

Generalizations of Continuous Mappings

BY

Sagayamary. J.

A DISSERTATION SUBMITTED TO THE AVINASHILINGAM INSTITUTE FOR HOME SCIENCE
AND HIGHER EDUCATION FOR WOMEN (DEEMED UNIVERSITY) COIMBATORE-641 043,
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE IN MATHEMATICS

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CERTIFIED AS BONAFIDE RESEARCH WORK

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Introduction

INTRODUCTION

Mathematics is of profound significance in the universe, not because it exhibits principle, that we obey, but because it exhibits principles that we impose

- J.N.N. Sullivan.

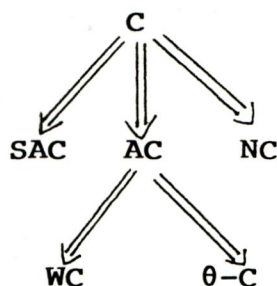
In this thesis we present a brief survey of the following three papers (1). Almost-continuous mappings by M.K. Singal and Asharani Singal [10], (2). On θ -continuous mappings by Shashi Prabha Arya and Mamata Deb [9], (3). A characterization of completely regular spaces by Sunder Lal [12].

The main ideas discussed in the first two papers are different generalizations of the concept of continuity. These are called Almost-continuity, weak - continuity, θ -continuity, Near - continuity, Almost-continuity in the sense of Stallings.

In the third paper a characterization of completely regular spaces is obtained in terms of a suitable topology defined in the space of convergent filters.

A mapping $f: X \rightarrow Y$ is said to be almost-continuous in the sense of Stallings, if for every open set W containing the graph of f , there exists a continuous mapping $g: X \rightarrow Y$ such that the graph of g is a subset of W .

The above concepts are connected as follows.



Here C =continuous, SAC =almost-continuous in the sense of Stallings, AC =almost-continuous NC =nearly continuous and θ - c = θ -continuous.

The important results proved in this connection are the following.

- (1) An open mapping is almost-continuous if and only if it is weakly-continuous.
- (2) Let $f_\alpha: X_\alpha \rightarrow X_\alpha^*$ be almost-continuous (θ -continuous) for each $\alpha \in \Lambda$ and let $f: \prod_{\alpha \in \Lambda} X_\alpha \rightarrow \prod_{\alpha \in \Lambda} X_\alpha^*$ be defined by setting $(f(x_\alpha)) = (f_\alpha(x_\alpha))$ for each point $(x_\alpha) \in \prod_{\alpha \in \Lambda} X_\alpha$. Then f is almost-continuous (θ -continuous).

- (3) Let $h: X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$ be almost-continuous (θ -continuous) for each $\alpha \in I$, define $f_\alpha : X \rightarrow X_\alpha$ by setting $f_\alpha(x) = (h(x))_\alpha$. Then f is almost-continuous (θ -continuous) for all $\alpha \in I$.
- (4) The composition of two θ -continuous mappings is θ -continuous.
- (5) The graph mapping of a θ -continuous mapping is θ -continuous.
- (6) For a mapping $f: X \rightarrow Y$, the following are equivalent.
- f is almost-continuous
 - Inverse image of every regularly-open subset of Y is an open subset of X .
 - Inverse image of every regularly-closed subset of Y is a closed subset of X .
 - For each point $x \in X$ and for each regularly open neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(N) \subset M$.
 - $f^{-1}(A) \subset \text{int}(f^{-1}(\text{int cl } A))$ for every open set A of Y .
 - $\text{cl}(f^{-1}(\text{int cl } B)) \subset f^{-1}(B)$ for every closed subset B of Y .
 - For any point $x \in X$ and for any net $\{x_\lambda\}_{\lambda \in D}$ which converges to x , the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in each regularly open set containing $f(x)$.

clearly $\sigma \leq \tau^*$ and for completely regular spaces, they are identical.

The main results proved here are the following.

- (1) (X, τ) is completely regular if and only if $f: (X^*, \sigma) \rightarrow (X, \tau)$ is a continuous mapping.
- (2) (X, τ) is almost regular if and only if $f: (X^*, \tau^*) \rightarrow (X, \tau)$ is an almost-continuous mapping.
- (3) (X, τ) is almost-completely regular if and only if $f: (X^*, \sigma) \rightarrow (X, \tau)$ is an almost continuous mapping.

Chapter I

CHAPTER - I

PRELIMINARY DEFINITIONS

Definition : 1.1

(X, τ) is a Hausdorff space if and only if $x, y \in X$ with $x \neq y$ implies that there exists open sets U and V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$

Defintion : 1.2

(X, τ) is compact if and only if each open cover of (X, τ) contains a finite (open) subcover.

Definition : 1.3

(X, τ) is regular if and only if $p \in X$ and $F \subset X - p$ with F is closed implies that there exists open sets U and V with $p \in U$, $F \subset V$ and $U \cap V = \emptyset$

Definition : 1.4

(X, τ) is completely regular if and only if $p \in X$ and $F \subset X - \{p\}$ with F is closed implies that the existence of a continuous function $f: X \rightarrow I^1$ ($I^1 = [0, 1]$) with $f(p) = \{0\}$ and $f(F) = \{1\}$.

Definition : 1.5

A set A is called Regularly - open if it is the interior of its own closure or equivalently if it is the interior of some closed set.

Definition : 1.6

A set A is called regularly-closed if it is the closure of its own interior or equivalently if it is the closure of some open set.

Definition : 1.7

A subset A of a space X is said to be discrete set if and only if every point in A has a isolated points.

Definition : 1.8

A space X is said to be locally compact and x if there is some compact subset C of X that contains a neighbourhood of x . If X is locally compact at each of its points, then X is simply said to be locally compact.

Definition : 1.9

Let X be any set and consider an arbitrary subfamily C^1 of \mathcal{R}^X . The weak topology induced by C^1 is defined to be the smallest topology on X . Such that all function in C^1 are continuous.

Definition : 1.10

A subset A of a space X is called a G_δ -set in X if it equals the intersection of a countable collection of open subsets of X .

Definition : 1.11

Let $f:(S, \tau_1) \rightarrow (T, \tau_2)$ be a function then f is open if and only if $G \in \tau_1$ implies $f(G) \in \tau_2$

Definition : 1.12

Let $f:(S, \tau_1) \rightarrow (T, \tau_2)$ be a function then f is closed if and only if C is closed in S implies $f(C)$ is closed in T .

Definition : 1.13

A topological space (X, τ) is said to be almost-regular if for each $x \in X$ and for each regular open set U containing x , there exists an open set V such that $x \in V \subset \text{cl } V \subset U$.

Definition : 1.14

A mapping $f:X \rightarrow Y$ is said to be almost-open if the image of every regularly-open subset X is an open subset of Y .

Definition : 1.15

A mapping $f:X \rightarrow Y$ is said to be almost-closed if the image of every regularly-closed subset of X is a closed subset of Y .

Definition : 1.16

A space is called Uryshon space if for every pair of distinct points x and y , there exists open sets U and V such that $x \in U$, $y \in V$ and $(clU) \cap (clV) = \emptyset$.

Definition : 1.17

A space is said to be almost-compact if each open cover has a finite subfamily whose closures cover the space.

Chapter II

CHAPTER - II

SECTION - 1

DIFFERENT GENERALIZATIONS OF CONTINUITY

Definition : 2:1

A mapping $f:X \rightarrow Y$ is said to be continuous at a point $x \in X$ if for every neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(N) \subset M$.

Example : 2:2

- (1) A constant map $f:X \rightarrow X$ is always continuous.
- (2) Let X be any set, τ_1, τ_2 two topologies on X . Then the map $i:(X, \tau_1) \rightarrow (X, \tau_2)$ is continuous.
- (3) Let $A \subset X$ and $i:A \rightarrow X$ be the inclusion map $a \rightarrow a$ then i is continuous.
- (4) Let f be the function defined on R by setting $f(x) = x^2$ for all $x \in R$. Then f is continuous on R .
- (5) Let f be the function defined on R by setting $f(x) = |x|$ for all $x \in R$. Then f is continuous on R .

Definition : 2:3

A Mapping $f:X \rightarrow Y$ is said to be almost-continuous at a point $x \in X$, if for every neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(N) \subset \text{int } \text{cl } M$. f is said to be almost-continuous if it is almost-continuous at every point of X .

Note:

Obviously, every continuous map is almost-continuous. But the converse of this statement need not be true, as the following example shows.

Example : 2:4

Let R be the set of real numbers. Let τ_1 consists of ϕ, R and complements of all countable subsets of R . Let $X = \{a, b\}$ and $\tau_2 = \{X, \phi, \{a\}\}$. Let $f: (R, \tau_1) \rightarrow (X, \tau_2)$ defined by $f(x) = a$ if x is rational
 $= b$ if x is irrational.

