider $Int(cl(Fr(U))) = Int(Fr(U)) = Int(\phi) = \phi$

[since Fr(A) is always a closed set].

Therefore, Fr(U) is nowhere dense.

Hence Fr(A) is nowhere dense.

Therefore, A is an NDB set.

Conversely assume that A is semi-preopen and an NDB set.

A is semi-preopen $\Rightarrow A \subset cl(Int(cl(A)))$.

$$\begin{aligned} Fr(A) \text{ is nowhere dense } &\Rightarrow Int(cl(A)) \subset cl(Int(A)) \\ &\Rightarrow A \subset cl(Int(A)). \end{aligned}$$

Therefore A is semi-open.

(2) Assume that A is semi-preopen and semi-closed.

A is semi-preopen $\Rightarrow$ $A \subset cl(Int(cl(A)))$.

A is semi-closed $\Rightarrow$ $Int(cl(A)) = Int(A)$

[by proposition 1.1.6]

$\Rightarrow$ $A \subset cl(Int(A))$.

Hence A is semi-open.

Hence A is semi-regular. The reverse is trivial.

(3) Assume that A is semi-preopen and α-closed.

Then $A \subset cl(Int(cl(A)))$ and $cl(Int(cl(A))) \subset A$.

Thus $A = cl(Int(cl(A)))$ and hence regular closed.

The reverse is trivial.

CHARACTERIZATIONS IN TERMS OF SEMI-PREOPEN SETS

The following series of theorems give characterizations of various spaces in terms of semi-preopen sets.

THEOREM 1.3.3 [6]

For a space X the following are equivalent.

(1) X is irreducible.

(2) Every non-void semi-preopen subset is dense.

PROOF :

(1) $\Rightarrow$ (2).

Let A be a non-void semi-preopen subset of X.

Then $A \subset cl(Int(cl(A)))$.

Hence $Int(cl(A)) \neq \phi$.

If U were a non-void open set, then by (1),

$U \cap Int(cl(A)) \neq \phi$.

$\Rightarrow U \cap cl(A) \neq \phi \Rightarrow U \cap A \neq \phi$ [since $U \cap cl(A) \neq \phi$

$\Rightarrow$ there exists $x \in U \cap cl(A) \Rightarrow x \in U$

and $x \in cl(A) \Rightarrow U \cap A \neq \phi$].

Therefore A is dense.

(2) $\Rightarrow$ (1)

Let U and V be two non-void open subsets.

Then $\phi \neq V \subset Int(cl(V))$ [since every open set is preopen]

Since every regular open set is semi-preopen, by (2), every regular open set is dense in X.

Therefore, we have $U \cap Int(cl(V)) \neq \phi$.

Therefore, $U \cap cl(V) \neq \phi$. Hence $U \cap V \neq \phi$.

Hence X is irreducible.

REMARK 1.3.4

It is shown that a space is irreducible iff the intersection of two non-void semi-open subsets is non-void. This condition does not hold however for semi-preopen sets, for an indiscrete space (with cardinality at least 2) shows.

THEOREM 1.3.5

[T.Aho and T.Nieminen, Q and A in General Topology 11 (1993), 93-104.]

For a space X the following are equivalent:

(1) X is semi-irreducible.

(2) Every semi-preopen subset is semi-irreducible.

THEOREM 1.3.6 [6]

For a space X the following are equivalent:

(1) X is extremally disconnected.

(2) The closure of every semi-preopen subset is open.

(3) Every semi-preopen subset is preopen.

(4) If A is a semi-preopen subset of X, then $scl(A) = cl(A)$.

(5) If A is a semi-preopen subset of X, then $scl(A) = \delta\text{-}cl(A)$.

(6) If $A \cap B = \phi$, where A is semi-preopen and B is semi-open, then $cl(A) \cap cl(B) = \phi$ (or $\delta\text{-}cl(A) \cap \delta\text{-}cl(B) = \phi$).

(7) If A is semi-open and B is semi-preopen, then $cl(A \cap B) = cl(A) \cap cl(B)$.

(8) The intersection of a semi-open and a semi-preopen set is always semi-preopen.

(9) Every semi-preopen subset is a CO-set.

PROOF :

(1) $\Rightarrow$ (2)

If $A \subset X$ is semi-preopen, then cl(A) is regular closed.

That is, $cl(A) = cl(Int(cl(A)))$

= closure of an open set.

Hence by (1), cl(A) is an open set.

(2) $\Rightarrow$ (3).

Let A be a semi-preopen subset.

By (2), cl(A) is open $\Rightarrow Int(cl(A)) = cl(A)$.

Since $A \subset cl(A) = Int(cl(A))$, A is preopen.

(3) $\Rightarrow$ (1)

Let A be an open set.

Then cl(A) is semi-open [by proposition 1.1.3]

$\Rightarrow$ cl(A) is semi-preopen.

By (3), cl(A) is preopen $\Rightarrow$ cl(A)$\subset$ Int(cl(A))

Therefore, cl(A) = Int(cl(A))

$\Rightarrow$ cl(A) is open. Hence X is extremally disconnected.

(4) $\Rightarrow$ (5) follows from Theorem 1.2.14.

(5) $\Rightarrow$ (6) follows from Theorems 1.2.16 and 1.2.17.

(6) $\Rightarrow$ (1) follows from Theorem 1.2.17.

(1) $\Rightarrow$ (5) follows from Theorem 1.2.16.

(6) $\Rightarrow$ (7)

Since (6) $\Rightarrow$ (1), X_α is extremally disconnected (7) follows from Theorem 1.2.18.

(7) $\Rightarrow$ (8) follows from Theorem 1.2.18

(7) $\Rightarrow$ (6) follows from Theorem 1.2.17.

(2) $\Leftrightarrow$ (9) is obvious from the definition of a CO-set.

THEOREM 1.3.7 [6]

For a space X the following are equivalent:

(1) X is a PS-space.

(2) Every semi-preopen subset is semi-open.

(3) Every non-empty semi-preopen subset is irresolvable.

(4) If A is semi-preopen, then cl(A) = A $\cup$ cl(Int(A)) = pcl(A).

(5) Every semi-preopen subset is a non-residual set.

PROOF :

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is proved in T.Aho and T.Nieminen, Ricerche Mat., 1994.

(2) $\Rightarrow$ (4)

Let A be a semi-preopen subset.

By (2), A is semi-open.

Therefore, $A \subset cl(Int(A)) \Rightarrow cl(A) \subset cl(Int(A))$ and

$A \cup cl(Int(A)) = cl(Int(A))$

Therefore $cl(A) = cl(Int(A))$ and

$A \cup cl(Int(A)) = pcl(A)$ [by (b) of proposition 1.1.19.]

