CHAPTER 3

Topological Insights on Neutrosophic Set Variants

(3.1) Topological Structures of Neutrosophic Variants

(3.2) Fermatean Neutrosophic Gradation of Openness

(3.3) Fermatean Temporal Neutrosophic Topology

This chapter explores the topology of neutrosophic set variants through the frameworks proposed by Chang, Sostak, and Lowen. In Chang's framework, the neutrosophic variants defines whole set as the maximum value of the considered set, and the null set is zero. Whereas in the Lowen approach, the maximum value among the assigned membership values replaces the whole set, and the minimum value replaces the empty set. Sostak's approach defines grades for open sets. By examining these diverse viewpoints, the concepts of Pythagorean, Spherical, and Fermatean neutrosophic topologies are developed. It also proposes a new class called Fermatean neutrosophic temporal topology. It also introduces key concepts such as Fermatean neutrosophic gradation of openness, subspace, gradation-preserving maps, base and subbase, Fermatean product neutrosophic topological space, Fermatean neutrosophic compactness, and Tychonoff's theorem.

3.1 Topological Structures of Neutrosophic variants

This section, examines the concepts of Pythagorean neutrosophic topology, neutrosophic spherical topology, and Fermatean neutrosophic topology within the foundational frameworks established by Chang and Lowen.

Pythagorean Neutrosophic Topology

Definition 3.1.1

Let X be a non empty set. Then τ is known as a Pythagorean Neutrosophic Topology (PNT) in the sense of **Chang** if it satisfies the following axioms:

(T1) $0_{PN}, \sqrt{2}_{PN} \in \tau$,

(T2) for any $P_1, P_2 \in \tau$, implies $P_1 \cap P_2 \in \tau$.

(T3) for any $\{P_n | n \in J\}$, implies $\bigcup_{n \in J} P_n \in \tau$.

the pair $(X,\ \tau)$ is called a Pythagorean neutrosophic topological space (PNTS).

Example 3.1.2

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set, and let A_P, B_P, C_P, D_P be PNSs on X,

$$A_P = \langle (\tilde{x}, 0.9,0.6,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5) \rangle.$$
$$B_P = \langle (\tilde{x}, 0.5,0.7,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8) \rangle.$$
$$C_P = \langle (\tilde{x}, 0.9,0.7,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5) \rangle.$$
$$D_P = \langle (\tilde{x}, 0.5,0.6,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8) \rangle.$$

$\tau = \{0_{PN}, \sqrt{2}_{PN}, A_P, B_P, C_P, D_P\}$. The collection τ is called a PNT on X, the pair (X, τ) a PNTS.

Definition 3.1.3

Let X be a non empty set. Then τ is known as a PNT in the sense of **Lowen** if it satisfies the following axioms:

(T1) $A_P \in \delta$ for each $T_{A_P}, I_{A_P}, F_{A_P} \in \delta \subseteq [0, \sqrt{2}]$ with

$$0 \leq {T_{A_P}}^2 + {I_{A_P}}^2 + {F_{A_P}}^2 \leq 2$$

(T2) for any $A_{P_1}, A_{P_2} \in \tau$, implies $A_{P_1} \cap A_{P_2} \in \tau$.

(T3) for any $\{A_{P_n} | n \in J\}$, implies $\cup_{n \in J} A_{P_n} \in \tau$.

the pair (X, τ) is called a PNTS.

Example 3.1.4

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set, and let A_P, B_P, C_P, D_P be PNSs on X,

$$A_P = \langle(\tilde{x}, 0.9,0.6,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5)\rangle.$$
$$B_P = \langle(\tilde{x}, 0.5,0.7,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8)\rangle.$$
$$C_P = \langle(\tilde{x}, 0.9,0.7,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5)\rangle.$$
$$D_P = \langle(\tilde{x}, 0.5,0.6,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8)\rangle.$$

$\tau = \{A_P, B_P, C_P, D_P\}$. The collection τ is called a PNT on X, the pair (X, τ) a Pythagorean neutrosophic topological space (PNTS).

Definition 3.1.5

The complement $C(A_P)$ of a PNS A_P in an PNTS (X, τ) is called a Pythagorean neutrosophic closed set (PNCS) in X.

Proposition 3.1.6

Let (X, τ) be a PNTS on X. Then construct PNT on X in the following way

$$\tau_1 = \{\bullet\, G : G \in \tau\}$$

where

$$\bullet\, G = \{\langle x, F_G(x), I_G(x), T_G(x)\rangle | x \in X\},$$

Proof:

(T1) $0_{PN}, \sqrt{2}_{PN} \in \tau_1$ is obvious.

(T2) Let $\bullet\, G_1, \bullet\, G_2 \in \tau_1$. Since $G_1, G_2 \in \tau$

therefore $G_1 \cap G_2 = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle \in \tau$. This implies that

$$(\bullet\, G_1 \cap \bullet\, G_2) = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, (T_{G1}) \vee (T_{G2})\rangle$$
$$= \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, (T_{G1} \wedge T_{G2})\rangle \in \tau_1$$

(T_3) Let $\{\bullet\, G_i, i \in J, G_i \in \tau\} \subseteq \tau_1$.

Since $\cup\, G_i = \langle x, \vee\, T_{G_i}, \vee\, I_{G_i}, \wedge\, F_{G_i} \rangle \in \tau$

$$\cup\,(\bullet\, G_i) = \langle x, \vee\, T_{G_i}, \vee\, I_{G_i}, \wedge\,(T_{G_i}) \rangle$$

$$= \langle x, \vee\, T_{G_i}, \vee\, I_{G_i}, (\vee\, T_{G_i}) \rangle \in \tau_1.$$

Definition 3.1.7

Let $(X, \tau_1), (X, \tau_2)$ be two PNTSs on X. Then τ_1 is said to be contained in τ_2 if $G \in \tau_2$ for each $G \in \tau_1$. Then τ_1 is coarser than τ_2.

Proposition 3.1.8

Let $\{\tau_i : i \in J\}$ be a family of PNTS on X. Then $\cap\, \tau_i$ is also a PNT on X. Furthermore, $\cap\, \tau_i$ is the coarsest PNT on X containing all τ_i.

i. $0_{PN} \in \tau_i$ for every $i \in J$. From this it follows that $0_{PN} \in \cap\, \tau_i$. Similarly, $\sqrt{2}_{PN} \in \cap\, \tau_i$.

ii. Let $G_1, G_2 \in \cap\, \tau_i$. Then $G_1, G_2 \in \tau_i$, for every $i \in J$ and hence, $G_1 \cap G_2 \in \tau_i$, $\forall i \in J$. Thus, $G_1 \cap G_2 \in \cap\, \tau_i$.

iii. Let $\{G_j : j \in K\} \subseteq \cap\, \tau_i$. Then $\{G_j : j \in K\} \subseteq \tau_i$, for every $i \in J$ and hence, $\underset{j \in K}{\cup} G_j \in \tau_i$, $\forall i \in J$. Thus, $\underset{j \in K}{\cup} G_j \in \cap\, \tau_i$.

Clearly, it is the coarsest topology on X containing all τ_i's. Since if τ' is any other PNT on X which contains every τ_i, then obviously it will also contain $\cap\, \tau_i$.

Definition 3.1.9

Let $\alpha, \beta, \gamma \in (0, \sqrt{2})$ and $\alpha^2 + \beta^2 + \gamma^2 \leq 2$. A Pythagorean Neutrosophic Point (PNP) $p^x_{(\alpha,\beta,\gamma)}$ of X is a PNS of X defined by $p^x_{(\alpha,\beta,\gamma)} = \langle x, T_p, I_p, F_p \rangle$, where for $x \in X$

$$T_p(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases},$$

$$I_p(y) = \begin{cases} \beta & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases},$$

$$F_p(y) = \begin{cases} \gamma & \text{if } \ y = x \\ \sqrt{2} & \text{if } \ y \neq x \end{cases}$$

In this case, x is called the support of $p^x_{(\alpha,\beta,\gamma)}$. A PNP $p^x_{(\alpha,\beta,\gamma)}$ is said to belong to a PNS $A = \langle x, T_A, I_A, F_A \rangle$ of X, denoted by $p^x_{(\alpha,\beta,\gamma)} \in A$, if $\alpha \leq T_A(x)$, $\beta \leq I_A(x)$ and $\gamma \geq F_A(x)$.

Proposition 3.1.10

Every PNS A_P in X can be expressed as the union of all PNP contained in A_P.

Definition 3.1.11

A collection $\mathfrak{B}$ of PNS on a set X is said to be basis (or base) for a PNT on X, if

i. For every $p^x_{(\alpha,\beta,\gamma)}$ in X, there exists $B \in \mathfrak{B}$ such that $p^x_{(\alpha,\beta,\gamma)} \in B$.

ii. If $p^x_{(\alpha,\beta,\gamma)} \in B_1 \cap B_2$, where $B_1, B_2 \in \mathfrak{B}$ then there exist $B_3 \in \mathfrak{B}$ such that $p^x_{(\alpha,\beta,\gamma)} \in B_3 \subseteq B_1 \cap B_2$.

Proposition 3.1.12

Let $\mathfrak{B}$ be a basis for a PNT on X. Let τ contains those PNS G of X for which corresponding to each $p^x_{(\alpha,\beta,\gamma)} \in G$, there exist $B \in \mathfrak{B}$ such that $p^x_{(\alpha,\beta,\gamma)} \in B \subseteq G$. Then τ is a PNT on X.

Proposition 3.1.13

Let τ be an PNT on a set X, generated by a basis $\mathfrak{B}$. Then members of τ are precisely the union of members of $\mathfrak{B}$, that is, $G \in \tau$ if and only if $G = \bigcup_{\alpha \in A} B_\alpha$, where $B_\alpha \in \mathfrak{B}, \forall \alpha \in X$.

Definition 3.1.14

Let (X, τ) be a PNTS. Then a subfamily $B_s \subseteq \tau$ is called a sub-basis for τ if the family of finite intersections of members of B_s forms a base for τ.

Definition 3.1.15

Let (X,τ) be a PNTS and $A_P = \{\langle x: T_{A_P}(x), I_{A_P}(x), F_{A_P}(x)\rangle / x \in X\}$ be an PNS in X. Then the Pythagorean neutrosophic closure of A_P are defined by

$$Cl(A_P) = \cap \{K: K \text{ is a PNCS in } X \text{ and } A_P \subseteq K\}.$$

Definition 3.1.16

Let (X,τ) be a PNTS and $A_P = \{\langle x: T_{A_P}(x), I_{A_P}(x), F_{A_P}(x)\rangle / x \in X\}$ be an PNS in X. Then the Pythagorean neutrosophic interior of A_P are defined by

$$int(A_P) = \cup \{G: G \text{ is a PNOS in } X \text{ and } G \subseteq A_P\}.$$

Note that $Cl(A_P)$ is a Pythagorean neutrosophic closed sets (PNCSs) and $int(A_P)$ is a Pythagorean neutrosophic open sets (PNOSs) in X.

Further,

(a) A_P is a PNCS in X iff $Cl(A_P) = A_P$;

(b) A_P is a PNOS in X iff $int(A_P) = A_P$.

Example 3.1.17

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ be a set, and let A_P, B_P, C_P, D_P be PNSs on X,

$$A_P = \langle(\tilde{x}, 0.9,0.6,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5)\rangle.$$

$$B_P = \langle(\tilde{x}, 0.5,0.7,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8)\rangle.$$

$$C_P = \langle(\tilde{x}, 0.9,0.7,0.3), (\tilde{y}, 0.8,0.8,0.3), (\tilde{z}, 0.7,0.8,0.5)\rangle.$$

$$D_P = \langle(\tilde{x}, 0.5,0.6,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.7,0.8,0.8)\rangle.$$

$$\tau = \{0_{PN}, \sqrt{2}_{PN}, A_P, B_P, C_P, D_P\}$$

If

$$P_P = \langle(\tilde{x}, 0.8,0.5,0.4), (\tilde{y}, 0.8,0.6,0.2), (\tilde{z}, 0.6,0.7,0.7)\rangle$$

$$int(P_P) = \cup \{G: G \text{ is a PNOS in } X \text{ and } G \subseteq P_P\}$$

$$= max\left\{\begin{matrix}\langle(\tilde{x}, 0.5,0.7,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8)\rangle, \\ \langle(\tilde{x}, 0.5,0.6,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.7,0.8,0.8)\rangle\end{matrix}\right\}$$

$$= \langle(\tilde{x}, 0.5,0.7,0.8), (\tilde{y}, 0.3,0.7,0.9), (\tilde{z}, 0.2,0.8,0.8)\rangle = B_P$$

$$\text{PNCS in } X = \left\{ \begin{array}{c} \langle(\tilde{x},\xi,\psi,0),(\tilde{y},\xi,\psi,0),(\tilde{z},\xi,\psi,0)\rangle, \\ \langle(\tilde{x},0,0,\zeta),(\tilde{y},0,0,\zeta),(\tilde{z},0,0,\zeta)\rangle, \\ \langle(\tilde{x},0.3,0.6,0.9),(\tilde{y},0.3,0.8,0.8),(\tilde{z},0.5,0.8,0.7)\rangle, \\ \langle(\tilde{x},0.8,0.7,0.5),(\tilde{y},0.9,0.7,0.3),(\tilde{z},0.8,0.8,0.2)\rangle, \\ \langle(\tilde{x},0.3,0.7,0.9),(\tilde{y},0.3,0.8,0.8),(\tilde{z},0.5,0.8,0.7)\rangle, \\ \langle(\tilde{x},0.8,0.6,0.5),(\tilde{y},0.9,0.7,0.3),(\tilde{z},0.8,0.8,0.7)\rangle \end{array} \right\}$$

$$Cl(P_P) = \cap \{K : K \text{ is a PNCS in } X \text{ and } P_P \subseteq K\}$$
$$= min\{\sqrt{2}_{PN}\} = \sqrt{2}_{PN}$$

Proposition 3.1.18

For any PNS A_P in (X,τ),

(a) $Cl\left(\overline{A_P}\right) = \overline{int(A_P)}$

(b) $int\left(\overline{A_P}\right) = \overline{Cl(A_P)}$

Proof:

(a) Let $A_P = \langle x, T_{A_P}, I_{A_P}, F_{A_P}\rangle$ and suppose that the PNOS's contained in A_P are indexed by the family $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i}\rangle : i \in J\}$. Then, $int(A_P) = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge F_{G_i}\rangle$ and hence

$$\overline{int(A_P)} = \langle x, \wedge F_{G_i}, \vee (I)_{G_i}, \vee T_{G_i}\rangle \qquad (3.1.1)$$

Since $\overline{(A_P)} = \langle x,\ \wedge F_{A_P},\ \vee (I)_{A_P},\ \vee T_{A_P}\rangle$ and $T_{G_i} \le T_{A_P}, I_{G_i} \le I_{A_P},\ F_{G_i} \ge F_{A_P}$ for every $i \in J$, $\{\langle x,\ F_{G_i}, (I)_{G_i},\ T_{G_i}\rangle : i \in J\}$ is the family of PNCS's containing $\overline{(A_P)}$, that is,

$$Cl\left(\overline{A_P}\right) = \langle x, \wedge F_{G_i}, \vee (I)_{G_i}, \vee T_{G_i}\rangle \qquad (3.1.2)$$

Hence from equation (3.1.1) and (3.1.2) and $Cl\left(\overline{A_P}\right) = \overline{int(A_P)}$.

(b) Let $A_P = \langle x, T_{A_P}, I_{A_P}, F_{A_P}\rangle$ and suppose that the family of PNCS's containing A_P is given by $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i}\rangle : i \in J\}$. Then, $Cl(A_P) = \langle x, \wedge T_{G_i}, \wedge I_{G_i}, \vee F_{G_i}\rangle$ and hence,

$$\overline{Cl(A_P)} = \langle x, \vee F_{G_i}, \wedge (I)_{G_i}, \wedge T_{G_i} \rangle \tag{3.1.3}$$

Since $\overline{(A_P)} = \langle x, \wedge F_{A_P}, \vee (I)_{A_P}, \vee T_{A_P} \rangle$ and $T_{A_P} \leq T_{G_i}, I_{A_P} \leq I_{G_i}, F_{A_P} \geq F_{G_i}$, for each $i \in J$, $\{\langle x, F_{G_i}, (I)_{G_i}, T_{G_i} \rangle : i \in J\}$ is the family of PNOS's contained in $\overline{(A_P)}$, that is,

$$int\left(\overline{A_P}\right) = \langle x, \vee F_{G_i}, \wedge (I)_{G_i}, \wedge T_{G_i} \rangle \tag{3.1.4}$$

Hence, from equation (3.1.3) and (3.1.4), $int\left(\overline{A_P}\right) = \overline{Cl(A_P)}$.

Proposition 3.1.19

Let (X, τ) be a PNTS and $\tilde{A}_P, \tilde{B}_P$ be PNSs in X. Then the following properties holds

a) $int(\tilde{A}_P) \subseteq \tilde{A}_P$
b) $\tilde{A}_P \subseteq Cl(\tilde{A}_P)$
c) $\tilde{A}_P \subseteq \tilde{B}_P \Rightarrow int(\tilde{A}_P) \subseteq int(\tilde{B}_P)$
d) $\tilde{A}_P \subseteq \tilde{B}_P \Rightarrow Cl(\tilde{A}_P) \subseteq Cl(\tilde{B}_P)$
e) $int(int(\tilde{A}_P)) \subseteq int(\tilde{A}_P)$
f) $Cl\left(Cl(\tilde{A}_P)\right) \subseteq Cl(\tilde{A}_P)$
g) $int(\tilde{A}_P \cap \tilde{B}_P) \Rightarrow int(\tilde{A}_P) \cap int(\tilde{B}_P)$
h) $Cl(\tilde{A}_P \cup \tilde{B}_P) \Rightarrow Cl(\tilde{A}_P) \cup Cl(\tilde{B}_P)$
i) $int(\sqrt{2}_{PN}) = \sqrt{2}_{PN}$
j) $Cl(0_{PN}) = 0_{PN}$

Definition 3.1.20

Let $p^{x}_{(T,I,F)}$ be a PNP of a PNTS (X, τ). A PNS A of X is called a Pythagorean Neutrosophic Neighborhood (PNN) of $p^{x}_{(T,I,F)}$ if there is a PNOS B in X such that $p^{x}_{(T,I,F)} \in B \subseteq A$.

Theorem 3.1.21

Let (X,τ) be a PNTS. Then a PNS A of X is a PNOS iff A is an PNN of $p^{x}_{(T,I,F)}$ for every PNP $p^{x}_{(T,I,F)} \in A$.

Proof:

Let A be a PNOS of X. Clearly, A is a PNN of every $p^{x}_{(T,I,F)} \in A$. Conversely, consider that A is a PNN of every PNP that belongs to A.

Let $p^{x}_{(T,I,F)} \in A$. Since A is a PNN of $p^{x}_{(T,I,F)}$, there is a PNOS $B_{p^{x}_{(T,I,F)}}$ in X such that $p^{x}_{(T,I,F)} \in B_{p^{x}_{(T,I,F)}} \subseteq A$. So

$$A = \cup \{p^{x}_{(T,I,F)} : p^{x}_{(T,I,F)} \in A\} \subseteq \cup \left\{B_{p^{x}_{(T,I,F)}} : p^{x}_{(T,I,F)} \in A\right\} \subseteq A$$

Hence $A = \cup \left\{B_{p^{x}_{(T,I,F)}} : p^{x}_{(T,I,F)} \in A\right\}$. Since each $B_{p^{x}_{(T,I,F)}}$ is a PNOS, A is also a PNOS in X.