Then f is almost-continuous at each point of R but not continuous at $x \in R$ if x is rational.

Definition : 2:5

A mapping $f: X \rightarrow Y$ is said to be weakly-continuous if for each point $x \in X$ and each neighbourhood M of $f(x)$ there exists a neighbourhood N of x such that $f(N) \subset \text{cl } M$.

Theorem : 2:6

Every almost-continuous mapping is weakly - continucus.

Proof:

Let $f: X \rightarrow Y$ be almost-continuous. Then for $x \in X$ and for every neighbourhood M of $f(x)$ there is a neighbourhood N of x such that $f(N) \subset \text{int cl } M$. This implies $f(N) \subset \text{int cl } M \subset \text{cl } M$. Here f is weakly-continuous.

Clearly, every continuous map is weakly-continuous. The converse of the above statement is need not true. This is shown by the following example.

Example : 2:7

Let R be the set of real numbers and $\tau = \{\phi, R, U/R-U \text{ is countable}\}$. Let $X = \{a, b, c\}$ and let $\tau^* = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$

Let $f: (R, \tau) \rightarrow (X, \tau^*)$ be defined as,

$$\begin{aligned} f(x) &= a && \text{if } x \text{ is rational.} \\ &= b && \text{if } x \text{ is irrational.} \end{aligned}$$

Then f is a weakly-continuous mapping which not almost-continutinous (at any rational point).

The map f is weakly-continuous as $\text{cl } a = \{a, b\}$, and not almost-continuous as $\text{int } \text{cl } a = \{a\}$. Hence it is not continuous.

Definition : 2:8

A mapping $f: X \rightarrow Y$ is said to be θ -continuous at a point $x \in X$ if for each neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(\text{cl } N) \subset \text{cl } M$. f is said to be a θ -continuous if it is θ -continuous at each point of X .

Note:

From the above definition it follows that every continuous mapping is θ -continuous. But the coverse of this statement need not be true, as the following example shows.

Example : 2:9

Let R be the set of all real numbers. The topology τ_1 on R consists of R, ϕ and the complements of all countable subsets of R . Let $X = \{a, b, c\}$ and let $\tau_2 = \{\phi, X, \{a\}, \{c\}, \{a, c\}\}$. Let $f: (R, \tau_1) \rightarrow (X, \tau_2)$ be defined as follows

$$\begin{aligned} f(x) &= a \text{ if } x \text{ is rational} \\ &= b \text{ if } x \text{ is irrational.} \end{aligned}$$

Then f is a θ -continuous mapping which is not continuous.

Definition : 2:10

The mappings $f: X \rightarrow Y$ and $g: X \rightarrow Y$ are said to be α -near for an open covering α of Y for every $x \in X$, $f(x)$ and $g(x)$ belong to the same set $U \in \alpha$. A mapping $f: X \rightarrow Y$ is said to be nearly-continuous, if for every open covering α of Y , there exists a continuous mapping $g: X \rightarrow Y$ which is α -near to f .

Obviously, every continuous mapping is nearly-continuous. The converse of the above statement is need not true. This is shown by the following example.

Example : 2:11

Let $X = \{a, b, c, d\}$, $\tau_1 = \{\phi, X, \{a, d\}, \{c\}, \{a, c, d\}\}$ and let $\tau_2 = \{\phi, X, \{a\}, \{b, c\}, \{a, b, c\}\}$. Let $f: (X, \tau_1) \rightarrow (X, \tau_2)$ be the mapping defined by $f(a) = f(b) = f(c) = a$, $f(d) = b$. Then f is nearly-continuous but it is continuous.

Definition : 2:12

A mapping $f:X \rightarrow Y$ is said to be almost-continuous in the sense of Stallings if for every open set W containing the graph of f , there exists a continuous mapping $g:X \rightarrow Y$ such that the graph of g is a subset of W .

Note :

From the above definition it follows that every continuous map is almost-continuous in the sense of Stallings. But the converse of this statement need not be true, as the following example shows.

Example : 2:13

Let X be the set of all natural numbers and let $\mathcal{B} = \{\emptyset, X, \{2n-1, 2n\} / n=1, 2, \dots\}$. Let τ be the topology for X for which \mathcal{B} is a base. Let Y be the set of all real numbers and \mathcal{U} be the usual topology for Y . Let $f:(X, \tau) \rightarrow (Y, \mathcal{U})$ be the mapping defined by $f(m) = \frac{1}{m}$ for all $m \in X$.

Then obviously f is not continuous. However f is almost-continuous in the sense of Stallings. Since if G be any open set containing the graph of f , then the graph of the continuous mapping $g:(X, \tau) \rightarrow (Y, \mathcal{U})$ defined by $g(2n) = g(2n-1) = \frac{1}{2n-1}$, $n=1, 2, \dots$ is contained in G . Hence f is almost-continuous in the sense of Stallings.

Remark : 2:14

The inverse of an almost-continuous one-to-one mapping need not be almost-continuous.

Let R be the set of real numbers. Let τ consists of \emptyset, R and the complement of all countable subsets of R . Let U denote the usual topology for R . Let i be the identity mapping of (R, U) onto (R, τ) . Then i is almost-continuous.

The inverse of the mapping is not almost-continuous. To prove $i^{-1}: (R, \tau) \rightarrow (R, U)$ is not almost-continuous. Let $x \in R$, Let $U=(a, b)$ be the neighbourhood of $i^{-1}(x)=x$. There exists no neighbourhood V of x in (R, τ) such that $i^{-1}(V) \subset \text{int cl } U$. Hence i^{-1} is not almost-continuous.

Next we prove an important theorem connected with almost-continuous mappings.

Theorem : 2:15

For a mapping $f: X \rightarrow Y$, the following are equivalent:

- (a) f is almost-continuous.
- (b) Inverse image of every regularly-open subset of Y is an open subset of X .
- (c) Inverse image of every regularly-closed subset of Y is a closed subset of X .

- (d) For each point $x \in X$ and for each regularly-open neighbourhood M of $f(x)$, there is a neighbourhood N of x such that $f(N) \subset M$.
- (e) $f^{-1}(A) \subset \text{int}(f^{-1}(\text{int cl } A))$ for every open subset A of Y .
- (f) $\text{cl}(f^{-1}(\text{int cl } B)) \subset f^{-1}(B)$ for every closed subset B of Y .
- (g) For any point $x \in X$ and for any net $\{x_\lambda\}_{\lambda \in D}$ which converges to x , the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in each regularly-open set containing $f(x)$.

Proof:

(a) \Rightarrow (b)

Let U be any regularly-open subset of Y . Let $x \in f^{-1}(U)$. Then $f(x) \in U$. Since f is almost-continuous, there exists an open set V in X such that $x \in V$ and $f(V) \subset \text{int cl } U = U$. Thus $x \in V \subset f^{-1}(U)$ and therefore $f^{-1}(U)$ is a neighbourhood of x . Hence $f^{-1}(U)$ is a neighbourhood of x . Hence $f^{-1}(U)$ is open.

(b) \Rightarrow (c)

Let A be any regularly-closed subset of Y . Then $Y-A$ is regularly-open and therefore $f^{-1}(Y-A)$ is open [by (b)]

Claim : $f^{-1}(Y-A) = X - f^{-1}(A)$.

Let $x \in f^{-1}(Y-A)$
iff $f(x) \in Y-A$
iff $f(x) \notin A$
iff $x \in f^{-1}(A)$
iff $x \in X - f^{-1}(A)$

Hence $f^{-1}(Y-A) = X - f^{-1}(A)$ is open.

Hence $f^{-1}(A)$ is closed.

(c) \Rightarrow (d)

Let M be regularly-open. Then $Y-M$ is regularly-closed. Therefore $f^{-1}(Y-M)$ is closed [by(c)]. Since $f^{-1}(M)$ is open. Let $x \in f^{-1}(M) = N$ (say). Then N is a neighbourhood of x such that $f(N) \subset M$.

(d) \Rightarrow (e)

Let $x \in f^{-1}(A)$. Then $\text{int cl } A$ is a regularly-open neighbourhood of $f(x)$. Since A is open. Then there exists an open neighbourhood N of x such that $f(N) \subset \text{int cl } A$. Thus $x \in N \subset f^{-1}(\text{int cl } A)$.

Also since N is open, $N = \text{int } N \subset \text{int } f^{-1}(\text{int cl } A)$.
Hence $x \in \text{int}(f^{-1}(\text{int cl } A))$. Hence $f^{-1}(A) \subset \text{int}(f^{-1}(\text{int cl } A))$.

