A comparison of Unreliable $M^X/G/1$ queueing system under multiple adapted vacation policy with classical vacation policies

JYOTHI N

(16PMA008)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641 043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

April, 2018

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CHAPTER I

INTRODUCTION

Queueing theory is the mathematical study of queues or waiting lines. There are many day to day life situations where the queue is formed. The queueing theory is not only used to solve the daily life problems but also used in traffic flow, scheduling patients in hospitals, computer programming, bank sectors, toll plaza, railway stations, military logistics, industrial engineering etc. Queueing theory is a branch of Operations Research because the results are used for making decisions about the resources needed to provide good service. Oueueing theory was developed to provide models to predict the behaviour of the systems that attempt to provide service for randomly arising demands.

Agner Krarup Erlang (1878-1929) Danish Engineer who is the father of Queueing theory,in 1909's has published "The Theory of Probabilities and Telephone Conversations". Many researchers are motivated by his work and developed the queueing theory for practical applications.

Queueing theory, as a stochastic process, contributes vital informations required in decisions associated with waiting lines, by predicting various performance measures such as mean queue length, mean busy period, etc,. Therefore, the ultimate goal is to achieve an economic balance between the cost and the cost associated with waiting time for that service. Queueing theory formulae help us to take all variables into consideration. And this makes it easier in decision making.

1.1 The basic Characteristics of queueing process

In designing a good queueing system, it is necessary to have good informations about the model. The basic characteristics of queueing systems are the following.

1.1.1 The input process or arrival pattern of customers

The arrival pattern describes the manner in which the arrival occurs. It is specified by the inter-arrival time between any two consecutive arrivals or by the mean arrival rate. The input pattern also indicates whether the arrivals occur single or in batches or in random size. The inter-arrival time may be deterministic or stochastic. When it is stochastic the probability distribution associated with it is required. In case of bulk arrivals, not only the time between successive arrivals may be probabilistic but also the number of customers in a batch.

1.1.2 The service pattern

The service pattern can be measured by the number of customers served per unit of time or the time taken to complete a service. The service time may also be constant (deterministic) or stochastic. if it is stochastic , the probability distribution associated with it will be required. The service can be provided in single or batch.

1.1.3 Queueing discipline

It refers to the manner by which customers are selected for service when a queue has formed.The most common discipline that can be observed in everyday life is first come, first served (FCFS) or it is something called first in, first out (FIFO). Another important discipline, which is very common in the inventory system, is the last in, first out (LIFO). Besides from these two, there are other queueing disciplines such as random selection of service (RSS)and a variety of priority schemes (very common in hospital casualty), where customers are selected for service on the basis of their priorities.

1.1.4 System capacity

The number of customers in the queue and in the service together is called the system capacity. The system may have a queue of finite capacity or effectively infinite capacity.

1.1.5 The number of service channels

A system may have a single server or a number of parallel or series of channels. In parallel channels, each and every channel provides an identical facilities, so that several customers may be served simultaneously. Also a queueing system may have only a single stage of service or it may have several stages operated by a single server.

1.1.6 Kendall’s Notation

Any queueing system is presented by the notation introduced by the notation introduced by Kendall (1951): A/B/C/Y/Z, where A represents the inter arrival time distribution of the customer, B denotes the service time distribution, C denotes the number of parallel srevers, Y denotes the capacity of the system and Z denotes the queueing discipline.

For example, the notation M/G/1 denotes a queueing system with Poisson (Markovian) input, generally distributed service time with single server following FCFS queue discipline and the system capacity is infinite.

The following are some of the special characteristics of queueing systems considered in the thesis.

1.1.7 Bulk or batch Arrival Queueing models

A queueing system where arrivals or service or both takes place in batches of fixed or random sizes is called a bulk queueing system. Batch arrival queueing models can be used in many practical situations such as the analysis of message packetization in data communication systems.

The basis queueing system considered in the present work is $M^X/G/1$. In this queueing system, customers arrive in batches in accordance with a time-homogeneous Poisson process with parameter λ. The batch size X is a Random Variable with probability distribution Pr (X=k) = g_k, k=1, 2, 3,… that is, the probability that a batch of size 'k' arrives in an infinitesimal interval (t, t + h) is $\lambda\, g_k\, h + o(h)$.

Let $X(z) = \sum_{k=1}^{\infty} g_k z^k$ be the PGF of X and $X'(1) = E(X)$ be the mean of X. Then, the arrival process is said to be compound Poisson process with mean arrival rate $\lambda E(X)$.

1.1.8 Markovian Queueing Model

Queueing models with exponential inter-arrival time and exponential service time are called Markovian queueing models.

1.1.9 Non Markovian Queueing Models

Queues in which inter arrival and /or service time distribution and or any other probability distribution in the system are other than exponential are known as Non Markovian queues.

1.1.10 Queue with server's vacation

In classical queueing models, servers are always available in the service facility. However in many practical situations servers may become unavailable for a period of time due to a variety of reasons. This period of server absence is called server vacation. i.e., vacation in queueing models represents the period of temporary server absence. There are two major types of vacation mechanism, namely exhaustive and non-exhaustive services. With an exhaustive service, the server cannot take a vacation until the system becomes empty. On the other hand non-exhaustive system, the server can take a vacation between two services, during busy period. In either case, the rules for resuming service vacation completion instant are numerous. Based on these rules the two main vacation policies, framed under the exhaustive service discipline are single and multiple vacation policies.

(i) Multiple vacation policy

In multiple vacation policy whenever the system becomes empty, (or at the end of each busy period) the server leaves the system for vacation. On returning from the vacation, if the server finds less than the required number of customers then, he may immediately take another vacation. He will continue in this manner, until he finds, upon returning from the vacation, the required number of waiting customers.

(ii) Single vacation policy

In case of single vacation policy when the server returns from the first vacation even if the server finds less than the minimum number of customers required for service, he joins the system and stays idle in the system awaiting the queue length to reach the minimum number for starting his next service.

(iii) Randomized J-vacation or < p, J > -Vacation policy

According to <p, J>-vacation policy, if all the customers are served in the queue exhaustively, the server then immediately takes a vacation. Upon returning from a vacation if the queue does not contain required number of customers, then the server either joins the system and remains idle with probability **p** or leaves for another vacation with probability **q (p + q = 1)**. This pattern continues until the number of vacations reaches J. At the end of the J^{th} vacation the server necessarily joins the service facility.

The classical single and multiple vacation policies are the two extreme cases of J-vacation policy (respectively with J = 1 and J tends to infinity), with p=1.

(iv) Multiple adapted vacation (MAV) policy

At the end of each busy period the server either remains idle with probability β_0 or takes a vacation with probability $1 - \beta_0$. If, at the end of the first vacation no customer has arrived then the server, independently of everything else, takes a new vacation with probability β_1, or remains idle and available to serve the first customer that arrives, with probability $1 - \beta_1$. The process is repeated until atleast one customer is found in the system. That is the policy is determined by the sequence of probabilities $\{ \beta_k\}$, $k = 0, 1, 2, . .$

Multiple adapted vacation policy reduces to the randomized J-vacation policy with suitable selection of β_k : (i.e.) $\beta\ 0 = 1$; β_k =p, for $1\leq k \leq J-1$, $\beta_k = 0$, $k \geq J$. The suitable selection of β_j's (shown in chapter IV) will reduce, multiple adapted vacation policy to single vacation, multiple vacation and J-vacation policies.

1.1.11 Queueing systems with server's Breakdown (or) service Interruption

In queueing situations, servers may breakdown while providing service and the service of the customer being served is then interrupted and cannot resume service until the server is repaired. The period during which the system is in breakdown state is termed as breakdown period. The server breakdown can be classified into operation and time dependent failures. The operation dependent server breakdown can occur only when the server is in operation. The time dependent server breakdown can occur at any time, independent of whether the server is rendering service or not. In queueing models with server breakdown, once failure occurs, the repair process starts immediately and after completion of the repair, service is continued according to the repeat or the resume rule.

1.2 REVIEW OF LITERATURE

1.2.1 Classical Queues

The pioneer investigator of queueing theory was the Denish Mathematician Erlang (1909) who published "The theory of probabilities and telephone conversations" and modelled the telephone traffic systems. Due to its wide applications in many areas, queueing theory has been one of the most active research topics in Operation Research and Management Science for the past several years. Kendall (1951, 1957) was the pioneer who viewed and developed queueing theory from the perspective of stochastic process. Some excellent books on classical queueing theory have been published; they include Takacs (1962), Cooper (1981), Cohen (1982), Gross and Haris (1985), Satty (1961), Wolff (1989), Prabhu (1997), etc.

1.2.2 Vacation Queues

Server vacations are useful for the systems in which the servers want to utilize their idle time for different purposes. Vacation queues have attracted many attentions from numerous researchers since Levi and Yechalli (1975) introduced the two standard vacation policies (single and multiple). The comprehensive survey on vacation queues can be found in Doshi (1986 and 1990), Takagi (1991), in the book by Tian and Zhang (2006) and in recent years Ke et al (2010). The amount of literature relating to queueing models with vacations is growing rapidly and the analysis of the queueing systems with vacations has been discussed through a considerable amount of work recently. The early work on vacation queueing models focused on exhaustive service policy, where the server takes vacations only if the system becomes empty.

However, multifarious vacation policies with non-exhaustive service have important application values in communication network and computer systems. To adopt diversified application background, some new vacation policies were presented by Keilson and Servi (1986).

A wide class of policies for governing the vacation mechanism has been discussed in literature. Takagi (1991) first proposed the concept of variant vacation for the M/G/1 regular system which generalizes the single and multiple vacation policies. Zhang and Tian (2001) investigated a Geo/G/1 queue with modified vacation policy, where the server can take at most J vacations continuously. This modified vacation policy can be reduced to the classical single and multiple vacation policy by setting J to be one or infinity. Ke and Chu (2006) studied a batch arrival system under this modified vacation policy.

Recently, Mytalas and Zazanis (2015) considered the multiple adapted vacation (MAV) policy for a M^x/G/1 queueing system with disaster and repair. This policy is determined by a sequence of probabilities $\{\beta_k\}$, k = 0, 1,2,……which provides additional flexibility and includes, the cases of other vacation policies in the same framework. The vacation policy considered by them (MAV) first introduced by Takagi (1991) and termed by Tian and Zhang (2006).

1.2.3 Queueing Models with server Breakdowns

The study of queueing systems, wherein the server (service channel) is subjected to unpredictable break downs is a most popular subject that has received a lot of attention for the past 60 years. Queueing models with service interruptions have proved to be useful abstraction in situations, where breakdowns occur while providing service and the service of the customer being served cannot resume service until the server is fixed (repaired). Service interruptions in queueing systems have been investigated by several researchers. Gaver (1962) seems to be the first to analyse a queueing process with service interruptions in which he studied the effect of interruptions on the distributions of important measures such as busy period queue length and waiting time for M/G/1 queueing model.

The study of service interruptions mainly assumes that the interruptions occur only when the server is busy and the server in an interrupted state will not be affected by the process of further interruption. Whenever server breaks down while servicing a customer, the server will be sent immediately to repair facility and the customer resumes service only when the server is fixed. Regarding the resumption of services, there are several possible scenarios to restoring an interrupted service. They include (i) starting a service from the very beginning independent of the earlier service (ii) starting a service from where it got interrupted and (iii) denying service of the one whose service got interrupted.

Keilson (1962) treats the first two cases of interruptions. Yue and Tu (2001) studied M/G/1 queueing system with server interruptions in which the interrupted service is repeated from the beginning and investigated the completion period of jobs. Wang (2004) analysed an M/G/1 queue with two phases of service and obtained the transient and steady– state solution for interesting system measures for the case where the interrupted service is resumed when the server is fixed. In a recent survey paper, Krishnamoorthy et al. (2012) gives a detailed description of research on queueing models with interruptions that occur due to many reasons and highlighted the rules for resumptions.

1.3 THESIS ORGANISATION:

Queueing systems with vacations have been studied extensively in the past due to its wide applications in production / inventory systems, communication systems, computer systems etc., The survey work of Doshi (1986) and the monograph of Takagi (1991) explain the concept of variant vacation policies and their generalizations. The variations and extensions of the vacation queuing models can also be reformed to Choundhury (2002), J.C,Ke (2007), Ke et al (2010), Wang et al (2009) etc., In the present work, the author considers a batch arrival queueing system $M^x/G/1$ in which an unreliable server operates multiple adapted vacation policy during the idle time. According to the MAV policy, the server makes the decision to go for vacation or stays idle in the system whenever the system becomes empty. (i.e.) the server takes vacation with probability β_0 (called first vacation) or stays in system idle with probability $1-\beta_0$. Upon returning from a vacation, if the sever finds atleast one customer waiting in the queue, the server will be immediately activated for service. Otherwise, if no customers are waiting for service at the end of vacation the server either remains ideal in the system with probability $1-\beta_1$ or leaves for second vacation with probability β_1. This pattern continues until atleast one customer is found in the system.

By unreliable server, we mean that the server is typically subjected to unpredictable breakdowns while working and sent immediately to a repair facility. The customer whose service is interrupted wait in the system to resume his service until the server returns from the repair facility.Depending on the situations there are several possible scenarios to restoring an interrupted service. Two of the main scenarios are,

1) The service interrupted customers join the head of the queue to repeat the service from the very beginning.
2) Resuming service from where it got interrupted.

In Chapters II and III, the author analyses $M^X/G/1$ queueing model under MAV policy by considering the above two scenarios of resuming service during breakdown period. The server returning from the repair facility is treated as good as before. The service times, vacation time and repair time are assumed to follow arbitrary distributions. The systems are analysed in steady-state using supplementary variable technique by introducing the remaining service time, remaining vacation time and remaining repair time as supplementary variables.

The total probability generating function of the system size is derived through the partial generating functions of the system size probabilities when the system is in different states. The system size probabilities and the average number of customers waiting in the system when the server is busy or in vacation or in breakdown state are also calculated.

The performance measures are analysed through numerical analysis and it is shown through numerical computations that the mean queue length increases with arrival rate (λ), mean vacation time (EVI), mean repair time etc. for both the models (in chapter IV).