Therefore, $cl(A) = A \cup cl(Int(A)) = pcl(A)$.

(4) $\Rightarrow$ (1)

Let $A \subset X$ be preopen.

Then by (4), $A \subset Int(cl(A)) = Int(pcl(A)) = Int(cl(Int(A)))$.

Thus A is α-open.

Therefore, X is a PS-space, since every α-open set is semi-open.

(2) $\Rightarrow$ (5)

Let A be a non-void semi-preopen set.

By (2), A is semi-open $\Rightarrow A \subset cl(Int(A))$.

To prove $Int(A) \neq \phi$.

Suppose $Int(A) = \phi$.

$\Rightarrow cl(Int(A)) = \phi$

$\Rightarrow A = \phi$, which is a contradiction.

Hence $Int(A) \neq \phi$.

(5) $\Rightarrow$ (1)

Let A be a preopen set.

Let $U = Int(cl(A)) \setminus cl(Int(A))$ which is open.

Therefore, $U \cap A$ is open in A and hence preopen in A.

Therefore, by transitivity $U \cap A$ is preopen in X.

$\Rightarrow U \cap A$ is semi-preopen in X.

Since $Int(U \cap A) = Int(U) \cap Int(A) = U \cap Int(A) = \phi$,

by the definition of U.

Hence by (5), $U \cap A = \phi$

We claim that $A \setminus cl(Int(A)) = \phi$.

Suppose $A \setminus cl(Int(A)) \neq \phi$.

Therefore, there exists x such that $x \in A \setminus cl(Int(A))$.

That is, $x \in A$ and $x \notin cl(Int(A))$.

Since A is preopen, $A \subset Int(cl(A))$ and hence $x \in Int(cl(A))$.

Therefore $x \in Int(cl(A)) \setminus cl(Int(A))$.

That is $x \in U$. Hence $x \in U \cap A \Rightarrow U \cap A \neq \phi$,

which is a contradiction.

Therefore, $A \setminus cl(Int(A)) = \phi$.

Therefore, $A \subset cl(Int(A))$.

Hence A is semi-open in X.

The proofs of the following two theorems are found in T.Aho and T.Nieminen, Ricerche Mat., 1994.

THEOREM 1.3.8 [6]

For a space X the following are equivalent.

(1) X is an extremal PS-space.

(2) Every semi-preopen subspace is α-open.

THEOREM 1.3.9 [6]

For a space X the following are equivalent.

(1) X is an irreducible PS-space.

(2) The intersection of two non-void semi-preopen subsets is always non-void.

(3) X is not the disjoint union of two non-void semi-preopen subsets.

THEOREM 1.3.10 [5]

Let x be a point of (X, τ). Then either $\{x\}$ is nowhere dense or $(x) \subset \mathrm{Int}(\mathrm{cl}\,\{x\}) = \mathrm{scl}\,\{x\}$.

PROOF :

Suppose that $\{x\}$ is not nowhere dense.

Then $\mathrm{Int}(\mathrm{cl}\,\{x\}) \neq \phi$.

Therefore $x \in \mathrm{Int}(\mathrm{cl}\,\{x\})$.

By (a) of proposition 1.1.19 we get,

$\mathrm{scl}\,\{x\} = \{x\} \cup \mathrm{Int}(\mathrm{cl}\,\{x\}) = \mathrm{Int}(\mathrm{cl}\,\{x\})$

Thus, $\{x\} \subset \mathrm{Int}(\mathrm{cl}\,\{x\}) = \mathrm{scl}\,\{x\}$.

THEOREM 1.3.11 [6]

For a space X the following are equivalent:

(1) X is a partition space.

(2) Every nowhere dense subset is semi-preopen.

(3) Every singleton is semi-preopen.

(4) Every subset is semi-preopen.

(5) Every closed subset is semi-preopen.

PROOF :

(1) $\Rightarrow$ (4)

By (1), every subset of X is preopen and hence semi-preopen. Hence (4).

(4) $\Rightarrow$ (3) obvious.

(3) $\Rightarrow$ (1) Let $A \subset X$.

To show that A is preopen.

By Theorem 1.3.10, we have each singleton is either preopen or nowhere dense.

Let $\{a\} \subset X$. If $\{a\}$ is nowhere dense, $Int(cl\{a\}) = \phi$.

By (3), $\{a\} \subset cl(Int(cl\{a\})) = \phi$, which is impossible.

Thus each singleton in X is preopen and hence $A = \bigcup_{a \in A} \{a\}$, union of preopen sets.

Therefore A is preopen. Hence X is a partition space.

(4) $\Rightarrow$ (2) , (4) $\Rightarrow$ (5) obvious.

(2) $\Rightarrow$ (3) Let $\{a\} \subset X$.

If $\{a\}$ is preopen, then clearly it is semi-preopen. If it is nowhere dense, then by (2) it is semi-preopen.

(5) $\Rightarrow$ (4) Let $A \subset X$.

BY (5), cl(A) is semi-preopen.

Therefore, $cl(A) = cl(Int(cl(A)))$.

That is, $A \subset cl(Int(cl(A)))$.

Therefore, A is semi-preopen.

THEOREM 1.3.12 [6]

For a space the following are equivalent:

(1) $A \subset X$ is an S-set.

(2) Every semi-preopen cover of A has a finite dense subsystem.

PROOF: (1) $\Rightarrow$ (2)

Let $\{A_i\}_{i \in I}$ be a semi-preopen cover of A.

Then $A \subset \bigcup_{i \in I} A_i$ and each A_i is semi-preopen.

Since A_i is semi-preopen, $cl(A_i)$ is regular closed.

Therefore $A \subset \bigcup_{i \in I} A_i \Rightarrow A \subset \bigcup_{i \in I} cl(A_i)$.

Then it follows from the definition of an S-set, $A \subset \bigcup_{i\in J} cl(A_i)$, where J is finite.

Therefore, every semi-preopen cover of A has a finite dense subsystem.

(2) ⇒ (1)

Let $A \subset \bigcup_{i\in I} A_i$, where each A_i is regular closed.

Since every regular closed set is semi-preopen, we get $\{A_i\}_{i\in I}$ is a semi-preopen cover of A.

Therefore by (2), $A \subset \bigcup_{i\in J} cl(A_i)$, where J is finite.

Since a regular closed set is closed, $cl(A_i) = A_i$.

Therefore, $A \subset \bigcup_{i\in J} A_i$. Therefore A is an S-set.

THEOREM 1.3.13 [6]

For a space X, the following are equivalent:

(1) $A \subset X$ is an I-set.

(2) Every semi-preopen cover of A has a finite interior dense subsystem.