(e) \Rightarrow (f)

Let $(Y-B)$ be open.

Then, $f^{-1}(Y-B) \subset \text{int}[f^{-1}(\text{int cl}(Y-B))]$ [by (e)]

$f^{-1}(Y-B) = X - f^{-1}(B) \subset \text{int}[f^{-1}(\text{int cl } Y-B)]$

$X - [X - f^{-1}(B)] \supset X - \text{int}[f^{-1}(\text{int cl } (Y-B))]$

$f^{-1}(B) \supset X - \text{int}[f^{-1}(\text{int cl } (Y-B))]$

That is, $\text{cl} [f^{-1} (\text{int cl } B)] \subset f^{-1} (B)$.

(f) \Rightarrow (g)

Let N be any regularly-open set containing $f(x)$. Then $Y-N$ being closed.

$\text{cl}[f^{-1}(\text{int cl}(Y-N))] \subset f^{-1}(Y-N)$ [by (f)].

Since $Y-N$ is regularly-closed therefore

$[f^{-1}(Y-N)]^{-1} \subset X-f^{-1}(N)$. This means that $f^{-1}(N) \subset \text{int}[f^{-1}(N)]$.

Thus $f^{-1}(N)$ is an open set containing x . Since the net $\{x_\lambda\}_{\lambda \in D}$ converges to x , there exists $\lambda_0 \in D$ such that for all $\lambda \gg \lambda_0$ (D is directed by \gg) $x_\lambda \in f^{-1}(N)$. This means that $f(x_\lambda) \in N$ for all $\lambda \gg \lambda_0$, that is the net $\{f(x_\lambda)\}_{\lambda \in D}$ is eventually in N .

(g) \Rightarrow (a)

Suppose that f is not almost-continuous. Then there is an open set U containing $f(x)$ such that for every open set V containing x , $f(U) \cap (Y-\text{int cl } V) \neq \emptyset$. This implies that $U \cap f^{-1}(Y-\text{int cl } V) \neq \emptyset$, for every open set V containing x directed by the set inclusion.

(ie) $\mathcal{A} = \{U/x \in U\}$.

Let U_1, U_2, \dots be open sets such that,
 $U_1 \supset U_2 \supset \dots$

Since $U \cap f^{-1}(Y-\text{int cl } V) \neq \emptyset$,

there exists points x_1, x_2, \dots such that

$x_1 \in U_1 \cap f^{-1}(Y-\text{int cl } V)$

$x_2 \in U_2 \cap f^{-1}(Y-\text{int cl } V)$

.....

Similarly $x_3 \in U_3 \cap f^{-1}(Y\text{-int cl } V)$.

$$\Rightarrow f(x_3) \in f(U_3) \cap (Y\text{-int cl } V)$$

$$\Rightarrow f(x_3) \in f(U_3) \quad \text{and} \quad f(x_3) \in (Y\text{-int cl } V).$$

$$\Rightarrow f(x_3) \in f(U_3) \quad \text{and} \quad f(x_3) \notin \text{int cl } V.$$

Then $\{x_U\}_{U \in \mathcal{U}}$ is a net in X which converges to x and is such that no $f(x_U)$ is in $\text{int cl } V$. Thus $\{f(x_U)\}_{U \in \mathcal{U}}$ is not eventually in the regularly-open set $\text{int cl } V$, which is a contradiction. Hence our assumption is wrong. Therefore f is almost-continuous.

Theorem : 2.16

Every almost-continuous mapping is θ -continuous.

Proof:

Let $f: X \rightarrow Y$ be an almost-continuous mapping of X into Y . Let $x \in X$, and let U be any neighbourhood of $f(x)$. There exists an open set G such that $f(x) \in G \subseteq U$.

Since f is almost-continuous, the set $V = f^{-1}(\text{int cl } G)$ is open in X . Thus $x \in f^{-1}(\text{int cl } G) \subseteq f^{-1}(\text{cl } G)$. The inverse image of every regularly-closed set under an almost-continuous mapping is closed by theorem [2.15]. We assert that $f^{-1}(\text{cl } G)$ is closed and therefore $\text{cl } V \subseteq f^{-1}(\text{cl } G)$. That is $f^{-1}(\text{cl } V) \subseteq \text{cl } G \subseteq \text{cl } U$. Thus f is θ -continuous. Hence every almost-continuous map is θ -continuous.

Note :

The converse of the above theorem is not true. That is every θ -continuous mapping is need not be almost-continuous. The mapping given in example [2.9] is a θ -continuous mapping which is not almost-continuous.

The next theorem gives a condition under which a θ -continuous mapping is almost-continuous.

Theorem : 2.17

If $f:X \rightarrow Y$ is a θ -continuous almost-open mapping then f is almost-continuous.

Proof :

Let $f:X \rightarrow Y$ is a θ -continuous almost-open mapping. To prove f is almost-continuous. Let $x \in X$ and let M be any neighbourhood of $f(x)$. Since f is θ -continuous, there exists an open neighbourhood N of x such that $f(\text{cl } N) \subset \text{cl } M$. Since, f is almost-open, the image of every regularly-open subset of X is an open subset of Y . Therefore $f(\text{int } \text{cl } N)$ is open. Thus $f(\text{int } \text{cl } N) \subseteq \text{int } [f(\text{cl } N)] \subseteq \text{int } \text{cl } M$ and consequently f is almost-continuous.

It is interesting to note from the following examples that a mapping may be almost-continuous in the sense of Stallings without being θ -continuous and nearly-continuous; a mapping may be nearly-continuous without being

θ -continuous and almost-continuous in the sense of Stallings, and that a mapping may be θ -continuous without being nearly-continuous and almost-continuous in the sense of Stallings.

Example : 2.18

Let $X = \{p, q\}$ and $\tau_1 = \{\emptyset, \{p\}, X\}$ and let $Y = \{a, b\}$ and τ_2 be the discrete topology for Y . Then the mapping $f: (X, \tau_1) \rightarrow (Y, \tau_2)$ defined by $f(p) = a$, $f(q) = b$ is neither θ -continuous nor nearly-continuous but it is almost-continuous in the sense of Stallings.

Example : 2.19

Let $X = [0, 1]$ and let \mathcal{U} be the usual topology for X . Let $Y = [0, 1]$ and τ be the topology for Y , which is the usual topology at all points except at 0. Let the neighbourhood system of 0 consists of complements of finite subsets of Y . Let $f: X \rightarrow Y$ be defined as $f(x) = x$ if $x \neq 0$ and $f(x) = 1$ if $x = 0$. Then f is θ -continuous but neither nearly-continuous nor almost-continuous in the sense of Stallings.

Example : 2.20

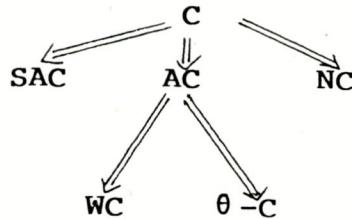
Let $X = [0, 1]$ and let \mathcal{U} be the usual topology for X . Let $Y = \{a, b, c\}$ and let $\tau = \{Y, \emptyset, \{c\}, \{a\}, \{a, c\}\}$. Let $f: (X, \mathcal{U}) \rightarrow (Y, \tau)$ be the mapping defined as $f(x) = a$ if $0 < x < 1/2$ and $f(x) = c$ if $1/2 \leq x \leq 1$. Then f is nearly-continuous but not θ -continuous. f is not almost-continuous in the sense of Stallings, since the set $W = AXB$ where

$A=[0, 1/2+\epsilon] \times \{a\}$ and $B=[1/2-\epsilon, 1] \times \{c\}$ is a neighbourhood of the graph of f such that the graph of no continuous function from X to Y is contained in W .

Note :

The map defined in example [2.9] is θ -continuous, almost-continuous in the sense of Stallings and nearly-continuous but not almost-continuous.

We now have the following implication diagram:



Here C = continuous, SAC = almost-continuous in the sense of Stallings, AC = almost-continuous, NC = nearly-continuous and $\theta-C$ = θ -continuous.

Theorem : 2.21

If $f: X \rightarrow Y$ is weakly-continuous open mapping, then f is almost-continuous.

Proof :

Let $x \in X$ and M be any neighbourhood of x . Since f is weakly-continuous, there is an open neighbourhood N of x such that $f(N) \subset \text{cl } M$. Since f is open, $f(N)$ is open. Then $f(N) \subset \text{int } \text{cl } M$ and hence f is almost-continuous.

Corollary : 2.22

An open mapping is almost-continuous if and only if it is weakly-continuous.

Theorem : 2.23

Every restriction of an almost-continuous mapping is almost-continuous.