The main object of the present work is to demonstrate that, the MAV policy generalises the other vacation policies. In Chapter IV the corresponding results of other vacation policies (single vacation, multiple vacation, J-vacation) are derived from the MAV policy model discussed in Chapters II and III.

It is also shown that, when the service time is exponential, then the steady-state results of both models derived in chapters II and III coincide. The present work helps the researchers to consider more general vacation queueing models under MAV policy.

1.4 METHODOLOGY:

The techniques generally used in studying non-Markovian queues are Supplementary Variable Technique and Embedded Markovian Chain. In the present work the supplementary variable technique is used to analyse the models.

Supplementary Variables Technique (SVT) (Alfa and Srinivasa Rao (2000)

The first step in analysing a non-Markovian queueing system is to set it up as a Markov process. In most practical queueing systems, supplementary variables are usually needed to achieve this. The alternative to that is an embedded Markov chain method. In the queueing literature, we have two kinds of supplementary variables, in general. They are elapsed time and the remaining time of random variables. For both cases, the approaches of deriving the queueing characteristics are different. The main reason for adding supplementary variables to a stochastic process variable is to make the system Markovian. One can assert that the technique of remaining time as the supplementary variable is simple and elegant. The present work uses this technique. The supplementary variable technique analysis for the queueing problems by considering the remaining time as supplementary variables involves, probability density function and its LST and partial differential equations.

1.5 PRELIMINARIES

1.5.1 Transient and steady-state solution:

Let N(t) denote the number of customers in the system at time t and its probability distribution be denoted by $P_n(t) = \Pr(N(t) = n/N(0) = 0)$.For a complete description of the queueing process, the transient or time-dependent solutions are necessary. But it is often difficult to obtain such solutions. Further in many practical situations, we need to know the behaviour in steady-state, i.e., when the system reaches an equilibrium state, after being in operation for a pretty long time. The time-dependent solutions are called transient solutions. And the solutions obtained as $t \to \infty$, are called steady-state solutions.

There are several methods of solving the difference-equations and presenting the probability distributions of the system size and hence calculate important performance measures of the model. Very often the transforms such as the probability generating function $P(z,t) = \sum_{n=o}^{\infty} P_n(t) z^n$ and the Laplace transforms of the function $L(P_n(t)) = \int_0^{\infty} e^{-\theta t} P_n(t)\, dt$ are used to solve the differential equation.

1.5.2 Probability Generating Function (PGF) of the Random Variable X:

In probability theory, the probability generating function of a discrete random variable is a power series representation of the probability mass function of the random variable. The probability generating functions are often employed for their succinct description of the sequence of probabilities Pr (X = i) and to make available the well-developed theory of power series with non-negative coefficients.

Definition

Suppose that X is a random variable that assumes non-negative integral values 0, 1, 2, 3, …. And that Pr {X = k} = p_k , k = 0, 1, 2, 3, ….with $\sum_{k=0}^{\infty} p_k = 1$, then the corresponding generating function $P(z) = \sum_{k=0}^{\infty} p_k z^k$ of the sequence of probabilities {p_k} is known as the probability generating function (PGF) of the random variable X. Also called as the z-transform of the random variable X.

We have p(1) = 1; the series P(z) converges for atleast – 1 ≤ z ≤ 1 and is infinitely differentiable. The function P(z) is defined by {p_k} and in turn defines {p_k} uniquely, i.e., a PGF determines a distribution uniquely.

The k^{th} factorial moment of X is given by

$$E(X(x-1),\dots\dots,(x-k+1)) = \left[\frac{d^k}{dz^k} P(z)\right]_{z=1}, \qquad \text{for } k = 1, 2, \dots\dots$$

1.5.3 Laplace Transform:

Laplace transforms serve as very powerful tools in many situations and provide an effective means for the solution of many problems arising in queueing theory. The transforms are very effective for solving linear differential equations and reduce a linear differential equation to an algebraic equation. In the study of some probability distributions, this method could be used to find the Laplace transform of a probability distribution rather than the distribution itself.

Laplace Stieltjes Transform (LST)

The Laplace-Stieltjes transform of a non-negative random variable X with distribution function F(.), is defined as $F^*(\theta) = \int_{x=0}^{\infty} e^{-\theta X}\, dF(x), \theta \geq 0$ When the random variable X has a density then the transform simplifies to $F^*(\theta) = \int_{x=0}^{\infty} e^{-\theta X}\, dF(x), \theta \geq 0$. Note that $|F^*(\theta)| \leq 1$ for all $\theta \geq 0$. Further $F(0) = 1, F'(0) = -E(X), F^{(k)}(0) = (-1)^k\, E(X^k)$

For numerical study, the algorithms were implemented in computer programmes written in C^{++}, using objective oriented tools.

The results used in the present works are as follows:

1.5.4. Identities:

1. $$\sum_{n=1}^{\infty} z^n \left(\sum_{k=1}^{n} QI_{n-k}^*(\theta) g_k\right) = \left(\sum_{k=1}^{\infty} g_k z^k\right)\left(\sum_{k=1}^{\infty} QI_n^*(\theta) z^n\right) = X(z) QI^*(z,\theta)$$
2. $$\sum_{n=2}^{\infty} z^n \left(\sum_{k=1}^{n-1} P_{n-k}^*(\theta) g_k\right) = \left(\sum_{k=1}^{\infty} g_k z^k\right)\left(\sum_{k=1}^{\infty} P_n^*(\theta) z^n\right) = X(z) P^*(z,\theta)$$

1.5.5. Results using L'hospital rule:

If $f(1) = g(1)$, then $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right)_{z=1} = \frac{g'(1)f''(1) - f'(1)g''(1)}{2\big(g'(1)\big)^2}$,

where the dashes represent the derivatives of the functions.

Let $w_x(z) = \lambda\big(1 - X(z)\big)$, $g_\alpha\big(w_x(z)\big) = \alpha + w_x(z)$

and $h_\alpha\,(w_x(z)) = g_\alpha\big(w_x(z)\big) - \alpha\, R^*\big(w_x(z)\big)$. Then

1. $\lim_{z\to 1}\left[\frac{1-R^*\left(w_x(z)\right)}{w_x(z)}\right] = E(R)$

2. $\lim_{z\to 1}\left[\frac{1-V^*\left(w_x(z)\right)}{w_x(z)}\right] = E(V)$

3. $\lim_{z\to 1}\left[\frac{-\left(w_x(z)\right)}{D(z)}\right] = \frac{\lambda\, E(X)}{D'(1)}$

If D (1) = 0 and $D'(1) \neq 0$.

4. $\lim_{z\to 1}\left[\frac{1-S^*\left(w_x(z)\right)}{h_\alpha(w_x(z))}\right] = E(S)$

5. $\lim_{z\to 1} S^*(h_\alpha(w_x(z))) = 1$

6. $\lim_{z\to 1}\left[\frac{-w_x(z)}{z - S^*\left(h_\alpha\left(w_x(z)\right)\right)}\right] = \frac{\lambda\, E(X)}{1-\rho}$

7. $\frac{d}{dz}\left[\frac{1-R^{*1}\left(w_x(z)\right)}{w_x(z)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(R^2)}{2}$

8. $\frac{d}{dz}\left[\frac{1-V^*\left(w_x(z)\right)}{w_x(z)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(V^2)}{2}$

9. $\frac{d}{dz}[S^*(w_x(z))]_{z=1} = \lambda\, E(X)\; E(S)$

10. $\frac{d^2}{dz^2}[S^*(w_x(z))]_{z=1} = (\lambda\, E(X))^2\; E(S^2) + \lambda\, E(X(X-1))\; E(S)$

11. $\frac{d}{dz}\left[S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]_{z=1} = \lambda\, E(X)\; E(S)[1+\alpha\, E(R)]$

12. $\frac{d}{dz}\left[\frac{1-S^*\left(w_x(z)\right)}{w_x(z)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(S^2)}{2}$

13. $\frac{d}{dz}\left[\frac{1-S^*\left(h_\alpha\left(w_x(z)\right)\right)}{h_\alpha\left(w_x(z)\right)}\right]_{z=1} = \lambda\, E(X)\,\frac{E(S^2)}{2}\,[1+\alpha\, E(R)]$

14. $\frac{d}{dz}\left[\frac{z-1}{z-S^*\left(w_x(z)\right)}\right]_{z=1} = \frac{(\lambda\, E(X))^2\; E(S^2) + \lambda\, E(X(X-1))\; E(S)}{2(1-\rho)^2}$

CHAPTER II

AN UNRELIABLE $M^X/G/1$ QUEUEING SYSTEM UNDER MULTIPLE ADAPTED VACATION POLICY

Introduction:

The main object of the present work is to analyse bulk arrival queueing model ($M^X/G/1$) under server's vacation. In classical queueing systems, busy period starts as soon as a customer enters to the idle state and it ends, when the system becomes empty again. This is possible only when, the server is readily available in the system. But, keeping the server unnecessarily idle in the system is not very much appreciable and hence the concept of vacation was introduced in the early 1970's. Later the models are analysed under different types of vacation policies. Notable among them are single vacation and multiple vacations.

Later, J.C.Ke (2007) introduced the J-vacation policy in which the server can take finite number of vacations (atmost J) during an idle period. Recently Mytalas and Zazanis (2015) introduced the multiple adapted vacation policy in which the server can take decision whether to go for vacation or stay idle in the system at the end of every vacations. It is characterized by a sequence of probabilities and hence generalizes the other vacation policies with the suitable selection of the parameters. The following chapters II and III, deal with $M^X/G/1$ system in which the server adopts multiple adaptive vacation policy, during idle period.

Generally, in queueing systems, the server (human or machine) is subjected to unpredictable breakdowns. As soon as the server breaks down, it will be sent for repair immediately and the system will not work for a short period of time. In the literature of queueing systems with server break downs, two types of behaviours of the customers whose service is interrupted are considered. The first type of behaviour is that, the customers will lose the service completely and will start a new service as soon as the server is fixed. The second behaviour is the customer will stay in the service facility until the server gets repaired and complete the remaining service. These two behaviours are analysed in chapters II and III respectively, and compared when the service time is exponential.

For both the models, the systems are analysed at steady- state, using supplementary variable technique and the PGF of the steady state system size probabilities are obtained. The total PGF is derived and the decomposition property is verified. The important system measures such as, the system size probabilities, the mean system size are obtained. Numerical computations are made to justify the formulae and to derive some conclusions. The results corresponding to other vacation policies are obtained as special cases.

2.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

2.1.1 MODEL DESCRIPTION

Arrival Pattern

The present chapter considers an $M^X/G/1$ queueing system in which the arrivals occur in batches in accordance with a time homogeneous Poisson process with random batch size **X**, group arrival rate λ and probability distribution $Pr(X=k)=g_k$, **k=1,2,3…** The customers are served one by one according to the order in the queue.

Multiple Adapted Vacation Policy (MAV)

A cycle starts whenever the system becomes empty and the server is deactivated. The deactivated server either leaves the system for a vacation (first vacation) of random length (**V**) with probability β_0 or remains idle in the system with probability $(1- \beta_0)$. Upon returning from the vacation, if the server finds atleast one customer waiting in the system, then the server starts a busy period immediately. Otherwise, if there are no customers found waiting in the queue then the server either joins the system with the probability $(1- \beta_1)$ or takes a new vacation (second vacation) with probability β_1. This pattern continues until atleast one customer is found in the system. (i.e) Thus the vacation policy is determined by the sequence of probabilities $\{\beta_i\}$, i=0, 1, 2, … The vacations have independent duration with common distribution function **V(t)** and density function **v(t)** of finite moments.

Busy Period and Breakdown Period

During busy period, the server provides single service. The service times follow general (arbitrary) distributions with distribution functions **S(t)** and the density functions **s(t)** of finite moments $E(S^k)$, **k=1,2.**

The server may breakdown at any time while providing service and the server is sent for repair instantaneously and the service channels will not function for a short interval of time. The service interrupted customer then joins the queue and repeats his service from the beginning when the server is fixed. The breakdowns occur according to the Poisson process with rate α.

As soon as the server fails, the service is stopped for the customer until the channel is repaired. The customer whose service is interrupted joins the head of the queue and waits for the server to return from the repair facility to start a new service. The repair times (denoted by **R**) of the server is assumed to be arbitrarily distributed with distribution function **R(t),** density function **r(t)** for **t≥0.**

The customers continue to arrive according to the Compound Poisson process, independent of the state of the system and wait in the queue during the busy period, breakdown period and vacation period. The service completion period of a customer consists of the service time and repair time of the server. Thus a cycle is made up of idle vacation period, setup period and completion period.

We denote the model by **M^X/G/1/MAV/breakdown**, where MAV denotes the Multiple Adapted Vacation policy. Various stochastic processes involved in the queueing system are assumed to be independent to each other. Using supplementary variable technique the steady state system equations under the steady state condition are analyzed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

To analyse the steady-state results of the model, the following notations are introduced.

Notations:

N (t) : The system size at time t

λ : Group arrival rate

X : Group size random variable

g_k : Pr(X = k), k = 1, 2, 3, …

X(z) : Probability generating function of X.

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RVs are listed in table (2.1).

Table (2.1)

	RV	**CDF**	**PDF**	**LST**	**k^{th} moments**
Service time	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$
Repair time	R	R(x)	r(x)	$R^*(\theta)$	$E(R^k)$
Vacation Time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$

If f(x) is the density function of the probability distribution F(x) then

$$F^*(\theta) = \int_0^\infty e^{-\theta x} f(x)dx = \int_0^\infty e^{-\theta x} d(F(x))$$

Let $V^0(t), S^0(t)$ and $R^0(t)$ denote the remaining times of the random variables namely Idle vacation time, the service time and Repair time respectively.

Let the state of the system at time **t** be given by Y(t) = 0, 1, 2, 3 according as the server is idle in the system, on vacation during idle time, busy state and break down state respectively. The supplementary variables are introduced in order to obtain a bivariate Markov Process **{N(t),δ(t)}** where **N(t)** denotes the system size at time t and $\delta(t) = (0, V^0(t),\ S^0(t), R^0(t))$ according as Y (t) = (0, 1, 2, 3) respectively.