PROOF: (1) ⇒ (2)

Let $A \subset X$ be an I-set.

Let $A \subset \bigcup_{i\in I} A_i$, where each A_i is semi-preopen in X.

Since A_i is semi-preopen, $cl(A_i)$ is regular closed.

$\therefore A \subset \bigcup_{i \in I} A_i \Rightarrow A \subset \bigcup_{i \in I} cl(A_i)$.

Then it follows from the definition of an I-set,

$A \subset \bigcup_{i \in J} Int(cl(A_i))$, where J is finite.

$\therefore$ Every semi-preopen cover of A has a finite interior dense subsystem.

(2) $\Rightarrow$ (1)

Let $A \subset \bigcup_{i \in I} A_i$, where each A_i is regular closed.

$\therefore A_i$ is semi-preopen.

By (2), $A \subset \bigcup_{i \in J} Int(cl(A_i))$, where J is finite.

$\therefore A \subset \bigcup_{i \in J} Int(A_i)$, ($cl(A_i) = A_i$, since A_i is regular closed.)

$\therefore A \subset X$ is an I-set.

THEOREM 1.3.14 [6]

For a space X the following are equivalent:

(1) X is lightly compact.

(2) Every non-void locally finite family $(A_i)_{i \in I}$ of semi-preopen sets is finite.

PROOF:

Let $(A_i)_{i \in I}$ be a non-void locally finite family of semi-preopen sets.

Since $\phi \neq A_i \subset cl(Int(cl(A_i))$ (since A_i is semi-preopen) then $Int(cl(A_i) \neq \phi$, for each index i.

Consider any $x \in X$. By the definition of locally finite, there exists an open neighbourhood U of x and a finite subset I_0 of I such that $U \cap A_j \neq \phi$ for $j \in I_0$ and $U \cap A_j = \phi$ for $j \in I$.

CLAIM : $U \cap cl(A_j) = \phi$ for $j \notin I_0$.

Suppose $U \cap cl(A_j) \neq \phi$.

Let $y \in U \cap cl(A_j) \Rightarrow y \in U$ and $y \in cl(A_j)$.

$y \in cl(A_j) \Rightarrow$ every neighbourhood of y intersects A_j.

That is, $U \cap A_j \neq \phi$, which is a contradiction.

Therefore $U \cap cl(A_j) = \phi$ for $j \notin I_0$.

Hence $\{cl(A_i)\}_{i \in I}$ is a locally finite family and since $Int(cl(A_j)) \subset cl(A_j)$, $U \cap Int(cl(A_j)) = \phi$ for $j \notin I_0 \Rightarrow Int(cl(A_j))_{j \in I}$ is locally finite.

Since X is lightly compact, I is finite.

(2) $\Rightarrow$ (1).

Let $(A_i)_{i \in I}$ be a locally finite family of non-void open sets.

Then $(cl(A_i))_{i \in I}$ is a locally finite family of non-void regular open sets.

Since every regular open set is semi-preopen, by (2), I is finite.

Hence X is lightly compact.

THEOREM 1.3.15 [6]

For a space the following are equivalent:

(1) X is an almost regular space.

(2) If A is semi-preopen, then $\theta\text{-}cl(A) = cl(A)$.

PROOF :

(1) ⟹ (2)

By (1), X is an almost regular space.

That is, for every regular closed subset $A \subset X$, we have

θ - cl(A) = cl(A).

Let A be a semi-preopen set.

Then cl(A) is regular closed [by Theorem 1.3.1].

Therefore by (1), cl(A) = θ-cl(cl(A)).

Since $cl(A) \subset \theta\text{-}cl(A) \subset \theta\text{-}cl(cl(A)) = cl(A)$, $cl(A) = \theta\text{-}cl(A)$.

(2) ⟹ (1)

Let A be any regular closed set.

Since every regular closed set is semi-preopen, by (2), θ-cl(A) = cl(A). Hence X is an almost regular space.

THEOREM 1.3.16 [6]

For a space X the following are equivalent:

(1) X is β- compact.

(2) Every semi-preopen cover of X has a finite subcover.

PROOF :

Since semi-preopen sets are nothing else but β-open sets, the result is obvious.

SECTION 1.4 CHARACTERIZATIONS OF MAPS IN TERMS OF SEMI-PREOPEN SETS

First let us give definitions of various maps.

DEFINITION 1.4.1

A map $f: X \rightarrow Y$ is called **almost continuous** iff the preimage of every regular open set is open.

DEFINITION 1.4.2

A map $f: X \rightarrow Y$ is called **α-continuous** iff the preimage of every open set is an α-set (or) iff $f(\alpha\text{-cl}(A)) \subset \text{cl}(f(A))$ for each $A \subset X$ (or) iff $f(\text{cl}(\text{Int}(\text{cl}(A)))) \subset \text{cl}(f(A))$ for each $A \subset X$.

DEFINITION 1.4.3

A map $f: X \rightarrow Y$ is called **almost weakly continuous** iff for every open $B \subset Y$, we have,

$f^{-1}(B) \subset \text{Int}(\text{cl}(f^{-1}(B)))$ (or) iff

$\text{pcl}(f^{-1}(B)) \subset f^{-1}(\text{cl}(B))$.

DEFINITION 1.4.4

A map is called **almost closed** iff the image of every regular closed set is closed.

DEFINITION 1.4.5

A map $f: X \rightarrow Y$ is called **semi-preclosed** iff the image of every closed set is semi-preclosed.

DEFINITION 1.4.6

A map $f: X \rightarrow Y$ is called **quasi-irresolute** iff when $x \in X$ and $f(x) \in V \in SO(Y)$, then for some $U \in SO(X)$ such that $x \in U$, we have $f(U) \subset \text{scl}(V)$ or equivalently iff the preimage of every semi-regular set is semi-regular.

THEOREM 1.4.7 [6]

For a map f: $X \to Y$ the following are equivalent:

(1) f is almost continuous.

(2) If $B \subset Y$ is semi-preopen, then $cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

PROOF :

(1) $\Rightarrow$ (2)

Let $B \in SPO(Y)$, then cl(B) is regular closed.

Therefore $f^{-1}(cl(B))$ is closed, since f is almost continuous.

That is, $cl(f^{-1}(B)) \subset cl(f^{-1}(cl(B))) = f^{-1}(cl(B))$.

$\Rightarrow$ $cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

(2) $\Rightarrow$ (1)

Let B be any regular closed set in Y.

Then B is semi-open and hence semi-preopen.

Therefore by (2), $cl(f^{-1}(B)) \subset f^{-1}(cl(B))$.

Since B is regular closed, B is closed.

Therefore $cl(f^{-1}(B)) \subset f^{-1}(cl(B)) = f^{-1}(B)$.