Proof :

Let f be an almost-continuous mapping of X into Y . Let A be any subset of X . Let S be any regularly-open subset of Y .

Claim : $(f/A)^{-1}(S) = A \cap f^{-1}(S)$.

Let $y \in A \cap f^{-1}(S)$.

iff $y \in A$ and $y \in f^{-1}(S)$

iff $y \in A$ and $f(y) \in S$

iff $(f/A)(y) \in S$

iff $y \in (f/A)^{-1}(S)$

Hence $(f/A)^{-1}(S) = A \cap f^{-1}(S)$.

But f being almost-continuous, $f^{-1}(S)$ is open by theorem [2.15]. Hence $A \cap f^{-1}(S)$ is a relatively open subset of A . That is $(f/A)^{-1}(S)$ is open subset of A . Hence (f/A) is almost-continuous.

Theorem : 2.24:

Every restriction of a θ -continuous mapping is θ -continuous.

Proof :

Let $f: (X, \tau) \rightarrow (Y, \mathcal{U})$ be a θ -continuous mapping and let A be any subset of X . Let $x \in A$ and let V be a neighbourhood of $f(x)$. Since f is θ -continuous, there exists an open set U such that $x \in U \subseteq \text{cl } U \subseteq f^{-1}(\text{cl } V)$. Since $x \in U$ and $x \in A$, then $x \in U \cap A \subseteq \text{cl}_A(U \cap A) \subseteq f^{-1}(\text{cl } V) \subseteq f^{-1}(\text{cl}_A V) \cap A$. If $f|_A = g$, then $x \in U \cap A \subseteq \text{cl}_A(U \cap A) \subseteq g^{-1}(\text{cl } V)$. It follows that g is θ -continuous at every point of A and hence g is θ -continuous.

Theorem : 2.25

Let $f_\alpha : X_\alpha \rightarrow X_\alpha^*$ be almost-continuous for each $\alpha \in \Lambda$ and let $f: \prod X_\alpha \rightarrow \prod X_\alpha^*$ be defined by setting $(f(x_\alpha)) = (f_\alpha(x_\alpha))$ for each point $(x_\alpha) \in \prod_{\alpha \in \Lambda} X_\alpha$. Then f is almost-continuous.

Proof :

Let $(x_\alpha) \in \prod X_\alpha$. Let O^* be a regularly-open subset of $\prod X_\alpha^*$ containing $(f(x_\alpha))$. Then there is a member $\prod_{\alpha \in \Lambda} O_\alpha^*$ of the base of the product topology on $\prod X_\alpha^*$ such that $(f(x_\alpha)) \in \prod_{\alpha \in \Lambda} O_\alpha^* \subset O^*$ and $O_\alpha^* = X_\alpha^*$ for all $\alpha \in \Lambda$ except for a finite number of indices α_i , $i = 1, 2, \dots, n$ (say) and $O_{\alpha_i}^*$ is an open subset of $X_{\alpha_i}^*$, $i = 1, 2, \dots, n$.

Since O^* is regularly-open.

$\text{int cl } \prod_{\alpha} O_\alpha^* \subset O^*$. Thus for each α_i , $f_{\alpha_i}(x_{\alpha_i}) \in O_{\alpha_i}^* \subset \text{cl } O_{\alpha_i}^*$ and f_{α_i} being almost-continuous there is an open subset U_{α_i} of X_{α_i} such that $x_{\alpha_i} \in U_{\alpha_i}$ and $f_{\alpha_i}(x_{\alpha_i}) \in f_{\alpha_i}(U_{\alpha_i}) \subset \text{int cl } O_{\alpha_i}^*$. Thus $\prod U_\alpha$ containing (x_α) such that $f(\prod_{\alpha \in \Lambda} U_\alpha) \subset O^*$. Hence f is almost-continuous.

Theorem : 2.26

Let $f_\alpha : X_\alpha \rightarrow X_\alpha^*$ be θ -continuous for each $\alpha \in \Lambda$ and let $f : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow \prod_{\alpha \in \Lambda} X_\alpha^*$ be defined by $(f(x_\alpha)) = (f_\alpha(x_\alpha))$ for each $(x_\alpha) \in \prod_{\alpha \in \Lambda} X_\alpha$. Then f is $a\theta$ -continuous.

Proof :

Let $(x_\alpha) \in \prod_{\alpha \in \Lambda} X_\alpha$ and V^* be an open subset of $\prod_{\alpha \in \Lambda} X_\alpha^*$ containing $(f(x_\alpha))$. Then there is a member $\prod_{\alpha \in \Lambda} V_\alpha^*$ of the defining base for the product topology on $\prod_{\alpha \in \Lambda} X_\alpha^*$ such that

$(f(x_\alpha)) \in \prod_{\alpha \in \Lambda} V_\alpha^* \subseteq V^*$ and $V_\alpha^* = X_\alpha^*$ for all $\alpha \in \Lambda$ except for a finite number of indices $\alpha_i, i=1,2,\dots,n$ (say) and $V_{\alpha_i}^*$ is an open subset of $X_{\alpha_i}^*$, $i=1,2,\dots,n$

Now for each α_i , $f_{\alpha_i}(x_{\alpha_i}) \in V_{\alpha_i}^*$ since f_{α_i} is θ -continuous, there is an open subset U_{α_i} of X_{α_i} such that $x_{\alpha_i} \in U_{\alpha_i}$ and $f_{\alpha_i}(\text{cl } U_{\alpha_i}) \subseteq \text{cl } V_{\alpha_i}^*$. It follows that the set $U_\alpha = \prod_{\alpha \in \Lambda} U_\alpha$, where $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_i, i=1,2,\dots,n$ is an open set containing (x_α) such that $f(\text{cl } \prod_{\alpha \in \Lambda} U_\alpha) \subseteq \text{cl } V^*$. Hence f is θ -continuous. $\alpha \in \Lambda$

Theorem : 2.27

Let $h : X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$ be almost-continuous. For each $\alpha \in \Lambda$ define $f_\alpha : X \rightarrow X_\alpha$ by setting $f_\alpha(x) = (h(x))_\alpha$. Then f_α is almost-continuous for all $\alpha \in \Lambda$.

Proof :

Let $P_\alpha : \prod_{\alpha \in \Lambda} X_\alpha \rightarrow X_\alpha$ be the projection map. Let $h : X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$ be a map defined by $f_\alpha(x) = (h(x))_\alpha$. Then $P_\alpha \circ h = f_\alpha$ for

each α . Now P_α is open and continuous for each α and h is almost-continuous. Since the composition of two almost-continuous mappings is almost-continuous. Therefore $P_\alpha \circ h$ is almost-continuous. That is f_α is almost-continuous for each α .

Theorem : 2.28

The composition of two θ -continuous mapping is θ -continuous.

Proof :

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two θ -continuous mappings. We shall show that the composite $g \circ f: X \rightarrow Z$ is a θ -continuous mapping. Let $x \in X$ and let W be an open subset of Z containing $(g \circ f)(x)$. Since g is θ -continuous. There exists an open subset V of Y such that $f(x) \in V \subseteq \text{cl } V \subseteq g^{-1}(\text{cl } W)$. Again, since f is θ -continuous, there exists an open subset U of X such that $x \in U \subseteq \text{cl } U \subseteq f^{-1}(\text{cl } V)$. Thus $x \in U \subseteq \text{cl } U \subseteq f^{-1}(g^{-1}(\text{cl } W)) = (g \circ f)^{-1}(\text{cl } W)$. It follows that $g \circ f$ is θ -continuous.

Corollary : 2.29

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be any two mappings. If $g \circ f$ is θ -continuous and one of f and g is an open, one-to-one, onto mapping, then the other is θ -continuous.

Proof :

If f be an open one-to-one, onto mapping, then f^{-1} is continuous. Since every continuous mapping is θ -continuous

than f^{-1} is θ -continuous. Take $g = (g \circ f) \circ f^{-1}$, then $g \circ f$ is θ -continuous by theorem [2.28]. Again by theorem [2.28] g is θ -continuous. Similarly it can be proved that f is θ -continuous in case g is an open, one-to-one, on-to mapping.

Theorem : 2.30

Let $h: X \rightarrow \prod X_\alpha$ be θ -continuous. For each $\alpha \in \Lambda$ define $f_\alpha: X \rightarrow X_\alpha$ by $f_\alpha(x) = (h(x))_\alpha$. Then each f_α is θ -continuous.