Let $PI^\circ(t) = Pr\{Y(t) = 0, N(t) = 0\}$

$QI_{n,j}(x, t)\, dt = Pr\{N(t) = n, x < V^\circ(t) \le x + dt, Y(t) = 1\}$; $n \ge 0$

$P_n(x, t)\, dt = Pr\{N(t) = n, x < S^\circ(t) \le x + dt, Y(t) = 2\}$; $n \ge 1$

$B_n(x, t)\, dt = Pr\{N(t) = n, x < R^\circ(t) \le x + dt, Y(t)=3\}$; $n \ge 1$

Then, $QI_{n,j}(x, t)\, dt$ is the joint probability that at time **t**, there are n customers in the system, and the remaining **vacation** time of the server is between x and x + dt, where $n \ge 0$.

$P_n(x, t)$ dt is the joint probability that at time t, there are n customers in the system, the server is **busy** and the customer being served in the service channel with remaining service time lies between x and x + dt, where n≥1.

$B_n(x, t)$ denotes the probability that there are n-customers in the system at time **t**, the server is under repair and the remaining repair time x lies in the interval x and x + dt and the customer whose service is terminated due to breakdown joins the head of the queue to repeat the service, where ; n ≥ 1.

PI(t) denotes that the server is idle in the empty system at time **t**.

Assuming that at steady-state, the probabilities are independent of time t we have

$$\lim_{t\to\infty}\frac{\partial}{\partial x}P_n(x,t) = \frac{d}{dx}P_n(x)\ ; \qquad \lim_{t\to\infty}\frac{\partial}{\partial x}QI_{n,j}(x,t) = \frac{d}{dx}QI_{n,j}(x)$$

$$\lim_{t\to\infty}\frac{\partial}{\partial x}B_n(x,t) = \frac{d}{dx}B_n(x)$$

$$\lim_{t\to\infty}\left(\frac{\partial}{\partial t}P_n(x,t) = \frac{\partial}{\partial t}QI_{n,j}(x,t) = \frac{\partial}{\partial t}B_n(x,t)\right) = 0$$

At steady state $P_n(0), QI_{n,j}(0),$ and $B_n(0)$ denote the probability that there are n customers in the system at the termination of service period, vacation (during idle) period and repair period respectively.

With these notations, the steady state equations satisfied by the steady state probabilities are given in the following sections.

2.2. THE STEADY STATE SYSTEM SIZE EQUATIONS

Idle state

$$\lambda PI = \sum_{j=1}^{\infty}\left(1-\beta_j\right)QI_{0,j}(0) + P_1(0)(1-\beta_0) \tag{2.1}$$

Vacation during idle state

$$-\frac{d}{dx}QI_{0,1}(x) = -\lambda QI_{0,1}(x) + P_1(0)\beta_0 v(x) \tag{2.2}$$

$$-\frac{d}{dx}QI_{0,j}(x) = -\lambda QI_{0,j}(x) + QI_{0,j-1}(0)\beta_{j-1}v(x), \quad j \geq 2 \tag{2.3}$$

$$-\frac{d}{dx}QI_{n,j}(x) = -\lambda QI_{n,j}(x) + \sum_{k=1}^{n} QI_{n-k,j}(x)\, g_k \qquad n \geq 1, j \geq 1 \tag{2.4}$$

Busy state

$$-\frac{d}{dx}P_1(x) = -(\lambda+\alpha)P_1(x) + B_1(0)s(x) + P_2(0)s(x)$$
$$+ \sum_{j=1}^{\infty} QI_{1,j}\,(0)\, s(x) + \lambda PI\, g_1\, s(x) \tag{2.5}$$

$$-\frac{d}{dx}P_n(x) = -(\lambda+\alpha)P_n(x) + \lambda \sum_{k=1}^{n-1} P_{n-k}(x) g_k + P_{n+1}(0)s(x) + B_n(0)\, s(x)$$
$$+ \sum_{j=1}^{\infty} QI_{n,j}\,(0)\, s(x) + \lambda PI\, g_n s(x) \qquad n \geq 2 \tag{2.6}$$

Breakdown state

$$-\frac{d}{dx}B_1(x) = -\lambda B_1(x) + \alpha \int_0^{\infty} P_1(w)\, dw\, r(x) \tag{2.7}$$

$$-\frac{d}{dx}B_n(x) = -\lambda B_n(x) + \lambda \sum_{k=1}^{\infty} B_{n-k}\,(x) g_k + \alpha \int_0^{\infty} P_n(w)\, dw\, r(x), \quad n \geq 2 \tag{2.8}$$

Thus the L.S.T of the equations with respect to x and y are given by

$$\theta QI^*_{0,1}(\theta) - QI_{0,1}(0) = \lambda QI^*_{0,1}(\theta) - P_1(0)\beta_0 V^*(\theta) \tag{2.9}$$

$$\theta QI^*_{0,j}(\theta) - QI_{0,j}(0) = \lambda QI^*_{n,j}(\theta) - QI_{0,j-1}(0)\beta_{j-1} V^*(\theta) \quad , j \geq 2 \tag{2.10}$$

$$\theta QI^*_{n,j}(\theta) - QI_{n,j}(0) = \lambda QI^*_{0,j}(\theta) - \lambda \sum_{k=1}^{n} QI^*_{n-k,j}(\theta)\, g_k \quad , n \geq 1 \text{ and } j \geq 2 \tag{2.11}$$

$$\theta P^*_1(\theta) - P_1(0) = (\lambda+\alpha)P^*_1(\theta) - B_1(0)S^*(\theta)$$
$$-P_2(0)S^*(\theta) - \sum_{j=1}^{\infty} QI_{1,j}\,(0)\, S^*(\theta) - \lambda\, PI\, g_1\, S^*(\theta) \tag{2.12}$$

$$\theta P^*_n(\theta) - P_n(0) = (\lambda+\alpha)P^*_n(\theta) - \lambda \sum_{k=1}^{n-1} P^*_{n-k}(\theta)\, g_k - B_n(0)S^*(\theta)$$
$$-P_{n+1}(0)S^*(\theta) - \sum_{j=1}^{\infty} QI_{n,j}\,(0)S^*(\theta) - \lambda PI\, g_n\, S^*(\theta), n \geq 2 \tag{2.13}$$

$$\theta B_1^*(\theta) - B_1(0) = \lambda B_1^*(\theta) - \alpha\, P_1{}^*(0)\, R^*(\theta) \quad (2.14)$$

$$\theta B_n^*(\theta) - B_n(0) = \lambda B_n^*(\theta) - \lambda \sum_{k=1}^{n-1} B_{n-k}^*(\theta)\, g_k - \alpha\, P_n{}^*(0) R^*(\theta), n \geq 2 \quad (2.15)$$

The following PGFs are defined to solve the equations.

2.3. THE PROBABILITY GENERATING FUNCTIONS

$$P^*(z,\theta) = \sum_{n=1}^{\infty} P_n^*(\theta) z^n \qquad P(z,0) = \sum_{n=1}^{\infty} P_n(0) z^n$$

$$B^*(z,\theta) = \sum_{n=1}^{\infty} B_n^*(\theta) z^n \qquad B(z,0) = \sum_{n=1}^{\infty} B_n(0) z^n$$

$$QI_j^*(z,\theta) = \sum_{n=0}^{\infty} QI_{n,j}^*(\theta) z^n \qquad QI_j(z,0) = \sum_{n=0}^{\infty} QI_{n,j}(0) z^n$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, PGFs are derived, through some algebraic manipulations.

Multiplying the equations (2.9) and (2.11) at j=1 by z^n and adding over n=0 to ∞

$$\theta QI_1^*(z,\theta) - QI_1(z,0) = \lambda QI_1^*(z,\theta) - \lambda QI_1^*(z,\theta) X(z) - P_1(0)\beta_0 V^*(\theta) \quad n \geq 1$$

$$[\theta - \lambda + \lambda\, x(z)]\, QI_1^*(z,\theta) = QI_1(z,0) - P_1(0)\beta_0 V^*(\theta)$$

$$[\theta - \lambda(1 - x(z))]\, QI_1^*(z,\theta) = QI_1(z,0) - P_1(0)\beta_0 V^*(\theta)$$

$$(\theta - w_x(z)) QI_1^*(z,\theta) = QI_1(z,0) - P_1(0)\beta_0\, V^*(\theta) \quad (2.16)$$

Where $w_x(z) = \lambda(1 - x(z))$

At $\theta = w_x(z)$,

$$QI_1(z,0) = P_1(0)\beta_0 V^*(w_x(z)) \quad (2.17)$$

Substituting for $QI_1(z,0)$ in (16)

$$QI_1^*(z,0) = P_1(0)\beta_0\, \frac{1 - V^*(w_x(z))}{w_x(z)} \quad (2.18)$$

Similarly, equations (2.10) and (2.11) imply for $j \geq 2$

$$\theta QI_j^*(z,\theta) - QI_j(z,0) \quad = \lambda QI_j^*(z,\theta) - \lambda QI_j^*(z,\theta)X(z) - QI_{0,j-1}(0)\beta_{j-1}V^*(\theta)$$

$$\text{ie., } (\theta - w_x(z))QI_j^*(z,\theta) \ = QI_j(z,0) - QI_{0,j-1}(0)\beta_{j-1}V^*(\theta) \tag{2.19}$$

At $\theta = w_x(z)$,

$$QI_j(z,0) = QI_{0,j-1}(0)\ \beta_{j-1}V^*(w_x(z)) \tag{2.20}$$

Substituting $QI_j(z,0)$ in (2.19)

$$QI_j^*(z,0) = QI_{0,j-1}(0)\ \beta_{j-1}\frac{1 - V^*(w_x(z))}{w_x(z)} \quad \forall\, j \geq 2 \tag{2.21}$$

If αI_n denotes the probability that n- customers arrive during a vacation time

$$\text{then } V^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha I_n\, z^n \qquad [\text{Gross and Harry}(1981)] \tag{2.22}$$

$$\text{where } \alpha I_n = \int_0^{\infty} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i g^i}{\lfloor i}\, dv(t) \tag{2.23}$$

Thus the equations (2.17) and (2.21) imply respectively,

$$\sum_{n=0}^{\infty} QI_{n,1}(0)\, z^n = P_1(0)\ \beta_0 \sum_{n=0}^{\infty} \alpha I_n\, z^n \tag{2.24}$$

and

$$\sum_{n=0}^{\infty} QI_{n,j}(0)\, z^n = QI_{0,j-1}(0)\beta_{j-1} \sum_{n=0}^{\infty} \alpha I_n\, z^n \tag{2.25}$$

Thus equating the coefficients of z^n gives,

$$QI_{n,1}(0) = P_1(0)\ \beta_0 \alpha I_n \quad , n \geq 0 \tag{2.26}$$

$$QI_{n,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_n \ \ n \geq 0 \text{and } j \geq 2 \tag{2.27}$$

At n=0 equations (2.26) and (2.27) give,

$$QI_{0,1}(0) = P_1(0)\ \beta_0 \alpha I_0\ ; \tag{2.28}$$

$$QI_{0,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_0 \quad , \forall\, j \geq 2 \tag{2.29}$$

By recursion

$$QI_{0,j}(0) = QI_{0,1}(0)\prod_{i=0}^{j-1}\beta_i\alpha I_0^{j-1} \tag{2.30}$$

$$= P_1(0)\prod_{i=0}^{j-1}\beta_i\alpha I_0^{j} \quad ,\forall\, j \geq 1 \tag{2.31}$$

Thus, $\quad QI_1^*(z,0) = P_1(0)\,\beta_0\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right) \qquad$ [from(2.18)]

$$QI_j^*(z,0) = P_1(0)\prod_{i=0}^{j-1}\beta_i\alpha I_0^{j-1}\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right) \quad ,\forall\, j \geq 2 \tag{2.32}$$

$$QI^*(z,0) = \sum_{j=0}^{\infty} QI_j^*(z,0)$$

Then from equation (2.21)

$$QI^*(z,0) = P_1(0)\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\left[\beta_0 + \sum_{j=2}^{\infty}\alpha I_0^{j-1}\prod_{i=0}^{j-1}\beta_i\right]$$

$$QI^*(z,0) = P_1(0)\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^{j}\prod_{i=0}^{j}\beta_i \tag{2.33}$$

similarly,

$$QI_1(z,0) = P_1(0)\beta_0 V^*\big(w_x(z)\big) \qquad \text{and}$$

$$QI_j(z,0) = V^*(w_x(z))\,P_1(0)\sum_{j=0}^{\infty}\alpha I_0^{j-1}\prod_{i=0}^{j-1}\beta_i \qquad ,j \geq 2$$

Adding the above equations

$$QI(z,0) = \sum_{j=0}^{\infty} QI_j(z,0) \qquad \text{implies,}$$

$$QI(z,0) = V^*\big(w_x(z)\big)\,P_1(0)\sum_{j=0}^{\infty}\alpha I_0^{j}\prod_{i=0}^{j}\beta_i \tag{2.34}$$

Substituting for $QI_{0,j}(0)$ from equation (2.31), in equation (2.1) implies,

$$\lambda PI = P_1(0)\sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + P_1(0)(1-\beta_0)$$

$$= \left[\sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1-\beta_0)\right] P_1(0)$$

$$\lambda PI = \varphi P_1(0) \tag{2.35}$$

where,

$$\varphi = [\sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^j + (1-\beta_0)] \tag{2.36}$$

The generating functions of the system size when the server is in breakdown states are calculated by using the equations (2.14) and (2.15).

$$\theta B^*(z,\theta) - B(z,0) = \lambda B^*(z,\theta) - \lambda B^*(z,\theta)X(z) - \alpha P^*(z,0)R^*(\theta)$$

$$[\theta - \lambda(1 - x(z))]B^*(z,\theta) = B(z,0) - \alpha P^*(z,0)R^*(\theta)$$

$$\big(\theta - w_x(z)\big)B^*(z,\theta) = B(z,0) - \alpha P^*(z,0)R^*(\theta) \tag{2.37}$$

At $\theta = w_x(z)$,

$$B(z,0) = R^*(w_x(z))[\alpha\, P^*(z,0)] \tag{2.38}$$

And substituting for $B(z,0)$ in equation (2.37), at $\theta = 0$,

$$-w_x(z)\, B^*(z,0) = [R^*\big(w_x(z)\big) - 1]\,\alpha\, P^*(z,0)$$

$$B^*(z,0) = \frac{[R^*\big(w_x(z)\big) - 1]}{-w_x(z)}\ \alpha\, P^*(z,0)$$

$$B^*(z,0) = \frac{1 - R^*(w_x(z))}{w_x(z)}\alpha P^*(z,0) \tag{2.39}$$

Next to calculate the PGF of system size corresponding to busy period, equations (2.12) and (2.13) are used.