Hence $f^{-1}(B)$ is closed. Hence f is almost continuous.

THEOREM 1.4.8 [6]

For a map f: $X \to Y$ the following are equivalent:

(1) f is α - continuous.

(2) If $A \subset X$ is semi-preopen, then $f(cl(A)) \subset cl(f(A))$.

PROOF :

(1) $\Rightarrow$ (2)

Since A is semi-preopen, $cl(A) = cl(Int(cl(A)))$.

Therefore, $f(cl(A)) = f(cl(Int(cl(A)))) \subset cl(f(A))$.

Therefore, $f(cl(A)) \subset cl(f(A))$.

(2) $\Rightarrow$ (1)

Since cl(pint(A)) = cl(Int(cl(A))), it is enough to prove that f(cl(pint(A))) $\subset$ cl(f(A)).

Since pint(A) is preopen, it is semi-preopen.

Therefore by (2), f(cl(pint(A)))$\subset$cl(f(pint(A)))$\subset$cl(f(A)).

Hence f is α- continuous.

THEOREM 1.4.9 [6]

For a map f: X$\rightarrow$Y the following are equivalent:

(1) f is almost weakly continuous.

(2) If B$\subset$Y is semi-preopen, then pcl(f^{-1}(Int(cl(B)))) $\subset$ f^{-1}(cl(B)).

PROOF :

(1) $\Rightarrow$ (2)

Let B ε SPO(Y) $\Rightarrow$ cl(B) = cl(Int(cl(B))).

Since Int(cl(B)) is open, by (1),

pcl(f^{-1}(Int(cl(B)))) $\subset$ f^{-1}(cl(Int(cl(B)))) = f^{-1}(cl(B)).

Hence (2).

(2) $\Rightarrow$ (1)

Let B be an open set in y.

Then B is preopen $\Rightarrow$ B$\subset$Int(cl(B)). ....(i)

As B is also semi-preopen, by (2),

Pcl f^{-1}(Int(cl(B)))$\subset$$f^{-1}$(cl(B)). ....(ii)

Combining (i) and (ii), we get,

pcl(f^{-1}(B)) $\subset$ pcl(f^{-1}(Int(cl(B)))) $\subset$ f^{-1}(cl(B)).

Hence pcl(f^{-1}(B)) $\subset$ f^{-1}(cl(B)).

Hence f is almost weakly continuous.

THEOREM 1.4.10 [6]

For a map f: $X \to Y$ the following are equivalent:

(1) f is almost closed.

(2) If $A \subset X$ is semi-preopen, then $cl(f(A)) \subset f(cl(A))$.

PROOF :

(1) ⇒ (2)

Let A be a semi-preopen set.

Then cl(A) is regular closed.

Therefore by (1), f(cl(A)) is closed.

Therefore $cl(f(A)) \subset cl(f(cl(A))) = f(cl(A))$.

Hence $cl(f(A)) \subset f(cl(A))$.

(2) ⇒ (1)

Let $A \subset X$ be any regular closed set.

Therefore A is semi-preopen.

By (2), $cl(f(A)) \subset f(cl(A)) = f(A)$ (since every regular closed set is closed).

Therefore, f(A) is closed.

THEOREM 1.4.11 [6]

For a map f: $X \to Y$ the following are equivalent:

(1) f is α - continuous and almost closed.

(2) If $A \subset X$ is semi-preopen, then $cl(f(A)) = f(cl(A))$.

PROOF :

(1) ⇒ (2)

f is α - continuous ⇒ $f(cl(A)) \subset cl(f(A))$, for every $A \in SPO(Y)$ (by Theorem 1.4.8).

f is almost closed ⇒ $cl(f(A)) \subset f(cl(A))$, for every $A \in SPO(Y)$ (by Theorem 1.4.10).

Therefore $cl(f(A)) = f(cl(A))$ for every $A \in SPO(X)$.

(2) $\Rightarrow$ (1).

By (2), if $A \subset X$ is semi-preopen, then $cl(f(A)) = f(cl(A))$.

Hence by Theorems 1.4.8 and 1.4.10, f is α- continuous and almost closed.

THEOREM 1.4.12 [6]

For a map $f: X \to Y$ the following are equivalent:

(1) f is semi-preclosed.

(2) If $f^{-1}\{y\} \subset U$, where U is open and $y \in Y$, then there exists a semi-preopen set $V \subset Y$ such that $y \in V$ and $f^{-1}(V) \subset U$.

PROOF :

(1) $\Rightarrow$ (2)

Let U be an open set, $y \in Y$ and $f^{-1}\{y\} \subset U$.

Then $X \setminus U$ is closed and hence by (1), $f(X \setminus U)$ is semi-preclosed.

Therefore $Y \setminus f(X \setminus U) = V$ is semi-preopen.

Since $f^{-1}\{y\} \subset U$, $X \setminus U \subset X \setminus f^{-1}\{y\} = f^{-1}(Y \setminus \{y\})$

and hence $f(X \setminus U) \subset f(f^{-1}(Y \setminus \{y\})) \subset Y \setminus \{Y\}$.

That is, $\{y\} \subset Y \setminus f(X \setminus U) = V$.

On the other hand, $X \setminus U \subset f^{-1}(f(X \setminus U)) = f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$.

That is $f^{-1}(V) \subset U$.

(2) $\Rightarrow$ (1)

Let $A \subset X$ be closed.

To prove f(A) is semi-preclosed.

That is, to prove $Y \setminus f(A)$ is semi-preopen.

Let $y \in Y \setminus f(A)$.

Then $f^{-1}\{y\} \subset f^{-1}(Y \setminus f(A)) = X \setminus f^{-1}(f(A)) \subset X \setminus A$.

$f^{-1}\{y\} \subset X \setminus A$, where $X \setminus A$ is open.

By (2), there is a semi-preopen set V such that

$y \in V$ and $f^{-1}(V) \subset X \setminus A$.

Thus $f^{-1}(V) \cap A = \phi$ and hence $V \cap f(A) = \phi$.

Therefore $y \in V \subset Y \setminus f(A)$.

$\Rightarrow$ $Y \setminus f(A)$ is semi-preopen.

Therefore f(A) is semi-preclosed.

THEOREM 1.4.13 [6]

For a map $f: X \to Y$ the following are equivalent:

(1) f is quasi - irresolute

(2) If $x \in X$ and $f(x) \in B \in SO(Y)$, Then there exists $A \in SPO(X)$ such that $x \in A$ and $f(A) \subset scl(B)$.

PROOF :

(1) $\Rightarrow$ (2) follow from the fact that every semi-open set is semi-preopen.