Proof :

Let $P_\alpha: \prod X_\alpha \rightarrow X_\alpha$ be a projection map. Let $h: X \rightarrow \prod X_\alpha$ be a θ -continuous map. Let $f_\alpha: X \rightarrow X_\alpha$ be a map defined by $f_\alpha(x) = (h(x))_\alpha$. Then $P_\alpha \circ h = f_\alpha$ for each α . Now P_α is open and continuous. Since every continuous mappings are θ -continuous. Hence P_α is θ -continuous mapping. Hence f_α is θ -continuous by theorem [2.28].

Theorem : 2.31

Let f map X into Y and let x be a point of X . If there exists a neighbourhood N of x such that the restriction of f to N is almost-continuous at x .

Proof :

Let U be any regularly-open set containing $f(x)$. Since f/N is almost-continuous at x , there is an open set V_1 such that $x \in N \cap V_1$ and $f(N \cap V_1) \subset U$. Obviously $N \cap V_1$ is a neighbourhood of x . Hence f is almost-continuous.

Corollary : 2.32

Let f map X into Y and let $\{G_\lambda : \lambda \in \Lambda\}$ be an open cover of X . If for each $\lambda \in \Lambda$, $f|_{G_\lambda}$ is almost-continuous at each point of G_λ then f is almost-continuous.

Proof :

Let x be an arbitrary point in G_λ for some $\lambda \in \Lambda$. Since $f|_{G_\lambda}$ is almost-continuous at $x \in G_\lambda$ by theorem [2.31] f is continuous at x . Since x is arbitrary f is almost-continuous at every point $x \in G_\lambda$. Hence f is almost-continuous.

Theorem : 2.33

If f is a mapping from X into Y and $X = X_1 \cup X_2$ where X_1 and X_2 are closed and $f|_{X_1}$ and $f|_{X_2}$ are almost-continuous, then f is almost-continuous.

Proof :

Let A be a regularly-closed subset of Y . Then, Since $f|_{X_1}$ and $f|_{X_2}$ are both almost-continuous, therefore $(f|_{X_1})^{-1}(A)$ and $(f|_{X_2})^{-1}(A)$ are both closed in X_1 and X_2 respectively by theorem [2.15]. Since X_1 and X_2 are closed subsets of X , $(f|_{X_1})^{-1}(A)$ and $(f|_{X_2})^{-1}(A)$ are closed subsets of X .

Also $(f|_{X_1})^{-1}(A) \cup (f|_{X_2})^{-1}(A) = f^{-1}(A)$. Thus $f^{-1}(A)$ is the union of two closed sets and is therefore closed. Hence f is almost-continuous.

Theorem : 2.34

If f is a mapping from X into Y and $X = X_1 \cup X_2$ and if $f|_{X_1}$ and $f|_{X_2}$ are both almost-continuous at a point x belonging to $X_1 \cap X_2$, then f is almost-continuous at x .

Proof :

Let U be any regularly-open set containing $f(x)$. Since $x \in X_1 \cap X_2$ and $f|_{X_1}$ and $f|_{X_2}$ are both almost-continuous at x . Therefore there exists open sets V_1 and V_2 such that $x \in X_1 \cap V_1$ and $f(X_1 \cap V_1) \subset U$ and $x \in X_2 \cap V_2$ and $f(X_2 \cap V_2) \subset U$. Now, since $X = X_1 \cup X_2$, therefore

$$\begin{aligned} f(V_1 \cap V_2) &= f(X_1 \cap V_1 \cap V_2) \cup f(X_2 \cap V_1 \cap V_2) \\ &\subset f(X_1 \cap V_1) \cup f(X_2 \cap V_2) \\ &\subset U. \end{aligned}$$

Thus, $V_1 \cap V_2 = V$ is an open set containing x such that $f(V) \subset U$ and hence f is almost-continuous at x .

Theorem : 2.35

If f is an almost-continuous, closed mapping of a regular space X onto a space Y such that $f^{-1}(y)$ is compact for each point $y \in Y$, then Y is almost-regular.

Proof :

Let A be a regularly-closed subset of Y . Let $y \notin A$. Then $f^{-1}(y) \cap f^{-1}(A) = \emptyset$. Since f is almost-continuous $f^{-1}(A)$ is closed by theorem [2.15]. Let $f^{-1}(y)$ be compact. Since X is regular, there exist disjoint open sets G and H such that $f^{-1}(A) \subset G$, $f^{-1}(y) \subset H$.

Let $P = \{z: f^{-1}(z) \subset G\}$ and $Q = \{z: f^{-1}(z) \subset H\}$. Then $Y \notin P$, $A \subset P$, $Y \in Q$ and $P \cap Q = \emptyset$. Since f is closed, P and Q are open. Hence Y is almost-regular.

Definition : 2.36

For any mapping $f: X \rightarrow Y$, the mapping $g: X \rightarrow X \times Y$ defined by setting $g(x) = (x, f(x))$ is called the graph mapping of f .

Theorem : 2.37

The graph mapping of a θ -continuous mapping is θ -continuous.

Proof :

Let $f: X \rightarrow Y$ be θ -continuous mapping and let $g: X \rightarrow X \times Y$ be the graph mapping of f . To prove g is θ -continuous. Let M be a neighbourhood of $g(x)$, there exists an open sets U_1 and U_2 of X and Y respectively, such that $g(x) \in U_1 \times U_2 \subset M$.

Since $g(x) = (x, f(x))$ therefore this implies that $f(x) \in U_2$. Since f is θ -continuous there exists an open set V of X such that $x \in V$ and $f(\text{cl } V) \subset U_2$. Let $N = V \cap U_1$, then N is a neighbourhood of x such that $g(\text{cl } N) \subset U_1 \times U_2 \subset \text{cl}(U_1 \times U_2) \subset \text{cl } M$. Hence $g(\text{cl } N) \subset \text{cl } M$. Hence g is θ -continuous. Therefore the graph mapping of a θ -continuous mapping is θ -continuous.

Definition : 2.38

A space X is said to be an E_1 -space if every point $x \in X$ is the intersection of a countable number of closed neighbourhoods of x . X is said to be an E_0 -space if every point $x \in X$ is a G_δ .

Theorem : 2.39

If $f: X \rightarrow Y$ is a θ -continuous mapping of a space X onto a space Y such that $f^{-1}(y)$ is a discrete subset of X for all $y \in Y$ and if Y is an E_1 -space, then X is an E_0 -space.

Proof :

Let $x \in X$. Since Y is an E_1 -space, there exists a countable family $\{U_i / i=1, 2, \dots\}$ of open neighbourhoods of $f(x)$ such that $\{f(x)\} = \bigcap_{i=1}^{\infty} U_i$. Since f is θ -continuous, there exists for each i , an open subset V_i of X such that $x \in V_i \subseteq \text{cl } V_i \subseteq f^{-1}(\text{cl } U_i)$. Then $f^{-1}(\{f(x)\}) = \bigcap_{i=1}^{\infty} f^{-1}(\text{cl } U_i) \supseteq \bigcap_{i=1}^{\infty} V_i$

Since $f^{-1}(\{f(x)\})$ is a discrete subspace, there exists a closed subset F of X such that $\{x\} = f^{-1}(\{f(x)\}) - F \supseteq \left(\bigcap_{i=1}^{\infty} V_i\right) - F = \bigcap_{i=1}^{\infty} (V_i - F)$. It follows that $\{x\} = \bigcap_{i=1}^{\infty} (V_i - F)$. consequently, $\{x\}$ is a G_δ and hence X is an E_0 -space.

SECTION : 2

RETRACTION OF TOPOLOGICAL SPACES BY CONTINUOUS MAPPINGS
AND θ -CONTINUOUS MAPPINGS.Definition : 2.40

A subset A of a space X is said to be continuous retract of X if there exists a continuous mapping $f: X \rightarrow A$ such that f is the identity on A . (that is $f(x)=x$ for all $x \in A$).

Definition : 2.41

A subset A of a space X is said to be θ -continuous retract of X if there exists a θ -continuous mapping $f: X \rightarrow A$ such that f is identity on A (that is $f(x)=x$ for all $x \in A$).

Theorem : 2.42

Every continuous retract is a θ -continuous retract.

Proof :

Let A be a subset of X . Since A is continuous retract there exists a continuous mapping $f: X \rightarrow A$, such that f is identity on A . That is $f(x)=x$. Since every continuous map is θ -continuous. Therefore $f: X \rightarrow A$ is θ -continuous, such that f is identity on A , where A is a subset of X . Hence A is θ -continuous retract. Hence every continuous retract is a θ -continuous retract.

Note :

The converse of the above theorem need not be true. That every θ -continuous retract need not be a continuous retract. This is shown by the following example.

Example : 2.43

Let $X=[0,1]$ and let τ be the topology for X which is the usual topology at each point of X except at 0 and the neighbourhood system of 0 consists of sets which are complement of finite subsets of X . Then the subset $A=\{0,1\}$ of X is a θ -continuous retract of X as much as the mapping $f: X \rightarrow A$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a θ -continuous mapping which is the identity on A . However, A is not a continuous retract of X .