Thus multiplying the corresponding equations by z^n and adding n=1 to ∞; it is found that

$$\theta P^*(z,\theta) - P(z,0) = (\lambda+\alpha)P^*(z,\theta) - \frac{S^*(\theta)}{z}\left[\sum_{n=2}^{\infty} P_{n+1}z^{n+1} + P_1(0)z - P_1(0)z\right]$$

$$-S^*(\theta)B(z,0) - \left[\sum_{j=1}^{\infty} QI_{n,j}(0) + QI_{0,j}(0) - QI_{0,j}(0)\right] S^*(\theta)$$

$$-S^*(\theta)\,\lambda\,PI\,X(Z) - \lambda\,X(z)\,P^*(z,\theta)$$

$$[\theta - (\alpha + w_x(z))]P^*(z,\theta)$$

$$= P(z,0) + S^*(\theta)\left[-B(z,0) - \frac{P(z,0)}{z} + P_1(0) - QI(z,0)\right.$$

$$\left. + \sum_{j=1}^{\infty} QI_{0,j}(0) + \lambda\,PI\,X(Z)\right] \qquad (2.40)$$

Multiplying the equation (2.35) by $S^*(\theta)$ and add with equation (2.40)

$$[\theta - (\alpha + w_x(z))]P^*(z,\theta)$$

$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right]$$

$$- S^*(\theta)\left[B(z,0) - P_1(0) + QI(z,0) - QI(z,0)\right.$$

$$- \sum_{j=1}^{\infty} QI_{0,j}(0) + \lambda\,PI\,X(Z) - \lambda PI$$

$$\left. + \sum_{j=1}^{\infty} QI_{0,j}(0) - \sum_{j=1}^{\infty} \beta_J\,QI_{0,j}(0) + P_1(0) - \beta_0 P_1(0)\right]$$

$$[\theta - g_\alpha(w_x(z))]P^*(z,\theta)$$

$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right]$$

$$- S^*(\theta)\left[B(z,0) + QI(z,0) - [\lambda - \lambda\,x(z)]\,PI - \sum_{j=1}^{\infty} \beta_J\,QI_{0,j}(0) - \beta_0 P_1(0)\right]$$

Where, $g_\alpha(w_x(z)) = \alpha + w_x(z)$

$$[\theta - g_{\alpha}\,(w_x(z))]P^*(z,\theta)$$

$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] - S^*(\theta)\left[B(z,0) + QI(z,0) - w_x(z)PI - \sum_{j=1}^{\infty}\beta_J\, QI_{0,j}\,(0) - \beta_0 P_1(0)\right] \quad (2.41)$$

Adding equations (2.34) and (2.38), the following result is obtained.

$$B(z,0) + QI(z,0) - w_x(z)PI - \sum_{j=1}^{\infty}\beta_J\, QI_{0,j}\,(0) - \beta_0 P_1(0)$$

$$= \alpha\, P^*(z,0)\;\; R^*\big(w_x(z)\big) \;\; + P_1(0) \sum_{j=0}^{\infty} \alpha {I_0}^j \prod_{i=0}^{j} \beta_i\;\; v^*\big(w_x(z)\big) - \;PI\; w_x(z)$$

$$-P_1(0)\left[\sum_{j=0}^{\infty} \alpha {I_0}^j \prod_{i=0}^{j-1} \beta_i + \beta_0\right]$$

$$= \alpha\, P^*(z,0)\;\; R^*\big(w_x(z)\big) \;\; + P_1(0)\; w_x(z)\left(\left[\frac{v^*\big(w_x(z)\big) - 1}{w_x(z)}\right]\sum_{j=0}^{\infty} \alpha {I_0}^j \prod_{i=0}^{j} \beta_i - \frac{\varphi}{\lambda}\right)$$

$$= \alpha\, P^*(z,0)\;\; R^*\big(w_x(z)\big) \;\; + P_1(0)\; w_x(z)\;\; [-I(z)]$$

Where,

$$I(z) \;= \left[\frac{v^*\big(w_x(z)\big) - 1}{w_x(z)}\right]\sum_{j=0}^{\infty} \alpha {I_0}^j \prod_{i=0}^{j} \beta_i - \frac{\varphi}{\lambda}$$

Substituting the above equation in equation (2.41).

$$[\theta - g_{\alpha}\,(w_x(z))]P^*(z,\theta)$$

$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] - S^*(\theta)\left[\alpha\, P^*(z,0)\;\; R^*\big(w_x(z)\big) + P_1(0)\; w_x(z)\;\; [-I(z)]\right] \quad (2.42)$$

At $\theta = 0$,

$$[\alpha\, R^*\big(w_x(z)\big) - g_{\alpha}\,(w_x(z))]\; P^*(z,0) = \; P(z,0)\left[\frac{z-1}{z}\right] + P_1(0)\; w_x(z)\; I(z)$$

$$P^*(z,0) = \frac{P_1(0)\; w_x(z)\; I(z) + P(z,0)\left[\frac{z-1}{z}\right]}{[\alpha\, R^*\big(w_x(z)\big) - g_{\alpha}\,(w_x(z))]} \quad (2.43)$$

Then Substituting for $P^*(z,0)$ in equation (2.42), we have

$$[\theta - g_\alpha\,(w_x(z))]P^*(z,\theta)$$

$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right]$$

$$-\ S^*(\theta)\left[\left[\frac{P_1(0)\ w_x(z)\ I(z) + P(z,0)\left[\frac{z-1}{z}\right]}{[\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]}\right]\alpha\, R^*\big(w_x(z)\big)\ \ + P_1(0)\ w_x(z)\ \ [-I(z)]\right]$$

$$= \frac{P(z,0)}{z}\left[z - S^*(\theta) - \frac{S^*(\theta)\ \alpha\, R^*\big(w_x(z)\big)\ (z-1)}{[\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]})\right]$$

$$- S^*(\theta)P_1(0)\ w_x(z)\ \ I(z)\left[\frac{\alpha\, R^*\big(w_x(z)\big)}{[\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]} - 1\right]$$

$$[\theta - g_\alpha\,(w_x(z))]P^*(z,\theta)$$

$$= \frac{P(z,0)}{z\ \ [\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]}\Big[z\Big(\ \alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))$$

$$-\ \ \alpha\, R^*\big(w_x(z)\big)S^*(\theta)\Big) + S^*(\theta)\ g_\alpha\,(w_x(z)\Big]$$

$$+\left[\frac{S^*(\theta)P_1(0)\ w_x(z)\ \ I(z)g_\alpha\,(w_x(z))}{[\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]}\right] \qquad (2.44)$$

If $\theta = g_\alpha\,(w_x(z)$

$$\frac{P(z,0)}{z\ \ [\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]}\Big[z\Big(\ \alpha\, R^*\big(w_x(z)\big)(1 - S^*\big(g_\alpha\,(w_x(z)\big) - g_\alpha\,(w_x(z)\Big)$$

$$+ S^*\big(g_\alpha\,(w_x(z)\big)g_\alpha\,(w_x(z)\Big]$$

$$= -\left[\frac{S^*\big(g_\alpha\,(w_x(z)\big)P_1(0)\ w_x(z)\ \ I(z)g_\alpha\,(w_x(z))}{[\alpha\, R^*\big(w_x(z)\big) - g_\alpha\,(w_x(z))]}\right]$$

$$\frac{P(z,0)}{z} = \frac{S^*\big(g_\alpha\,(w_x(z)\big)P_1(0)\ w_x(z)\ \ I(z)}{D(z)} \qquad (2.45)$$

Where,

$$D(z) =\ z\left[1 - \frac{\alpha\, R^*\big(w_x(z)\big)(1 - S^*\big(g_\alpha\,(w_x(z)\big)}{g_\alpha\,(w_x(z)}\right] - S^*\big(g_\alpha\,(w_x(z)\big) \qquad (2.46)$$

Using equation (2.45) in equation (2.44),

$$[\theta - g_{\alpha}\,(w_x(z))]P^*(z,\theta) = \frac{S^*\big(g_{\alpha}\,(w_x(z)\big)P_1(0)\,w_x(z)\;I(z)}{D(z)}$$

$$\left[\frac{z\Big(\alpha\,R^*\big(w_x(z)\big) - g_{\alpha}\,(w_x(z)) - \alpha\,R^*\big(w_x(z)\big)S^*(\theta)\Big) + S^*(\theta)\,g_{\alpha}\,(w_x(z)}{\alpha\,R^*\big(w_x(z)\big) - g_{\alpha}\,(w_x(z))}\right]$$

$$+\left[\frac{S^*(\theta)P_1(0)\,w_x(z)\;I(z)g_{\alpha}\,(w_x(z))}{[\alpha\,R^*\big(w_x(z)\big) - g_{\alpha}\,(w_x(z))]}\right]$$

$$[\theta - g_{\alpha}\,(w_x(z))]P^*(z,\theta) = \frac{P_1(0)\,w_x(z)\;I(z)\,z}{D(z)}\left[S^*\big(g_{\alpha}\,(w_x(z)\big) - S^*(\theta)\right]$$

$$P^*(z,\theta) = \frac{P_1(0)\,w_x(z)\;I(z)\,z}{D(z)[\theta - g_{\alpha}\,(w_x(z))]}\left[S^*\big(g_{\alpha}\,(w_x(z)\big) - S^*(\theta)\right] \tag{2.47}$$

At $\theta = 0$,

$$P^*(z,0) = \frac{P_1(0)\,w_x(z)\;I(z)\,z}{D(z)[g_{\alpha}\,(w_x(z))]}\left[S^*\big(g_{\alpha}\,(w_x(z)\big) - 1\right] \tag{2.48}$$

And

$$P(z,0) = \frac{P_1(0)\,w_x(z)\;I(z)\,z}{D(z)}\left[S^*\big(g_{\alpha}\,(w_x(z)\big)\right] \tag{2.49}$$

Thus the partial probability generating functions of the system size probabilities at arbitrary epoch when the server is in different states are obtained and are listed below

$$QI^*(z,0) = P_1(0)\left(\frac{1 - V^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i \tag{2.50}$$

$$P^*(z,0) = \frac{P_1(0)\,w_x(z)\;I(z)\,z}{D(z)[g_{\alpha}\,(w_x(z))]}\left[S^*\big(g_{\alpha}\,(w_x(z)\big) - 1\right] \tag{2.51}$$

$$B^*(z,0) = \frac{1 - R^*(w_x(z))}{w_x(z)}\alpha P^*(z,0) \tag{2.52}$$

Next to derive the total PGF of the system size distribution, the following generating functions are considered.

Let $P_{comp}(z)$ = The PGF of the system size when the server is in busy or in breakdown state.

$$P_{comp}(z) = P^*(z,0) + B^*(z,0) \tag{2.53}$$

Then, $P_I(z)$= The PGF of the system size when the server is idle state.

$$P_1(z) = QI^*(z,0) + PI$$

$$= P_1(0)\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i + P_1(0)\frac{\varphi}{\lambda}$$

$$= P_1(0)\left[\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i + \frac{\varphi}{\lambda}\right]$$

$$= P_1(0)\, I(z) \tag{2.54}$$

Thus by equation (2.53)

$$P_{comp}(z) = P^*(z,0) + \frac{1-R^*(w_x(z))}{w_x(z)}\alpha\, P^*(z,0)$$

$$= \frac{P_1(0)\, w_x(z)\;\; I(z)\, z}{D(z)[g_\alpha\,(w_x(z))]}\left[S^*\big(g_\alpha\,(w_x(z)\big)-1\right]\left[\frac{w_x(z)+\alpha-\alpha R^*(w_x(z))}{w_x(z)}\right]$$

$$= \frac{P_1(0)\;\; I(z)\, z}{D(z)}\left[S^*\big(g_\alpha\,(w_x(z)\big)-1\right]\left[1-\frac{\alpha R^*(w_x(z))}{g_\alpha\,(w_x(z)}\right] \tag{2.55}$$

Therefore the total PGF of the system size is given by

$$P(z) = P_I(z) + P_{comp}(z)$$

$$= \frac{P_1(0)\;\; I(z)}{D(z)}\left[D(z) + z\left[S^*\big(g_\alpha\,(w_x(z)\big)-1\right]\left[1-\frac{\alpha R^*(w_x(z))}{g_\alpha\,(w_x(z)}\right]\right]$$

$$= \frac{P_1(0)\;\; I(z)}{D(z)}\left[S^*\big(g_\alpha\,(w_x(z)\big)(z-1)\right] \tag{2.56}$$

2.3.1. STABILITY CONDITION

To evaluate $P_1(0)$, The normalizing condition $1 = \lim_{z\to 1} P(z)$ is used. For this we note that,

$$\lim_{z\to 1}\left(\frac{z-1}{D(z)}\right) = \frac{1}{D'(1)} \qquad \text{[using L'Hospital rule]}$$

And

$$D'(1) = (1-\rho)\, S^*(\propto) \qquad \text{[from equation (2.46)]}$$

Where,

$$\rho = \frac{\lambda\, E(X)}{\alpha\, S^*(\propto)}[1 - S^*(\propto)][1 + \alpha\, E(R)] \tag{2.57}$$

Then

$\lim_{z\to 1} P(z) = 1$ Implies,

$$P_1(0)\ \ I(1)\ \ S^*(\propto) \lim_{z\to 1}\left(\frac{z-1}{D(z)}\right) = 1$$

$$P_1(0) = \frac{(1-\rho)}{I(1)} \tag{2.58}$$

Where,

$$I(1) = \frac{\varphi}{\lambda} + \sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \beta_i\ \ E(V) \tag{2.59}$$

Substituting for $P_1(0)$ in equation (2.56)

$$P(z) = \frac{(1-\rho)}{I(1)} \frac{I(z)}{D(z)} \left[S^*\big(g_\alpha\,(w_x(z)\big)(z-1)\right] \tag{2.60}$$

Where,

$$D(z) = z\left[1 - \frac{\alpha\, R^*\big(w_x(z)\big)\big(1 - S^*\big(g_\alpha\,(w_x(z)\big)\big)}{g_\alpha\,(w_x(z)}\right] - S^*\big(g_\alpha\,(w_x(z)\big)$$

Thus the total PGF of the steady-state system size probabilities is derived in a compact form.