(2) $\Rightarrow$ (1).

Let $f(x) \in V \in SO(Y)$. Thus $V \in SPO(Y)$. The set scl(V) is semi-open and hence semi-regular, since it is also semi-closed.

By (2), $f^{-1}(scl(V)) = U$ is semi-regular, hence semi-open, such that $x \in U$ and $f(U) \subset scl(V)$.

Hence (1).

Chapter II

CHAPTER - II

A STUDY OF AN INTERESTING GENERALIZATION OF PARACOMPACTNESS

This chapter is devoted to the study of some properties of α-paracompact and α-nearly paracompact subsets and almost closed mappings. The results are due to Kovacevic [2,3].

SECTION 2.1 PRELIMINARY DEFINITIONS AND RESULTS

DEFINITION 2.1.1

Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be **locally finite at a point** x in X if it has a neighbourhood that intersects only finitely many elements of $\mathcal{A}$. The collection $\mathcal{A}$ is said to be **locally finite** if $\mathcal{A}$ is locally finite at each and every point of X.

DEFINITION 2.1.2

A subset A of a space X is **α-paracompact (α-nearly paracompact)** iff for every open (regularly open) cover $\mathcal{U}$ of A, there exists an open locally finite family $\mathcal{V}$ which refines $\mathcal{U}$ and covers A.

REMARK 2.1.3

Every α - paracompact set is α-nearly paracompact.

This result follows easily from the fact that every regular open cover is an open cover.

DEFINITION 2.1.4

A subset A of a space X is **α-paracompact (α-nearly paracompact) with respect to a subset B** iff for every open (regular open) cover $\mathcal{U} = \{U_i : i \in I\}$ of A, there is an open family $\mathcal{V} = \{V_j : j \in J\}$ such that:

(a) $\mathcal{V}$ refines $\mathcal{U}$, (b) $A \subset \cup \{V_j : j \in J\}$, and (c) $\mathcal{V}$ is locally finite at each point, $x \in B$.

DEFINITION 2.1.5

Subsets A and B of a space X are **mutually α-paracompact (mutually α-nearly paracompact)** iff the subset A is α-paracompact (α-nearly paracompact) with respect to the subset B and B is α-paracompact (α-nearly paracompact) with respect to the subset A.

DEFINITION 2.1.6

A subset A of a space X is **α-Hausdorff** iff for any two points, a,b of a space X, where $a \in A$ and $b \in X \setminus A$, there are disjoint open sets U and V containing a and b respectively.

DEFINITION 2.1.7

A subset A of a space X is **α-regular (α-almost regular)** iff for any point $a \in A$ and any open (regularly open) subset U containing a there exists an open subset V of X such that $a \in V \subset cl(V) \subset U$.

DEFINITION 2.1.8

A mapping f: X → Y is **almost closed (almost open)** iff for any regularly closed (regularly open) subset F of X, f(F) is closed (open) in Y.

DEFINITION 2.1.9

A mapping $f: X \rightarrow Y$ has a **closed graph** $G(f)$ iff $G(f) = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$.

SECTION 2.2 PROPERTIES OF α-PARACOMPACT AND α-NEARLY PARACOMPACT SUBSETS.

THEOREM 2.2.1 [2]

Let $\mathcal{U} = \{U_i : i \in I\}$ be any family of subsets of a space X. If $x \in cl(\cup\{U_i : i \in I\}) \setminus \cup\{cl(U_i) : i \in I\}$, then $\mathcal{U}$ is not locally finite at x.

PROOF:

Suppose that $\mathcal{U}$ is locally finite at x.

Then there exists an open set M such that $x \in M$ and M intersects finitely many members of $\mathcal{U}$.

Let I_0 be a finite subset of I such that,

$M \cap U_i \neq \phi,\ i \in I_0\ ;\ M \cap U_i = \phi,\ i \in I \setminus I_0$.

Let $U_1 = \cup\{U_i : i \in I_0\}\ ;\ V_1 = \cup\{U_i : i \in I \setminus I_0\}$.

$cl(\cup\{U_i : i \in I\}) = cl(U_1) \cup cl(V_1)$.

Since $M \cap V_1 = \phi$, $x \notin cl(V_1)$.

$$\text{Hence } x \in cl(U_1) = cl(\cup\{U_i : i \in I_0\}) = \cup\{cl(U_i) : i \in I_0\},$$

which is a contradiction,

$\therefore \mathcal{U}$ is not locally finite at x.

THEOREM 2.2.2 [2]

Let A be an α-Hausdorff α-nearly paracompact subset relative to a point $x \in X \setminus A$. Then there exist disjoint regularly open neighbourhoods of x and A. Consequently, if an α-Hausdorff subset A is α-nearly paracompact relative to each point of $X\setminus A$, then A is closed.

PROOF:

Since A is α-Hausdorff, for each a in A there exists disjoint open sets U_a and V_a such that $a \in U_a$ and $x \in V_a$.

Then $\mathcal{U} = \{ \alpha(U_a) : a \in A \}$, $(\alpha(U) = \text{Int}(\text{cl}(U)))$ is a regularly open covering of A.

Since A is α-nearly paracompact relative to x there exists an open family $\mathcal{V} = \{ V_i : i \in I\}$ such that

(a) $\mathcal{V}$ refines $\mathcal{U}$ (b) $\mathcal{V}$ covers A (c) $\mathcal{V}$ is locally finite at the point x.

By (c), there exists an open subset M_x conttaining x such that M_x intersects finitely many members of $\mathcal{V}$.

Let $I_0 \subset I$ be a finite subset of I such that

$M_x \cap V_i \neq \phi$, $i \in I_0$; $M_x \cap V_i = \phi$; $i \in I \setminus I_0$.

For each $i \in I$ there exists $a_i \in A$ such that $V_i \subset \alpha(U_{a_i})$.

Let $U = \cup \{ V_i : i \in I\}$ and $V = M_x \cap (\cap \{V_{a_i} : i \in I_0\})$.

Then $\alpha(U)$ and $\alpha(V)$ are regularly open disjoint neighbourhoods of A and x respectively.

COROLLARY 2.2.3 [2]

Let A be an α-Hausdorff α-paracompact subset relative to a point $x \in X\setminus A$. Then there exist disjoint regularly open neighbourhoods of x and A. Consequently if an α-Hausdorff subset A is α-paracompact relative to each point of $X\setminus A$, then A is closed.

PROOF:

The proof follows from Remark 2.1.3.

COROLLARY 2.2.4 [2]

Every α-Hausdorff α-nearly paracompact (α-paracompact) subset is closed.