Neutrosophic Spherical Topology

Definition 3.1.22

Let X be a non empty set. Then τ is called a Neutrosophic Spherical Topology (NST) in the sense of **Chang** if it satisfies the following axioms:

(T1) $0_{NS}, 1_{NS} \in \tau$,

(T2) for any $S_1, S_2 \in \tau$, implies $S_1 \cap S_2 \in \tau$.

(T3) for any $\{S_n | n \in J\}$, implies $\bigcup_{n\in J} S_n \in \tau$.

the pair (X,τ) is called a Neutrosophic Spherical Topological Space (NSTS).

Example 3.1.23

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ and let A_S, B_S, C_S, D_S be NSSs.

$$A_S = \langle(\tilde{x}, 0.9,0.8,0.4), (\tilde{y}, 0.6,0.8,0.6), (\tilde{z}, 0.7,0.9,0.8)\rangle.$$
$$B_S = \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.9,0.8), (\tilde{z}, 0.6,0.9,0.6)\rangle.$$
$$C_S = \langle(\tilde{x}, 0.9,0.8,0.4), (\tilde{y}, 0.6,0.9,0.6), (\tilde{z}, 0.7,0.9,0.6)\rangle.$$
$$D_S = \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.8,0.8), (\tilde{z}, 0.6,0.9,0.8)\rangle.$$

$\tau = \{0_{NS}, 1_{NS}, A_S, B_S, C_S, D_S\}$. The collection τ is called a NST on X, the pair (X, τ) a NSTS.

Definition 3.1.24

Let X be a non-empty. Then τ is called a NST in the sense of **Lowen** if it satisfies the following axioms:

(T1) $A_S \in \delta$ for each $T_{A_S}, I_{A_S}, F_{A_S} \in \delta \subseteq [0,1]$ with

$0 \le {T_{A_S}}^2 + {I_{A_S}}^2 + {F_{A_S}}^2 \le \sqrt{3}$

(T2) for any $A_{S_1}, A_{S_2} \in \tau$, implies $A_{S_1} \cap A_{S_2} \in \tau$.

(T3) for any $\{A_{S_n} | n \in J\}$, implies $\bigcup_{n \in J} A_{S_n} \in \tau$.

the pair (X, τ) is called a NSTS.

Example 3.1.25

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ and let A_S, B_S, C_S, D_S be NSSs.

$A_S = \langle(\tilde{x}, 0.9,0.8,0.4), (\tilde{y}, 0.6,0.8,0.6), (\tilde{z}, 0.7,0.9,0.8)\rangle.$

$B_S = \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.9,0.8), (\tilde{z}, 0.6,0.9,0.6)\rangle.$

$C_S = \langle(\tilde{x}, 0.9,0.8,0.4), (\tilde{y}, 0.6,0.9,0.6), (\tilde{z}, 0.7,0.9,0.6)\rangle.$

$D_S = \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.8,0.8), (\tilde{z}, 0.6,0.9,0.8)\rangle.$

$\tau = \{A_S, B_S, C_S, D_S\}$. The collection τ is called a NST on X, the pair (X, τ) a NSTS.

Definition 3.1.26

The complement $C(A_S)$ of a NSS A_S in an NSTS (X, τ) is called a neutrosophic spherical closed set (NSCS) in X.

Proposition 3.1.27

Let (X, τ) be a NSTS on X. Then construct NST on X in the following way

$$\tau_1 = \{\bullet\, G : G \in \tau\}$$

where

$$\bullet\, G = \{\langle x, F_G(x), I_G(x), T_G(x)\rangle | x \in X\},$$

Proof:

(T1) $0_{NS}, 1_{NS} \in \tau_1$ is obvious.

(T2) Let $\bullet\, G_1, \bullet\, G_2 \in \tau_1$.

Since $G_1, G_2 \in \tau$ therefore $G_1 \cap G_2 = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle \in \tau$. This implies that

$$(\bullet\, G_1, \bullet\, G_2) = \langle x, F_{G1} \wedge F_{G2}, I_{G1} \wedge I_{G2}, T_{G1} \vee T_{G2}\rangle$$
$$= \langle x, F_{G1} \wedge F_{G2}, I_{G1} \wedge I_{G2}, T_{G1} \wedge T_{G2}\rangle \in \tau_1$$

(T3) Let $\{\bullet\, G_i, i \in J, G_i \in \tau\} \subseteq \tau_1$.

Since $\cup\, G_i = \left\langle x,\ \vee F_{G_i},\ \vee I_{G_i},\ \wedge T_{G_i}\right\rangle \in \tau$

$$\cup\, (\bullet\, G_i) = \left\langle x, \vee F_{G_i}, \vee I_{G_i}, \wedge T_{G_i}\right\rangle$$
$$= \left\langle x, \vee F_{G_i}, \vee I_{G_i}, \vee T_{G_i}\right\rangle \in \tau_1.$$

Definition 3.1.28

Let $(X, \tau_1), (X, \tau_2)$ be two NSTSs on X. Then τ_1 is said to be contained in τ_2 if $G \in \tau_2$ for each $G \in \tau_1$. In this case, τ_1 is coarser than τ_2.

Proposition 3.1.29

Let $\{\tau_i : i \in J\}$ be a family of NSTS on X. Then $\cap\, \tau_i$ is also an NST on X. Furthermore, $\cap\, \tau_i$ is the coarsest NST on X containing all τ_i.

i. $0_{NS} \in \tau_i$ for every $i \in J$. From this it follows that $0_{NS} \in \cap\, \tau_i$. Similarly, $1_{NS} \in \cap\, \tau_i$.
ii. Let $G_1, G_2 \in \cap\, \tau_i$. Then $G_1, G_2 \in \tau_i$, for every $i \in J$ and hence, $G_1 \cap G_2 \in \tau_i$, $\forall i \in J$. Thus, $G_1 \cap G_2 \in \cap\, \tau_i$.
iii. Let $\{G_j : j \in K\} \subseteq \cap\, \tau_i$. Then $\{G_j : j \in K\} \subseteq \tau_i$, for every $i \in J$ and hence, $\bigcup_{j \in K} G_j \in \tau_i \forall i \in J$. Thus, $\bigcup_{j \in K} G_j \in \cap\, \tau_i$.

Clearly, it is the coarsest topology on X containing all τ_i's. Since if τ' is any other NST on X which contains every τ_i, then obviously it will also contain $\cap\, \tau_i$.

Definition 3.1.30

Let $\alpha, \beta, \gamma \in (0,1)$ and $\alpha^2 + \beta^2 + \gamma^2 \leq \sqrt{3}$. A neutrosophic spherical point (NSP) $p^x_{(\alpha,\beta,\gamma)}$ of X is a NSS of X defined by $p^x_{(\alpha,\beta,\gamma)} = \langle x, T_p, I_p, F_p \rangle$, where for $x \in X$.

$$T_p(y) = \begin{cases} \alpha & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}, \; I_p(y) = \begin{cases} \beta & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}, \; F_p(y) = \begin{cases} \gamma & \text{if } y = x \\ 1 & \text{if } y \neq x \end{cases}$$

In this case, x is called the support of $p^x_{(\alpha,\beta,\gamma)}$. A NSP $p^x_{(\alpha,\beta,\gamma)}$ is said to belong to a NSS $A = \langle x, T_A, I_A, F_A \rangle$ of X, denoted by $p^x_{(\alpha,\beta,\gamma)} \in A$, if $\alpha \leq T_A(x), \beta \leq I_A(x)$ and $\gamma \geq F_A(x)$.

Proposition 3.1.31

Every NSS A_S in X can be expressed as the union of all PNP contained in A_S

Definition 3.1.32

A collection B of NSS on a set X is said to be basis (or base) for a NST on X, if

i. For every $p^x_{(\alpha,\beta,\gamma)}$ in X, there exists $B \in B$ such that $p^x_{(\alpha,\beta,\gamma)} \in B$.

ii. If $p^x_{(\alpha,\beta,\gamma)} \in B_1 \cap B_2$, where $B_1, B_2 \in B$ then there exist $B_3 \in B$ such that $p^x_{(\alpha,\beta,\gamma)} \in B_3 \subseteq B_1 \cap B_2$.

Proposition 3.1.33

Let $\mathfrak{B}$ be a basis for a NST on X. Let τ contains those NSS G of X for which corresponding to each $p^x_{(\alpha,\beta,\gamma)} \in G$, there exist $B \in \mathfrak{B}$ such that $p^x_{(\alpha,\beta,\gamma)} \in B \subseteq G$. Then τ is an NST on X.

Proposition 3.1.34

Let τ be an NST on a set X, generated by a basis $\mathfrak{B}$. Then members of τ are precisely the union of members of B, that is, $G \in \tau$ iff $G = \bigcup_{\alpha \in A} B_\alpha$, where $B_\alpha \in \mathfrak{B}$, $\forall \alpha \in A$.

Definition 3.1.35

Let (X, τ) be a NSTS. Then a subfamily $B_s \subseteq \tau$ is called a sub basis for τ if the family of finite intersections of members of B_s forms a base for τ.

Definition 3.1.36

Let (X, τ) be a NSTS and $A_S = \{\langle x: T_S(x), I_S(x), F_S(x)\rangle / x \in X\}$ be an NSS in X. Then the neutrosophic spherical closure of A_S are defined by

$$Cl(A_S) = \cap \{K: K \text{ is a NSCS in } X \text{ and } A_S \subseteq K\}.$$

Definition 3.1.37

Let (X, τ) be a NSTS and $A_S = \{\langle x: T_S(x), I_S(x), F_S(x)\rangle / x \in X\}$ be an NSS in X. Then the neutrosophic spherical interior of A_S are defined by

$$int(A_S) = \cup \{G: G \text{ is a NSOS in } X \text{ and } G \subseteq A_S\}.$$

Note that $Cl(A_S)$ is a neutrosophic spherical closed sets (NSCSs) and $int(A_S)$ is a neutrosophic spherical open Sets (NSOSs) in X.

Further,

(a) A_S is a NSCS in X iff $Cl(A_S) = A_S$;

(b) A_S is a NSOS in X iff $int(A_S) = A_S$.

Example 3.1.38

Let $X = \{\tilde{x}, \tilde{y}, \tilde{z}\}$ and let A_S, B_S, C_S, D_S be NSSs.

$$A_S = \langle(\tilde{x}, 0.9, 0.8, 0.4), (\tilde{y}, 0.6, 0.8, 0.6), (\tilde{z}, 0.7, 0.9, 0.8)\rangle.$$

$$B_S = \langle(\tilde{x}, 0.5, 0.3, 0.9), (\tilde{y}, 0.3, 0.2, 0.8), (\tilde{z}, 0.6, 0.4, 0.7)\rangle.$$

$$C_S = \langle(\tilde{x}, 0.9, 0.8, 0.4), (\tilde{y}, 0.6, 0.9, 0.6), (\tilde{z}, 0.7, 0.9, 0.6)\rangle.$$

$$D_S = \langle(\tilde{x}, 0.5, 0.7, 0.9), (\tilde{y}, 0.3, 0.8, 0.8), (\tilde{z}, 0.6, 0.6, 0.6)\rangle.$$

$$\tau = \{0_{NS}, 1_{NS}, A_S, B_S, C_S, D_S\}.$$

If

$$P_S = \langle(\tilde{x}, 0.8, 0.7, 0.6), (\tilde{y}, 0.6, 0.8, 0.3), (\tilde{z}, 0.7, 0.6, 0.6)\rangle.$$

$int(P_S) = \cup \{G: G \text{ is a PNOS in } X \text{ and } G \subseteq P_S\}$

$$= max\begin{Bmatrix}\langle(\tilde{x}, 0.5,0.3,0.9), (\tilde{y}, 0.3,0.2,0.8), (\tilde{z}, 0.6,0.4,0.7)\rangle, \\ \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.8,0.8), (\tilde{z}, 0.6,0.6,0.6)\rangle\end{Bmatrix}$$

$$= \langle(\tilde{x}, 0.5,0.7,0.9), (\tilde{y}, 0.3,0.8,0.8), (\tilde{z}, 0.6,0.6,0.6)\rangle = D_S$$

$Cl(P_S) = \cap \{K: K$ is a PNCS in X and $P_S \subseteq K\}$

$$= min\{\langle(\tilde{x}, \xi, \psi, 0), (\tilde{y}, \xi, \psi, 0), (\tilde{z}, \xi, \psi, 0)\rangle\} = \sqrt{2}_{PN}$$

Proposition 3.1.39

For any NSS A_S in (X, τ),

(a) $Cl(\overline{A_S}) = \overline{int(A_S)}$

(b) $int(\overline{A_S}) = \overline{Cl(A_S)}$

Proof: (a) Let $A_S = \langle x, T_{A_P}, I_{A_P}, F_{A_P}\rangle$ and suppose that the NSOS's contained in A_S are indexed by the family $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i}\rangle : i \in J\}$. Then, $int(A_S) = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge F_{G_i}\rangle$ and hence

$$\overline{int(A_S)} = \langle x, \wedge F_{G_i}, \vee I_{G_i}, \vee T_{G_i}\rangle \tag{3.1.5}$$

Since $\overline{(A_S)} = \langle x, \wedge F_{A_P}, \vee (I)_{A_P}, \vee T_{A_P}\rangle$ and $T_{G_i} \leq T_{A_P}, I_{G_i} \leq I_{A_P}, F_{G_i} \geq F_{A_P}$, for every $i \in J$, $\{\langle x, F_{G_i}, (I)_{G_i}, T_{G_i}\rangle : i \in J\}$ is the family of NSCS's containing $\overline{(A_S)}$, that is,

$$Cl(\overline{A_S}) = \langle x, \wedge F_{G_i}, \vee (I)_{G_i}, \vee T_{G_i}\rangle \tag{3.1.6}$$

Hence from equation (3.1.5) and (3.1.6) and $Cl(\overline{A_S}) = \overline{int(A_S)}$.

(b) Let $A_S = \langle x, T_{A_S}, I_{A_S}, F_{A_S}\rangle$ and suppose that the family of NSCS's containing A_S is given by $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i}\rangle : i \in J\}$. Then, $Cl(A_S) = \langle x, \wedge T_{G_i}, \wedge I_{G_i}, \vee F_{G_i}\rangle$ and hence,

$$\overline{Cl(A_S)} = \langle x, \vee F_{G_i}, \wedge (I)_{G_i}, \wedge T_{G_i}\rangle \tag{3.1.7}$$

Since $\overline{(A_S)} = \langle x, \wedge F_{A_S}, \vee (I_{A_S}), \vee T_{A_S} \rangle$ and $T_{A_S} \le T_{G_i}, I_{A_S} \le I_{G_i}, F_{A_S} \ge F_{G_i}$, for each $i \in J$, $\{\langle x, F_{G_i}, (I)_{G_i}, T_{G_i} \rangle : i \in J\}$is the family of PNOS's contained in $\overline{(A_S)}$, that is,

$$int(\overline{A_S}) = \langle x, \vee F_{G_i}, \wedge (I)_{G_i}, \wedge T_{G_i} \rangle \tag{3.1.8}$$

Hence, from equation (3.1.7) and (3.1.8) and $int(\overline{A_S}) = \overline{Cl(A_S)}$.

Proposition 3.1.40

Let (X, τ) be a NSTS and $\tilde{A}, \tilde{B}$ be NSSs in X. Then the following properties holds

a) $int(\tilde{A}_S) \subseteq \tilde{A}_S$
b) $\tilde{A}_S \subseteq Cl(\tilde{A}_S)$
c) $\tilde{A}_S \subseteq \tilde{B}_S \Rightarrow int(\tilde{A}_S) \subseteq int(\tilde{B}_S)$
d) $\tilde{A}_S \subseteq \tilde{B}_S \Rightarrow Cl(\tilde{A}_S) \subseteq Cl(\tilde{B}_S)$
e) $int(int(\tilde{A}_S)) \subseteq int(\tilde{A}_S)$
f) $Cl\left(Cl(\tilde{A}_S)\right) \subseteq Cl(\tilde{A}_S)$
g) $int(\tilde{A}_S \cap \tilde{B}_S) \Rightarrow int(\tilde{A}_S) \cap int(\tilde{B}_S)$
h) $Cl(\tilde{A}_S \cup \tilde{B}_S) \Rightarrow Cl(\tilde{A}_S) \cup Cl(\tilde{B}_S)$
i) $int(1_{NS}) = 1_{NS}$
j) $Cl(0_{NS}) = 0_{NS}$

Definition 3.1.41

Let $p^x_{(T,I,F)}$ be a neutrosophic spherical point of a NSTS (X, τ). A NSS A of X is called an neutrosophic spherical neighbourhood (NSN) of $p^x_{(T,I,F)}$ if there is a NSOS B in X such that $p^x_{(T,I,F)} \in B \subseteq A$.

Theorem 3.1.42

Let (X, τ) be a NSTS. Then a NSS A of X is a NSOS if and only if A is an NSN of $p^x_{(T,I,F)}$ for every NSP $p^x_{(T,I,F)} \in A$.

Proof:

Let A be a NSOS of X. Clearly, A is a NSN of every $p^{x}_{(T,I,F)} \in A$. Conversely, suppose that A is a NSN of every NSP belonging to A. Let $p^{x}_{(T,I,F)} \in A$. Since A is a NSN of $p^{x}_{(T,I,F)}$, there is a NSOS $B_{p^{x}_{(T,I,F)}}$ in X such that $p^{x}_{(T,I,F)} \in B_{p^{x}_{(T,I,F)}} \subseteq A$.

$$A = \cup \{p^{x}_{(T,I,F)} : p^{x}_{(T,I,F)} \in A\} \subseteq \cup \left\{B_{p^{x}_{(T,I,F)}} : p^{x}_{(T,I,F)} \in A\right\} \subseteq A$$

hence $A = \cup \left\{B_{p^{x}_{(T,I,F)}} : p^{x}_{(T,I,F)} \in A\right\}$.

Since each $B_{p^{x}_{(T,I,F)}}$ is an NSOS, A is also a NSOS in X.

Fermatean Neutrosophic Topology

Definition 3.1.43

Let X be a non empty set. Then τ is called a Fermatean Neutrosophic Topology (FNT) in the sense of Chang if it satisfies the following axioms:

(T1) $0_{FN}, 1_{FN} \in \tau$

(T2) for any $F_1, F_2 \in \tau$, implies $F_1 \cap F_2 \in \tau$.

(T3) for any $\{F_n | n \in I\}$, implies $\bigcup_{n \in I} F_n \in \tau$.

the pair (X, τ) is called a Fermatean neutrosophic topological space (FNTS).

Example 3.1.44

Let $X = \{\tilde{x}, \tilde{y}, z\}$ and let A_F, B_F, C_F, and D_F are FNSs.

$A_F = \langle (\tilde{x}, 0.85, 0.9, 0.7), (\tilde{y}, 0.9, 0.8, 0.6), (\tilde{z}, 0.5, 0.8, 0.9) \rangle$.

$B_F = \langle (\tilde{x}, 0.8, 0.9, 0.7), (\tilde{y}, 0.6, 0.9, 0.8), (\tilde{z}, 0.9, 0.7, 0.6) \rangle$.

$C_F = \langle (\tilde{x}, 0.85, 0.9, 0.7), (\tilde{y}, 0.9, 0.9, 0.6), (\tilde{z}, 0.9, 0.8, 0.6) \rangle$.