2.3.2. DECOMPOSITION PROPERTY:

The equation (2.60) can be written as

$$P(z) = (1-\rho)\ \left[\frac{S^*\big(g_\alpha\,(w_x(z)\big)(z-1)}{D(z)}\right] \frac{I(z)}{I(1)} \tag{2.61}$$

The above equation (2.61) shows that the PGF of the system size of the model under consideration is decomposed into the product of two probability generating functions, one of which is the PGF of the single server queueing model with server breakdown on the other PGF of the conditional system size distribution $\frac{I(z)}{I(1)}$ during the server idle period under the condition $\rho < 1$.

2.4. PERFORMANCE MEASURES

In this section some useful performance measures of the proposed model are presented.

2.4.1 THE STEADY STATE SYSTEM SIZE PROBABILITIES:

Let P_V , P_{busy} and P_{br} denote the probability that the server is on vacation, busy state and breakdown state respectively. Then the corresponding probabilities are obtained, by considering the equations in ((2.50), (2.51), (2.52)) at z=1.

Thus,

(i) $P_V = \lim_{z\to1} Q^*(z,0)$

$$= \frac{(1-\rho)}{I(1)} E(V) \sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \beta_i$$

(ii) $P_{busy} = \lim_{z\to1} P^*(z,0)$

$$= \frac{\lambda E(X)(1 - S^*(\alpha))}{\alpha\, S^*(\alpha)}$$

(iii) $P_{br} = \lim_{z\to1} B^*(z,0)$

$$= \alpha\, E(R)\, P_{busy}$$

And

$$PI = P_1(0)\frac{\varphi}{\lambda} = \frac{\varphi}{\lambda}\frac{(1-\rho)}{I(1)}$$

2.4.2 MEAN SYSTEM SIZE:

In this section the average number of customers waiting in the system, when the server is in different states are calculated.

Let L_V , L_{busy} and L_{br} denote the expected system size when the server is in vacation state, busy state and breakdown state respectively. Then the derivatives of equations in ((2.50) to (2.52)) at z=1 give the required measures.

Thus the mean system size corresponding to different states are given by,

$$\text{(i)}L_V = \left(\frac{d}{dz}QI^*(z,0)\right)_{z=1}$$

$$= \frac{\lambda E(X)(1-\rho)}{I(1)}\frac{E(V^2)}{2}\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i \quad (2.61)$$

$$\text{(ii)}L_{busy} = \left(\frac{d}{dz}P^*(z,0)\right)_{z=1}$$

$$= P_{busy} + \lambda\, E(X)\frac{(1-S^*(\alpha))}{\alpha\, S^*(\alpha)}\frac{I'(1)}{I(1)} + \frac{\left(\lambda E(X)\right)^2}{S^*(\alpha)}\left(\frac{S^{*\prime}(\alpha)}{\alpha} + \frac{1-S^*(\alpha)}{\alpha^2}\right)$$
$$+ \frac{1-S^*(\alpha)}{2\alpha\, S^*(\alpha)}\left(\lambda E\big(X(X-1)\big) + \frac{\lambda\, E(X)}{D'(1)}(-D''(1))\right) \quad (2.62)$$

Where,

$$I'(1) = \lambda E(X)\left(\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i\left(\frac{E(V^2)}{2}\right)\right)$$

$$(-D''(1)) = \lambda E\big(X(X-1)\big)\left(\big(1-S^*(\alpha)\big)\left(E(R)+\frac{1}{\alpha}\right)\right)$$
$$+ \left(\lambda E(X)\right)^2\left[\big(1-S^*(\alpha)\big)\left(E(R^2)+\frac{2}{\alpha^2}+\frac{2E(R)}{\alpha}\right) + 2S^{*\prime}(\alpha)\left(E(R)+\frac{1}{\alpha}\right)\right.$$
$$\left. + 2\lambda\, E(X)\left(\big(1-S^*(\alpha)\big)\left(E(R)+\frac{1}{\alpha}\right) + S^{*\prime}(\alpha)\right)\right]$$

$$\text{(iii)}L_{br} = \left(\frac{d}{dz}B^*(z,0)\right)_{z=1}$$

$$= \alpha\left(L_{busy}\, E(R) + P_{busy}\,\lambda\, E(X)\frac{E(R^2)}{2}\right) \quad (2.63)$$

The total expected system size for the model can be obtained using equation (2.60)

$$L = \left[\frac{d}{dz}P(z)\right]_{z=1} = \frac{1-\rho}{I(1)}\;\frac{d}{dz}\left[\frac{z-1}{D(z)}S^*\left(g_\alpha\big(w_x(z)\big)\right)I(z)\right]$$

$$L = (1-\rho)\, S^*(\alpha) \left(\frac{-D''(1)}{2\, D'^2(1)} \right) + \frac{I'(1)}{I(1)} + \frac{{S^*}'(\alpha)}{S^*(\alpha)} \big(-\lambda\, E(X)\big) \qquad (2.64)$$

Then, it is verified that

$$L_V + L_{br} + L_{busy} = \frac{I'(1)}{I(1)} + \frac{-D''(1)}{2\, S^*(\alpha)(1-\rho)} - \lambda\, E(X) \frac{{S^*}'(\alpha)}{S^*(\alpha)} \qquad (2.65)$$

$$= L$$

CONCLUSION:

In the present chapter, the bulk arrival queueing model in which the server is subjected to unpredictable breakdown and takes multiple adapted vacations when the system becomes idle is considered. It is assumed that the customer whose service is interrupted will join the head of the queue and repeats his service from the beginning as soon as the server is fixed. In the following chapter III, the model is analysed under the condition that the service interrupted customers, instead of repeating the service from the beginning, will stay in the service facility to complete the remaining services.

CHAPTER III

AN UNRELIABLE $M^X/G/1$ QUEUEING SYSTEM WITH SERVICE RESUMPTION UNDER MULTIPLE ADAPTED VACATION POLICY

In chapter II, whenever breakdowns occur, the service interrupted customers lose the service and starts a new service as soon as the server is fixed. In the present chapter, the model is discussed under the condition that if the service is interrupted then, the customer will stay in the service facility to complete the remaining service.

3.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

3.1.1 MODEL DESCRIPTION

Arrival Pattern

The present chapter considers an $M^X/G/1$ queueing system in which the arrivals occur in batches in accordance with a time homogeneous Poisson process with random batch size **X**, group arrival rate **λ** and probability distribution $\mathbf{Pr(X=k)=g_k}$**, k=1,2,3…** The customers are served one by one according to the order in the queue.

Multiple Adapted Vacation Policy (MAV)

A cycle starts whenever the system becomes empty and the server is deactivated. The deactivated server either leaves the system for a vacation (first vacation) of random length (**V**) with probability β_0 or remains idle in the system with probability **(1- β_0).** Upon returning from the vacation, if the server finds atleast one customer waiting in the system, then the server starts a busy period immediately. Otherwise, if there are no customers found waiting in the queue then the server either joins the system with the probability (**1- β_1**) or takes a new vacation (second vacation) with probability β_1. This pattern continues until atleast one customer is found in the system. Thus the vacation policy is determined by the sequence of probabilities {β_i}, i=0, 1, 2, … (i,e.) The vacations have independent duration with common distribution function **V(t)** and density function **v(t)** of finite moments.

Busy Period and Breakdown Period

During busy period, the server provides single service. The service times follow general (arbitrary) distributions with distribution functions **S(t)** and the density functions **s(t)** of finite moments $E(S^k)$**, k=1,2.**

The server may breakdown at any time while providing service and the server is sent for repair instantaneously. The service channels will not function for a short interval of time. The service interrupted customer stays in the service facility and server starts the service from where it got interrupted. The breakdowns occur according to the Poisson process with rate $\boldsymbol{\alpha}$ and the repair times (denoted by **R**) of the server is assumed to be arbitrarily distributed with distribution function **R(t),** density function **r(t)** for $\mathbf{t \geq 0}$**.**

The customers continue to arrive according to the Compound Poisson process, independent of the state of the system and wait in the queue during the busy period, breakdown period and vacation period. The service completion period of a customer consists of the service time and repair time of the server. Thus a cycle is made up of idle vacation period, busy period and breakdown period.

We denote the model by $\mathbf{M^X/G/1/MAV/breakdown}$(Resuming service), where MAV denotes the Multiple Adapted Vacation policy. Various stochastic processes involved in the queueing system are assumed to be independent to each other. Using supplementary variable technique the steady state system equations under the steady state condition are analyzed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

To analyse the steady state results of the model the following notations are introduced.

Notations:

N (t) : The system size at time t

λ : Group arrival rate

X : Group size random variable

g_k : Pr(X = k), k = 1, 2, 3, …

X(z) : Probability generating function of X.

The notations of Random Variables (RV), Cumulative Distribution Functions (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RVs are listed in table (3.1).

Table (3.1)

	RV	CDF	PDF	LST	k^{th} moments
Service time	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$
Repair time	R	R(x)	r(x)	$R^*(\theta)$	$E(R^k)$
Idle Vacation Time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$

If f(x) is the density function of the probability distribution F(x) then

$$F^*(\theta) = \int_0^\infty e^{-\theta x} f(x)dx = \int_0^\infty e^{-\theta x} d\big(F(x)\big)$$

Let $V^0(t), S^0(t)$ and $R^0(t)$ denote the remaining times of the random variables namely Idle vacation time, the service time and Repair time respectively.

Let the state of the system at time **t** be given by Y(t) = 0, 1, 2, 3 according as the server is idle in the system, on vacation during idle time, busy state and break down state respectively. The supplementary variables are introduced in order to obtain a bivariate Markov Process **{N(t),δ(t)}** where **N(t)** denotes the system size at time t and $\delta(t) = (0, V^0(t),\ S^0(t), R^0(t))$ according as Y (t) = (0, 1, 2, 3) respectively.

Let $PI^\circ(t) = Pr\{Y(t) = 0, N(t) = 0\}$

$QI_{n,j}(x, t)\, dt = Pr\{N(t) = n, x < V^\circ(t) \le x + dt, Y(t) = 1\}$; $n \ge 0$

$P_n(x, t)\, dt = Pr\{N(t) = n, x < S^\circ(t) \le x + dt, Y(t) = 2\}$; $n \ge 1$

$B_n(x, y, t)\, dt = Pr\{N(t) = n, x < R^\circ(t) \le x + dt, Y(t)=3\}$; $n \ge 1$

Then, $QI_{n,j}(x, t)\, dt$ is the joint probability that at time **t**, there are n customers in the system, and the remaining **vacation** time of the server is between x and x + dt, where n≥0.

$P_n(x, t)\,dt$ is the joint probability that at time t, there are n customers in the system, the server is **busy** and the customer being served in the service channel with remaining service time lies between x and x + dt, where $n \geq 1$.

$B_n(x, y, t)$ denotes the probability that there are n-customers in the system at time **t**, the server is under repair and the remaining service time for the customer is equal to x, and the server is being repaired with the remaining repair time between y and y + dt.

PI(t) denotes that the server is idle in the empty system at time **t**.

Assuming that at steady-state, the probabilities are independent of time t we have

$$\lim_{t\to\infty} \frac{\partial}{\partial x} P_n(x,t) = \frac{d}{dx} P_n(x) \; ; \qquad \lim_{t\to\infty} \frac{\partial}{\partial x} QI_{n,j}(x,t) = \frac{d}{dx} QI_{n,j}(x)$$

$$\lim_{t\to\infty} \frac{\partial}{\partial x} B_n(x,y,t) = \frac{d}{dx} B_n(x,y)$$

$$\lim_{t\to\infty} \left(\frac{\partial}{\partial t} P_n(x,t) = \frac{\partial}{\partial t} QI_{n,j}(x,t) = \frac{\partial}{\partial t} B_n(x,y,t) \right) = 0$$

At steady state $P_n(0)$,$QI_{n,j}(0)$, and $B_n(0)$ denote the probability that there are n customers in the system at the termination of service period, vacation period and repair period respectively.

With these notations, the steady state equations satisfied by the steady state probabilities are given in the following sections.