THEOREM 2.2.5 [2]

For any two disjoint subsets A and B of a space X, where the subset A is α-Hausdorff α-nearly paracompact relative to each point of B, and the subset B is α-nearly paracompact relative to the subset A, there exist disjoint regularly open neighbourhoods of A and B respectively.

PROOF:

By Theorem 2.2.2, for every $x \in B$ there exists disjoint open sets U_x and V_x such that $x \in U_x$ and $A \subset V_x$.

Consider the family, $\mathcal{U} = \{ \alpha(U_x) : x \in B\}$.

It is a regularly open covering of the subset B.

Since B is α- nearly paracompact relative to the subset A, there exists an open family $\mathcal{V}$ such that

(a) $\mathcal{V}$ refines $\mathcal{U}$ (b) $\mathcal{V}$ covers B (c) $\mathcal{V}$ is locally finite at each point of the subset A.

Let $V = \cup \{ V : V \in \mathcal{V}\}$.

Then $B \subset V$, $A \subset X \setminus cl(V) = U$.

$A \subset U = Int(U) \subset Int(cl(U)) = \alpha(U)$.

Similarly, $B \subset \alpha(V)$.

$\therefore$ $\alpha(U)$ and $\alpha(V)$ are disjoint regularly open neighbourhoods of A and B respectively.

THEOREM 2.2.6 [2]

Let A be an α-regular subset and U be an open neighbourhood of A. If the subset A is α-paracompact relative to $X \setminus U$, then there exists an open neighbourhood V of A such that $A \subset V \subset cl(V) \subset U$.

PROOF :

Since A is an α-regular subset, for each point $x \in A$ there exists an open set W_x such that $x \in W_x \subset cl(W_x) \subset U$.

Consider the family, $\mathcal{W} = \{ W_x : x \in A \}$.

Clearly this is an open covering of A.

Since A is α-paracompact relative to $X \setminus U$, there exists an open family $\mathcal{V}$ such that (a) $\mathcal{V}$ refines $\mathcal{W}$ (b) $\mathcal{V}$ covers A (c) $\mathcal{V}$ is locally finite at each point of $X \setminus U$.

Let $V = \cup \{V_i : V_i \in \mathcal{V}\}$.

Then $A \subset V \subset cl(V) \subset U$.

THEOREM 2.2.7 [2]

Let A be an α-almost regular subset of a space X. Let U be any regularly open neighbourhood of A. If the subset A is α-nearly paracompact relative to $X \setminus U$, then there exists an open neighbourhood V of A such that $A \subset V \subset cl(V) \subset U$.

PROOF :

It is similar to the proof of Theorem 2.2.6.

COROLLARY 2.2.8 [2]

If U is an open (α regularly open) α-regular (α - almost regular) subset which is α-paracompact (α-nearly paracompact) to $X \setminus U$, then U is closed.

PROOF :

Taking $A = U$ in the above theorems we get an open set V such that $U \subset V \subset cl(V) \subset U$.

$\therefore U = cl(V)$.

Hence U is closed. ◆

SECTION 2.3 ALMOST CLOSED MAPPINGS

THEOREM 2.3.1 [2]

Let $f:X \rightarrow Y$ be an almost closed mapping of a space X onto a space Y, such that the family $\{ f^{-1}(y) : y \in Y \}$ consists of α - Hausdorff subsets which are mutually α-nearly paracompact. Then Y is Hausdorff.

PROOF :

Let y_1 and y_2 be any two distinct points of Y. By Theorem 2.2.2, there exists disjoint regularly open neighbourhoods U_1 and U_2 of $f^{-1}(y_1)$ and $f^{-1}(y_2)$ respectively.

Since f is almost closed, there exist open sets V_1 and V_2 containing y_1 and y_2 respectively such that $f^{-1}(y_1) \subset f^{-1}(V_1) \subset U_1$; $f^{-1}(y_2) \subset f^{-1}(V_2) \subset U_2$. Since f is onto, $y_1 \in V_1$, $y_2 \in V_2$, Y is Hausdorff.

The following result which has been proved by Kovacevic [1] is used to prove theorem 2.3.3.

THEOREM 2.3.2 [2]

If $f: X \rightarrow Y$ is an almost closed mapping of a space X onto a compact space Y such that $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset for each point $y \in Y$, then f is continuous.

THEOREM 2.3.3 [2]

Let X be any topological space such that every closed subset is α-Hausdorff. Let $f: X \rightarrow Y$ be a closed mapping of a space X on to a compact space Y such that for each point $y \in Y$, $f^{-1}(y)$ is an α-Hausdorff α-paracompact subset of a space X. Then X is regular.

PROOF :

Since every closed mapping is almost closed and by Remark 2.1.3 and Theorem 2.3.1 we get that, the space Y is Hausdorff.

Since every compact Hausdorff space is regular, Y is regular.

Since $f^{-1}(y)$ is an α-Hausdorff α-paracompact subset of a space X, by Theorem 2.3.2, f is continuous. Now, let A be any closed subset of the space X and x be any point such that $x \notin A$.

$\Longrightarrow$ Either $f(X) \notin f(A)$ or $f(x) \in f(A)$.

Suppose $f(x) \notin f(A)$.

Since Y is regular, there exist disjoint open sets U and V containing $f(x)$ and $f(A)$ respectively. Now $f^{-1}(U)$ and $f^{-1}(V)$ are open sets such that $x \in f^{-1}(U)$, $A \subset f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$.

Hence X is regular.

Suppose $f(x) \in f(A)$.

CLAIM : $A \cap f^{-1}(f(x)) \neq \phi$

$f(x) \in f(A)$

$\Rightarrow$ $f(x) = f(a)$, for some $a \in A$.

$\Rightarrow$ $a \in f^{-1}(f(x))$

$\Rightarrow$ $A \cap f^{-1}(f(x)) \neq \phi$.

Since $x \notin A$, $x \notin A \cap f^{-1}(f(x))$.

$f^{-1}(f(x))$ is an α-Hausdorff α-paracompact subset of a space X.

Hence $f^{-1}(f(x))$ is closed.

Then $A \cap f^{-1}(f(x))$ is a closed α-Hausdorff α-paracompact subset of X.

Hence by Theorem 2.2.2, there exist disjoint open sets U_1 and V_1 such that $x \in U_1$, $A \cap f^{-1}(f(x)) \subset V_1$.

Since $A \setminus V_1$ is closed, then $f(A \setminus V_1)$ is closed subset such that $f(x) \notin f(A \setminus V_1)$.

There exist disjoint open sets U_2 and V_2 such that

$f(x) \in U_2$ and $f(A \setminus V_1) \subset V_2$.

Let $U = U_1 \cap f^{-1}(U_2)$ and $V = V_1 \cap f^{-1}(V_2)$.