$D_F = \langle (\tilde{x}, 0.8, 0.9, 0.7), (\tilde{y}, 0.6, 0.8, 0.8), (\tilde{z}, 0.5, 0.7, 0.9) \rangle$.

$\tau = \{0_{FN}, 1_{FN}, A_F, B_F, C_F, D_F\}$. The collection τ is called a FNT on X, the pair (X, τ) a FNTS.

Definition 3.1.45

Let X be a non empty set and $\tau \subset FNTS$. Then τ is called a FNT in the sense of Lowen if it satisfies the following axioms:

(T1) $A_F \in \delta$, for each $T_{A_F}, I_{A_F}, F_{A_F} \in \delta \subseteq [0,1]$ with

$$T_{A_F}{}^3(x) + I_{A_F}{}^3(x) + F_{A_F}{}^3(x) \leq 2$$

(T2) for any $A_{F_1}, A_{F_2} \in \tau$, implies $A_{P_1} \cap A_{P_2} \in \tau$.

(T3) for any $\{A_{F_n} | n \in I\}$, implies $\bigcup_{n \in I} A_{F_n} \in \tau$.

In this case the pair (X, τ) is called a FNTS.

Example 3.1.46

Let $X = \{\tilde{x}, \tilde{y}, z\}$ and let A_F, B_F, C_F, and D_F are FNSs.

$A_F = \langle(\tilde{x}, 0.85,0.9,0.7), (\tilde{y}, 0.9,0.8,0.6), (\tilde{z}, 0.5,0.8,0.9)\rangle.$

$B_F = \langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.6,0.9,0.8), (\tilde{z}, 0.9,0.7,0.6)\rangle.$

$C_F = \langle(\tilde{x}, 0.85,0.9,0.7), (\tilde{y}, 0.9,0.9,0.6), (\tilde{z}, 0.9,0.8,0.6)\rangle.$

$D_F = \langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.6,0.8,0.8), (\tilde{z}, 0.5,0.7,0.9)\rangle.$

$\tau = \{A_F, B_F, C_F, D_F\}$. The collection τ is called a FNT on X, the pair (X, τ) a FNTS.

Definition 3.1.47

The complement $C(A_F)$ of a Fermatean neutrosophic open set (FNOS) A_F in a FNTS (X, τ) is called a Fermatean neutrosophic closed set (NSCS) in$X$.

Proposition 3.1.48

Let (X, τ) be a FNTS on X . Then construct several FNT on X in the following way

a) $\tau_1 = \{\bullet\, G : G \in \tau\}$

b) $\tau_2 = \{\circ\, G : G \in \tau\}$

where

$$\bullet\, G = \{\langle x, T_G(x), I_G(x), 1 - T_G(x)\rangle | x \in X\},$$

$$\circ G = \{\langle x, 1 - F_G(x), I_G(x), F_G(x)\rangle | x \in X\}$$

Proof:

(a) (T_1) $0_{FN}, 1_{FN} \in \tau_1$ is obvious.

(T_2) Let $\bullet G_1, \bullet G_2 \in \tau_1$. Since $G_1, G_2 \in \tau$ therefore

$$G_1 \cap G_2 = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle \in \tau.$$

This implies that

$$(\bullet G_1, \bullet G_2) = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, (1 - T_{G1}) \vee (1 - T_{G2})\rangle$$

$$= \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, 1 - (T_{G1} \wedge T_{G2})\rangle \in \tau_1$$

(T_3) Let $\{\bullet G_i, i \in J, G_i \in \tau\} \subseteq \tau_1$. Since $\cup G_i = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge F_{G_i}\rangle \in \tau$

$$\cup (\bullet G_i) = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge (1 - T_{G_i})\rangle$$

$$= \langle x, \vee T_{G_i}, \vee I_{G_i}, 1 - (\vee T_{G_i})\rangle \in \tau_1.$$

(b) (T_1) $0_{FN}, 1_{FNS} \in \tau_1$ is obvious.

(T_2) Let $\circ G_1, \circ G_2 \in \tau_2$. Since $G_1, G_2 \in \tau$ therefore

$$G_1 \cap G_2 = \langle x, T_{G1} \wedge T_{G2}, I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle \in \tau.$$

This implies that

$$(\circ G_1, \circ G_2) = \langle x, (1 - F_{G1}) \wedge (1 - F_{G2}), I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle$$

$$= \langle x, 1 - (F_{G1} \vee F_{G2}), I_{G1} \wedge I_{G2}, F_{G1} \vee F_{G2}\rangle \in \tau_2$$

(T_3) Let $\{\circ G_i, i \in J, G_i \in \tau\} \subseteq \tau_2$. Since $\cup G_i = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge F_{G_i}\rangle \in \tau$

$$\cup (\circ G_i) = \langle x, \vee (1 - F_{G_i}), \vee I_{G_i}, \wedge F_{G_i}\rangle$$

$$= \langle x, 1 - (\wedge F_{G_i}), \vee I_{G_i}, \wedge F_{G_i}\rangle \in \tau_1.$$

Definition 3.1.49

Let $(X, \tau_1), (X, \tau_2)$ be two FNTSs on X. Then τ_1 is said to be contained in τ_2 if $G \in \tau_2$ for each $G \in \tau_1$. In this case, τ_1 is coarser than τ_2.

Proposition 3.1.50

Let $\{\tau_i : i \in J\}$ be a family of FNTS on X. Then $\cap \tau_i$ is also a FNT on X. Furthermore, $\cap \tau_i$ is the coarsest FNT on X containing all τ_i's.

i. $0_{FN} \in \tau_i$ for every $i \in J$.From this it follows that $0_{FN} \in \cap \tau_i$. Similarly, $1_{FN} \in \cap \tau_i$.

ii. Let $G_1, G_2 \in \cap \tau_i$. Then $G_1, G_2 \in \tau_i$, for every $i \in J$ and hence, $G_1 \cap G_2 \in \tau_i, \forall i \in J$. Thus, $G_1 \cap G_2 \in \cap \tau_i$.

iii. Let $\{G_j : j \in K\} \subseteq \cap \tau_i$. Then $\{G_j : j \in K\} \subseteq \tau_i$, for every $i \in J$ and hence, $\bigcup_{j \in K} G_j \in \tau_i \forall i \in J$ .Thus, $\bigcup_{j \in K} G_j \in \cap \tau_i$.

Clearly, it is the coarsest topology on X containing all τ_i's. Since if τ' is any other FNT on X which contains every τ_i then obviously it will also contain $\cap \tau_i$.

Definition 3.1.51

Let $\alpha, \beta, \gamma \in (0,1)$ and $\alpha^3 + \beta^3 + \gamma^3 \leq 2$. A Fermatean neutrosophic point (FNP) $p^x_{(\alpha,\beta,\gamma)}$ of X is a FNS of X defined by

$p^x_{(\alpha,\beta,\gamma)} = \langle x, T_p, I_p, F_p \rangle$, where for $x \in X$.

$$T_p(y) = \begin{cases} \alpha \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases},$$

$$I_p(y) = \begin{cases} \beta \text{ if } y = x \\ 0 \text{ if } y \neq x \end{cases},$$

$$F_p(y) = \begin{cases} \gamma \text{ if } y = x \\ 1 \text{ if } y \neq x \end{cases}$$

In this case, x is called the support of $p^x_{(\alpha,\beta,\gamma)}$.

A FNP $p^x_{(\alpha,\beta,\gamma)}$ is said to belong to a FNS $A = \langle x, T_A, I_A, F_A \rangle$ of X, denoted by $p^x_{(\alpha,\beta,\gamma)} \in A$, if $\alpha \leq T_A(x), \beta \leq I_A(x)$ and $\gamma \geq F_A(x)$.

Proposition 3.1.52

Every FNS A_F in X can be expressed as the union of all PNP contained in A_F.

Definition 3.1.53

A collection $\mathfrak{B}$ of FNS on a set X is said to be basis (or base) for a FNT on X, if

i. For every $p^x_{(\alpha,\beta,\gamma)}$ in X, there exists $B \in \mathfrak{B}$ such that $p^x_{(\alpha,\beta,\gamma)} \in B$.

ii. If $p^{x}_{(\alpha,\beta,\gamma)} \in B_1 \cap B_2$, where $B_1, B_2 \in \mathfrak{B}$ then there exist $B_3 \in \mathfrak{B}$ such that $p^{x}_{(\alpha,\beta,\gamma)} \in B_3 \subseteq B_1 \cap B_2$.

Proposition 3.1.54

Let $\mathfrak{B}$ be a basis for a FNT on X. Let τ contains those FNS G of X for which corresponding to each $p^{x}_{(\alpha,\beta,\gamma)} \in G$, there exist $B \in \mathfrak{B}$ such that $p^{x}_{(\alpha,\beta,\gamma)} \in B \subseteq G$. Then τ is a FNT on X.

Proposition 3.1.55

Let τ be a FNT on a set X, generated by a basis $\mathfrak{B}$. Then members of τ are precisely the union of members of $\mathfrak{B}$, that is, $G \in \tau$ if and only if $G = \bigcup_{\alpha \in A} B_\alpha$, where $B_\alpha \in \mathfrak{B}, \forall \alpha \in A$.

Definition 3.1.56

Let (X, τ) be a FNTS. Then a subfamily $B_S \subseteq \tau$ is called a sub basis for τ if the family of finite intersections of members of B_S forms a base for τ.

Definition 3.1.57

The complement $C(A_F)$ of a FNOS A_F in a FNTS (X, τ) is called a Fermatean neutrosophic closed set (FNCS) in X.

$$C(A_F) = \{\langle x, F_{A_F}(x), 1 - I_{A_F}(x), T_{A_F}(x)\rangle | x \in X\}$$

Definition 3.1.58

Let (X, τ) be a FNTS and $A_F = \{\langle x: T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)\rangle / x \in X\}$ be an FNS in X. Then the Fermatean neutrosophic closure of A_F are defined by

$$Cl(A_F) = \cap \{K: K \text{ is a FNCS in } X \text{ and } A_F \subseteq K\}.$$

Definition 3.1.59

Let (X, τ) be a FNTS and $A_F = \{\langle x: T_{A_F}(x), I_{A_F}(x), F_{A_F}(x)\rangle / x \in X\}$ be an FNS in X. Then the Fermatean neutrosophic interior of A_F are defined by

$$int(A_F) = \cup \{G: G\ is\ a\ FNOS\ in\ X\ and\ G \subseteq A_F\}.$$

Note that $Cl(A_F)$ is a FNCSs and $int(A_F)$is a FNOSs in X.

Further,

(a) A_F is a FNCS in X iff $Cl(A_F) = A_F$.

(b) A_F is a FNOS in X iff $int(A_F) = A_F$.

Example 3.1.60

Let $X = \{\tilde{x}, \tilde{y}, z\}$ and let A_F, B_F, C_F, and D_F are FNSs.

$$A_F = \langle(\tilde{x}, 0.85,0.9,0.7), (\tilde{y}, 0.9,0.8,0.6), (\tilde{z}, 0.5,0.8,0.9)\rangle.$$

$$B_F = \langle(\tilde{x}, 0.8,0.9,0.6), (\tilde{y}, 0.6,0.9,0.8), (\tilde{z}, 0.9,0.7,0.6)\rangle.$$

$$C_F = \langle(\tilde{x}, 0.85,0.9,0.6), (\tilde{y}, 0.9,0.9,0.6), (\tilde{z}, 0.9,0.8,0.6)\rangle.$$

$$D_F = \langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.6,0.8,0.8), (\tilde{z}, 0.5,0.7,0.9)\rangle.$$

$$\tau = \{0_{FN}, 1_{FN}, A_F, B_F, C_F, D_F\}$$

If $P_F = \langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.8,0.8,0.7), (\tilde{z}, 0.5,0.8,0.5)\rangle$.

$$\begin{aligned} int(A_F) &= \cup \{G: G \text{ is } a \text{ FNOS in } X \text{ and } G \subseteq P_F\} \\ &= max\{\langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.6,0.8,0.8), (\tilde{z}, 0.5,0.7,0.9)\rangle.\} \\ &= \langle(\tilde{x}, 0.8,0.9,0.7), (\tilde{y}, 0.6,0.8,0.8), (\tilde{z}, 0.5,0.7,0.9)\rangle = D_F \end{aligned}$$

$$\begin{aligned} Cl(A_P) &= \cap \{K: K \text{ is } a \text{ PNCS in } X \text{ and } P_F \subseteq K\} \\ &= min\{\langle(\tilde{x}, 1,1,0), (\tilde{y}, 1,1,0), (\tilde{z}, 1,1,0)\rangle\} = 1_{FN}. \end{aligned}$$

Proposition 3.1.61

For any FNS A_F in (X, τ),

(a) $Cl(\overline{A_F}) = \overline{int(A_F)}$

(b) $int(\overline{A_F}) = \overline{Cl(A_F)}$

Proof:

(a) Let $A_F = \langle x, T_{A_F}, I_{A_F}, F_{A_F}\rangle$ and suppose that the FNOS's contained in A_F are indexed by the family $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i}\rangle: i \in J\}$. Then, $int(A_F) = \langle x, \vee T_{G_i}, \vee I_{G_i}, \wedge F_{G_i}\rangle$ and hence

$$\overline{int(A_F)} = \langle x, \wedge F_{G_i}, \vee (1-I)_{G_i}, \vee T_{G_i} \rangle \quad (3.19)$$

Since $\overline{(A_F)} = \langle x, \wedge F_{A_F}, \vee (1-I)_{A_F}, \vee T_{A_F} \rangle$ and $T_{G_i} \le T_{A_P}, I_{G_i} \le I_{A_P}, F_{G_i} \ge F_{A_P}$, for every $i \in J$, $\{\langle x, F_{G_i}, (1-I)_{G_i}, T_{G_i} \rangle : i \in J\}$ is the family of FNCS's containing $\overline{(A_F)}$, that is,

$$Cl(\overline{A_F}) = \langle x, \wedge F_{G_i}, \vee (1-I)_{G_i}, \vee T_{G_i} \rangle \quad (3.1.10)$$

Hence from equation (3.1.9) and (3.1.10) and $Cl(\overline{A_F}) = \overline{int(A_F)}$.

(b) Let $A_F = \langle x, T_{A_F}, I_{A_F}, F_{A_F} \rangle$ and suppose that the family of FNCS's containing A_F is given by $\{\langle x, T_{G_i}, I_{G_i}, F_{G_i} \rangle : i \in J\}$. Then, $Cl(A_F) = \langle x, \wedge T_{G_i}, \wedge I_{G_i}, \vee F_{G_i} \rangle$ and hence,

$$\overline{Cl(A_F)} = \langle x, \vee F_{G_i}, \wedge (1-I)_{G_i}, \wedge T_{G_i} \rangle \quad (3.1.11)$$

Since $\overline{(A_F)} = \langle x, \wedge F_{A_F}, \vee (1-I)_{A_F}, \vee T_{A_F} \rangle$ and $T_{A_F} \le T_{G_i}, I_{A_F} \le I_{G_i}, F_{A_F} \ge F_{G_i}$, for each $i \in J$, $\{\langle x, F_{G_i}, (1-I)_{G_i}, T_{G_i} \rangle : i \in J\}$is the family of FNOS's contained in $\overline{(A_F)}$, that is,

$$int(\overline{A_F}) = \langle x, \vee F_{G_i}, \wedge (1-I)_{G_i}, \wedge T_{G_i} \rangle \quad (3.1.12)$$

Hence, from equation (3.1.11) and (3.1.12) and $int(\overline{A_F}) = \overline{Cl(A_F)}$.

Proposition 3.1.62

Let (X, τ) be a FNTS and $\tilde{A}, \tilde{B}$ be FNSs in X. Then the following properties holds

a) $int(\tilde{A}_F) \subseteq \tilde{A}_F$
b) $\tilde{A}_F \subseteq Cl(\tilde{A}_F)$
c) $\tilde{A}_F \subseteq \tilde{B}_F \Rightarrow int(\tilde{A}_F) \subseteq int(\tilde{B}_F)$
d) $\tilde{A}_F \subseteq \tilde{B}_F \Rightarrow Cl(\tilde{A}_F) \subseteq Cl(\tilde{B}_F)$
e) $int(int(\tilde{A}_F)) \subseteq int(\tilde{A}_F)$

f) $Cl\left(Cl(\tilde{A}_F)\right) \subseteq Cl(\tilde{A}_F)$

g) $int(\tilde{A}_F \cap \tilde{B}_F) \Rightarrow int(\tilde{A}_F) \cap int(\tilde{B}_F)$

h) $Cl(\tilde{A}_F \cup \tilde{B}_F) \Rightarrow Cl(\tilde{A}_F) \cup Cl(\tilde{B}_F)$

i) $int(1_{FN}) = 1_{FN}$

j) $Cl(0_{FN}) = 0_{FN}$

Definition 3.1.63

Let $p^x_{(T,I,F)}$ be a Fermatean neutrosophic point of a FNTS (X, τ). A FNS A of X is called a Fermatean neutrosophic neighborhood (FNN) of $p^x_{(T,I,F)}$ if there is a FNOS B in X such that $p^x_{(T,I,F)} \in B \subseteq A$.

Theorem 3.1.64

Let (X, τ) be a FNTS. Then a FNS A of X is a FNOS iff A is an FNN of $p^x_{(T,I,F)}$ for every FNP $p^x_{(T,I,F)} \in A$.

Proof:

Let A be a FNOS of X. Clearly, A is a FNN of every $p^x_{(T,I,F)} \in A$. Conversely, suppose that A is a FNN of every FNP belonging to A. Let $p^x_{(T,I,F)} \in A$.

Since A is a FNN of $p^x_{(T,I,F)}$, there is a FNOS $B_{p^x_{(T,I,F)}}$ in X such that $p^x_{(T,I,F)} \in B_{p^x_{(T,I,F)}} \subseteq A$.

And $A = \cup \{p^x_{(T,I,F)} : p^x_{(T,I,F)} \in A\} \subseteq \cup \left\{B_{p^x_{(T,I,F)}} : p^x_{(T,I,F)} \in A\right\} \subseteq A$ and hence $A = \cup \left\{B_{p^x_{(T,I,F)}} : p^x_{(T,I,F)} \in A\right\}$.

Since each $B_{p^x_{(T,I,F)}}$ is a FNOS, A is also a FNOS in X.

3.2 Fermatean Neutrosophic gradation of openness

This section, defines Fermatean neutrosophic gradation of openness, subspace, gradation-preserving maps, base and subbase, Fermatean product neutrosophic topological space, Fermatean neutrosophic compactness, and Tychonoff's theorem.

Here let X be a nonempty set, I to denote $[0,1]$, I_0 to denote $(0,1]$, I_1 to denote $[0,1)$, and I^X to be the set of all Fermatean neutrosophic subsets defined on X. The family of all FNSs in X will be represented by $FN(X)$. Fermatean neutrosophic subsets will be represented by symbols such as $\lambda, \mu, \nu, \eta, \chi$, and so on throughout this section.