3.2 THE STEADY STATE SYSTEM SIZE EQUATIONS

Idle state

$$\lambda PI = \sum_{j=1}^{\infty} (1 - \beta_j)\, QI_{0,j}(0) + P_1(0)(1 - \beta_0) \tag{3.1}$$

Vacation during idle state

$$-\frac{d}{dx} QI_{0,1}(x) = -\lambda QI_{0,1}(x) + P_1(0)\beta_0 v(x) \tag{3.2}$$

$$-\frac{d}{dx} QI_{0,j}(x) = -\lambda QI_{0,j}(x) + QI_{0,j-1}(0)\beta_{j-1} v(x), \quad j \geq 2 \tag{3.3}$$

$$-\frac{d}{dx}QI_{n,j}(x) = -\lambda QI_{n,j}(x) + \sum_{k=1}^{n} QI_{n-k,j}(x)\, g_k \qquad n \geq 1, j \geq 1 \tag{3.4}$$

Busy state

$$-\frac{d}{dx}P_1(x) = -(\lambda+\alpha)P_1(x)+B_1(x,0) + P_2(0)s(x)$$

$$+\sum_{j=1}^{\infty} QI_{1,j}\,(0)\; s(x) + \lambda PI\; g_1\; s(x) \tag{3.5}$$

$$-\frac{d}{dx}P_n(x) = -(\lambda+\alpha)P_n(x) + \lambda\sum_{k=1}^{n-1} P_{n-k}(x)g_k\;\; P_{n+1}(0)\; s(x) + B_n(x,0)$$

$$+\sum_{j=1}^{\infty} QI_{n,j}\,(0)\; s(x) + \lambda PI\; g_n s(x) \qquad , n \geq 2 \tag{3.6}$$

Breakdown state

$$-\frac{\partial}{\partial x}B_1(x,y) = -\,\lambda B_1(x,y) + \alpha P_1(x)\; r(y) \tag{3.7}$$

$$-\frac{\partial}{\partial x}B_n(x,y) = -\,\lambda B_n(x,y) + \lambda\sum_{k=1}^{\infty} B_{n-k}\,(x,y)\; r(y)\; g_k \; + \alpha\; P_n(x)\; r(y), n \geq 2 \tag{3.8}$$

Thus the L.S.T of the equations with respect to x are given by

$$\theta QI^*_{0,1}(\theta) - QI_{0,1}(0) = \lambda QI^*_{0,1}(\theta) - P_1(0)\beta_0 V^*(\theta) \tag{3.9}$$

$$\theta QI^*_{0,j}(\theta) - QI_{0,j}(0) = \lambda QI^*_{n,j}(\theta) - QI_{0,j-1}(0)\beta_{j-1}V^*(\theta) \quad , j \geq 2 \tag{3.10}$$

$$\theta QI^*_{n,j}(\theta) - QI_{n,j}(0) = \lambda QI^*_{0,j}(\theta) - \lambda\sum_{k=1}^{n} QI^*_{n-k,j}(\theta)\, g_k \quad , n \geq 1 \text{ and } j \geq 2 \tag{3.11}$$

$$\theta P^*_1(\theta) - P_1(0) = (\lambda+\alpha)P^*_1(\theta) - {B_1}^*(\theta,0)$$

$$-P_2(0)\; S^*(\theta) - \sum_{j=1}^{\infty} QI_{1,j}\,(0)\; S^*(\theta) - \lambda\; PI\; g_1\; S^*(\theta) \tag{3.12}$$

$$\theta P_n^*(\theta) - P_n(0) = (\lambda + \alpha)\, P_n^*(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k}(\theta)\, g_k - B_n{}^*(\theta, 0)$$

$$-P_{n+1}(0)\, S^*(\theta) - \sum_{j=1}^{\infty} QI_{n,j}\,(0) S^*(\theta) - \lambda\, PI\, g_n\, S^*(\theta), n \geq 2 \qquad (3.13)$$

$$-\frac{\partial}{\partial y} B_1^*(\theta, y) = -\lambda\, B_1^*(\theta, y) + \alpha\, P_1{}^*(\theta)\, r(y) \qquad (3.14)$$

$$-\frac{\partial}{\partial y} B_n^*(\theta, y) = -\lambda\, B_n^*(\theta, y) + \lambda \sum_{k=1}^{n-1} B_{n-k}^*(\theta, y)\, g_k + \alpha\, P_n{}^*(\theta)\, r(y)\,, n \geq 2 \qquad (3.15)$$

Taking L.S.T with respect to y for breakdown equations

$$\theta_1 B^{*\,*\,1}(\theta, \theta_1) - B^*(\theta, 0) = \lambda\, B^{*\,*\,1}(\theta, \theta_1) - \alpha\, P_n{}^*(\theta)\, R^{*1}(\theta_1) \qquad (3.16)$$

$$\theta_1 B^{*\,*\,1}(\theta, \theta_1) - B^*(\theta, 0) = \lambda\, B^{*\,*\,1}(\theta, \theta_1) - \alpha\, P_n{}^*(\theta)\, R^{*1}(\theta_1)$$

$$-\lambda \sum_{k=1}^{n} B_{n-k}{}^{**1}(\theta, \theta_1) g_k \;, \qquad n \geq 0 \qquad (3.17)$$

The following PGF’s are defined to solve the equations.

3.3. THE PROBABILITY GENERATING FUNCTIONS

$$PI^*(z, \theta) = \sum_{n=1}^{\infty} PI_n^*\,(\theta) z^n \qquad PI(z, 0) = \sum_{n=1}^{\infty} PI_n\,(0) z^n$$

$$P^*(z, \theta) = \sum_{n=0}^{\infty} P_n^*\,(\theta) z^n \qquad P(z, 0) = \sum_{n=0}^{\infty} P_n\,(0) z^n$$

$$QI_j^*(z, \theta) = \sum_{n=0}^{\infty} QI_{n,j}^*\,(\theta) z^n \qquad QI_j(z, 0) = \sum_{n=0}^{\infty} QI_{n,j}\,(0) z^n$$

$$B^{**1}(z, \theta, \theta_1) = \sum_{n=0}^{\infty} B_n{}^{**1}(\theta, \theta_1)\, z^n \qquad B^*(z, \theta, 0) = \sum_{n=0}^{\infty} B_n{}^*(\theta, 0) z^n$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, PGFs are derived, through some algebraic manipulations.

Multiplying the equations (3.9) and (3.11) at j=1 by z^n and adding over n=0 to ∞

$$\theta QI_1^*(z,\theta) - QI_1(z,0) \; = \lambda QI_1^*(z,\theta) - \lambda QI_1^*(z,\theta)X(z) - P_1(0)\beta_0 V^*(\theta) \quad n \geq 1$$

$$[\theta - \lambda + \lambda\, x(z)]\, QI_1^*(z,\theta) = QI_1(z,0) \; - P_1(0)\beta_0 V^*(\theta)$$

$$[\theta - \lambda(1 - \, x(z))]\, QI_1^*(z,\theta) = QI_1(z,0) \; - P_1(0)\beta_0 V^*(\theta)$$

$$(\theta - w_x(z))QI_1^*(z,\theta) \quad = QI_1(z,0) - P_1(0)\beta_0\, V^*(\theta) \qquad (3.18)$$

Where $w_x(z) = \lambda(1 - \, x(z))$

At $\theta = w_x(z)$,

$$QI_1(z,0) = P_1(0)\beta_0 V^*(w_x(z)) \qquad (3.19)$$

Substituting for $QI_1(z,0)$in equation(3.16)

$$QI_1^*(z,0) = P_1(0)\beta_0 \, \frac{1 - V^*(w_x(z))}{w_x(z)} \qquad (3.20)$$

Similarly, equations (3.10) and (3.11) imply for $j \geq 2$

$$\theta QI_j^*(z,\theta) - QI_j(z,0) \quad = \lambda QI_j^*(z,\theta) - \lambda QI_j^*(z,\theta)X(z) - QI_{0,j-1}(0)\beta_{j-1}V^*(\theta)$$

ie., $(\theta - w_x(z))QI_j^*(z,\theta) \; = QI_j(z,0) - QI_{0,j-1}(0)\beta_{j-1}V^*(\theta)$ (3.21)

At $\theta = w_x(z)$,

$$QI_j(z,0) = QI_{0,j-1}(0)\;\beta_{j-1}V^*(w_x(z)) \qquad (3.22)$$

Substituting $QI_j(z,0)$ in equation (3.21)

$$QI_j^*(z,0) = QI_{0,j-1}(0)\;\beta_{j-1}\frac{1 - V^*(w_x(z))}{w_x(z)} \quad \forall\, j \geq 2 \qquad (3.23)$$

If αI_n denotes the probability that n- customers arrive during a vacation time

Then, $$V^*(w_x(z)) = \sum_{n=0}^{\infty} \alpha I_n\, z^n \qquad \text{[Gross and Harry(1981)]} \qquad (3.24)$$

where, $$\alpha I_n = \int_0^{\infty} e^{-\lambda t} \sum_{i=0}^{\infty} \frac{(\lambda t)^i g^i}{\lfloor i} \, dv(t) \tag{3.25}$$

Thus the equations (3.22) and (3.23) imply respectively,

$$\sum_{n=0}^{\infty} QI_{n,1}(0)\, z^n = P_1(0)\, \beta_0 \sum_{n=0}^{\infty} \alpha I_n\, z^n \tag{3.26}$$

and

$$\sum_{n=0}^{\infty} QI_{n,j}(0)\, z^n = QI_{0,j-1}(0)\beta_{j-1} \sum_{n=0}^{\infty} \alpha I_n\, z^n \tag{3.27}$$

Thus equating the coefficients of z^n gives,

$$QI_{n,1}(0) = P_1(0)\, \beta_0 \alpha I_n \quad , n \geq 0 \tag{3.28}$$

$$QI_{n,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_n \quad n \geq 0 \text{and } j \geq 2 \tag{3.29}$$

At n=0 equations (3.28) and (3.29) give,

$$QI_{0,1}(0) = P_1(0)\, \beta_0 \alpha I_0 \, ; \tag{3.30}$$

$$QI_{0,j}(0) = QI_{0,j-1}(0)\beta_{j-1}\alpha I_0 \quad , \forall\, j \geq 2 \tag{3.31}$$

By recursion

$$QI_{0,j}(0) = QI_{0,1}(0) \prod_{i=0}^{j-1} \beta_i \alpha I_0^{j-1} \tag{3.32}$$

$$= P_1(0) \prod_{i=0}^{j-1} \beta_i \alpha I_0^{j} \quad , \forall\, j \geq 1 \tag{3.33}$$

Thus, $$QI_1^*(z,0) = P_1(0)\, \beta_0 \left(\frac{1 - V^*\big(w_x(z)\big)}{w_x(z)}\right) \qquad \text{from}(3.20)$$

$$QI_j^*(z,0) = P_1(0) \prod_{i=0}^{j-1} \beta_i \alpha I_0^{j-1} \left(\frac{1 - V^*\big(w_x(z)\big)}{w_x(z)}\right) \quad , \forall\, j \geq 2 \tag{3.34}$$

$$QI^*(z,0) = \sum_{j=0}^{\infty} QI_j^*(z,0)$$

Thus, from equation (3.23)

$$QI^*(z,0) = P_1(0)\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\left[\beta_0 + \sum_{j=2}^{\infty} \alpha I_0^{j-1} \prod_{i=0}^{j-1} \beta_i\right]$$

$$QI^*(z,0) = P_1(0)\left(\frac{1-V^*\big(w_x(z)\big)}{w_x(z)}\right)\sum_{j=0}^{\infty} \alpha I_0^{j} \prod_{i=0}^{j} \beta_i \qquad ,\forall\, j \geq 2 \tag{3.35}$$

similarly,

$$QI_1(z,0) = P_1(0)\beta_0 V^*\big(w_x(z)\big) \qquad \text{and}$$

$$QI_j(z,0) = V^*(w_x(z))\, P_1(0)\sum_{j=0}^{\infty} \alpha I_0^{j-1} \prod_{i=0}^{j-1} \beta_i \qquad ,\forall\, j \geq 2$$

Adding the above equations

$$QI(z,0) = \sum_{j=0}^{\infty} QI_j(z,0) \qquad \text{implies,}$$

$$QI(z,0) = V^*\big(w_x(z)\big)\, P_1(0)\sum_{j=0}^{\infty} \alpha I_0^{j} \prod_{i=0}^{j} \beta_i \qquad ,\forall\, j \geq 2 \tag{3.36}$$

Substituting for $QI_{0,j}(0)$ in equation (3.1) implies,

$$\begin{aligned}\lambda PI &= P_1(0)\sum_{j=1}^{\infty}\big(1-\beta_j\big)\prod_{i=0}^{j-1}\beta_i\, \alpha I_0^{j-1} V^*\big(w_x(z)\big) + P_1(0)(1-\beta_0)\\ &= \left[\sum_{j=1}^{\infty}\big(1-\beta_j\big)\prod_{i=0}^{j-1}\beta_i\, \alpha I_0^{j-1} V^*\big(w_x(z)\big) + (1-\beta_0)\right] P_1(0)\end{aligned}$$

$$\lambda PI = \varphi P_1(0) \tag{3.37}$$

Where,

$$\varphi = \sum_{j=1}^{\infty}\big(1-\beta_j\big)\prod_{i=0}^{j-1}\beta_i\, \alpha I_0^{j} V^*\big(w_x(z)\big) + (1-\beta_0) \tag{3.38}$$

The generating functions of the system size when the server is in breakdown states are calculated by using the equations (3.16) and (3.17).

$$\theta_1 B^{**1}(z,\theta,\theta_1) - B^*(z,\theta,0) = \lambda B^{**1}(z,\theta,\theta_1) - \alpha P^*(z,\theta)\, R^{*1}(\theta_1) - \lambda\, x(z)\, B^{**1}(z,\theta,\theta_1)$$

$$\big(\theta_1 - w_x(z)\big) B^{**1}(z,\theta,\theta_1) = B^*(z,\theta,0) - \alpha P^*(z,\theta)\, R^{*1}(\theta_1) \tag{3.39}$$

At $\theta_1 = w_x(z)$

$$B^*(z,\theta,0) = \alpha P^*(z,\theta)\, R^{*1}\big(w_x(z)\big) \tag{3.40}$$

Now,

$$\big(\theta_1 - w_x(z)\big) B^{**1}(z,\theta,\theta_1) = \alpha\, P^*(z,\theta)[\, R^{*1}\big(w_x(z)\big) - R^{*1}(\theta_1)]$$

$$B^{**1}(z,\theta,\theta_1) = \frac{\alpha P^*(z,\theta)[\, R^{*1}\big(w_x(z)\big) - R^{*1}(\theta_1)]}{\big(\theta_1 - w_x(z)\big)} \tag{3.41}$$

At $\theta_1 = 0$, and $\theta = 0$,

$$B^{**1}(z,0,0) = \frac{\alpha\, P^*(z,0)[1 - R^{*1}\big(w_x(z)\big)]}{w_x(z)} \tag{3.42}$$

Next to calculate the PGF of system size corresponding to busy period, equations (3.12) and (3.13) are used.