Theorem : 2.44

θ -continuous retract B of a θ -continuous retract A of X is a θ -continuous retract of X .

Proof:

Since A is a θ -continuous retract of X , there exists a θ -continuous mapping $f: X \rightarrow A$ such that f is the identity on A . Since B is a θ -continuous retract of A , there exists a θ -continuous mapping $g: A \rightarrow B$ such that g

is the identity on B . Then the composite $g \circ f: X \rightarrow B$ is a θ -continuous mapping by theorem [2.28]. If x be any point of B , then $x \in A$ and thus $(g \circ f)(x) = g(f(x)) = g(x) = x$. It follows that $g \circ f$ is the identity on B and hence B is a θ -continuous retract of X .

Theorem : 2.45

Let X be any space and $A \subset X$. A necessary and sufficient condition that A be a retract of X is that for every space Y each continuous $f: A \rightarrow Y$ is extendable over X .

Proof :

First, let A be a retract of X . Let $g: X \rightarrow A$ be a continuous mapping such that g is the identity on A . Let $f: A \rightarrow Y$ be a continuous mapping. Then the composite $f \circ g: X \rightarrow Y$ is a continuous mapping. If x be any point of A , then $(f \circ g)(x) = f(g(x)) = f(x)$. Thus $f \circ g$ is an extension of f .

Conversely, suppose that for every space Y , every continuous mapping $f: A \rightarrow Y$ can be extended to a continuous mapping $g: X \rightarrow Y$. Let $f: A \rightarrow A$ be the mapping defined by $f(a) = a, \forall a \in A$. Then f is continuous. By hypothesis, there exists a continuous mapping $g: X \rightarrow A$ such that $g|_A = f$. Let x be any point of A . Then $g(x) = f(x) = x$. Hence A is a retract of X .

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Theorem : 2.46

A subset A of a space X is a θ -continuous retract of X if and only if for every space Y , every θ -continuous mapping $f:A \rightarrow Y$ can be extended to a θ -continuous mapping on X into Y .

Proof :

Assume A is a θ -continuous retract of X . To prove the given condition. Since A is θ -continuous retract of X , there exists $g:X \rightarrow A$ be a θ -continuous mapping such that g is identity on A . Given $f:A \rightarrow Y$ is a θ -continuous mapping. Then the composite $f \circ g:X \rightarrow Y$ is a θ -continuous mapping by theorem [2.28]. If x be any point of A then $(f \circ g)(x) = f(g(x)) = f(x)$. Thus $f \circ g$ is an extension of f . Hence $f:A \rightarrow Y$ can be extended to a θ -continuous mapping on X into Y .

Conversely, assume that for every space Y , every θ -continuous mapping $f:A \rightarrow Y$ can be extended to a θ -continuous mapping $g:X \rightarrow Y$. To prove A is a θ -continuous retract of X . Let $f:A \rightarrow A$ be the mapping defined by $f(a) = a$ for all $a \in A$. Then, f is continuous and hence f is θ -continuous. Then by our hypothesis there exists a θ -continuous mapping $g:X \rightarrow A$ such that $g|_A = f$. Let x be any point of A . Hence $g(x) = f(x) = x$. This implies g is identity on A . Hence A is a θ -continuous retract of X .

SECTION : 3.**FIXED POINT PROPERTY IN RELATION TO CONTINUOUS MAPPINGS
AND θ -CONTINUOUS MAPPINGS.**Definition : 2.47

A space X is said to have the continuous fixed point property if for every continuous mapping $f:X \rightarrow X$ there exists an $x \in X$ such that $f(x)=x$.

Definition : 2.48

A space X is said to have the θ -continuous fixed point property if for every θ -continuous mapping $f:X \rightarrow X$ there exists an $x \in X$ such that $f(x)=x$.

Theorem : 2.49

If a space has the θ -continuous fixed point property, then it also has the fixed point property.

Proof :

Assume X has a θ -continuous fixed point property. To prove X has a fixed point property. Since X has a θ -continuous fixed point property, there exists a θ -continuous $f:X \rightarrow X$ such that $x \in X$, $f(x)=x$.

Let $g: X \rightarrow X$ be a identity mapping. Since, the identity mapping is continuous hence g is θ -continuous. Therefore there exists $x \in X$ such that $g(x)=x$. Hence X has a fixed point property.

Note :

The converse of the above statement is not true. This is shown by the following example.

Example : 2.50

Let $X = \{a, b, c\}$ and let $\mathcal{A} = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. The mapping $f: X \rightarrow X$ defined by $f(a)=b=f(c)$, $f(b)=c$ is a θ -continuous mapping which does not map any point of X into itself. Hence X does not have the θ -continuous fixed point property, however X has the fixed point property.

Theorem : 2.51

If a space X has the θ -continuous fixed point property and A is a θ -continuous retract of X , then A has the θ -continuous fixed point property.

Proof :

Let $f: A \rightarrow A$ be any θ -continuous mapping. Since A is a θ -continuous retract of X . Then f can be extended to a θ -continuous mapping $g: X \rightarrow A$. Since X has the

θ -continuous fixed point property there exists $x \in X$ such that $g(x)=x$.

Since g is a mapping from X to A , there exists $x \in A$. It follows that $f(x)=g(x)=x$ and so X is a fixed point of f . Hence A has the θ -continuous fixed point property.

Theorem : 2.52

If (X, τ) is a regular space with the fixed point property and if τ^* is a topology for X stronger than such that for each $G \in \tau^*$, $\tau\text{-cl } G = \tau^*\text{-cl } G$, then (X, τ^*) has the θ -continuous fixed point property.

Proof :

Let $f: (X, \tau^*) \rightarrow (X, \tau^*)$ be any θ -continuous mapping. Let $g: (X, \tau^*) \rightarrow (X, \tau)$ and $h: (X, \tau) \rightarrow (X, \tau)$ be the mappings defined by $h(x) = g(x)$ for all $x \in X$. Let $i: (X, \tau^*) \rightarrow (X, \tau^*)$ be the identity mapping on X . Obviously, i is an open, one-to-one and onto mapping. Also, since $f=i \circ g$ and f is θ -continuous, therefore g is θ -continuous by corollary [2.29]. Since every θ -continuous mapping into a regular space is continuous therefore it follows that g is a continuous mapping. We shall now show that h is also a continuous mapping. Let $x \in X$ and let G be any τ -open set containing $h(x)$.

Since (X, τ) is regular, there exists a τ -open set G , such that $h(x) \in G, \tau\text{-cl } G \subseteq G$.

Now, G_1 is a τ -open set and g is continuous. Therefore $g^{-1}(G_1)$ is a τ^* -open subset of X . Since $h^{-1}(G_1) = g^{-1}(G_1)$, this means that $h^{-1}(G_1)$ is a τ^* -open subset of X . Thus $\tau\text{-cl } [h^{-1}(G_1)] = \tau^*\text{-cl } [h^{-1}(G_1)]$. It follows that the set $G_2 = X \sim \tau\text{-cl } [h^{-1}(G_1)]$ is τ -open as well as τ^* -open. Consequently, $\tau\text{-cl } G_2 = \tau^*\text{-cl } G_2 (=G_3 \text{ say})$ and $G_2 \subseteq G_3$. Now, the set $X \sim G_3$ is τ -open as well as τ^* -open and is contained in $\tau\text{-cl } h^{-1}(G_1)$.

Also, $x \in X \sim G_3$. Thus $X \sim G_3$ is a τ -open set such that $x \in X \sim G_3 \subseteq \tau\text{-cl } h^{-1}(G_1) \subseteq h^{-1}(\tau\text{-cl } G_1) \subseteq h^{-1}(G)$. So for each $x \in X$ and for each τ -open set G containing $h(x)$, there exists a τ -open set containing x whose image under h is contained in G . Hence h is a continuous mapping. Since (X, τ) has the fixed point property, there exists a point $x \in X$ such that $h(x) = x$. Since $h(x) = f(x)$ for all x , therefore $f(x) = x$. Thus every θ -continuous mapping $f: (X, \tau^*) \rightarrow (X, \tau^*)$ has a fixed point and consequently (X, τ^*) has the θ -continuous fixed point property.

Following is an example of a Hausdorff non-regular space which has the θ -continuous fixed point property.