Let X be a FNS. Let Y be a subset of X and $\mu \in I^X$; the restriction of μ on Y is denoted by μ/Y. For each $\mu \in I^Y$ the extension of μ on X, denoted by μ_X, is defined by

$$\mu_X(x) = \begin{cases} \mu(x) & \text{if} \quad x \in Y \\ 0_{FN} & \text{if} \quad x \in X - Y \end{cases}$$

Definition 3.2.1

Let X be a non empty set. A Fermatean Neutrosophic Gradation of Openness (FNGO) of Fermatean neutrosophic subsets of X, is a triplet (τ^T, τ^I, τ^F) of functions $\tau^T, \tau^I, \tau^F : I^X \to I$ such that

(FNGO1) $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2, \forall \lambda \in I^X$

(FNGO2) $\tau^T(0_{FN}) = \tau^T(1_{FN}) = 1, \tau^I(0_{FN}) = \tau^I(1_{FN}) = 1,$
$\tau^F(0_{FN}) = \tau^F(1) = 0$

(FNGO3) $\tau^T(\lambda_1 \cap \lambda_2) \geq \tau^T(\lambda_1) \wedge \tau^T(\lambda_2),$
$\tau^I(\lambda_1 \cap \lambda_2) \geq \tau^I(\lambda_1) \wedge \tau^I(\lambda_2),$
$\tau^F(\lambda_1 \cap \lambda_2) \leq \tau^F(\lambda_1) \vee \tau^F(\lambda_2), \lambda_i \in I^X, i = 1,2$

(FNGO4) $\tau^T(\cup_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} \tau^T(\lambda_i),$
$\tau^I(\cup_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} \tau^I(\lambda_i)$
$\tau^F(\cup_{i\in\Delta} \lambda_i) \leq \vee_{i\in\Delta} \tau^F(\lambda_i), \lambda_i \in I^X, i \in \Delta.$

Where τ^T, τ^I are independent components and τ^F is dependent component with respect to τ^T. τ^T represent the degree of openness, τ^I is the indeterminacy of openness, and τ^F

non-openness respectively. The triplet (τ^T, τ^I, τ^F) is a FNGO X. Then the collections $(X, \tau^T, \tau^I, \tau^F)$ is known as FNTS.

Definition 3.2.3

Let X be a non empty set and $F, F^*: I^X \to I$ be two mappings satisfying

(FNGC1) $\left(F^T(\lambda)\right)^3 + \left(F^I(\lambda)\right)^3 + \left(F^I(\lambda)\right)^3 \leq 2, \forall \lambda \in I^X$

(FNGC2) $F^T(0_{FN}) = F^T(1_{FN}) = 1, F^I(0_{FN}) = F^I(1_{FN}) = 1,$

$F^F(0_{FN}) = F^F(1_{FN}) = 0$

(FNGC3) $F^T(\lambda_1 \cup \lambda_2) \geq F^T(\lambda_1) \wedge F^T(\lambda_2),$

$F^I(\lambda_1 \cup \lambda_2) \geq F^I(\lambda_1) \wedge F^I(\lambda_2)$

$F^F(\lambda_1 \cup \lambda_2) \leq F^F(\lambda_1) \vee F^F(\lambda_2), \lambda_i \in I^X, i = 1,2$

(FNGC4) $F^T(\bigcap_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} F^T(\lambda_i),$

$F^I(\bigcap_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} F^I(\lambda_i),$

$F^F(\bigcap_{i\in\Delta} \lambda_i) \leq \vee_{i\in\Delta} F^F(\lambda_i), \lambda_i \in I^X, i \in \Delta.$

Then the triplet (F^T, F^I, F^F) is a Fermatean neutrosophic gradation of closedness (FNGC) on X.

Example 3.2.4

Let $X = R$, the set of all real numbers, T be the topology strictly finer than the usual topology generated by T_0. Let T be the topology generated by Ƀ= $\{(p, q]; p, q \in R, p < q\}$ as a subbase and T_0 be the family of all open sets of R with respect to the usual topology of R. Define τ^T, τ^I, τ^F on crisp FNS χ_A, where $A \subseteq R$. Define $\tau^T, \tau^I, \tau^F: I^X \to I$ by

$$\tau^T(\chi_A) = \begin{cases} 1 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 1 & \text{if} \quad A \in T_0 \\ 0.6 & \text{if} \quad A \in T \backslash T_0 \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

$$\tau^I(\chi_A) = \begin{cases} 1 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 0.6 & \text{if} \quad A \in T_0 \\ 0.4 & \text{if} \quad A \in T \backslash T_0 \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

and

$$\tau^F(\chi_A) = \begin{cases} 0 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 0.2 & \text{if} \quad A \in T_0 \\ 0.3 & \text{if} \quad A \in T \setminus T_0 \\ 1 & \text{otherwise} \end{cases}$$

Then (τ^T, τ^I, τ^F) is an FNGO on X.

Example 3.2.5

Let $X = \{R, G, Y\}$ be the set. X_{FN} be FNS.

$X_{FN} = \{(R, 0.85,0.9,0.5), (G, 0.9,0.8,0.6), (Y, 0.7,0.6,0.8)\}$

$Y_{FN} = \{\langle(R, 0.7,0.6,0.8), (G, 0.7,0.9,0.7), (Y, 0.7,0.8,0.6)\rangle\}$

Let B be the collection containing the FN-tuples. Let τ be the topology generated by B. Let τ_0 denote the usual topology on the finite set X (so $\phi, X \in \tau_0$).

$$B = \begin{Bmatrix} (R, 0.85,0.9,0.5), (G, 0.9,0.8,0.6), (Y, 0.7,0.6,0.8) \\ \langle(R, 0.7,0.6,0.8), (G, 0.7,0.9,0.7), (Y, 0.7,0.8,0.6)\rangle \\ \langle(R, 0.7,0.6,0.8), (G, 0.7,0.8,0.7), (Y, 0.7,0.6,0.8)\rangle \end{Bmatrix}$$

Let τ be the topology generated by bases

$$\tau = \begin{Bmatrix} (R, 0.85,0.9,0.5), (G, 0.9,0.8,0.6), (Y, 0.7,0.6,0.8) \\ \langle(R, 0.7,0.6,0.8), (G, 0.7,0.9,0.7), (Y, 0.7,0.8,0.6)\rangle \\ \langle(R, 0.7,0.6,0.8), (G, 0.7,0.8,0.7), (Y, 0.7,0.6,0.8)\rangle \\ \langle(R, 0.85,0.9,0.5), (G, 0.9,0.9,0.6), (Y, 0.7,0.8,0.6)\rangle \end{Bmatrix}$$

For any crisp FNS χ_A (with $A \subseteq X$), define $\tau^T, \tau^I, \tau^F : I^X \to I$ by

$$\tau^T(\{(R, 0.85,0.9,0.5)\}) = \begin{cases} 1 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 1 & \text{if} \quad A \in T_0 \setminus \{0_{FN}, 1_{FN}\} \\ 0.7 & \text{if} \quad A \in \tau \setminus \tau_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tau^I(\{(R, 0.85,0.9,0.5)\}) = \begin{cases} 1 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 0.7 & \text{if} \quad A \in T_0 \setminus \{0_{FN}, 1_{FN}\} \\ 0.5 & \text{if} \quad A \in \tau \setminus \tau_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\tau^F(\{(R, 0.85,0.9,0.5)\}) = \begin{cases} 0 & \text{if} \quad A \in \{0_{FN}, 1_{FN}\} \\ 0 & \text{if} \quad A \in T_0 \setminus \{0_{FN}, 1_{FN}\} \\ 0.3 & \text{if} \quad A \in \tau \setminus \tau_0 \\ 1 & \text{otherwise} \end{cases}$$

Then (τ^T, τ^I, τ^F) is an FNGO on X.

Definition 3.2.6

For two pairs of mappings of FNGO (τ^T, τ^I, τ^F) and FNGC (F^T, F^I, F^F) from $I^X \to I$ define

$$\tau^T{}_{F^T}(\lambda) = F^T(\lambda^C), \tau^I{}_{F^I}(\lambda) = F^I(\lambda^C), \tau^F{}_{F^F}(\lambda) = F^F(\lambda^C)$$
$$F^T{}_{\tau^T}(\lambda) = \tau^T(\lambda^C), F^I{}_{\tau^I}(\lambda) = \tau^I(\lambda^C), F^F{}_{\tau^F}(\lambda) = \tau^F(\lambda^C)$$

Theorem 3.2.7

(a) (τ^T, τ^I, τ^F) is a FNGO on X iff $\left(F^T{}_{\tau^T}, F^I{}_{\tau^I}, F^F{}_{\tau^F}\right)$ is a FNGC on X,

(b) (F^T, F^I, F^F) is a FNGC on X iff $(\tau^T{}_{F^T}, \tau^I{}_{F^I}, \tau^F{}_{F^F})$ is a FNGO on X,

(c)$\tau^T{}_{F^T{}_{\tau^T}} = \tau^T, \tau^I{}_{F^I{}_{\tau^I}} = \tau^I, \tau^F{}_{F^F{}_{\tau^F}} = \tau^F, F^T{}_{\tau^T{}_{F^T}} = F^T, F^I{}_{\tau^I{}_{F^I}} = F^I, F^F{}_{\tau^F{}_{F^F}} = F^F$

Proof:

(a)

$\left(F^T{}_{\tau^T}(\mu)\right)^3 + \left(F^I{}_{\tau^I}(\mu)\right)^3 + \left(F^F{}_{\tau^F}(\mu)\right)^3 = \left(\tau^T(\mu^C)\right)^3 + \left(\tau^I(\mu^C)\right)^3 + \left(\tau^F(\mu^C)\right)^3$, $\forall \mu \in I^X$.

so $\left(F^T{}_{\tau^T}(\mu)\right)^3 + \left(F^I{}_{\tau^I}(\mu)\right)^3 + \left(F^F{}_{\tau^F}(\mu)\right)^3 \leq 2$ iff

$$\left(\tau^T(\mu^C)\right)^3 + \left(\tau^I(\mu^C)\right)^3 + \left(\tau^F(\mu^C)\right)^3 \leq 2, \forall \mu \in I^X \qquad (3.2.1)$$

Clearly,

$$F^T{}_{\tau^T}(0_{FN}) = F^T{}_{\tau^T}(1_{FN}) = 1 \Leftrightarrow \tau^T(1_{FN}) = \tau^T(0_{FN}) = 1$$
$$F^I{}_{\tau^I}(0_{FN}) = F^I{}_{\tau^I}(1_{FN}) = 1 \Leftrightarrow \tau^I(1_{FN}) = \tau^I(0_{FN}) = 1$$
$$F^F{}_{\tau^F}(0_{FN}) = F^F{}_{\tau^F}(1_{FN}) = 0 \Leftrightarrow \tau^F(0_{FN}) = \tau^F(1_{FN}) = 0$$
$$F^T{}_{\tau^T}(\mu_1 \cup \mu_2) = \tau^T[(\mu_1 \cup \mu_2)^C] = \tau^T({\mu_1}^C \cap {\mu_2}^C) \qquad (3.2.2)$$

so

$$F^T{}_{\tau^T}(\mu_1 \cup \mu_2) \geq F^T{}_{\tau^T}(\mu_1) \wedge F^T{}_{\tau^T}(\mu_2), \forall \mu_1, \mu_2 \in I^X$$
$$\Leftrightarrow \tau^T({\mu_1}^C \cap {\mu_2}^C) \geq \tau^T({\mu_1}^C) \wedge \tau^T({\mu_2}^C), \forall \mu_1, \mu_2 \in I^X$$
$$\Leftrightarrow \tau^T(\mu_1 \cap \mu_2) \geq \tau^T(\mu_1) \wedge \tau^T(\mu_2), \forall \mu_1, \mu_2 \in I^X$$
$$(\mu^C)^C = \mu \qquad (3.2.3A)$$

Similarly

$$F^{I}{}_{\tau^{I}}(\mu_1 \cup \mu_2) \geq F^{I}{}_{\tau^{I}}(\mu_1) \wedge F^{I}{}_{\tau^{I}}(\mu_2)$$

$$\Leftrightarrow \tau^{I}(\mu_1 \cap \mu_2) \geq \tau^{I}(\mu_1) \wedge \tau^{T}(\mu_2), \forall \mu_1, \mu_2 \in I^{X} \qquad (3.2.3B)$$

$$F^{F}{}_{\tau^{F}}(\mu_1 \cap \mu_2) \geq F^{F}{}_{\tau^{F}}(\mu_1) \vee F^{F}{}_{\tau^{F}}(\mu_2)$$

$$\Leftrightarrow \tau^{F}(\mu_1 \cup \mu_2) \leq \tau^{F}(\mu_1) \vee \tau^{F}(\mu_2), \forall \mu_1, \mu_2 \in I^{X} \qquad (3.2.3C)$$

$$F^{T}{}_{\tau^{T}}\left(\bigcap_{i \in \Delta} \mu_i\right) = \tau^{T}\left[\left(\bigcap_{i \in \Delta} \mu_i\right)^{C}\right] = \tau^{T}\left(\bigcup_{i \in \Delta} \mu_i{}^{C}\right)$$

$$F^{T}{}_{\tau^{T}}\left(\bigcap_{i \in \Delta} \mu_i\right) \geq \bigwedge_{i \in \Delta} F^{T}{}_{\tau^{T}}(\mu_i), \mu_i \in I^{X}, i \in \Delta$$

$$\Leftrightarrow \tau^{T}\left(\bigcup_{i \in \Delta} \mu_i{}^{C}\right) \geq \bigwedge_{i \in \Delta} \tau^{T}(\mu_i{}^{C}), \mu_i \in I^{X}, i \in \Delta$$

$$\Leftrightarrow \tau^{T}\left(\bigcup_{i \in \Delta} \mu_i\right) \geq \bigwedge_{i \in \Delta} \tau^{T}(\mu_i), \mu_i \in I^{X}, i \in \Delta \qquad (3.2.4A)$$

Similarly

$$F^{I}{}_{\tau^{I}}\left(\bigcap_{i \in \Delta} \mu_i\right) \geq \bigwedge_{i \in \Delta} F^{I}{}_{\tau^{I}}(\mu_i)$$

$$\Leftrightarrow \tau^{I}\left(\bigcup_{i \in \Delta} \mu_i\right) \geq \bigwedge_{i \in \Delta} \tau^{I}(\mu_i), \mu_i \in I^{X}, i \in \Delta \qquad (3.2.4B)$$

$$F^{F}{}_{\tau^{F}}\left(\bigcap_{i \in \Delta} \mu_i\right) \leq \bigvee_{i \in \Delta} F^{F}{}_{\tau^{F}}(\mu_i)$$

$$\Leftrightarrow \tau^{F}\left(\bigcup_{i \in \Delta} \mu_i\right) \leq \bigvee_{i \in \Delta} \tau^{F}(\mu_i), \mu_i \in I^{X}, i \in \Delta \qquad (3.2.4C)$$

Hence by (3.2.1), (3.2.2), (3.2.3A), (3.2.3B), (3.2.3C), (3.2.4A), (3.2.4B) and (3.2.4C) (a) hold.

(b) The proof of (b) is similar to (a).

(c) The proof is straightforward.

Definition 3.2.8

Consider $\{(\tau^T, \tau^I, \tau^F)\}_{i\in\Delta}$ be a family of FNGOs on X. Then their intersection is defined by

$$\bigcap_{i\in\Delta}(\tau^T, \tau^I, \tau^F) = \left(\underset{i\in\Delta}{\wedge} {\tau_i}^T, \underset{i\in\Delta}{\wedge} {\tau_i}^I, \underset{i\in\Delta}{\vee} {\tau_i}^T\right)$$

where

$(\wedge_{i\in\Delta} {\tau_i}^T)(\mu) = \wedge_{i\in\Delta}({\tau_i}^T(\mu))$, $(\wedge_{i\in\Delta} {\tau_i}^I)(\mu) = \wedge_{i\in\Delta}({\tau_i}^I(\mu))$,

$(\vee_{i\in\Delta} {\tau_i}^F)(\mu) = \vee_{i\in\Delta}({\tau_i}^F(\mu)), \mu \in I^X$.

Theorem 3.2.9

An arbitrary intersection of FNGOs is an FNGO.

Definition 3.2.10

Let $({\tau_1}^T, {\tau_1}^I, {\tau_1}^F)$ and $({\tau_2}^T, {\tau_2}^I, {\tau_2}^F)$ be two FNGOs on X. Then, define a relation ‘ $\leq$ ’ by

$$({\tau_1}^T, {\tau_1}^I, {\tau_1}^F) \leq ({\tau_2}^T, {\tau_2}^I, {\tau_2}^F) \Leftrightarrow {\tau_1}^T \leq {\tau_2}^T, {\tau_1}^I \leq {\tau_2}^I, {\tau_1}^F \geq {\tau_2}^F$$

Remark 3.2.11

Let X be a nonempty set. Define ${\tau_p}^T, {\tau_p}^I, {\tau_p}^F, {\tau_q}^T, {\tau_q}^I, {\tau_q}^F : I^X \to I$ by the rule

$${\tau_p}^T(0_{FN}) = {\tau_p}^T(1_{FN}) = 1,$$

$${\tau_p}^I(0_{FN}) = {\tau_p}^I(1_{FN}) = 1,$$

$${\tau_p}^F(0_{FN}) = {\tau_p}^F(1_{FN}) = 0,$$

$${\tau_p}^T(\mu) = 0, {\tau_p}^I(\mu) = 0, {\tau_p}^F(\mu) = 1, \forall\mu \in I^X - \{0_{FN}, 1_{FN}\} \text{ and}$$

$${\tau_q}^T(\mu) = 1, {\tau_q}^I(\mu) = 1, {\tau_q}^F(\mu) = 0, \forall\mu \in I^X$$

Then $({\tau_p}^T, {\tau_p}^I, {\tau_p}^F)$ and $({\tau_q}^T, {\tau_q}^I, {\tau_q}^F)$ are two FNGOs on X such that for any FNGO (τ^T, τ^I, τ^F) on X, $({\tau_p}^T, {\tau_p}^I, {\tau_p}^F) \leq (\tau^T, \tau^I, \tau^F) \leq ({\tau_q}^T, {\tau_q}^I, {\tau_q}^F)$.

Theorem 3.2.12

The collection G of all FNGOs on X forms a complete lattice with respect to ' $\leq$ ' of which $\left(\tau_p{}^T, \tau_p{}^I, \tau_p{}^F\right)$ and $\left(\tau_q{}^T, \tau_q{}^I, \tau_q{}^F\right)$ are, respectively, the smallest and largest element.

Proof: From the theorem 3.2.9 and remark 3.2.11, it is straightforward.

Definition 3.2.13

Let (τ^T, τ^I, τ^F) be a FNGO on X. For $s \in I_0$, define

$$\tau_s{}^T = (\tau^T)^{-1}[s,1], \tau_s{}^I = (\tau^I)^{-1}[s,1], \tau_s{}^F = (\tau^F)^{-1}[0,1-s].$$

Theorem 3.2.14

Let $(X, \tau^T, \tau^I, \tau^F)$ be a FNTS. Then $\{\tau^T{}_r\}_{r\in I_0}$, $\{\tau^I{}_r\}_{r\in I_0}$ and $\{\tau^F{}_r\}_{r\in I_0}$ are three descending families of topologies of Fermatean neutrosophic subsets on X such that

i. $\tau^T{}_r \subset \tau^F{}_r$

ii. $\tau^T{}_r = \bigcap_{s<r} \tau^T{}_s; \tau^I{}_r = \bigcap_{s<r} \tau^I{}_s; \tau^F{}_r = \bigcap_{s<r} \tau^F{}_s.$

Remark 3.2.15

If (τ^T, τ^I, τ^F) is a FNGO on X, then for $r \in I_0$, $(\tau_r{}^T, \tau_r{}^I, \tau_r{}^F)$ is an inclusive bitopologies of Fermatean neutrosophic subsets (BT-FN_{Subs}) on X , which will be called an r-level inclusive BT-FN_{Subs}.