Thus multiplying the corresponding equations by z^n and adding n=1 to ∞; it is found that

$$\theta P^*(z,\theta) - P(z,0) = (\lambda+\alpha)P^*(z,\theta) - \frac{S^*(\theta)}{z}\left[\sum_{n=2}^{\infty} P_{n+1} z^{n+1} + P_1(0)z - P_1(0)z\right]$$
$$-B^*(z,\theta,0) - \left[\sum_{j=1}^{\infty} QI_{n,j}\,(0) + QI_{0,j}\,(0) - QI_{0,j}\,(0)\right] S^*(\theta)$$
$$-S^*(\theta)\,\lambda\,\mathrm{PI}\, X(z) - \lambda\, X(z)\, P^*(z,\theta)$$

(i.e.) $[\theta - (\alpha + w_x(z))]P^*(z,\theta) = P(z,0) - \frac{S^*(\theta)}{z}[P(z,0) - P_1(0)z]$

$$-B^*(z,\theta,0) - \left[\sum_{j=1}^{\infty} QI_j\,(z,0) - QI_{0,j}\,(0)\right] S^*(\theta)$$
$$-S^*(\theta)\,\lambda\,\mathrm{PI}\, X(z)$$

$$[\theta - g_\alpha(w_x(z))]P^*(z,\theta) = P(z,0) - \frac{S^*(\theta)}{z}[P(z,0) - P_1(0)z]$$
$$-B^*(z,\theta,0) - \left[\sum_{j=1}^{\infty} QI_j(z,0) - QI_{0,j}(0)\right]S^*(\theta)$$
$$-S^*(\theta)\,\lambda\,PI\,X(z) \qquad (3.43)$$

Where, $g_\alpha(w_x(z)) = \alpha + w_x(z)$

Equation (3.1) multiplied by $(-S^*(\theta))$ gives,

$$-S^*(\theta)\,\lambda\,PI = \left[\sum_{j=1}^{\infty}(1-\beta_j)\,QI_{0,j}(0) + P_1(0)(1-\beta_0)\right](-S^*(\theta))$$
$$= -P_1(0)S^*(\theta)\left[\sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^{j-1}\,V^*(w_x(z)) + (1-\beta_0)\right]$$

[Using equations (3.30) and(3.31)]

(i.e.) $-S^*(\theta)\,\lambda\,PI = -P_1(0)S^*(\theta)\,\varphi$ (3.44)

Where,

$$\varphi = \sum_{j=1}^{\infty}(1-\beta_j)\prod_{i=0}^{j-1}\beta_i\,\alpha I_0^{j-1}\,V^*(w_x(z)) + (1-\beta_0)$$

Adding equations (3.43) and (3.44), the following result is obtained.

$$[\theta - g_\alpha(w_x(z))]P^*(z,\theta)$$
$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] + P_1(0)S^*(\theta) - B^*(z,\theta,0) - QI(z,0)S^*(\theta)$$
$$+\sum_{j=1}^{\infty} QI_{0,j}(0)\;S^*(\theta) - S^*(\theta)\,\lambda\,PI\,X(z) - \sum_{j=1}^{\infty} QI_{0,j}(0)\;S^*(\theta)$$
$$+S^*(\theta)\sum_{j=1}^{\infty}\beta_j\,QI_{0,j}(0) + S^*(\theta)\,\lambda\,PI - P_1(0)\,S^*(\theta)\,(1-\beta_0)$$

$$[\theta - h_\alpha(w_x(z))]P^*(z,\theta)$$
$$= P(z,0)\left[\frac{z - S^*(\theta)}{z}\right]$$
$$+ P_1(0)S^*(\theta)\, w_x(z)\left[\left(\frac{1 - V^*(w_x(z))}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j \prod_{i=0}^{j}\beta_i + \frac{\varphi}{\lambda}\right]$$

$$[\theta - h_\alpha(w_x(z))]P^*(z,\theta) = P(z,0)\left[\frac{z - S^*(\theta)}{z}\right] + P_1(0)\, S^*(\theta)\, w_x(z)\, I(z) \qquad (3.45)$$

Where

$$h_\alpha(w_x(z)) = g_\alpha(w_x(z)) - \alpha\, R^{*1}(\, w_x(z))$$

$$I(z) = \left(\frac{1 - V^*(w_x(z))}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j \prod_{i=0}^{j}\beta_i + \frac{\varphi}{\lambda} \qquad (3.46)$$

At $\theta = h_\alpha(w_x(z))$,

$$P(z,0) = -\left[\frac{z}{z - S^*\left(h_\alpha(w_x(z))\right)}\right]P_1(0)\, S^*\left(h_\alpha(w_x(z))\right)\, w_x(z)\, I(z) \qquad (3.47)$$

Substituting equation (3.47) in equation (3.45)

$$[\theta - h_\alpha(w_x(z))]P^*(z,\theta)$$
$$= -\left[\frac{z - S^*(\theta)}{z - S^*\left(h_\alpha(w_x(z))\right)}\right]P_1(0)\, S^*\left(h_\alpha(w_x(z))\right)\, w_x(z)\, I(z)$$
$$+ P_1(0)\, S^*(\theta)\, w_x(z)\, I(z)$$

$$[\theta - h_\alpha(w_x(z))]P^*(z,\theta) = \frac{z\, P_1(0)\, w_x(z)\, I(z)}{D(z)}\left[S^*(\theta) - S^*\left(h_\alpha(w_x(z))\right)\right] \qquad (3.48)$$

Where, $D(z) = z - S^*\left(h_\alpha(w_x(z))\right)$ (3.49)

At $\theta = 0$

$$P^*(z,0) = \frac{z\, P_1(0)(-w_x(z))\, I(z)}{D(z)[h_\alpha(w_x(z))]}\left[1 - S^*\left(h_\alpha(w_x(z))\right)\right] \qquad (3.50)$$

Thus the partial probability generating functions of the system size probabilities at arbitrary epoch when the server is in different states are obtained and are listed below

$$QI^*(z,0) = P_1(0)\left(\frac{1-V^*(w_x(z))}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i \qquad (3.51)$$

$$P^*(z,0) = \frac{z\,P_1(0)(-w_x(z))\,I(z)}{D(z)\left[h_\alpha(w_x(z))\right]}\left[1-S^*\left(h_\alpha(w_x(z))\right)\right] \qquad (3.52)$$

$$B^{**1}(z,0,0) = \frac{\alpha\,P^*(z,0)[1-R^{*1}(w_x(z))]}{w_x(z)} \qquad (3.53)$$

Next to derive the total PGF of the system size distribution, the following generating functions are considered.

Let $P_{comp}(z)$= The PGF of the system size when the server is in busy or in breakdown state.

$$P_{comp}(z) = P^*(z,0)+B^{**1}(z,0,0) \qquad (3.54)$$

Then, $P_I(z)$= The PGF of the system size when the server is idle state.

$$P_1(z) = QI^*(z,0) + PI$$

$$= P_1(0)\left(\frac{1-V^*(w_x(z))}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i + P_1(0)\frac{\varphi}{\lambda}$$

$$= P_1(0)\left[\left(\frac{1-V^*(w_x(z))}{w_x(z)}\right)\sum_{j=0}^{\infty}\alpha I_0^j\prod_{i=0}^{j}\beta_i + \frac{\varphi}{\lambda}\right]$$

$$= P_1(0)\,I(z) \qquad (3.55)$$

Thus by equation (3.54)

$$P_{comp}(z) = P^*(z,0) + \frac{\alpha\,P^*(z,0)[1-R^{*1}(w_x(z))]}{w_x(z)}$$

$$= P^*(z,0)\left[1+\frac{\alpha[1-R^{*1}(w_x(z))]}{w_x(z)}\right] \qquad (3.56)$$

Therefore the total PGF of the system size is given by

$$P(z) = P_I(z) + P_{comp}(z)$$

$$= P^*(z,0)\left[1+\frac{\alpha[1-R^{*1}(w_x(z))]}{w_x(z)}\right] + P_1(0)\,I(z)$$

$$= \frac{z\, P_1(0)(-w_x(z))\, I(z)\left[1 - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]}{\left[z - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]\left[h_\alpha\left(w_x(z)\right)\right]}\left[\frac{w_x(z) + \alpha\left[1 - R^{*1}\left(w_x(z)\right)\right]}{w_x(z)}\right]$$

$$+ P_1(0)\, I(z)$$

$$= P_1(0)\, I(z)\left[\frac{z\left[1 - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]}{z - S^*\left(h_\alpha\left(w_x(z)\right)\right)} + 1\right]$$

$$P(z) = P_1(0)\, I(z)(z-1)\left[\frac{S^*\left(h_\alpha\left(w_x(z)\right)\right)}{D(z)}\right] \tag{3.57}$$

3.3.1. STABILITY CONDITION

To evaluate $P_1(0)$, The normalizing condition $1 = \lim_{z\to 1} P(z)$ is used. For this we note that,

$$\lim_{z\to 1} P(z) = 1 \qquad \text{[using L'Hospital rule]}$$

$$1 = P_1(0)\, I(z) \lim_{z\to 1}\left[\frac{S^*\left(h_\alpha\left(w_x(z)\right)\right)}{D(z)}\right]$$

$$1 = \frac{P_1(0)\, I(1)}{D(z)} \tag{3.58}$$

where,

$$D'(1) = \left[z - S^*\left(h_\alpha\left(w_x(z)\right)\right)\right]'_{z=1} = 1 - \left[h_\alpha'\left(w_x(z)\right)\right]_{z=1} E(S)$$

$$= 1 - [1 + \propto E(R)]\, E(S)\, \lambda\, E(X) = 1 - \rho \tag{3.59}$$

and

$$I(1) = \frac{\varphi}{\lambda} + \sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \beta_i \; E(V) \tag{3.60}$$

Where, $\rho = [1 + \alpha E(R)]\, E(S)\, \lambda\, E(X)$ (3.61)

Thus the normalizing condition gives,

$$P_1(0) = \frac{(1-\rho)}{I(1)} \tag{3.62}$$

Substituting for $P_1(0)$ in equation (3.57)

$$P(z) = \frac{(1-\rho)}{I(1)} \frac{I(z)}{D(z)} \left[S^*\left(h_\alpha\,(w_x(z)\right)(z-1)\right] \qquad (3.63)$$

Where,

$$D(z) = z - S^*\left(h_\alpha\left(w_x(z)\right)\right)$$

3.3.2. DECOMPOSITION PROPERTY

The equation (3.63) can be written as

$$P(z) = (1-\rho)(z-1) \frac{I(z)}{I(1)} \left[\frac{S^*\left(h_\alpha\,(w_x(z)\right)}{D(z)}\right] \qquad (3.64)$$

The above equation (3.64) shows that the PGF of the system size of the model under consideration is decomposed into the product of two probability generating functions, one of which is the PGF of the single server queuing model with server breakdown without variation on the other PGF of the conditional system size distribution $\left(\frac{I(z)}{I(1)}\right)$ during the server idle period under the condition $\rho < 1$.

3.4. PERFORMANCE MEASURES

In this section some useful performance measures of the proposed model are presented.

3.4.1. THE STEADY STATE SYSTEM SIZE PROBABILITIES:

Let P_V , P_{busy} and P_{br} denote the probability that the server is on vacation, busy state and breakdown state respectively. Then the corresponding probabilities are obtained, by considering the equations in ((3.51), (3.52), (3.53)) at z=1. The results stated in section 1.5.5 of chapter I are used to obtain these measures. Thus,

(i) $P_V = \lim_{z\to 1} QI^*(z,0)$

$$= \frac{(1-\rho)}{I(1)} E(V) \sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \beta_i$$

$$\text{(ii)}P_{busy} = \lim_{z\to 1} P^*(z,0)$$

$$= \lambda\, E(X)E(S)$$

$$\text{(iii) } P_{br} = \lim_{z\to 1} B^*(z,,\theta,0)$$

$$= \alpha\, E(R)\, P_{busy}$$

And

$$PI = \frac{\varphi}{\lambda} P_1(0)$$

$$= \frac{\varphi}{\lambda}\, \frac{(1-\rho)}{I(1)}$$

3.4.2.MEAN SYSTEM SIZE:

In this section the average number of customers waiting in the system, when the server is in different states are calculated.

Let L_V , L_{busy} and L_{br} denote the expected system size when the server is in vacation state, busy state and breakdown state respectively. Then the derivatives of equations in ((3.51), (3.52), (3.53)) at z=1 give the required measures.

Thus the mean system size corresponding to different states are given by,

$$\text{(i)}L_V = \left(\frac{d}{dz} QI^*(z,0)\right)_{z=1}$$

$$= \frac{(1-\rho)}{I(1)} \frac{E(V^2)}{2} \sum_{j=0}^{\infty} \alpha I_0^j \prod_{i=0}^{j} \beta_i \tag{3.65}$$

$$\text{(ii)}L_{busy} = \left(\frac{d}{dz} P^*(z,0)\right)_{z=1}$$

$$= \frac{(1-\rho)}{I(z)}\, \frac{E\big(X(X-1)\big)}{2(1-\rho)} I(1)E(S) + \frac{\lambda\, E(X)}{2(1-\rho)} D''(1)E(S) + \lambda\, E(X)E(S) \frac{I'(1)}{I(1)}$$
$$+ [\lambda\, E(X)]^2 [1+\alpha\, E(R)] \frac{E(S^2)}{2} + \lambda\, E(X)E(S) \tag{3.66}$$

Where, $-D''(1) = \left[S^*\big(h_\alpha\, (w_x(z))\big)\right]''_{z=1} = \lambda\, E\big(X(X-1)\big)E(H) + [\lambda\, E(X)]^2\, E(H^2)$

Where, $E(H) = [1+\alpha\, E(R)]\, E(S)$

$$(iii) L_{br} = \left(\frac{d}{dz} B^*(z, 0,0)\right)_{z=1}$$

$$= \alpha \left[P_{busy}\, \lambda\, E(X) \frac{E(R^2)}{2} + L_{busy}\, E(R) \right] \tag{3.67}$$

The total expected system size for the model can be obtained using equation (3.63)

$$L = \left(\frac{d}{dz} P(z)\right)_{z=1}$$

$$= \frac{I'(1)}{I(1)} + \rho + \frac{(-D''(1))}{2D'(1)} \tag{3.68}$$

Then, it is also verified that

$$L_V + L_{br} + L_{busy} = \frac{I'(1)}{I(1)} + \rho + \frac{\lambda E\big(X(X-1)\big)E(H)}{2(1-\rho)} + \frac{\big(\lambda\, E(X)\big)^2}{2(1-\rho)}\, E(H^2) \tag{3.69}$$

$$= \frac{I'(1)}{I(1)} + \rho + \frac{(-D''(1))}{2D'(1)}$$

$$= L$$

Conclusion:

The steady-state system size probabilities and the mean system size when the server is in different states are calculated in a compact form, so that the numerical computations can be made easily corresponding to any distribution. A sample of computation work is presented in section 4.1 of chapter IV.