Example : 2.53

Let $X = [0,1] \times [0,1]$ and τ be the topology for X which is the usual topology for X except at $(0,0)$, A τ -neighbourhood of $(0,0)$ consists of the point $(0,0)$ and with the open radius on $y=0$ deleted. If U be the usual topology for X , then since (X, U) is a regular space with the fixed point property and the topology τ is stronger than U such that for each τ -open set G . $\text{cl } G = U\text{-cl } G$, it follows that (X, τ) must have the θ -continuous fixed point property by theorem [2.52].

SECTION : 4

HOMOTOPY AND θ -CONTINUOUS HOMOTOPY

Definition : 2.54

If f and g are continuous maps of the space X into the space Y , we say that f is homotopic to g if there is a continuous map $F: X \times I \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for each $x \in X$. The map F is called a homotopy between f and g . If f is homotopic to g we write $f \approx g$.

Definition : 2.55

A θ -continuous mapping $h: X \times I \rightarrow Y$ where X and Y are any spaces and I is the closed interval $[0, 1]$ with the usual topology is said to be θ -continuous homotopy.

Definition : 2.56

A subspace A of a space X is said to have the θ -continuous homotopy extension property in X with respect to a space Y if for every continuous mapping $f: X \rightarrow Y$ and every homotopy $h: A \times I \rightarrow Y$ such that $h(x, 0) = f(x)$ for all $x \in A$, there exists a θ -continuous homotopy $g: X \times I \rightarrow Y$ such that $g(x, t) = h(x, t)$ for all $(x, t) \in A \times I$ and $g(x, 0) = f(x)$ for all $x \in X$.

Definition : 2.57

Let ACX be given. We say that A has the absolute homotopy extension property (AHEP) in X if for each space Y and each continuous $F: X \rightarrow Y$, every homotopy of F/A is extendable to a homotopy of F .

Definition : 2.58

A subspace A of a space X is said to have the absolute θ -continuous homotopy extension property in X if A has the θ -continuous homotopy extension property in X with respect to every space Y .

Theorem : 2.59

- (1). A has the absolute homotopy extension property in X if and only if $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$.
- (2) $(X \times 0) \cup (A \times I) \cup (X \times 1)$ is a retract of $X \times I$ if and only if for each space Y and each pair of maps $F, G: X \rightarrow Y$ any given homotopy of F/A to G/A can be extended to a homotopy of F to G .

Proof :

- (1) Assume that $(X \times 0) \cup (A \times I)$ is a retract of $X \times I$. A given $F: X \rightarrow Y$ and a given homotopy $\Phi: A \times I \rightarrow Y$ of F/A determine a continuous $\Psi: (X \times 0) \cup (A \times I) \rightarrow Y$ by setting $\Psi(x, 0) = F(x)$,

$\psi(a,t) = \phi(a,t)$. According to theorem [2.45] ψ extends to a $\Psi^* : X \times I \rightarrow Y$ and Ψ^* is thus homotopy of F extending ϕ

Conversely, assume that A has the absolute homotopy extension property in X . choose $Y = (X \times 0) \cup (A \times I)$, let $F : X \rightarrow Y$ be the map $x \rightarrow (x, 0)$, and let the homotopy of F/A be $\phi(a,t) = (a,t)$. This extends to a homotopy ϕ of F , and $\phi : X \rightarrow Y$ is the desired retraction. The proof of (2) is similar.

Theorem : 2.60

If A is closed subset of a space X , then A has the absolute θ -continuous homotopy extension property in X if and only if $(X, 0) \cup (A \times I)$ is a θ -continuous retract of $X \times I$.

Proof :

First, suppose that $(X \times 0) \cup (A \times I)$ is a θ -continuous retract of $X \times I$. Let Y be any space, $f : X \rightarrow Y$ be any continuous mapping and let $h : A \times I \rightarrow Y$ be a homotopy such that $h(x, 0) = f(x)$ for all $x \in A$. Let $g : (X \times 0) \cup (A \times I) \rightarrow Y$ be the mapping defined by $g(x, t) = f(x)$ if $t = 0$ and $g(x, t) = h(x, t)$ if $t \neq 0$, obviously, g is a continuous mapping. Since $(X \times 0) \cup (A \times I)$ is a θ -continuous retract of $X \times I$, therefore g can be extended to a θ -continuous mapping $g^* : X \times I \rightarrow Y$ by theorem [2.46]. Since g^* is an extension of g , therefore $g(x, t) = g^*(x, t)$, for all $(x, t) \in A \times I$ and $g^*(x, 0) = f(x)$ for all $x \in X$. Hence A has the absolute

θ -continuous homotopy extension property in X .

Conversely, suppose A has the absolute θ -continuous homotopy extension property in X . Let $Y = (X \times 0) \cup (A \times I)$ and let $f: X \rightarrow Y$ be a mapping defined by $f(x) = (x, 0)$ for all $x \in X$. Then f is a continuous mapping. Let $g: A \times I \rightarrow Y$ be the homotopy defined by $h(a, t) = (a, t)$ for all $(a, t) \in A \times I$. Since A has the absolute θ -continuous homotopy extension property in X , therefore there exists a θ -continuous homotopy $g: X \times I \rightarrow Y$ such that $g(x, t) = (x, t)$ for all $(x, t) \in A \times I$ and $g(x, 0) = f(x)$ for all $x \in X$. Obviously, g is a θ -continuous mapping of $X \times I$ into $(X \times 0) \cup (A \times I)$. Let (x, y) be any point of $(X \times 0) \cup (A \times I)$. If $(x, y) \in X \times \{0\}$, then $Y = 0$ and therefore $g[(x, y)] = g[(x, 0)] = f(x) = (x, 0) = (x, y)$. Also, if $(x, y) \in A \times I$, then $x \in A$ and $g[(x, y)] = h(x, y) = (x, y)$. Thus g is the identity on $(X \times 0) \cup (A \times I)$. Hence $(X \times 0) \cup (A \times I)$ is a θ -continuous retract of $X \times I$. This completes the proof of the theorem.

Theorem : 2.61

If A is a closed subset of a space X , then $(X \times 0) \cup (A \times I) \cup (X \times \{1\})$ is a θ -continuous retract of $X \times I$ if and only if for each space Y and each pair of continuous functions $f: X \rightarrow Y$ and $g: X \rightarrow Y$, any given homotopy $f: X \rightarrow Y$ and $g: X \rightarrow Y$, any given homotopy between $f|_A$ and $g|_A$ can be extended to a θ -continuous homotopy between f and g .

Proof :

First, suppose that $(X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$ is a θ -continuous retract of $X \times I$. Let Y be any space and let $f: X \rightarrow Y$ and $g: X \rightarrow Y$ be continuous mappings. Let $h: A \times I \rightarrow Y$ be a homotopy between f/A and g/A . Define a mapping $h^*: (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\}) \rightarrow Y$ by

$$\begin{aligned} h^*(x,t) &= f(x) & \text{if } t=0 \\ &= h(x,t) & \text{if } 0 < t < 1, \\ &= g(x) & \text{if } t=1 \end{aligned}$$

Now h^* is a θ -continuous mapping. Also, since $(X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$ is a θ -continuous retract of $X \times I$, therefore h^* can be extended to $h^{**}: X \times I \rightarrow Y$, $h^{**}(x,0) = h^*(x,0) = f(x)$ and $h^{**}(x,1) = h^*(x,1) = g(x)$ for all $x \in X$. Thus h^{**} is a θ -continuous homotopy between f and g .

Conversely, suppose that if Y is any space and $f: X \rightarrow Y$ and $g: X \rightarrow Y$ are continuous mappings, then any given homotopy between f/A and g/A can be extended to a θ -continuous homotopy between f and g . Let $Y = (X \times \{0\}) \cup (A \times I) \cup (X \times \{1\})$ and let $f: X \rightarrow Y$ be the mapping defined by $f(x) = (x,0)$ for all $x \in X$. Then f and g are continuous mappings. Let $h: A \times I \rightarrow Y$ be the homotopy between f/A and g/A defined by $h(a,t) = (a,t)$ for all $(a,t) \in A \times I$. Then h can be extended to a θ -continuous homotopy $h^*: X \times I \rightarrow Y$ between f and g . Let (x,y) be any point of Y . If $(x,y) \in X \times \{0\}$, then $K(x,y) = K(x,0) = f(x) = (x,0) = (x,y)$. If $(x,y) \in X \times \{1\}$

then $K(x,y)=K(x,1)=g(x)=(x,1)=(x,y)$. If $(x,y) \in AxI$, then $K(x,y)=h(x,y)=(x,y)$. Thus K is the identity on $Y=(X \times 0) \cup (AxI) \cup (X \times 1)$. Hence $(X \times 0) \cup (AxI) \cup (X \times 1)$ is a θ -continuous retract of $X \times I$. This completes the proof of the theorem.