Theorem 3.2.16

Let $\{(\varphi_r{}^T, \varphi_r{}^I, \varphi_r{}^F); r \in I_0\}$ be a descending family of inclusive BT-FN_{Subs} on X. Define $\tau^T, \tau^I, \tau^F: I^X \to I$ by

$$\tau^T(\lambda) = \vee \{r; \lambda \in \varphi_r^T\}; \tau^I(\lambda) = \vee \{r; \lambda \in \varphi_r^I\}; \tau^F(\lambda) = \wedge \{1-r; \lambda \in \varphi_r^F\}$$

Then

a. (τ^T, τ^I, τ^F) is a FNGO on X.

b. $\tau_r{}^T = \varphi_r^T$ iff $\bigcap_{s<r} \varphi_s^T, r \in I_0$

c. $\tau_r{}^I = \varphi_r^I$ iff $\bigcap_{s<r} \varphi_s^I, r \in I_0$

d. $\tau_r{}^F = \varphi_r^F$ iff $\bigcap_{s<r} \varphi_s^F, r \in I_0$

(a) By definition 3.2.13, τ^T, τ^I satisfies all the properties of gradation of openness on X. Next consider the mapping $\tau^F: I^X \to I$. Clearly, $\tau^F(0_{FN}) = \tau^F(1_{FN}) = 0$.

Next let, $\tau^F(\lambda_1) = r_1, \tau^F(\lambda_2) = r_2$ and $r = max\{r_1, r_2\}$. If $r = 1$, then obviously $\tau^F(\lambda_1 \cap \lambda_2) \leq \tau^F(\lambda_1) \vee \tau^F(\lambda_2)$.

Suppose $r < 1$. Choose $\varepsilon > 0$ such that $r^3 + \varepsilon^3 < 1$. Then $\exists t_i \in (0,1)$, such that $t_i < {r_i}^3 + \varepsilon^3$ and $\lambda_i \in {\varphi_{1-t_i}}^F, i = 1,2$.

Let $t = max\{t_1, t_2\}$. Then, $\lambda_1, \lambda_2 \in \varphi_{1-t}^F$ (since $\{\varphi_r^F\}_{r \in I_0}$ is a descending family)

$\Rightarrow \lambda_1 \cap \lambda_2 \in \varphi_{1-t}^F$ (since φ_{1-t}^F is a BT-FN$_{\text{Subs}}$)

$\Rightarrow \tau^F(\lambda_1 \cap \lambda_2) \leq t \leq r^3 + \varepsilon^3$.

Since $\varepsilon > 0$ is arbitrary, therefore $\tau^F(\lambda_1 \cap \lambda_2) \leq r = \tau^F(\lambda_1) \vee \tau^F(\lambda_2)$.

Let $s_i = \tau^F(\lambda_i), i \in \Delta$ and $s = \vee_{i \in \Delta} s_i$. If $s = 1$, then obviously,

$$\tau^F\left(\bigcup_{i \in \Delta} \lambda_i\right) \leq s = \bigvee_{i \in \Delta} \tau^F(\lambda_i)$$

So, consider the case when $s < 1$. Choose $\varepsilon > 0$ such that $s^3 + \varepsilon^3 < 1$. For $i \in \Delta$, $\tau^F(\lambda_i) \leq s < s^3 + \varepsilon^3$. So, $\exists \varphi_r^F$ such that $\lambda_i \in \varphi_r^F$ and $1 - r < s^3 + \varepsilon^3$.

Therefore

$\lambda_i \in \varphi_r^F \subset \varphi_{1-s-\varepsilon}^F$, so $\lambda_i \in \varphi_{1-s-\varepsilon}^F,\ \forall i \in \Delta$.

Since $\varphi_{1-s-\varepsilon}^F$ is a FTNS, $\cup_{i \in \Delta} \lambda_i \in \varphi_{1-s-\varepsilon}^F$.

So,

$$\tau^F\left(\bigcup_{i \in \Delta} \lambda_i\right) \leq s^3 + \varepsilon^3.$$

Since $\varepsilon > 0$ is arbitrary

$$\tau^F\left(\bigcup_{i \in \Delta} \lambda_i\right) \leq s = \bigvee_{i \in \Delta} \tau^F(\lambda_i)$$

Let $\varphi^T(\lambda), \varphi^I(\lambda) = s$. If $s = 0$, then obviously $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2$.

If $s = 1$, then $\lambda \in \varphi_r^T, \forall r < 1$; $\lambda \in \varphi_r^I, \forall r < 1$ and $\lambda \in \varphi_r^F, \forall r < 1$ (since $\varphi_r^T \subset \varphi_r^F$). Therefore, $\tau^F(\lambda) = 0$. Then $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2$.

Next consider the case when $0 < s < 1$. Choose $\varepsilon > 0$ s.t $0 < s - \varepsilon < s + \varepsilon < 1$. Then, $\lambda \in \varphi_{s-\varepsilon}^T \subset \varphi_{s-\varepsilon}^F$. Therefore, $\tau^F(\lambda) \leq 1 - s + \varepsilon$.

Therefore $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2 + \varepsilon$

Since $\varepsilon > 0$ is arbitrary $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2$.

Hence (τ^T, τ^I, τ^F) is a FNGO on X.

(b) (c) The proof of (b) follows from definition 3.2.13.

(d) Suppose, $\bigcap_{s<r} \varphi_s^F = \varphi_r^T, \forall r \in I_0$. Now $\lambda \in \varphi_r^T \Rightarrow \tau^F(\lambda) \leq 1 - r \Rightarrow \lambda \in \tau_r^F$. Therefore, $\varphi_r^F \subset \tau_r^F$. Next $\mu \in \varphi_r^F \Rightarrow \tau^F(\mu) \leq 1 - r \Rightarrow$ for any $0 < s < r, \exists r_0$ such that $1 - r_0 < 1 - s$ and $\mu \in \varphi_{r_0}^F$.

Since, $\{\varphi_r^F\}_{r \in I_0}$ is a descending family, $\varphi_{r_0}^F \subset \varphi_s^F$. So, for any $0 < s < r$, $\mu \in \varphi_s^F$. Therefore, $\mu \in \bigcap_{0<s<r} \varphi_s^F$.

Therefore, $\tau_r^F \subset \bigcap_{s<r} \varphi_s^F = \varphi_r^F$. Therefore, $\tau_r^F = \varphi_r^F$ if $\bigcap_{s<r} \varphi_s^F = \varphi_r^F, r \in I_0$. Again, by (b) of Theorem 3.2.13, $\bigcap_{s<r} \tau_s^F = \tau_r^F, r \in I_0$. Hence, ${\tau_r}^F = \varphi_r^F$ iff $\bigcap_{s<r} \varphi_s^F = \varphi_r^F$.

Remark 3.2.17

The FNGO (τ^T, τ^I, τ^F) so obtained in Theorem 3.2.16 will be referred to as the FNGO generated by the descending family of inclusive BT-FNSubs $\{({\varphi_r}^T, {\varphi_r}^I, {\varphi_r}^F); r \in I_0\}$.

Definition 3.2.18

Let $(\varphi^T, \varphi^I, \varphi^F)$ be a BT-FNS on a nonempty set X. Define for each $r \in I_0$, three mappings ${\varphi_r}^T, {\varphi_r}^I, {\varphi_r}^F : I^X \to I$ by the rule

${\varphi_r}^T(0_{FN}) = {\varphi_r}^T(1_{FN}) = 1, {\varphi_r}^I(0_{FN}) = {\varphi_r}^I(1_{FN}) = 1, {\varphi_r}^F(0_{FN}) = {\varphi_r}^F(1_{FN}) = 0,$

$${\varphi_r}^T(\mu) = \begin{cases} r & \text{if } \mu \in \varphi^T - \{0_{FN}, 1_{FN}\}, \\ 0 & \text{if } \mu \in I^X - \varphi^T \end{cases}$$

$$\varphi_r{}^I(\mu) = \begin{cases} r & \text{if } \mu \in \varphi^I - \{0_{FN}, 1_{FN}\}, \\ 0 & \text{if } \mu \in I^X - \varphi^I \end{cases}$$

$$\varphi_r{}^F(\mu) = \begin{cases} 1-r & \text{if } \mu \in \varphi^F - \{0_{FN}, 1_{FN}\}, \\ 1 & \text{if } \mu \in I^X - \varphi^F \end{cases}$$

Theorem 3.2.19

Let $(\varphi^T, \varphi^I, \varphi^F)$ be an inclusive BT-FNSubs on X. Then $(\varphi_r{}^T, \varphi_r{}^I, \varphi_r{}^F)$ is an FNGO on X such that $(\varphi_r{}^T)_r = \varphi^T$, $(\varphi_r{}^I)_r = \varphi^I$ and $(\varphi_r{}^F)_r = \varphi^F$.

Proof:

Clearly, $\left(\varphi_r^T(\mu)\right)^3 + \left(\varphi_r^I(\mu)\right)^3 + \left(\varphi_r^F(\mu)\right)^3 \leq 2, \forall \mu \in I^X$. Therefore, (FNGO1) holds. By definition (FNGO2) holds.

Let $\lambda_i \in I^X, i = 1,2$. If one of λ_1, λ_2 (say λ_1) is 0_{FN}, then $\lambda_1 \cap \lambda_2 = 0_{FN}$.

Therefore $\varphi_r^T(\lambda_1 \cap \lambda_2) \geq \varphi_r^T(\lambda_1) \wedge \varphi_r^T(\lambda_2)$, $\varphi_r^I(\lambda_1 \cap \lambda_2) \geq \varphi_r^I(\lambda_1) \wedge \varphi_r^I(\lambda_2)$ and $\varphi_r^F(\lambda_1 \cap \lambda_2) \leq \varphi_r^F(\lambda_1) \vee \varphi_r^F(\lambda_2)$ holds.

If one of λ_1, λ_2 (say λ_1) is 1_{FN}, then $\lambda_1 \cap \lambda_2 = \lambda_2$.

Therefore $\varphi_r^T(\lambda_1 \cap \lambda_2) \geq \varphi_r^T(\lambda_1) \wedge \varphi_r^T(\lambda_2)$, $\varphi_r^I(\lambda_1 \cap \lambda_2) \geq \varphi_r^I(\lambda_1) \wedge \varphi_r^I(\lambda_2)$ and $\varphi_r^F(\lambda_1 \cap \lambda_2) \leq \varphi_r^F(\lambda_1) \vee \varphi_r^F(\lambda_2)$ holds.

If $\lambda_1, \lambda_2 \in \varphi^T - \{0_{FN}, 1_{FN}\}$, then $\lambda_1 \cap \lambda_2 \in \varphi^T$.

Hence $\varphi_r^T(\lambda_1 \cap \lambda_2) \geq r = \varphi^T(\lambda_1) \wedge \varphi^T(\lambda_2)$.

If one of λ_1, λ_2 (say λ_1) belongs to $I^X - \varphi^T$, then $\varphi^T(\lambda_1) = 0$ and hence $\varphi_r^T(\lambda_1) \wedge \varphi_r^T(\lambda_2) = 0 \leq \varphi_r^T(\lambda_1 \cap \lambda_2)$.

If $\lambda_1, \lambda_2 \in \varphi^I - \{0_{FN}, 1_{FN}\}$, then $\lambda_1 \cap \lambda_2 \in \varphi^I$ and hence $\varphi_r^I(\lambda_1 \cap \lambda_2) \geq r = \varphi^I(\lambda_1) \wedge \varphi^I(\lambda_2)$.

If one of λ_1, λ_2 (say λ_1) belongs to $I^X - \varphi^I$, then $\varphi^I(\lambda_1) = 0$ and hence $\varphi_r^I(\lambda_1) \wedge \varphi_r^I(\lambda_2) = 0 \leq \varphi_r^I(\lambda_1 \cap \lambda_2)$.

Let $\lambda_1, \lambda_2 \in \varphi^F - \{0_{FN}, 1_{FN}\}$, then $\lambda_1 \cap \lambda_2 \in \varphi^F$ and hence

$\varphi_r^F(\lambda_1 \cap \lambda_2) \geq 1 - r = \varphi^F(\lambda_1) \vee \varphi^F(\lambda_2)$.

If one of λ_1, λ_2 (say λ_1) belongs to $I^X - \varphi^I$, then $\varphi^F(\lambda_1) = 1$ an $\varphi_r^F(\lambda_1) \vee \varphi_r^F(\lambda_2) = 1 \geq \varphi_r^F(\lambda_1 \cap \lambda_2)$.

Hence (FNGO3) holds.

Let $\lambda_i \in I^X, i \in \Delta$. If $\lambda_i = 0_{FN}$, for some $i = i_0 \in \Delta$, then $\vee_{i\in\Delta} \lambda_i = \vee_{i\in\Delta(i\neq i_0)} \lambda_i$.

$\wedge_{i\in\Delta} \varphi_r^T(\lambda_i) = \wedge_{i\in\Delta(i\neq i_0)} \varphi_r^T(\lambda_i)$ (since $\varphi_r^T(\lambda_{i_0}) = 1$

$\wedge_{i\in\Delta} \varphi_r^I(\lambda_i) = \wedge_{i\in\Delta(i\neq i_0)} \varphi_r^I(\lambda_i)$ (since $\varphi_r^I(\lambda_{i_0}) = 1$

$\vee_{i\in\Delta} \varphi_r^F(\lambda_i) = \vee_{i\in\Delta(i\neq i_0)} \varphi_r^F(\lambda_i)$ (since $\varphi_r^F(\lambda_i) = 0$

So, without loss of generality, assume that $\lambda_i \neq \tilde{0}, \forall i \in \Delta$. If $\lambda_i = \tilde{1}$, , for some $i = i_0 \in \Delta$, then

$$\varphi_r^T\left(\bigcup_{i\in\Delta} \lambda_i\right) = \varphi_r^T(\tilde{1}) = 1 \geq \bigwedge_{i\in\Delta} \varphi_r^T(\lambda_i)$$

$$\varphi_r^I\left(\bigcup_{i\in\Delta} \lambda_i\right) = \varphi_r^I(\tilde{1}) = 1 \geq \bigwedge_{i\in\Delta} \varphi_r^I(\lambda_i)$$

$$\varphi_r^F\left(\bigcup_{i\in\Delta} \lambda_i\right) = \varphi_r^F(\tilde{1}) = 0 \geq \bigvee_{i\in\Delta} \varphi_r^F(\lambda_i)$$

If $\lambda_i \in \varphi^T - \{0_{FN}, 1_{FN}\}, i \in \Delta$ then $\cup_{i\in\Delta} \lambda_i \in \varphi^T$.

Therefore, $\varphi_r^T(\cup_{i\in\Delta} \lambda_i) \geq r = \wedge_{i\in\Delta} \varphi_r^T(\lambda_i)$.

If $\lambda_i \in I^X - \varphi^T$ for some $i = i_0$, then

$$\bigwedge_{i\in\Delta} \varphi_r^T(\lambda_i) = \varphi_r^T(\lambda_{i_0}) = 0 \leq \varphi_r^T\left(\bigcup_{i\in\Delta} \lambda_i\right)$$

Thus in all cases, $\varphi_r^T(\cup_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} \varphi_r^T(\lambda_i)$.

If $\lambda_i \in \varphi^I - \{0_{FN}, 1_{FN}\}, i \in \Delta$ then $\cup_{i\in\Delta} \lambda_i \in \varphi^I$.

Therefore, $\varphi_r^I(\cup_{i\in\Delta} \lambda_i) \geq r = \wedge_{i\in\Delta} \varphi_r^I(\lambda_i)$.

If $\lambda_i \in I^X - \varphi^I$ for some $i = i_0$, then

$$\bigwedge_{i\in\Delta} \varphi_r^I(\lambda_i) = \varphi_r^I(\lambda_{i_0}) = 0 \leq \varphi_r^I\left(\bigcup_{i\in\Delta} \lambda_i\right)$$

Thus in all cases, $\varphi_r^I(\cup_{i\in\Delta} \lambda_i) \geq \wedge_{i\in\Delta} \varphi_r^I(\lambda_i)$.

If $\lambda_i \in \varphi^F - \{0_{FN}, 1_{FN}\}, i \in \Delta$ then $\cup_{i\in\Delta} \lambda_i \in \varphi^F$.

Therefore, $\varphi_r^F(\cup_{i\in\Delta} \lambda_i) \leq 1 - r = \vee_{i\in\Delta} \varphi_r^F(\lambda_i)$.

If $\lambda_i \in I^X - \varphi^F$ for some $i = i_0$, then

$$\bigvee_{i\in\Delta} \varphi_r^F(\lambda_i) = \varphi_r^F(\lambda_{i_0}) = 1 \geq \varphi_r^F\left(\bigcup_{i\in\Delta} \lambda_i\right)$$

Thus in all cases, $\varphi_r^F(\cup_{i\in\Delta} \lambda_i) \leq \vee_{i\in\Delta} \varphi_r^F(\lambda_i)$.

Hence (FNGO4) holds.

By Theorem 3.2.14,

$(\varphi_r^T)_r = \{\mu \in I^X : \varphi_r^T(\mu) \geq r\} = \varphi^T$, $(\varphi_r^I)_r = \{\upsilon \in I^X : \varphi_r^I(\upsilon) \geq r\} = \varphi^I$ and $(\varphi_r^F)_r = \{\gamma \in I^X : \varphi_r^T(\gamma) \leq 1 - r\} = \varphi^F$.

Hence $({\varphi_r}^T, {\varphi_r}^I, {\varphi_r}^F)$ is a FNGO on X such that $({\varphi_r}^T)_r = \varphi^T$, $({\varphi_r}^I)_r = \varphi^I$ and $({\varphi_r}^F)_r = \varphi^F$.

Definition 3.2.20

If $(\varphi^T, \varphi^I, \varphi^F)$ is an inclusive BT-FN$_{\text{Subs}}$ on X, then $({\varphi_r}^T, {\varphi_r}^I, {\varphi_r}^F)$ is called r[th] Fermatean neutrosophic gradation of openness (briefly r[th] FNGO) on X and $(X, {\varphi_r}^T, {\varphi_r}^I, {\varphi_r}^F)$ is called r[th] graded FNTS.

Theorem 3.2.21

Let $(\varphi^T, \varphi^I, \varphi^F)$ be an inclusive BT-FN$_{\text{Subs}}$ on X and let (τ^T, τ^I, τ^F) be the corresponding r[th] FNGO on X , $r \in I_0$. Then $(\tau_r^T)^r = \tau^T, (\tau_r^I)^r = \tau^I, (\tau_r^F)^r = \tau^F$.

Proof.

By definition 3.2.20, for $r \in I_0$, $\tau^T = \varphi_r^T, \tau^I = \varphi_r^I, \tau^F = \varphi_r^F$.

So, by Theorem 3.2.19, $\tau_r^T = (\varphi_r^T)^r = \varphi^T$, $\tau_r^I = (\varphi_r^I)^r = \varphi^I$, $\tau_r^F = (\varphi_r^F)^r = \varphi^F$

Hence $(\tau_r^T)^r = \tau^T, (\tau_r^I)^r = \tau^I, (\tau_r^F)^r = \tau^F$.