CHAPTER IV

NUMERICAL ANALYSIS AND PARTICULAR CASES

4.1: NUMERICAL ANALYSIS:

In this chapter we consider two sections. In section 4.1, the measures of the models of chapters II and III are analysed through numerical computations. In section 4.2 some particular cases are derived. The following distributions are assumed for inter arrival time, service time, vacation time and repair time. The common parameters and the second moments used for the numerical tables are also listed below:

Random Variable (Y)	**Distribution**	**Mean E(Y)**	**Second order moments E(Y²)**
Vacation time (V)	Erlang-3 type $\eta = 0.5$	$\frac{1}{\eta}$	$\frac{4}{3\eta^2}$
Repair time(R)	Gamma(5, β_1) $\beta_1 = 5$	$\frac{5}{\beta_1}$	$\frac{30}{\beta^2{}_1}$
Breakdown(α)	Poisson	$\alpha = 1$	
Batch size (X)	Geometric p = (0.5)	$\frac{1}{(1-p)}$	$\frac{2p}{(1-p)}$
Service time (S)	2-stage hyper exponential $(a, \mu_1, \mu_2, \alpha) = (0.3,0.01,2,1)$	$S^*(\alpha) = \frac{a\mu_1}{\mu_1 + \alpha} + \frac{\mu_2(1-a)}{\mu_2 + \alpha}$	$S^{*\prime}(\alpha) = -\left(\frac{a\mu_1}{(\mu_1 + \alpha)^2} + \frac{\mu_2(1-a)}{(\mu_2 + \alpha)^2}\right)$

TABLE 4.1.1

The numerical computations are made, to calculate the measures using the formulae derived in earlier chapters. The values given in tables 4.1.2 (a) to 4.1.4 (a) show the value for the model of chapter II. The numerical value of the model of chapter III are given in tables 4.1.2 (b) to 4.1.4 (b)

In tables 4.1.2 (a) and 4.1.2(b) the system size probabilities when the system is in different states are calculated for the models of chapter II and III respectively, by increasing the values of the arrival rates $\boldsymbol{\lambda}$ and the utilization factor $\boldsymbol{\rho}$. In tables 4.1.2 (a) and (b), it is verified that the total probability is 1 for both the models.

ρ	λ	PI	P_V	P_{br}	P_{busy}
0.6139	0.151	0.2735	0.1125	0.2728	0.3410
0.6178	0.152	0.2700	0.1119	0.2746	0.3433
0.6220	0.153	0.2665	0.1113	0.2764	0.3455
0.6260	0.154	0.2631	0.1107	0.2782	0.3478
0.6301	0.155	0.2571	0.1101	0.2800	0.3500
0.6342	0.156	0.2562	0.1095	0.2818	0.3523
0.6382	0.157	0.2528	0.1088	0.2836	0.3546
0.6423	0.158	0.2494	0.1081	0.2854	0.3568
0.6464	0.159	0.24610	0.1074	0.2872	0.3591
0.6504	0.160	0.2427	0.1067	0.28910	0.3613

TABLE 4.1.2 (a)

ρ	λ	PI	P_V	P_{br}	P_{busy}
0.8736	0.91	0.0275	0.0988	0.3882	0.4853
0.8832	0.92	0.0252	0.0915	0.3925	0.4906
0.8928	0.93	0.0229	0.0842	0.3968	0.4960
0.9024	0.94	0.0206	0.0769	0.4010	0.5013
0.9120	0.95	0.0184	0.0695	0.4053	0.5066
0.9216	0.96	0.0162	0.0621	0.4096	0.5120
0.9312	0.97	0.0141	0.0546	0.4138	0.5173
0.9408	0.98	0.0120	0.0471	0.4181	0.5226
0.9504	0.99	0.0099	0.0396	0.4224	0.5280
0.9600	1.0	0.0079	0.0320	0.4266	0.5333

TABLE 4.1.2 (b)

In the tables 4.1.3 (a) and 4.1.3 (b) the number of customers waiting is the system when the server is idle or busy or in break down states are calculated corresponding to increasing values of the utilization factor ***ρ*** for both the models. It is shown that **L** increases as ***ρ*** increases.

ρ	λ	L_V	L_{br}	L_{busy}	L
0.6139	0.151	0.04533	0.0570	0.0135	0.1161
0.6178	0.152	0.04539	0.0696	0.0299	0.1199
0.6220	0.153	0.04545	0.0816	0.0457	0.1451
0.6260	0.154	0.04549	0.0932	0.0609	0.1729
0.6301	0.155	0.04552	0.1043	0.0755	0.2254
0.6342	0.156	0.04555	0.1150	0.0895	0.2501
0.6382	0.157	0.04556	0.1253	0.1030	0.2739
0.6423	0.158	0.04557	0.1351	0.1161	0.2967
0.6464	0.159	0.04558	0.1446	0.1286	0.3186
0.6504	0.160	0.04564	0.1537	0.1406	0.3396

TABLE 4.1.3(a)

ρ	λ	L_V	L_{br}	L_{busy}	L
0.8736	0.91	0.2398	6.8955	8.1777	15.3131
0.8832	0.92	0.2247	7.5048	8.9296	16.6591
0.8928	0.93	0.2090	8.2200	9.8137	18.2427
0.9024	0.94	.01929	9.7729	10.8691	20.1343
0.9120	0.95	0.1762	10.1064	12.1517	22.4344
0.9216	0.96	0.1590	11.3892	13.7450	25.2933
0.9312	0.97	0.1414	13.0245	15.7788	28.9448
0.9408	0.98	0.1232	15.1837	18.4674	33.7744
0.9504	0.99	0.1045	18.1709	22.1909	40.4668
0.9600	1.0	0.0853	22.5819	27.6940	50.3614

TABLE 4.1.3 (b)

In tables 4.1.4 (a) and 4.1.4 (b) it is shown that the system size increases as mean repair time increases or the mean vacation the mean variation time increases, for both the models.

E(R)	L	E(V)	L
21.05	32.9569	14.49	2.4536
22.22	34.6696	14.70	2.9233
23.52	36.7985	14.92	2.9699
25.00	39.4233	15.15	3.0179
26.66	42.6551	15.38	3.0674
28.57	46.6486	15.62	3.1184
30.76	51.6213	15.87	3.1711
33.33	57.8855	16.12	3.2250
36.36	65.8995	16.39	3.2817
40.00	76.3583	16.66	3.3397

TABLE 4.1.4 (a)

E(R)	L	E(V)	L
0.67	9.4150	1.69	13.7955
0.68	9.7375	1.72	13.8297
0.70	10.0932	1.75	13.8652
0.71	10.4871	1.78	13.9019
0.72	10.9250	1.82	13.9400
0.74	11.4174	1.85	13.9797
0.75	11.9715	1.88	14.8175
0.76	12.6009	1.92	14.8530
0.78	13.3217	1.96	14.8903
0.80	14.1546	2.00	14.9291

TABLE 4.1.4 (b)

4.2 PARTICULAR CASES:

In this section the steady state results of the bulk arrival queueing model discussed in chapters II and III are analysed corresponding to single vacation, multiple vacation and J- vacations policies. It is interesting to note from the decomposition property that the total PGF P(z) and the mean queue length L for both the models differ from that of other vacation policy models only in the expressions of φ, I(z) and I(1) mentioned in chapters II and III. Thus to obtain the total PGF and hence the mean queue length for different vacation policies it is enough to present φ, I(z) and I(1) for these models.

Let $P_1(z)$, $P_2(z)$ and L_1, L_2 respectively denote the total generating functions and the mean queue size for the models of chapters II and III. Then from the equations (2.61) and (2.64), we have,

$$P_1(z) = \frac{(1-\rho)}{I(1)}\frac{I(z)}{D(z)}\left[S^*\big(g_\alpha\,(w_x(z)\big)(z-1)\right]$$

$$L_1 = (1-\rho)S^*(\alpha)\left(\frac{-D''(1)}{2\,D'^2(1)}\right)+\frac{I'(1)}{I(1)}+\frac{{S^*}'(\alpha)}{S^*(\alpha)}(-\lambda\,E(X))$$

And from the equations (3.64) and (3.68),

$$P_2(z) = \frac{(1-\rho)}{I(1)}\frac{I(z)}{D(z)}\left[S^*\big(h_\alpha\,(w_x(z)\big)(z-1)\right]$$

$$L_2 = \frac{I'(1)}{I(1)}+\rho-\frac{D''(1)}{2D'(1)}$$

Case 1: Single vacation model:

When $\beta_0 = 1,\ \beta_j = 0$ for $j \geq 1$, the results for single vacation model are obtained. For this we note that

$$\varphi = \alpha_0$$

$$I(z) = \frac{\left(1 - V^*\big(w_x(z)\big)\right)}{w_x(z)} + \frac{\alpha_0}{\lambda}$$

$$I(1) = E(V) + \frac{\alpha_0}{\lambda}$$

Case 2: Multiple Vacation model:

With the selection of $\beta_i = 1$ for every $i \geq 0$ results are obtained for multiple vacation model. It is noted that

$$\varphi = 0$$

$$I(z) = \frac{\left(1 - V^*\left(w_x(z)\right)\right)}{w_x(z)(1-\alpha_0)}$$

$$I(1) = \frac{E(V)}{1-\alpha_0}$$

hold good for the multiple vacation policy.

Case 3: J-Vacation model :

For this policy the parameters are selected by, $\beta_0 = 1,\ \beta_j = p$ for $1 \leq j \leq J-1$ and $\beta_j = 0$ for $j \geq J$.

$$\varphi = \alpha_0(1-\rho)\left[\frac{1-(\alpha_0\rho)^{J-1}}{1-\alpha_0\rho}\right] + {\alpha_0}^j\rho^{j-1}$$

$$I(z) = \left[\frac{1-(\alpha_0\rho)^J}{1-\alpha_0\rho}\right]\frac{\left(1-V^*\left(w_x(z)\right)\right)}{w_x(z)} + \frac{1}{\lambda}\left[\alpha_0(1-\rho)\left[\frac{1-(\alpha_0\rho)^{J-1}}{1-\alpha_0\rho}\right] + {\alpha_0}^j\rho^{j-1}\right]$$

$$I(1) = \left[\frac{1-(\alpha_0\rho)^J}{1-\alpha_0\rho}\right]E(V) + \frac{1}{\lambda}\left[\alpha_0(1-\rho)\left[\frac{1-(\alpha_0\rho)^{J-1}}{1-\alpha_0\rho}\right] + {\alpha_0}^j\rho^{j-1}\right]$$

Case 4: Non-Vacation model :

To derive the results of non-vacation model, β_j's are set to 0, for every j.

$$\varphi = 1$$

$$I(z) = I(1) = \left(\frac{-1}{\lambda}\right)$$

It is interesting to note that, if the service times follow exponential distribution, then the results of both the models of chapters II and III coincide. For the exponential distribution, let service the density function $s(t) = e^{-\mu t}\mu$ and the LST of the service distribution $S^*(\theta) = \frac{\mu}{\mu+\theta}$. Then the PGFs $P_1(z)$ and $P_2(z)$ of the system size probabilities of chapters II and III coincide.

To verify this, we note that, the total PGF of the system size for, the model of chapter II is

$$P_1(z) = (z-1)(1-\rho_1)\frac{I(z)}{I(1)}\left[\frac{S^*\big(g_\alpha\,(w_x(z))\big)}{D_1(z)}\right]$$

Where,

$$\rho_1 = \frac{\lambda\, E(X)}{\alpha\, S^*(\propto)}[1 - S^*(\propto)][1+\alpha\, E(R)]$$

$$g_\alpha\,(w_x(z) = \alpha + w_x(z)$$

$$D_1(z) = z\left[1 - \frac{\alpha\, R^*\big(w_x(z)\big)(1 - S^*\big(g_\alpha\,(w_x(z))\big)}{g_\alpha\,(w_x(z)}\right] - S^*\big(g_\alpha\,(w_x(z))\big)$$

And for the model of chapter III, the total PGF is,

$$P_2(z) = (1-\rho_2)(z-1)\frac{I(z)}{I(1)}\left[\frac{S^*\big(h_\alpha\,(w_x(z))\big)}{D_2(z)}\right]$$

$$\rho_2 = [1+\alpha E(R)]\; E(S)\; \lambda\; E(X)$$

$$h_\alpha\,(w_x(z)) = g_\alpha\big(w_x(z)\big) - \alpha\, R^{*1}(w_x(z))$$

$$D_2(z) = z - S^*\Big(h_\alpha\big(w_x(z)\big)\Big)$$

For exponential service time

$$S^*\big(g_\alpha\,(w_x(z))\big) = \frac{\mu}{\mu + g_\alpha\,(w_x(z)}$$

$$S^*\Big(h_\alpha\big(w_x(z)\big)\Big) = \frac{\mu}{\mu + h_\alpha\big(w_x(z)\big)}$$

This implies, by calculation that $\rho_1 = \rho_2$ and

$$\frac{S^*\big(g_\alpha\,(w_x(z))\big)}{D_1(z)} = \frac{S^*\big(h_\alpha\,(w_x(z))\big)}{D_2(z)}$$

This shows $P_1(z) = P_2(z)$

SUMMARY AND CONCLUSION

In the present work the author has made an attempt to analyse the steady-state results for $M^X/G/1$ queueing systems under more general vacation policy namely Multiple Adapted Vacation policy. Two types of customer's behaviour during breakdown period are considered in chapters II and III. It is shown that if the vacation models are considered under multiple vacation policy during ideal period, then the results corresponding to other vacation policies as well as non-vacation case can be derived directly from the MAV policy models.

The formulae derived for performance measures are given in a closed form so that, the numerical computations can be made easy, corresponding to other system parameters as well as other distributions. It is realised that the two types of customers behaviour regarding their services during breakdown period namely repeating the interrupted service from the beginning or complete the remaining service from where it got interrupted are distinct only in the case of general distribution of service time.

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