Chapter III

CHAPTER - III

A CHARACTERIZATION OF COMPLETELY REGULAR SPACES

The set of all continuous functions from the topological space X into the topological space R is denoted by $C(x)$ or for short by C .

The sum or product of two continuous functions is again a continuous function. Suppose $f \in C$ then $(-f) \in C$. Therefore $C(x)$ is a commutative ring. Also the constant function (1) whose constant value is the real number 1 belongs to $C(x)$ and is its unity element.

Definition : 3:1

Consider $f \in C(x)$. Then the zero-set of f is denoted by $z_x(f)$ or $z(f)$ and is defined as $z(f) = \{x \in X / f(x) = 0\}$.

The family of all zero-set $\{z(f) / f \in C\}$ in X is denoted by $Z[C(x)]$ or simply $Z[x]$.

Definition: 3:2

Complements of zero-set in X are called co-zero sets.

Definition: 3:3

Let X be any set. A collection \mathcal{F} of non empty subsets of X is said to be a filter on X if

- (1) $\mathcal{F} \neq \emptyset$
- (2) if A and B are \mathcal{F} then $A \cap B$ is also in \mathcal{F}
- (3) if $A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$

Example : 3:4

If X is any set and Y is any nonempty subset of X , then the family \mathcal{F} of all subsets of X which contain Y is a filter on X .

Definition: 3:5

If (X, τ) is a topological space and \mathcal{F} is a filter on X , then \mathcal{F} is said to converge to x denoted by $\mathcal{F} \rightarrow x$, if every neighbourhood of x is a member of \mathcal{F}

Definition: 3:6

Let \mathcal{B} be a collection of closed sets. \mathcal{B} is said to be a base for the closed sets if every closed set in X is an intersection of members of \mathcal{B} .

Equivalently \mathcal{B} is a base if whenever F is closed in X , $x \in X - F$, there is a member of \mathcal{B} that contains F but not x .

Definition: 3:7

Let \mathcal{B} be a collection of open sets. \mathcal{B} is said to be a base for the open sets if every open set in X is an intersection of members of \mathcal{B} .

Equivalently \mathcal{B} is a base if whenever F is open X , $x \in X - F$, there is a member of \mathcal{B} that contains F but not x .

Let (X, τ) be a Hausdorff space, and let X^* be the set of all convergent filters on X . Then f , which assigns to each $\varphi \in X^*$, the point to which φ converges, is a mapping. For a subset $A \subset X$, we let $A^* = \{\varphi \in X^* / A \in \varphi\}$ and for $x \in X$, we denote by φ_x , the filter $\{F \subset X / x \in F\}$. Let τ^* be the topology on X^* generated by $\{U^* / U \text{ is open in } X\}$. To generate τ^* , it is enough to consider $\{B^* / B \in \mathcal{B}\}$ for any base for open sets in X . Let \mathcal{C} denote the family of all co-zero subsets of X^* and let σ be the topology on X^* generated by $\{C^* / C \in \mathcal{C}\}$. Clearly $\sigma \leq \tau^*$ and for completely regular spaces, they are identical. In all the theorems, the space (X, τ) is assumed to be Hausdorff.

Theorem: 3:8

A Hausdorff space X is completely regular if and only if the family $Z(X)$ of all zero-sets is a base for the closed sets.

Proof:

Suppose that X is a completely regular, then whenever F is a closed set and $x \in X - F$, there exists $f \in C(X)$ such that $f(x) = 1$ and $f(F) = 0$

Hence $F \in z(f)$ and $x \notin Z(f)$.

$Z(x)$ is a base.

Conversely assume that $Z(X)$ is a base. Then whenever F is a closed set and $x \in X - F$, there exists a zero-set say $Z(g)$ such that $Z(g) \supset F$ and $x \notin Z(g)$.

Let $r = g(x)$, then $r > 0$

Let $f = gr^{-1} \in C(X)$. Where r is a constant function whose constant value is r .

Note:

By the above theorem " A Hausdorff space X is completely regular if and only if the family of all co-zero sets is a base for the open sets".

Theorem: 3:9

(X, τ) is completely regular if and only if $f: (X, \sigma) \rightarrow (X, \tau)$ is a continuous mapping.

Proof:

Let (X, τ) be completely regular and let U be an open subset of X . Let $\varphi \in f^{-1}(U)$ so that $f(\varphi) \in U$ and

therefore there exists a co-zero set C . Since every completely regular space is regular. Therefore by using the result " (X, τ) is regular if and only if $p \in U \in \tau$ implies that $\forall \epsilon \tau$ such that $p \in V \subset \bar{V} \subset U$ ". Since $f(\varphi)$ is in U such that $f(\varphi) \in C \subset \bar{C} \subset U$. If $f(\varphi) \in C$ implies that $\varphi \in C^*$ and $\bar{C} = f(C^*) \subset U$ implies $C^* \subset f^{-1}(U)$. This proves that $f^{-1}(U) \in \sigma$ for all $U \in \tau$ and hence f is continuous.

Conversely assume that $f: (X^*, \sigma) \rightarrow (X, \tau)$ is continuous mapping. To prove (X, τ) is completely regular. Let F be a closed subset of X and let $x \notin F$. Since f is continuous and $\varphi_x \in f^{-1}(F)$, which implies that $\varphi_x \in X^* - f^{-1}(F)$. That is φ_x belonging to the open set in X^* . Therefore there exists some C^* , $C \in \mathcal{C}$ such that $\varphi_x \in C^*$ and $C^* \cap f^{-1}(F) = \emptyset$. It is clear from here that $x \in C$ and $\bar{C} \cap F = \emptyset$. Thus co-zero sets form a base for open sets by theorem [3.9] and therefore (X, τ) is completely regular.

Theorem: 3:10

(X, τ) is almost-regular if and only if $f: (X^*, \tau^*) \rightarrow (X, \tau)$ is an almost-continuous mapping.

Proof:

Let (X, τ) be almost-regular and U be a regular open subset of X . To prove f is almost-continuous. That is to prove $f^{-1}(U)$ is open. Let $\varphi \in f^{-1}(U)$. This implies $f(\varphi) \in U$.

Let V be an open subset of X such that $f(\varphi) \in V \subset \bar{V} \subset U$.
 Since $f(\varphi) \in V$, then $\varphi \in V^*$ and $\bar{V} = f(V^*) \subset U$. Implies that
 $V^* \subset f^{-1}(U)$, hence $f^{-1}(U) \in \tau^*$ Hence f is almost-continuous.

Conversely assume that f is almost-continuous mapping, to prove (X, τ) is almost-regular. let U is a regular-open subset of X , containing $x \in X$. Since f is continuous and U is open, then $f^{-1}(U)$ is open. This implies $f^{-1}(X-U)$ is closed subset of X^* not containing φ_x . Let $V \in \tau$ be such that $\varphi_x \in V$ and $V \cap f^{-1}(X-U) = \emptyset$. This implies $V \cap X-U = \emptyset$. It is clearly we have $x \in V \subset \bar{V} \subset U$. Hence (X, τ) is almost-regular.

Definition: 3:11

A topological space (X, τ) is said to be almost-completely regular if for each regular closed set (closure of an open set), F and for each $x \notin F$, there exists a continuous function $g: X \rightarrow [0, 1]$ such that $g(x) = 0$ and $g(F) = \{1\}$

Note:

Now the following weaker form of complete regularity is easy to characterize, and the proof follows from theorems 3.10 and 3.12.

Theorem: 3:12

(X, τ) is almost-completely regular if and only if $f: (X^*, \sigma) \rightarrow (X, \tau)$ is an almost-continuous mapping.

Definition: 3:13

A space X is said to be extremally-disconnected if every open set has an open closure.

Definition: 3:14

A space X is basically-disconnected if every open set has an open closure.

Note:

Any extremally-disconnected space is basically-disconnected. But the converse of the above statement is not true.

Given a topological space (X, τ) , let \mathcal{Q} be the collection of open sets U such that U is a compact subset of X . Let ρ be the topology generated by $\{R^*/R\mathcal{Q}\}$. Clearly $\rho \leq \tau^*$ and for a locally-compact Hausdorff space, they are identical.

In terms of ρ we can characterize local-compactness, extremally-disconnected, basically-disconnected.

Result: 3:15

(X, τ) is locally-compact if and only if $f: (X^*, \rho) \rightarrow (X, \tau)$ is a continuous mapping.

Result: 3:16

(X, τ) is extremally-disconnected if and only if $f: (X^*, \tau^*) \rightarrow (X, \tau)$ is an open mapping.

Result: 3:17

(X, τ) is basically-disconnected if and only if $f: (X^*, \sigma) \rightarrow (X, \tau)$ is an open mapping.

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