Fermatean Neutrosophic subspaces

Theorem 3.2.22

Let $(X, \tau^T, \tau^I, \tau^F)$ be a FNTS and $Y \subset X$. Define mappings ${\tau^T}_Y, {\tau^I}_Y, {\tau^F}_Y : I^Y \to I$ by the rule

$${\tau^T}_Y(\mu) = \vee \{\tau^T(\lambda); \lambda \in I^X, \lambda/Y = \mu\},$$

$${\tau^I}_Y(\mu) = \vee \{\tau^I(\lambda); \lambda \in I^X, \lambda/Y = \mu\},$$

$${\tau^F}_Y(\mu) = \wedge \{\tau^F(\lambda); \lambda \in I^X, \lambda/Y = \mu\} \forall \mu \in I^Y,$$

Then $({\tau^T}_Y, {\tau^I}_Y, {\tau^F}_Y)$ is a FNGO on Y and ${\tau^T}_Y(\mu) \geq \tau^T(\mu_X)$, ${\tau^I}_Y(\mu) \geq \tau^I(\mu_X)$, ${\tau^F}_Y(\mu) \leq \tau^F(\mu_X)$.

Proof:

For each $\lambda \in I^X$ with $\lambda/Y = \mu$, and $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2$, i. e $\left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 \leq 2 - \left[\left(\tau^F(\lambda)\right)^3\right]$.

Hence

$$\vee \left\{\left(\tau^T(\lambda)\right)^3; \lambda/Y = \mu\right\} + \vee \left\{\left(\tau^I(\lambda)\right)^3; \lambda/Y = \mu\right\} \leq \left[\vee \left\{2 - \left(\tau^F(\lambda)\right)^3; \lambda/Y = \mu\right\}\right]$$

$$\vee \left\{\left(\tau^T(\lambda)\right)^3; \lambda/Y = \mu\right\} + \vee \left\{\left(\tau^I(\lambda)\right)^3; \lambda/Y = \mu\right\} = 2 - \left[\wedge \left\{\left(\tau^F(\lambda)\right)^3; \lambda/Y = \mu\right\}\right]$$

$$\vee \left\{\left(\tau^T(\lambda)\right)^3; \lambda/Y = \mu\right\} + \left[\vee \left\{\left(\tau^I(\lambda)\right)^3; \lambda/Y = \mu\right\} + \wedge \left\{\left(\tau^F(\lambda)\right)^3; \lambda/Y = \mu\right\}\right] = 2$$

$$\Rightarrow \left(\tau^T(\lambda)\right)^3 + \left(\tau^I(\lambda)\right)^3 + \left(\tau^F(\lambda)\right)^3 \leq 2.$$

so (FNGO1) holds on Y

$${\tau_Y}^F(0_{FN\,Y}) = \wedge \{\tau^F(\lambda); \lambda \in I^X, \lambda/Y = 0_{FN\,Y}\}$$

$$\leq \tau^F(0_{FN\,X}) = 0.$$

$${\tau_Y}^F(0_{FN\,Y}) = 0.$$

$${\tau_Y}^F(1_{FN\,Y}) = \wedge \{\tau^F(\lambda); \lambda \in I^X, \lambda/Y = 1_{FN\,Y}\}$$

$$\leq \tau^F(1_{FN\,X}) = 0.$$

$${\tau_Y}^F(1_{FN\,Y}) = 0.$$

Now, ${\tau_Y}^F(\mu_1 \cap \mu_2) = \wedge \{\tau^F(\lambda); \lambda/Y = \mu_1 \cap \mu_2\}$. If ${\tau_Y}^F(\mu_1) \vee {\tau_Y}^F(\mu_2) = 1$, then $\tau^F(\mu_1 \cap \mu_2) \leq {\tau_Y}^F(\mu_1) \vee {\tau_Y}^F(\mu_2)$.

If $\tau_Y{}^F(\mu_1) \vee \tau_Y{}^F(\mu_2) < 1$ take $\tau_Y{}^F(\mu_1) \vee \tau_Y{}^F(\mu_2) < r < 1$ then $\exists \lambda_i \in I^X$ such that $\lambda_i \backslash Y = \mu_i$, $\tau^F(\lambda_i) < r, i = 1,2$. Then, $\tau^F(\lambda_1 \cap \lambda_2) \leq \tau^F(\lambda_1) \vee \tau^F(\lambda_2) < r$ and $(\lambda_1 \cap \lambda_2)/Y = (\lambda_1/Y) \wedge (\lambda_2/Y) = \mu_1 \wedge \mu$.

Therefore $\tau_Y{}^F(\mu_1 \cap \mu_2) \leq \tau^F(\mu_1 \cap \mu_2) < r$. So $\tau_Y{}^F(\mu_1) \vee \tau_Y{}^F(\mu_2) < r \Rightarrow \tau_Y{}^F(\mu_1 \cap \mu_2) < r$.

Hence $\tau_Y{}^F(\mu_1 \cap \mu_2) \leq \tau_Y{}^F(\mu_1) \vee \tau_Y{}^F(\mu_2)$.

$$\tau_Y{}^F\left(\bigcup_i \mu_i\right) = \wedge\left\{\tau^F(\lambda); \lambda/Y = \bigcup_i \mu_i\right\}$$

If $\vee_i \tau_Y{}^F(\mu_i) = 1$ then $\tau_Y{}^F(\cup_i \mu_i) \leq \vee_i \tau_Y{}^F(\mu_i)$

If $\vee_i \tau_Y{}^F(\mu_i) < 1$ take $\vee_i \tau_Y{}^F(\mu_i) < r < 1$, then $\tau_Y{}^F(\mu_i) < r, \forall i \in \Delta$.

Then $\exists \lambda_i \in I^X$ such that $\lambda_i/Y = \mu_i$, $\tau^F(\lambda_i) < r, i = 1,2$. Therefore $\tau^F(\bigcup_i \lambda_i) \leq r$.

Since

$$\left(\bigcup_i \lambda_i\right)/Y = \bigvee_i \lambda_i/Y = \bigvee_i \mu_i$$

it follows that, $\tau_Y{}^F(\bigcup_i \mu_i) \leq \vee_i \tau_Y{}^F(\mu_i)$.

Hence $(\tau^T{}_Y, \tau^I{}_Y, \tau^F{}_Y)$ is a FNGO on Y.

Clearly, $\tau^T{}_Y(\mu) \geq \tau^T(\mu_X)$, $\tau^I{}_Y(\mu) \geq \tau^I(\mu_X)$, $\tau^T{}_Y(\mu) \leq \tau^T(\mu_X)$ hold.

Definition 3.2.23

The Fermatean neutrosophic topological space $(Y, \tau^T{}_Y, \tau^I{}_Y, \tau^F{}_Y)$ is called a subspace of the Fermatean neutrosophic topological space $(X, \tau^T, \tau^I, \tau^F)$ then $(\tau^T{}_Y, \tau^I{}_Y, \tau^F{}_Y)$ is called the induced FNGO on Y from $(X, \tau^T, \tau^I, \tau^F)$.

Theorem 3.2.24

Let $(Y, \tau^T{}_Y, \tau^I{}_Y, \tau^F{}_Y)$ be a Fermatean neutrosophic subspace of the FNTS $(X, \tau^T, \tau^I, \tau^F)$ and $\mu \in I^Y$Then

i. $\varphi_{\tau_Y^T}(\mu) = \vee\left\{\varphi_{\tau^T}(\eta): \eta \in I^X, \eta/Y = \mu\right\}$

$$\varphi_{\tau_Y^I}(\mu) = \vee \{\varphi_{\tau^I}(\eta): \eta \in I^X, \eta/Y = \mu\}$$

$$\varphi_{\tau_Y^F}(\mu) = \wedge \{\varphi_{\tau^F}(\eta): \eta \in I^X, \eta/Y = \mu\}$$

ii. If $Z \subset Y \subset X$, then $\tau_Z^T = (\tau_Y^T)_Z$, $\tau_Z^I = (\tau_Y^I)_Z$,$\tau_Z^F = (\tau_Y^F)_Z$.

Proof:

The proof of $\varphi_{\tau_Y^T}(\mu) = \vee \{\varphi_{\tau^T}(\eta): \eta \in I^X, \eta/Y = \mu\}$

$$\varphi^F{}_{\tau_Y^F}(\mu) = \tau_Y^F(\mu^C)$$

$$= \wedge \{\tau^F(\lambda): \lambda \in I^X, \lambda/Y = \mu^C\}$$

$$= \wedge \{\tau^F(\lambda): \lambda^C \in I^X, \lambda^C/Y = \mu\}$$

$$= \wedge \{\varphi^F{}_{\tau^F}(\lambda^C): \lambda^C \in I^X, \lambda^C/Y = \mu\}$$

$$= \wedge \{\varphi^F{}_{\tau^F}(\eta): \eta \in I^X, \eta/Y = \mu\}$$

ii) If $\mu \in I^Z$, then $\tau_Z = (\tau_Y)_Z$,

$$(\tau_Y^F)_Z(\mu) = \wedge \{\tau_Y^F(\lambda): \lambda \in I^X, \lambda/Z = \mu\}$$

$$= \wedge \{\wedge \{\tau^F(\eta): \eta \in I^X, \eta/Y = \lambda\}: \lambda \in I^X, \lambda/Z = \mu\}$$

$$= \wedge \{\tau^F(\eta): \eta \in I^X, \eta/Z = \mu\} = \tau_Z^F(\mu).$$

Gradation preserving maps

Definition 3.2.25

Let $(X, \tau^T, \tau^I, \tau^F)$ and $(Y, \varphi^T, \varphi^I, \varphi^F)$ be two FNTSs and $f: X \to Y$ be a mapping. Then f is know as a gradation preserving map gp-map if for each $\mu \in I^Y$,

$\varphi^T(\mu) \le \tau^T(f^{-1}(\mu))$, $\varphi^I(\mu) \le \tau^I(f^{-1}(\mu))$ and $\varphi^F(\mu) \ge \tau^F(f^{-1}(\mu))$.

Definition 3.2.26

Let $f: (X, P^T, P^I, P^F) \to (Y, Q^T, Q^I, Q^F)$ be a mapping, where (X, P^T, P^I, P^F) and (Y, Q^T, Q^I, Q^F) are two BT-FN$_{\text{Subs}}$. Then f is said to be continuous if

$f: (X, P^T) \to (Y, Q^T)$, $f: (X, P^I) \to (Y, Q^I)$ and $f: (X, P^F) \to (Y, Q^F)$ are continuous.

Theorem 3.2.27

Let $(X, \tau^T, \tau^I, \tau^F)$ and $(Y, \varphi^T, \varphi^I, \varphi^F)$ be two FNTSs and $f: X \to Y$ be a mapping. Then

i. $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map iff

ii. $f: (X, \tau_r^T, \tau_r^I, \tau_r^F) \to (Y, \varphi_s^T, \varphi_s^I, \varphi_s^F)$ is continuous, $r \in I_0$.

Proof:

Suppose (i) holds. Then, $\varphi^T(\mu) \le \tau^T(f^{-1}(\mu))$, $\varphi^I(\mu) \le \tau^I(f^{-1}(\mu))$ and $\varphi^F(\mu) \ge \tau^F(f^{-1}(\mu)), \mu \in I^Y$.

$$\mu \in \varphi_r^T \Rightarrow r \le \varphi^T(\mu) \le \tau^T(f^{-1}(\mu))$$
$$\Rightarrow f^{-1}(\mu) \in \tau_r^T$$
$$\mu \in \varphi_r^I \Rightarrow r \le \varphi^I(\mu) \le \tau^I(f^{-1}(\mu))$$
$$\Rightarrow f^{-1}(\mu) \in \tau_r^I$$
$$\mu \in \varphi_r^F \Rightarrow 1 - r \ge \varphi^F(\mu) \ge \tau^F(f^{-1}(\mu))$$
$$\Rightarrow f^{-1}(\mu) \in \tau_r^F$$

So $(i) \Rightarrow (ii)$

Conversely, suppose (2) holds. Let $\mu \in I^Y$. If $\varphi^T(\mu) = 0$, then obviously, $\varphi^T(\mu) \le \tau^T(f^{-1}(\mu))$. If $\varphi^T(\mu) = r \in I_0$, then $\mu \in \varphi_r^T$. So, $f^{-1}(\mu) \in \tau_r$, by continuity of f. Hence $\tau^T(f^{-1}(\mu)) \ge r = \varphi^T(\mu)$.

Let $\mu \in I^Y$. If $\varphi^I(\mu) = 0$, then obviously, $\varphi^I(\mu) \le \tau^I(f^{-1}(\mu))$. If $\varphi^I(\mu) = r \in I_0$, then $\mu \in \varphi_r^I$. So, $f^{-1}(\mu) \in \tau_r$,by continuity of f. Hence $\tau^I(f^{-1}(\mu)) \ge r = \varphi^I(\mu)$.

Let, $\varphi^F(\mu) = s$. If $s = 1$, then obviously $\tau^F(f^{-1}(\mu)) \le \varphi^F(\mu)$. If $s < 1$, then $\varphi^F(\mu) = s = 1 - (1 - s), s \in (0,1)$

$$\Rightarrow \mu \in \varphi_{1-s}^F, (here 1 - s > 0)$$
$$\Rightarrow f^{-1}(\mu) \in \tau_{1-s}^F, (by (2))$$
$$\Rightarrow \tau^F(f^{-1}(\mu)) \le 1 - (1 - s) = s$$
$$\Rightarrow \tau^F(f^{-1}(\mu)) \le \varphi^F(\mu)$$

So $(ii) \Rightarrow (i)$. Therefore $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map.

Theorem 3.2.28

Let $(X, \tau^T, \tau^I, \tau^F)$, $(Y, \varphi^T, \varphi^I, \varphi^F)$ and $(Z, \psi^T, \psi^I, \psi^F)$ be three FNTSs. If $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$and $g: (Y, \varphi^T, \varphi^I, \varphi^F) \to (Z, \psi^T, \psi^I, \psi^F)$ be gp-maps, then $f \circ g: (X, \tau^T, \tau^I, \tau^F) \to (Z, \psi^T, \psi^I, \psi^F)$ is a gp-map.

Proof: the proof is straightforward.

Definition 3.2.29

Let (P^T, P^I, P^F) be a BT-FNs on X. Then $(Ƀ^T, Ƀ^I, Ƀ^F)$ is said to form a base of (P^T, P^I, P^F) if $Ƀ^T$, $Ƀ^I$ and $Ƀ^F$ are bases of P^T , P^I and P^F, respectively.

Definition 3.2.30

Let (P^T, P^I, P^F) be a BT-FNs on X. Then $(\wp^T, \wp^I, \wp^F)$ is said to form a subbase of (P^T, P^I, P^F) if $\wp^T$, $\wp^I$ and $\wp^F$ are subbases of P^T, P^I and P^F, respectively.

Theorem 3.2.31

Let $(X, \tau^T, \tau^I, \tau^F)$ be an FNTS and $f: X \to Y$ be a mapping.

Let $\{(P_r{}^T, P_r{}^I, P_r{}^F); r \in I_0\}$ be descending family of inclusive BT-FN$_{Subs}$ on Y. Let $(\varphi^T, \varphi^I, \varphi^F)$ be the FNGO on Y generated by this family. Further suppose, for each $r \in I_0$, (B^T, B^I, B^F) or $(\wp^T, \wp^I, \wp^F)$ is a base or a subbase respectively of $(P_r{}^T, P_r{}^I, P_r{}^F)$. Then the following statements are equivalent:

i. $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map.

ii. $\tau^T(f^{-1}(\mu)) \geq r, \forall \mu \in P_r{}^T, \forall r \in I_0;$

$\tau^I(f^{-1}(\mu)) \geq r, \forall \mu \in P_r{}^I, \forall r \in I_0;$

$\tau^F(f^{-1}(\mu)) \leq 1 - r, \forall \mu \in P_r{}^F, \forall r \in I_0;$

iii. $\tau^T(f^{-1}(\mu)) \geq r, \forall \mu \in B_r{}^T, \forall r \in I_0;$

$\tau^I(f^{-1}(\mu)) \geq r, \forall \mu \in B_r{}^I, \forall r \in I_0;$

$\tau^F(f^{-1}(\mu)) \leq 1 - r, \forall \mu \in B_r{}^F, \forall r \in I_0;$

iv. $\tau^T(f^{-1}(\mu)) \geq r, \forall \mu \in \wp_r{}^T, \forall r \in I_0;$

$$\tau^I\left(f^{-1}(\mu)\right) \geq r, \forall \mu \in \wp_r{}^I, \forall r \in I_0;$$

$$\tau^F\left(f^{-1}(\mu)\right) \leq 1 - r, \forall \mu \in \wp_r{}^F, \forall r \in I_0;$$

Proof:

Suppose (i) holds. Then

$\tau^T\left(f^{-1}(\mu)\right) \geq \varphi^T(\mu) \geq r, \forall \mu \in P_r^T, \forall r \in I_0$,

$\tau^I\left(f^{-1}(v)\right) \geq \varphi^I(v) \geq r, \forall v \in P_r^I, \forall r \in I_0$,

$\tau^F\left(f^{-1}(\nu)\right) \leq \varphi^F(\nu) \leq 1 - r, \forall \nu \in P_r^F, \forall r \in I_0$.

So $(i) \Rightarrow (ii)$

Condition $(ii) \Rightarrow (iii) \Rightarrow (iv)$ is obvious.

Suppose (iv) holds.

Let $\mu \in I^X$ and, without loss of generality, let $\varphi^T(\mu) = r > 0$. Then $\mu \in P_r^T$. Now, μ is of the form, $\mu = \bigcup_{i \in \Delta} \lambda_i$; where $\lambda_i \in B_r^T, \forall i \in \Delta$. Also, for each $i \in \Delta, \lambda_i$ is of the form, $\lambda_i = \bigcap_{j=1}^{n_i} v_{i,j}$, where $v_{i,j} \in \wp_r^T, \forall j = 1,2, \dots, n_i$. So

$$\tau\left(f^{-1}(\mu)\right) = \tau\left(f^{-1}\left(\bigcup_{i \in \Delta}\left(\bigcap_{j=1}^{n_i} v_{i,j}\right)\right)\right)$$

$$= \tau\left(\bigcup_{i \in \Delta}\left(\bigcap_{j=1}^{n_i} f^{-1}\left(v_{i,j}\right)\right)\right)$$

$$\geq \bigwedge_{i \in \Delta}\left(\bigwedge_{j=1}^{n_i} \tau^T\left(f^{-1}\left(v_{i,j}\right)\right)\right) \geq r. \left(by (iv)\right)$$

Therefore $\tau^T\left(f^{-1}(\mu)\right) \geq \varphi^T(\mu)$.

Similarly, $\tau^I\left(f^{-1}(v)\right) \geq \varphi^I(v)$,$\tau^F\left(f^{-1}(\nu)\right) \leq \varphi^F(\nu)$. So, $(iv) \Rightarrow (i)$

Theorem 3.2.32

Let (X, P^T, P^I, P^F) and (Y, Q^T, Q^I, Q^F) be two inclusive BT-FN$_{\text{Subs}}$ and $f: X \to Y$. Then,

i. $f: (X, P^T, P^I, P^F) \to (Y, Q^T, Q^I, Q^F)$ is continuous is

ii. $f:(X,(P^T)^r,(P^I)^r,(P^F)^r) \to (Y,(Q^T)^r,(Q^I)^r,(Q^F)^r)$ is a gp-map, $\forall r \in I_0$.

Proof:

Suppose (i) holds. Take $\mu \in I^Y$. Following possibilities arise:

a. $\mu \in \{\tilde{0}_Y, \tilde{1}_Y\}$

b. $\mu \in Q - \{\tilde{0}_Y, \tilde{1}_Y\}$

c. $\mu \in Q^F - Q^T$

d. $\mu \in \tilde{1}_Y - Q$

In case (a), $f^{-1}(\mu) \in \{\tilde{0}_X, \tilde{1}_X\}$ and hence $P_r^T(f^{-1}(\mu)) = 1 = Q_r^T(\mu)$, $P_r^I(f^{-1}(\mu)) = 1 = Q_r^I(\mu)$ and $P_r^F(f^{-1}(\mu)) = 0 = Q_r^F(\mu)$.