U and V are disjoint open sets such that $x \in U$ and $A \subset V$.

Hence the result.

THEOREM 2.3.4 [3]

Let $f: X \rightarrow Y$ be an almost closed mapping of a space X onto a space Y such that $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \setminus f^{-1}(y)$ for each $y \in Y$. Then f has a closed graph.

PROOF :

Let $(x,y) \notin G(f)$ be any point.

Since $y \neq f(x)$, $x \notin f^{-1}(y)$.

Given that $f^{-1}(y)$ is an α - Hausdorff α-nearly paracompact subset with respect to $X \setminus f^{-1}(y)$.

$\therefore$ By Theorem 2.2.2 there are regular open disjoint sets U and V, such that, $x \in U$, $f^{-1}(y) \subset V$. Since f is almost closed, there is an open neighbourhood H of y, such that, $f^{-1}(y) \subset f^{-1}(H) \subset V$.

CLAIM : $U \times H \cap G(f) = \phi$.

Suppose $U \times H \cap G(f) \neq \phi$.

$\therefore$ There exists $(x_0, y_0) \in U \times H \cap G(f)$.

$(x_0, f(x_0)) \in U \times H$.

$\Rightarrow$ $x_0 \in U$ and $f(x_0) \in H$

$\Rightarrow$ $x_0 \in U$ and $x_0 \in f^{-1}(H)$.

Since $f^{-1}(H) \subset V$, $x_0 \in U \cap V$, which is a contradiction.

Hence the claim.

$\therefore$ $(x,y) \notin G(f) \Rightarrow (x,y) \notin cl(G(f))$

$\therefore$ G(f) is closed.

COROLLARY 2.3.5 [3]

If $f: X \rightarrow Y$ is an almost closed mapping of a space X on to a compact space Y such that $f^{-1}(y)$ is an α-Hausdorff α-nearly paracompact subset with respect to $X \setminus f^{-1}(y)$ for each point $y \in Y$, then f is continuous.

PROOF :

Since $G(f)$ is closed (by theorem 2.3.4) and Y is compact, f is continuous.

THEOREM 2.3.6 [3]

Let $f: X \to Y$ be an almost closed mapping such that $f^{-1}(y)$ is closed for each point $y \in Y$. If A is an α-regular α-paracompact subset with respect to $X \setminus A$, then $f(A)$ is closed.

PROOF :

Let $y \notin f(A)$ be any point.

Since $f^{-1}(y)$ is closed, $X \setminus f^{-1}(y)$ is open.

Since A is α-regular α-paracompact with respect to $X \setminus A$, then by Theorem 2.2.6 there is an open set V such that

$A \subset V \subset cl(V) \subset X \setminus f^{-1}(y)$.

$\therefore f^{-1}(y) \subset X \setminus cl(V)$.

Since $X \setminus cl(V)$ is a regular open set and f is an almost closed mapping, there is an open neighbourhood H of y, such that $f^{-1}(y) \subset f^{-1}(H) \subset X \setminus cl(V)$.

CLAIM : $H \cap f(A) = \phi$.

Suppose $H \cap f(A) \neq \phi$.

Then there exist $y_0 \in H$ and $y_0 \in f(A)$.

$y_0 \in f(A) \Rightarrow y_0 = f(x_0)$ for some $x_0 \in A$.

$\therefore f(x_0) \in H$.

That is, $x_0 \in f^{-1}(H)$.

That is, $x_0 \in X \setminus cl(V)$ (since $f^{-1}(H) \subset X \setminus cl(V)$).

$\Rightarrow x_0 \notin cl(V)$.

This is a contradiction since $x_0 \in A$ and $A \subset cl(V)$.

$\therefore$ $H \cap f(A) = \phi$. Hence the claim.

$\therefore$ $y \notin cl(f(A))$

Hence f(A) is closed.

THEOREM 2.3.7 [3]

Let $f: X \to Y$ be an almost closed mapping such that G(f) is closed. If A is an α-regular α-paracompact subset with respect to $X \setminus A$, then f(A) is closed.

PROOF :

Given that G(f) is closed.

Let $x \notin f^{-1}(y) \Rightarrow f(x) \neq y$.

Since G(f) is closed, $(x,y) \notin G(f) = cl(G(f))$.

$\Rightarrow$ There exists an open $U \times V$ such that $(x,y) \in U \times V$ and $U \times V \cap G(f) = \phi$.

CLAIM : $U \cap f^{-1}(y) = \phi$.

Suppose $U \cap f^{-1}(y) \neq \phi$.

There exists $x_0 \in U \cap f^{-1}(y)$.

That is, $x_0 \in U$ and $f(x_0) = y$.

$\therefore$ $(x_0, f(x_0) \in G(f)$.

Since $y \in V$, $f(x_0) \in V$. $\therefore$ $(x_0, f(x_0)) \in U \times V$.

$\therefore$ $(x_0, f(x_0)) \in U \times V \cap G(f) \neq \phi$ which is a contradiction.

$\therefore$ There exists an open set U containing x such that $U \cap f^{-1}(y) = \phi$.

$\therefore$ $x \notin cl(f^{-1}(y))$. Hence $f^{-1}(y)$ is closed.

$\therefore$ By Theorem 2.3.6, $f(A)$ is closed.

COROLLARY 2.3.8 [3]

Let $f: X \to Y$ be an almost closed mapping such that $G(f)$ is closed. If every closed subset A of X is α-regular α-paracompact with respect to $X \setminus A$, then f is closed.

PROOF :

Let A be a closed subset of X.

Therefore, A is α-regular α-paracompact with respect to $X \setminus A$.

By Theorem 2.3.7, $f(A)$ is closed.

Therefore, we have for every closed subset A in X,

$f(A)$ is closed. Hence f is closed.

THEOREM 2.3.9 [3]

Let $f: X \to Y$ be an almost closed mapping such that $G(f)$ is closed. If $f(A)$ is an α-paracompact subset with respect to $Y \setminus f(A)$, then $f(A)$ is a closed α-Hausdorff subset of Y.

PROOF :

Let $y \notin f(A)$ be any point.

Given that $G(f)$ is closed.

That is, $(x,y) \notin G(f) = cl(G(f))$.

$\Rightarrow$ There exists a neighbourhood of UxV of (x,y) such that

$U_x \times V_y \cap G(f) = \phi$.

$\Rightarrow$ There exists $z \in U_x$ such that $f(z) \notin V_y$

$\Rightarrow$ $f(U_x) \cap V_y = \phi$.

To prove f(A) is α-Hausdorff.

Take any $z \in f(A)$ and $z' \in Y \setminus f(A)$.

$z \in f(A) \implies z = f(x)$ for some $x \in A$.