In case (b), by (i), $f^{-1}(\mu) \in P^T$ and hence $P_r^T(f^{-1}(\mu)) \geq r = Q_r^T(\mu)$, $P_r^I(f^{-1}(\mu)) \geq r = Q_r^I(\mu)$ and $P_r^F(f^{-1}(\mu)) \geq 1 - r = Q_r^T(\mu)$,(since $P^T \subset P^I \subset P^F$).

In case (c), by (i), $f^{-1}(\mu) \in P^F$ and hence $P_r^T(f^{-1}(\mu)) \geq 0 = Q_r^T(\mu)$, $P_r^F(f^{-1}(\mu)) \geq 0 = Q_r^F(\mu)$ and $P_r^F(f^{-1}(\mu)) \leq 1 - r = Q_r^T(\mu)$.

In case (d), $P_r^T(f^{-1}(\mu)) \geq 0 = Q_r^T(\mu)$, $P_r^F(f^{-1}(\mu)) \geq 0 = Q_r^F(\mu)$ and $P_r^F(f^{-1}(\mu)) \leq 1 = Q_r^T(\mu)$. So, (i) implies (ii).

Conversely, suppose (2) holds. Following possibilities arise:

i. $\mu \in \{\tilde{0}_Y, \tilde{1}_Y\}$

ii. $\mu \in Q - \{\tilde{0}_Y, \tilde{1}_Y\}$

iii. $\mu \in Q^F - Q^T$

In case (a), $f^{-1}(\mu) \in \{\tilde{0}_X, \tilde{1}_X\} \subset P^T, P^I, P^F$.

In case (b), $Q_r^T(\mu) = r$. Hence by (ii), $P_r^T(f^{-1}(\mu)) \geq Q_r^T(\mu) = r$ implies $f^{-1}(\mu) \in P^T$, $P_r^I(f^{-1}(\mu)) \geq Q_r^I(\mu) = r$ implies

$f^{-1}(\mu) \in P^I$ and $P_r^F(f^{-1}(\mu)) \leq Q_r^F(\mu) = 1 - r$ implies $f^{-1}(\mu) \in P^F$.

In case (c), $P_r^F(f^{-1}(\mu)) \leq Q_r^F(\mu) = 1 - r$ implies $f^{-1}(\mu) \in P^F$.

So (ii) implies (i).

Category of Fermatean Neutrosophic topological spaces

Let I_n-BTF denote the category of all-inclusive BT-FN_{Subs} and continuous functions. FN-top denotes the category of all Fermatean neutrosophic topological spaces and gp-maps. For each $r \in I_0$, FNr-top denote the category of rth graded FNTSs and gp-maps.

Theorem 3.2.33

a. FNr-top is a full subcategory of FN-top.
b. For each $r \in I_0$; I_n-BTF and FNr-top are isometric.
c. FNr-top is a bi-reflective full subcategory of FN-top, for all $r \in I_0$.

Proof.

Results (a) and (b) follow from the following facts: $({\tau_r}^T)_r = \tau^T$, $({\tau_r}^I)_r = \tau^I$,$({\tau_r}^F)_r = \tau^F$, if (τ^T, τ^I, τ^F) is r[th] FNGO (by Theorem 3.2.21); $\left({P_r}^T\right)_r = P^T$, $\left({P_r}^I\right)_r = P^I$,$\left({P_r}^F\right)_r = P^F$, if (P^T, P^I, P^F) is an inclusive BT-FN_S (by Theorem 3.2.19); and $f: (X, P^T, P^I, P^F) \to (Y, Q^T, Q^I, Q^F)$ is continuous iff $f: \left(X, {P_r}^T, {P_r}^I, {P_r}^F\right) \to \left(Y, {Q_r}^T, {Q_r}^I, {Q_r}^F\right)$ is a gp-map, for $r \in I_0$ by theorem 3.2.32

To prove (c), let take a member $(X, \tau^T, \tau^I, \tau^F)$ of FN-top. Then for $r \in I_0$; $(X, ({\tau_r}^T)_r, ({\tau_r}^I)_r, ({\tau_r}^F)_r)$ is a member of FNr-top and also $I_X: (X, \tau^T, \tau^I, \tau^F) \to (X, ({\tau_r}^T)_r, ({\tau_r}^I)_r, ({\tau_r}^F)_r)$ is a gp-map.

Let $(Y, \varphi^T, \varphi^I, \varphi^F)$ be a member of FNr-top and $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ be a gp-map. To complete the proof of (c) prove that $f: (X, ({\tau_r}^T)_r, ({\tau_r}^I)_r, ({\tau_r}^F)_r) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map.

$\varphi^T(0_{FN\,Y}) = \tau^T\left(f^{-1}(0_{FN\,Y})\right) = \tau^T(0_{FN\,X}) = ({\tau_r}^T)_r(0_{FN\,X}) = ({\tau_r}^T)_r\left(f^{-1}(0_{FN\,Y})\right) = 1.$

Similarly,

$$\varphi^T(1_{FN\,Y}) = \tau^T\left(f^{-1}(1_{FN\,Y})\right) = 1.$$

$\varphi^I(0_{FN\,Y}) = \tau^I\left(f^{-1}(0_{FN\,Y})\right) = \tau^I(0_{FN\,X}) = ({\tau_r}^I)_r(0_{FN\,X}) = ({\tau_r}^I)_r\left(f^{-1}(0_{FN\,Y})\right) = 1.$

Similarly,

$$\varphi^I(1_{FN\,Y}) = \tau^I\left(f^{-1}(1_{FN\,Y})\right) = 1 \text{ and}$$

$\varphi^F(0_{FN\,Y}) = \tau^F\big(f^{-1}(0_{FN\,Y})\big) = \tau^F(0_{FN\,X}) = ({\tau_r}^F)_r(0_{FN\,X}) = ({\tau_r}^F)_r\big(f^{-1}(0_{FN\,Y})\big) = 0.$
Similarly,

$\varphi^F(1_{FN\,Y}) = \tau^F\big(f^{-1}(1_{FN\,Y})\big) = 0.$

In case $\varphi^T(\mu) = 0$, then obviously, $\varphi^T(\mu) \le ({\tau_r}^T)_r\big(f^{-1}(\mu)\big)$.

If $\varphi^T(\mu) = r \in I_0$, then $\varphi^T(\mu) \le (\tau^T)\big(f^{-1}(\mu)\big)$ implies that $\big(f^{-1}(\mu)\big) \in {\tau_r}^T$ and hence $({\tau_r}^T)_r\big(f^{-1}(\mu)\big) \ge r = \varphi^T(\mu)$.

In case $\varphi^I(\mu) = 0$, then obviously, $\varphi^I(\mu) \le ({\tau_r}^I)_r\big(f^{-1}(\mu)\big)$.

If $\varphi^I(\mu) = r \in I_0$, then $\varphi^I(\mu) \le (\tau^I)\big(f^{-1}(\mu)\big)$ implies that $\big(f^{-1}(\mu)\big) \in {\tau_r}^I$ and hence $({\tau_r}^I)_r\big(f^{-1}(\mu)\big) \ge r = \varphi^I(\mu)$.

In case $\varphi^F(\mu) = 1$, then obviously, $\varphi^F(\mu) \ge ({\tau_r}^F)_r\big(f^{-1}(\mu)\big)$.

If $\varphi^I(\mu) = 1 - r$, then $\varphi^F(\mu) \ge (\tau^F)\big(f^{-1}(\mu)\big)$ implies that $\big(f^{-1}(\mu)\big) \in {\tau_r}^F$ and hence $({\tau_r}^F)_r\big(f^{-1}(\mu)\big) \le 1 - r = \varphi^F(\mu)$.

Thus $f: (X, ({\tau_r}^T)_r, ({\tau_r}^I)_r, ({\tau_r}^F)_r) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map.

Remark 3.2.34

Because of (b) and (c), henceforth, I_n-BTF may be called a bi-reflective full subcategory of FN-top.

Theorem 3.2.35

Let $\{(X_i, {\varphi_i}^T, {\varphi_i}^I, {\varphi_i}^F) : i \in \Delta\}$ be a class of FNTSs and X be a nonempty set. Let for each $i \in \Delta$, $f_i: X \to X_i$ be a map. Then there exists an FNGO (τ^T, τ^I, τ^F) on X such that the following conditions hold:

a. for each $i \in \Delta$, $f_i: (X, \tau^T, \tau^I, \tau^F) \to (X_i, {\varphi_i}^T, {\varphi_i}^I, {\varphi_i}^F)$ is a gp-map.
b. if $(Y, \mathfrak{I}^T, \mathfrak{I}^I, \mathfrak{I}^F)$ is a FNTS, then $g: (Y, \mathfrak{I}^T, \mathfrak{I}^I, \mathfrak{I}^F) \to (X, \tau^T, \tau^I, \tau^F)$ is a gp-map iff $f_i \circ g$ is a gp-map for each $i \in \Delta$.

Proof:

For each $r \in I_0$ and for each $j \in \Delta_0$, define, $\big({P_{j,r}}^T\big) = \Big\{f_j^{-1}(\mu) : \mu \in \big(\varphi_j^T\big)_r\Big\}$, $\big({P_{j,r}}^I\big) = \Big\{f_j^{-1}(\mu) : \mu \in \big(\varphi_j^I\big)_r\Big\}$, and $\big({P_{j,r}}^F\big) = \Big\{f_j^{-1}(\mu) : \mu \in \big(\varphi_j^F\big)_r\Big\}$ where

$\left((\varphi_j^T)_r, (\varphi_j^I)_r, (\varphi_j^F)_r\right)$ is the r-level inclusive BT-FNs on X_j corresponding to the FNGO $\left((\varphi_j^T), (\varphi_j^I), (\varphi_j^F)\right)$.

It can be proved that $\left(P_{j,r}{}^T, P_{j,r}{}^I, P_{j,r}{}^F\right)$ is an inclusive BT-FNs on X. For each $r \in I_0$, define,

$\rho_r{}^T = \bigcup_{j\in\Delta} P_{j,r}{}^T$, $\rho_r{}^I = \bigcup_{j\in\Delta} P_{j,r}{}^I$ and $\rho_r{}^F = \bigcup_{j\in\Delta} P_{j,r}{}^F$, $(j,r) \in \Delta \times I_0$.

Let $P_r{}^T, P_r{}^I$, and $P_r{}^F$ be the FNTSs on X generated by $\rho_r{}^T$, $\rho_r{}^I$and $\rho_r{}^F$, respectively, as subbases. It can be verified that $\{P_r{}^T : r \in I_0\}$, $\{P_r{}^I : r \in I_0\}$ and $\{P_r{}^F : r \in I_0\}$ are descending chains of FNTSs on X.

From construction $P_r{}^T \subset P_r{}^I \subset P_r{}^F, r \in I_0$. Now from Theorem 3.2.16, and an FNGO (τ^T, τ^I, τ^F) on X associated with $\{(P_r{}^T, P_r{}^I, P_r{}^F) : r \in I_0\}$, where $\tau^T(\lambda) = \vee \{r; \lambda \in P_r{}^T\}$, $\tau^I(\lambda) = \vee \{r; \lambda \in P_r{}^I\}$,$\tau^F(\lambda) = \wedge \{1 - r; \lambda \in P_r{}^T\}$.

Let $\mu \in I^{X_j}, \varphi_j^T(\mu) = r, r > 0$. Then $f_j^{-1}(\mu) \in P_{j,r}^T \subset \rho_r^T \subset P_r^T$.

Therefore, $\tau^T\left(f_j^{-1}(\mu)\right) \geq r = \varphi_j^T(\mu)$.

Let $\varphi_j^I(\mu) = r, r > 0$. Then $f_j^{-1}(\mu) \in P_{j,r}^I \subset \rho_r^I \subset P_r^I$.

Therefore, $\tau^I\left(f_j^{-1}(\mu)\right) \geq r = \varphi_j^I(\mu)$.

And let $\varphi_j^F(\mu) = s = 1 - (1 - s), 1 - s > 0$.Then $f_j^{-1}(\mu) \in P_{j,1-s}^T \subset \rho_{1-s}^T \subset P_{1-s}^T$.

Therefore, $\tau^F\left(f_j^{-1}(\mu)\right) \leq s = \varphi_j^F(\mu)$.

Hence $f_i : (X, \tau^T, \tau^I, \tau^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map.

Next let, $g : (Y, \mathfrak{J}^T, \mathfrak{J}^I, \mathfrak{J}^F) \to (X, \tau^T, \tau^I, \tau^F)$ be a gp-map. Since for each $j \in \Delta$, $f_i : (X, \tau^T, \tau^I, \tau^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map; so by Theorem 3.2.28, $f_i \circ g : g : (Y, \mathfrak{J}^T, \mathfrak{J}^I, \mathfrak{J}^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map.

Conversely, suppose for each $j \in \Delta$, $f_i \circ g : g : (Y, \mathfrak{J}^T, \mathfrak{J}^I, \mathfrak{J}^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map.

In order to show that $g : (Y, \mathfrak{J}^T, \mathfrak{J}^I, \mathfrak{J}^F) \to (X, \tau^T, \tau^I, \tau^F)$ is a gp-map, it is sufficient to verify that

$\Im^T(g^{-1}(\mu)) \geq r, \forall \mu \in \rho_r^T, \forall r \in I_0,$

$\Im^I(g^{-1}(\mu)) \geq r, \forall \mu \in \rho_r^I, r \in I_0$

and $\Im^F(g^{-1}(\mu)) \geq 1 - r, \forall \mu \in \rho_r^F, r \in I_0.$

Let, $r \in I_0, \mu \in \rho_r^T$. Then $\mu \in P_{j,r}^T$, for some $j \in \Delta$. So there is $\lambda \in (\varphi_j^T)_r$ such that $f_j^{-1}(\lambda) = \mu$,i.e., $\lambda \circ f_j = \mu$. Since for $j \in \Delta$, $(\varphi_j^T)_r : r \in I_0$ is a family of FNTS associated to φ_j^T on X_j and since $f_i \circ g: (Y, \Im^T, \Im^I, \Im^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map, by Theorem 3.2.31,

$\Im^T((f_i \circ g)^{-1}(\lambda)) \geq r$, i.e, $\Im^T\left(g^{-1}(f_j{}^{-1})(\lambda)\right) \geq r$, $\Im^T(g^{-1}(\mu)) \geq r$.

Let, $r \in I_0, \upsilon \in \rho_r^I$. Then $\upsilon \in P_{j,r}^I$, for some $j \in \Delta$. So there exists some $\lambda' \in (\varphi_j^I)_r$ such that $f_j^{-1}(\lambda') = \upsilon$,i.e., $\lambda' \circ f_j = \upsilon$. Since for $j \in \Delta$, $(\varphi_j^I)_r : r \in I_0$ is a family of FNTS associated to φ_j^I on X_j and since $f_i \circ g: (Y, \Im^T, \Im^I, \Im^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map, by Theorem 3.2.31,

$\Im^I((f_i \circ g)^{-1}(\lambda')) \geq r$, i.e, $\Im^I\left(g^{-1}(f_j{}^{-1})(\lambda')\right) \geq r$, $\Im^I(g^{-1}(\upsilon)) \geq r$.

Let, $r \in I_0, \nu \in \rho_r^F$. Then $\nu \in P_{j,r}^F$, for some $j \in \Delta$. So there is $\lambda'' \in (\varphi_j^F)_r$ such that $f_j^{-1}(\lambda'') = \nu$,i.e., $\lambda'' \circ f_j = \nu$. Since for $j \in \Delta$, $(\varphi_j^F)_r : r \in I_0$ is a family of FNTS associated to φ_j^F on X_j and since $f_i \circ g: (Y, \Im^T, \Im^I, \Im^F) \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ is a gp-map, by Theorem 3.2.31,

$\Im^F((f_i \circ g)^{-1}(\lambda'')) \leq 1 - r$, i.e,

$\Im^F\left(g^{-1}(f_j{}^{-1})(\lambda'')\right) \leq 1 - r$, $\Im^F(g^{-1}(\nu)) \leq 1 - r$.

Therefore, by Theorem 3.2.31 (iv) and (i), $g: (Y, \Im^T, \Im^I, \Im^F) \to (X, \tau^T, \tau^I, \tau^F)$ is a gp-map.

Definition 3.2.36

The FNGO (τ^T, τ^I, τ^F) on X so defined in Theorem 3.2.35 generated by the FNTSs $(X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)$ and the family of functions $\{f_i: X \to X_i\}_{i \in \Delta}$ is called the initial FNGO on X generated by the family $\{f_i\}_{i \in \Delta}$.

Definition 3.2.37

Let $\{(X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)\}_{i\in\Delta}$ be a family of FNTSs and $X = \pi_{i\in\Delta}X_i$ and $p_i: X \to X_i$, $i \in \Delta$ be the projection mapping. Then the initial FNGO on X generated by the family $\{p_i: X \to (X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)\}_{i\in\Delta}$ is called the product FNGO on X and is denoted by $(\pi_{i\in\Delta}\varphi^T{}_i, \pi_{i\in\Delta}\varphi^I{}_i, \pi_{i\in\Delta}\varphi^F{}_i)$. The triplet $(X, \pi_{i\in\Delta}\varphi^T{}_i, \pi_{i\in\Delta}\varphi^I{}_i, \pi_{i\in\Delta}\varphi^F{}_i)$ is called the product FNTS of the family $\{(X_i, \varphi_i{}^T, \varphi_i{}^I, \varphi_i{}^F)\}_{i\in\Delta}$.

Compactness

Several notions of compactness are introduced, initially within the BT-FNSubs framework and subsequently in the FNTS context. The degree of openness on X satisfies the extra requirement.

$\tau^T(\underline{\alpha}) = 1$, for all constant Fermatean neutrosophic sets $\underline{\alpha} \in I^X$, where $\underline{\alpha}: X \to I, \underline{\alpha}(x) = \alpha, \forall x \in X, \forall \alpha \in I$.

The FNGO (FNGC) in this section satisfies the additional condition

(FNGO5) $\tau^T(\underline{\alpha}) = 1, \tau^I(\underline{\alpha}) = 1$, and $\tau^F(\underline{\alpha}) = 0, \forall \alpha \in I$.

(FNGC5) $\varphi^T(\underline{\alpha}) = 1, \varphi^I(\underline{\alpha}) = 1$, and $\varphi^F(\underline{\alpha}) = 0, \forall \alpha \in I$.

Let $(X, \tau^T, \tau^I, \tau^F)$ be a BT-FNS and let $\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F \subset I^X$. Then the collection $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$ is said to be an (α, β, γ)- shading family $(\alpha, \beta, \gamma \in I_1)$ if for any $x \in X$, $\exists \mu^T \in \mathbb{C}^T$, $\mu^I \in \mathbb{C}^I$, $\mu^F \in \mathbb{C}^F$ such that $\mu^T(x) > \alpha, \mu^I(x) > \beta$, $\mu^F(x) > \gamma$.

An (α, β, γ)- shading family $(\wp^T, \wp^I, \wp^F)$ is said to be a subfamily of the (α, β, γ)- shading family $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$ if $\wp^T \subset \mathbb{C}^T$, $\wp^I \subset \mathbb{C}^I$, and $\wp^F \subset \mathbb{C}^F$.