Since $z' \notin f(A)$, $z' \neq f(x)$.

$\therefore (x, z') \notin G(f)$.

By the above argument, these exist open sets U and V such that $x \in U$ and $z' \in V$ such that $f(U) \cap V = \phi$.

Since f is open, f(U) is open and as $x \in U$, $z \in f(U)$

$\therefore$ For any two disjoint points $z \in f(A)$ and $z' \in Y \setminus f(A)$, there exist disjoint neighbourhoods f(U) and V containing z and z' respectively.

$\therefore$ f(A) is α- Hausdorff.

To prove f(A) is closed.

Let $y \notin f(A)$. Since f(A) is α-Hausdorff α-paracompact with respect to $Y \setminus f(A)$, there exist disjoint regular open sets U and V such that, $y \in U$ and $f(A) \subset V$, $U \cap V = \phi$.

Hence $y \in cl(V)$.

That is, $y \notin cl(f(A))$

$\implies$ f(A) is closed.

The chapter is concluded with an example to show that the assumption "f is an almost closed mapping" in Theorem 2.3.7 cannot be dropped.

EXAMPLE 2.3.10 [3]

Let $X = \{a_i : i \in I\}$ be a discrete space.

Let $Y = \{b\} \cup \{b_i : i \in N\}$.

Let each point b_i be isolated.

Let $\{ V^k(B) : k \in N \}$ be the fundamental system of neighbourhoods of b, where

$V^k(b) = \{b\} \cup \{b_i : i \geqslant k\}$

Let $V = \cup \{b_i : i \in N\}$

CLAIM 1: Y is a Hausdorff space such that cl(V) = Y.

Let x and y be any two distinct points of Y.

Case (i) $x = b_i$, $y = b_j$

In this case since b_i and b_j are isolated, the Hausdorff condition is satisfied.

Case (ii) : $x = b$, $y = b_j$

Since b_j is isolated, there exists a neighbourhood U_j containing b_j, which does not contain any other point.

Consider $V^{j+1}(b) = \{ b \} \cup \{ b_{j+1}, b_{j+2} \ldots\ldots \}$

$V^{j+1}(b)$ is a neighbourhood of b which does not contain b_j.

Therefore, $U_j \cap V^{j+1}(b) = \phi$.

Hence the Hausdorff condition is satisfied in this case also.

Therefore Y is a Hausdorff space.

To prove cl(V) = Y

It is enough to prove that $b \in cl(V)$.

Let U be an open set containing b.

Then by the fundamental system of neighbourhoods of b, there exist k such that $b \in V^k(b) \subset U$, where $V^k(b) = \{b, b_k, b_{k+1} \ldots\ldots\}$

$\therefore V^k(b) \cap V \neq \phi \Rightarrow U \cap V \neq \phi$.

$\therefore b \in cl(V)$. Hence claim 1.

CLAIM 2: The set $G(f)$ is closed in $X \times Y$.

Let $f: X \rightarrow Y$ be a mapping defined by $f(a_i) = b_i$, for every $i \in N$.

$G(f) = \{(a_i, f(a_i)) : i \in N\}$

$= \{(a_i, b_i) : i \in N\}$

Let $(x,y) \notin G(f)$.

Then $x = a_i$ for some i in N and $y \neq f(a_i)$.

$\therefore y \neq b_i$.

Hence $y = b_i$ for some $j \neq i$ (or) $y=b$.

Consider the case $y = b$.

$\therefore (x,y) = (a_i, b)$

Let $U = \{a_i\}$ and $V = V^{i+1}(b)$.

To prove $U \times V \cap G(f) = \phi$.

Suppose not, there exists $(x, y(x)) \in U \times V$.

That is, $(a_k, b_k) \in U \times V$.

Since $b_k \in V = V^{i+1}(b)$, $k \geqslant i+1$.

But since $a_k \in U$, $k=i$, a contradiction.

$\therefore U \times V \cap G(f) = \phi$

Consider the case $y=b_j$.

Let $U = \{a_i\}$, $V = \{b_j\}$.

As $b_j \neq f(a_i)$, $U \times V \cap G(f) = \phi$.

In either case, $(x,y) \notin cl(G(f))$.

Hence claim 2.

CLAIM 3: f is not almost closed.

Let $U = \{a_{2n} : n \in N\}$

Then $f(U) = \{b_{2n} : n \in N\}$

$\mathrm{Int}\,(U) = U$; $\mathrm{cl}(U) = U$.

$\mathrm{cl}(\mathrm{Int}(U)) = \mathrm{cl}(U) = U$.

Therefore, U is regular closed.

Clearly $b \notin f(U)$.

Let us show that $b \in \mathrm{cl}(f(U))$.

Let W be an open set such that $b \in W$.

$\therefore$ There exists k such that $b \in V^k(b) \subset W$.

Since $f(U) = \{ b_{2n} : n \in N \}$, we get, $V^k(b) \cap f(U) \neq \phi$.

$\therefore$ $W \cap f(U) \neq \phi$.

$\therefore$ $b \in \mathrm{cl}(f(U))$.

$\therefore$ $f(U) \neq \mathrm{cl}(f(U))$. Thus f(U) is not closoed.

Hence there exists a regular closed set U such that f(U) is not closed. Therefore f is not almost closed. Hence claim 3.

CLAIM 4: U is α-regular.

Let x be any point in U and W be an open set containing x.

Then $x = a_{2n}$ for some n.

Let $V = \{a_{2n}\} \Rightarrow \mathrm{cl}(V) = \{a_{2n}\}$

$x = a_{2n} \in V \subset \mathrm{cl}(V) = \{a_{2n}\} \subset W$.

Therefore U is α-regular.

CLAIM 5: U is α-paracompact with respect to $X \setminus U$.

Let $\mathcal{U} = \{ U_i : i \in I \}$ be an open cover of U.

Let $\mathcal{V} = \{ \{a_2\}, \{a_4,\} \{a_6\} \ldots\ldots \}$

(i) $\mathcal{V}$ refines $\mathcal{U}$

(ii) Obviously $\mathcal{V}$ covers U.

(iii) To prove $\mathcal{V}$ is locally finite at each $x \in X \setminus U$.

$x \in X \setminus U \Rightarrow x \notin U$.

$\therefore$ $x = a_{2n+1}$ for some n.

Let $W = \{a_{2n}, a_{2n+1}\}$. Then W is a neighbourhood of x and it intersects only two members of $\mathcal{V}$.

$\therefore$ $\mathcal{V}$ is locally finite.

Hence U is α-paracompact with respect to $X \setminus U$.

Thus there is an α-regular α-paracompact subset with respect to $X \setminus U$, such that f(U) is not closed.

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