An (α, β, γ)-shading family $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$ is said to be finite if $\mathbb{C}^T$, $\mathbb{C}^I$, and $\mathbb{C}^F$ are finite. For any set A, let $3^{(A)}$ denote the family of all finite subsets of A.

Let, $i_\alpha(P^T) = \{\lambda^{-1}\alpha, 1: \lambda \in P^T\}, \alpha \in I_1$, $i_\beta(P^I) = \{\eta^{-1}\beta, 1: \eta \in P^I\}, \beta \in I_1$, $i_\gamma(P^F) = \{\theta^{-1}\gamma, 1: \theta \in P^F\}, \gamma \in I_1$; and $i(P^T), i(P^I) i(P^F)$ be the topologies generated by $\bigcup_{\alpha\in I_1} i_\alpha(P^T)$, $\bigcup_{\beta\in I_1} i_\beta(P^I)$ and $\bigcup_{\gamma\in I_1} i_\gamma(P^F)$, respectively, as subbases.

Definition 3.2.38

For $\alpha, \beta, \gamma \in I_1$, $(X, \tau^T, \tau^I, \tau^F)$ is said to be (α, β, γ)-compact if each (α, β, γ)-shading family in (τ^T, τ^I, τ^F) has a finite (α, β, γ)-shading subfamily.

Definition 3.2.38

$(X, \tau^T, \tau^I, \tau^F)$ is said to be strongly Fermatean neutrosophic compact if it is (α, β, γ)-compact, $\forall \alpha, \beta, \gamma \in I_1$.

Definition 3.2.40

$(X, \tau^T, \tau^I, \tau^F)$ is said to be ultra- Fermatean neutrosophic compact if $(X, i(\tau^T)), (X, i(\tau^I))$ and $(X, i(\tau^F))$ are compact.

Definition 3.2.41

$(X, \tau^T, \tau^I, \tau^F)$ is said to be Fermatean neutrosophic compact if for three families $\mathbb{C}^T \subset \tau^T$, $\mathbb{C}^I \subset \tau^I$, $\mathbb{C}^F \subset \tau^F$ and for $\alpha, \beta, \gamma \in I$ such that $sup_{\mu \in \mathbb{C}^T} \mu \geq \alpha$, $sup_{\upsilon \in \mathbb{C}^I} \upsilon \geq \beta$, $sup_{\vartheta \in \mathbb{C}^F} \vartheta \geq \gamma$ and for $\varepsilon^T \in 0, \alpha$, $\varepsilon^I \in (0, \beta]$, $\varepsilon^F \in 0, \gamma$ there exist $\mathbb{C}_0^T \in 2^{(\mathbb{C}^T)}$, $\mathbb{C}_0^I \in 2^{(\mathbb{C}^I)}$, $\mathbb{C}_0^F \in 2^{(\mathbb{C}^F)}$such that $sup_{\mu \in \mathbb{C}_0^T} \mu \geq \alpha - \varepsilon^T$, $\sup_{\upsilon \in \mathbb{C}_0^I} \upsilon \geq \beta - \varepsilon^I$, $sup_{\vartheta \in \mathbb{C}_0^F} \vartheta \geq \gamma - \varepsilon^F$.

Definition 3.2.42

Let $(X, \tau^T, \tau^I, \tau^F)$ be an FNTS. Then $(X, \tau^T, \tau^I, \tau^F)$ is said to be (α, β, γ)-compact, strongly Fermatean neutrosophic compact, ultra- Fermatean neutrosophic compact and Fermatean neutrosophic compact, if $(X, {\tau_r}^T, {\tau_r}^I, {\tau_r}^F)$ is (α, β, γ)-compact, strongly Fermatean neutrosophic compact, ultra- Fermatean neutrosophic compact and Fermatean neutrosophic compact, respectively, $\forall r \in I_0$.

Theorem 3.2.43

If $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a surjective gp-map and if $(X, \tau^T, \tau^I, \tau^F)$ is Fermatean neutrosophic compact in any of the above senses, then $(Y, \varphi^T, \varphi^I, \varphi^F)$ is Fermatean neutrosophic compact in the same sense.

Proof:

Let $(X, \tau^T, \tau^I, \tau^F)$ be (α, β, γ)-compact. Then $(X, {\tau_r}^T, {\tau_r}^I, {\tau_r}^F)$ is (α, β, γ)-compact $\forall r \in I_0$. Now, to prove that $(Y, \varphi_r^T, \varphi_r^I, \varphi_r^F)$ is (α, β, γ)-compact.

Let $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$ be an (α, β, γ)- shading family $(\alpha, \beta, \gamma \in I_1)$ in $(\varphi_r^T, \varphi_r^I, \varphi_r^F)$. Since $f: (X, \tau^T, \tau^I, \tau^F) \to (Y, \varphi^T, \varphi^I, \varphi^F)$ is a gp-map,

by Theorem 3.2.27, $f^{-1}(\mu^T) \in \tau_r^T, \forall \mu^T \in \mathbb{C}^T$,$f^{-1}(\mu^I) \in \tau_r^I, \forall \mu^I \in \mathbb{C}^I$ and $f^{-1}(\mu^F) \in \tau_r^F, \forall \mu^F \in \mathbb{C}^F$.

Let $\wp^T = \{f^{-1}(\mu^T): \mu^T \in \mathbb{C}^T\}$, $\wp^I = \{f^{-1}(\mu^I): \mu^I \in \mathbb{C}^I\}$ and $\wp^F = \{f^{-1}(\mu^F): \mu^F \in \mathbb{C}^F\}$. Since $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$ is an (α, β, γ)- shading family in $({\tau_r}^T, {\tau_r}^I, {\tau_r}^F)$. By the (α, β, γ)-compactness of $(X, {\tau_r}^T, {\tau_r}^I, {\tau_r}^F)$,$(\wp^T, \wp^I, \wp^F)$ has a finite (α, β, γ)- shading subfamily, say $(\varepsilon^T, \varepsilon^I, \varepsilon^F)$ in $({\tau_r}^T, {\tau_r}^I, {\tau_r}^F)$.

Let $\mathbb{C}_0^T = \{f(\nu^T): \nu^T \in \varepsilon^T\}$, $\mathbb{C}_0^I = \{f(\nu^I): \nu^I \in \varepsilon^I\}$,$\mathbb{C}_0^F = \{f(\nu^F): \nu^F \in \varepsilon^F\}$. Since $f: X \to Y$ is surjective and $(\varepsilon^T, \varepsilon^I, \varepsilon^F)$is an$(\alpha, \beta, \gamma)$-shading, it follows that $(\mathbb{C}_0^T, \mathbb{C}_0^I, \mathbb{C}_0^F)$ is an (α, β, γ)-shading.

Also $(\mathbb{C}_0^T, \mathbb{C}_0^I, \mathbb{C}_0^F)$ is a finite subfamily of $(\mathbb{C}^T, \mathbb{C}^I, \mathbb{C}^F)$. Hence $(Y, \varphi_r^T, \varphi_r^I, \varphi_r^F)$ is (α, β, γ)-compact for each $r \in I_0$.Therefore $(Y, \varphi^T, \varphi^I, \varphi^F)$ is (α, β, γ)-compact.

Similarly, it can be easily shown that $(Y, \varphi^T, \varphi^I, \varphi^F)$ is Fermatean neutrosophic compact.

Theorem 3.2.44

Each FNTS of a family $\{(X, \tau^T, \tau^I, \tau^F)\}_{i \in \Delta}$ is Fermatean neutrosophic compact in any one of the above senses iff $(\pi_{i \in \Delta} X_i, \pi_{i \in \Delta} {\tau_i}^T, \pi_{i \in \Delta} {\tau_i}^I, \pi_{i \in \Delta} {\tau_i}^F)$ is so ($\pi_{i \in \Delta} X_i$ is assumed to be nonvoid).

Proof:

Since, $\forall i \in \Delta$, $pi: (\pi_{i \in \Delta} X_i, \pi_{i \in \Delta} {\tau_i}^T, \pi_{i \in \Delta} {\tau_i}^I, \pi_{i \in \Delta} {\tau_i}^F) \to (X_i, {\tau_i}^T, {\tau_i}^I, {\tau_i}^F)$ is a surjective gp-map, $(\pi_{i \in \Delta} X_i, \pi_{i \in \Delta} {\tau_i}^T, \pi_{i \in \Delta} {\tau_i}^I, \pi_{i \in \Delta} {\tau_i}^F)$ is Fermatean neutrosophic compact in any of the above senses implies that $(X_i, {\tau_i}^T, {\tau_i}^I, {\tau_i}^F)$,$i \in \Delta$; is also Fermatean neutrosophic compact in the same senses (by theorem 3.2.43)

Conversely, suppose that each $(X_i, \tau_i{}^T, \tau_i{}^I, \tau_i{}^F), i \in \Delta$, is Fermatean neutrosophic compact in any one of the above senses. Then, for each $r \in I_0$ and for each $i \in \Delta$, $(X_i, (\tau_i{}^T)_r, (\tau_i{}^I)_r, (\tau_i{}^F)_r)$ is Fermatean neutrosophic compact (as an inclusive BT-FNs).

So, by Lowen, $\left(\pi_{i\in\Delta} X_i, P_r{}^T, P_r{}^I, P_r{}^F\right)$ is compact (where $P_r{}^T, P_r{}^I, P_r{}^F$ are constructed as in Theorem 3.2.35).

Further, $P_r{}^T \subseteq \tau_r{}^T \subseteq P_s{}^T, P_r{}^I \subseteq \tau_r{}^I \subseteq P_s{}^I, P_r{}^F \subseteq \tau_r{}^F \subseteq P_s{}^F, \forall r, s \in I_0$ with $s < r$. So, $(\pi_{i\in\Delta} X_i, \tau_r{}^T, \tau_r{}^I, \tau_r{}^F)$ is compact, $\forall r \in I_0$.

Hence $(\pi_{i\in\Delta} X_i, \pi_{i\in\Delta} \tau_i{}^T, \pi_{i\in\Delta} \tau_i{}^I, \pi_{i\in\Delta} \tau_i{}^F)$ is Fermatean neutrosophic compact.

3.3 Fermatean Temporal Neutrosophic Topology

In this section, Fermatean temporal neutrosophic topology is defined in the context of the frameworks established by Sostak, Chang, and Lowen.

Fermatean Temporal Neutrosophic Topology in *Šostak's* sense

Fermatean temporal neutrosophic topological spaces (FT-NTS) take the form (X, τ_t) where $\tau_t = \left(T_{\tau_t}, I_{\tau_t}, F_{\tau_t}\right)$

Definition 3.3.1

Fermatean Temporal Neutrosophic Topology in **Šostak's** sense (briefly, SFT-NT) on a non-empty set X is a Fermatean temporal neutrosophic set, τ_t defined with $\tau_t(A) = \left(T_{\tau_t}(A), I_{\tau_t}(A), F_{\tau_t}(A)\right)$ on X satisfying the following axioms for each time moment *t*

i. $\tau_t\left(T_{0_{FN}}\right) = \tau_t\left(I_{0_{FN}}\right) = 0,\ \tau_t\left(T_{1_{FN}}\right) = \tau_t\left(I_{1_{FN}}\right) = 1$, and $\tau_t\left(F_{0_{FN}}\right) = 1, \tau_t\left(F_{1_{FN}}\right) = 0$

ii. $\tau_t(A_1 \cap A_2) \geq \tau_t(A_1) \wedge \tau_t(A_2)$ for any sets $A_1, A_2 \in NS(X)$.

iii. $\tau_t(\cup A_i) \geq \bigwedge_{i \in J} \left(\tau_t(A_i)\right)$ for $\{A_i; i \in J\} \subseteq TNS(X)$.

The pair (X, τ_t) is known as a SFT-NT. Any $A \in TNS(X)$, the numbers are truth $T_{\tau_t}(A)$, indeterminacy $I_{\tau_t}(A)$, and $F_{\tau_t}(A)$ falsity-membership functions of A. $T_{\tau_t}(A)$, $I_{\tau_t}(A)$ are independent components and $F_{\tau_t}(A)$ is dependent component with respect to $T_{\tau_t}(A)$.

Proposition 3.3.2

Let (X, τ_t) be a SFT-NTS on X and T be a time-moment set. Then $(X, \wedge \tau_t)$ defined by $\wedge \tau_t(A) = \left(\min_{t \in T} T_{\tau_t}(A), \min_{t \in T} I_{\tau_t}(A), \max_{t \in T} F_{\tau_t}(A)\right)$ is a Fermatean neutrosophic topology on $FTNS^{(X,T)}$ in Sostak's sense.

Definition 3.3.3

Let (X, τ_t) be a SFT-NTS and $A \in FTNS^{(X,T)}$. Then define instant (a specific point in time) closure and instant interior of A at time moment t_0 according to τ_t respectively as:

$$Cl^{t_0}(A) = \cap \left\{K \in FTNS(X); \tau_{t_0}(K) > 0_{FN}, A \subseteq K\right\}$$

$$int^{t_0}(A) = \cup \left\{K \in FTNS(X); \tau_{t_0}(K) > 0_{FN}, K \subseteq A\right\}$$

On the other hand, (α, β, γ)-instant closure and (α, β, γ)-instant interior of A are defined by:

$$Cl^{t_0}_{(\alpha,\beta,\gamma)}(A) = \cap \left\{K \in FTNS(X); \tau_{t_0}(K) > \langle \alpha, \beta, \gamma \rangle, A \subseteq K\right\}$$

$$int^{t_0}_{(\alpha,\beta,\gamma)}(A) = \cup \left\{K \in FTNS(X); \tau_{t_0}(K) > \langle \alpha, \beta, \gamma \rangle, K \subseteq A\right\}$$

where $\alpha \in 0,1, \beta \in [0,1], \gamma \in 0,1)$ with $\alpha^3 + \beta^3 + \gamma^3 \leq 2$.

Definition 3.3.4

Let (X, τ_t) and (Y, ϕ_t) be SFT-NTSs respectively for non-empty sets X, Y, time sets T' and T''. Let $f: X \to Y$ be a function. Then,

i. The preimage of $B \in FTNS^{(Y,T'')}$ under f at time moment t is defined as $f^{-1}(B) = \left\{\left(x, f\left(\overline{T}_B\right)(x,t), f\left(\overline{I}_B\right)(x,t), f\left(\overline{F}_B\right)(x,t)\right) : x \in X\right\}$ where

$$\overline{T}_B(f(x),t) = \begin{cases} T_B(f(x),t), & \text{if } t \in T'' \\ 0, & \text{if } t \in T' - T'' \end{cases}$$

$$\overline{I}_B(f(x),t) = \begin{cases} I_B(f(x),t), & \text{if } t \in T'' \\ 0, & \text{if } t \in T' - T'' \end{cases}$$

$$\overline{F}_B(f(x),t) = \begin{cases} F_B(f(x),t), & \text{if } t \in T'' \\ 1, & \text{if } t \in T' - T'' \end{cases}$$

ii. The image of $A \in FTNS^{(Y,T')}$ under f at time moment t is defined as $f(A) = \left\{\left(y, f\left(\overline{T}_A\right)(y,t), f\left(\overline{I}_A\right)(y,t), f\left(\overline{F}_A\right)(y,t)\right) : y \in Y\right\}$ where

$$f\left(\overline{T}_A\right)(y,t) = \begin{cases} f(T_A)(y,t), & \text{if } t \in T' \\ 0, & \text{if } t \in T'' - T' \end{cases}$$

$$f\left(\overline{I}_A\right)(y,t) = \begin{cases} f(I_A)(y,t), & \text{if } t \in T' \\ 0, & \text{if } t \in T'' - T' \end{cases}$$

$$f(\overline{F}_A)(y,t) = \begin{cases} 1 - f(T_A)(y,t), & \text{if } t \in T' \\ 0, & \text{if } t \in T'' - T' \end{cases}$$

If $T'' = T'$, It is clearly understood that

$$f^{-1}(B) = \{(x, T_B(f(x),t), I_B(f(x),t), F_B(f(x),t)) : x \in X\}$$

$$f(A) = \{(y, f(T_A)(y,t), f(I_A)(y,t), f(F_A)(y,t)) : y \in Y\}$$

Let (X, τ_t) and (Y, ϕ_t) be SFT-NTSs for non-empty sets X, Y and time set T. If $\tau_t(f^{-1}(B)) \geq \phi_t(B)$ for $t \in T$ and each $B \in FTNS^{(Y,T)}$, f is called Fermatean temporal neutrosophic continuous function at time moment t.

If f is Fermatean temporal neutrosophic continuous function at each time moment, f is called overall Fermatean neutrosophic continuous function.

On the other hand, if $\phi_t(f(A)) \geq \tau_t(A)$ for $t \in T$ and each $A \in FTNS^{(X,T')}$, f is called Fermatean temporal neutrosophic open function at time moment t. If f is Fermatean temporal neutrosophic open function at each time moment, f is called overall Fermatean neutrosophic open function.

Fermatean Temporal Neutrosophic Topology in *Chang's* sense

Definition 3.3.5

A Fermatean temporal neutrosophic topology in **Chang**'s sense (CFT-NT) on a non-empty set X is a family τ_{t_0} of FTNSs fulfilling the following axioms for fixed time moment t_0

i. ${0_{FN}}^{t_0} \in \tau_{t_0}$ and ${1_N}^{t_0} \in \tau_{t_0}$

ii. For each $A_1, A_2 \in \tau_{t_0}$, there exist a $F \in \tau_{t_0}$ such that $T_F(x,t_0) = T_{A_1 \cap_{t_0} A_2}(x,t_0)$, $I_F(x,t_0) = I_{A_1 \cap_{t_0} A_2}(x,t_0)$, $F_F(x,t_0) = F_{A_1 \cap_{t_0} A_2}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

iii. For any arbitrary family $\{A_i; i \in I\} \in \tau_{t_0}$, there exist a $D \in \tau_{t_0}$ such that $T_D(x,t_0) = T_{\cup_{i \in J}^{t_0} A_i}(x,t_0)$, $I_D(x,t_0) = I_{\cup_{i \in J}^{t_0} A_i}(x,t_0)$, $F_D(x,t_0) = F_{\cup_{i \in J}^{t_0} A_i}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

The pair $\left((X,T),\tau_{t_0}\right)$ is called CFT-NT. Any member of τ_{t_0} is called Fermatean temporal neutrosophic open set (FTN- open set).

On the other hand, the complement of any member of τ_{t_0} is called Fermatean temporal neutrosophic closed set (FTN-closed set).

Proposition 3.3.6

Let τ_{t_0} is an Fermatean temporal neutrosophic topological space in Chang's sense on non-empty set X and time moment set T, Then, define FNS's from every $A \in \tau_{t_0}$ FTNSs by following way: $T_{\hat{A}}(x) = T_{A_i}(x,t_0)$, $I_{\hat{A}}(x) = I_{A_i}(x,t_0)$ and $F_{\hat{A}}(x) = F_{A_i}(x,t_0)$. So that the new family $\tau^{t_0} = \{\hat{A}: A \in \tau_{t_0}\}$ obtained from τ_{t_0} is a Fermatean neutrosophic topology in Chang's sense.

Definition 3.3.7

A Fermatean Temporal neutrosophic topology in **Lowen**'s sense (LFT-NT) on a non-empty set X is a family τ_{t_0} of FTNSs satisfying the following axioms for fixed time moment t_0

i. For any fixed FN-pair (α,β,γ), let define FTNSs
$$A = (x, T_A(x,t), I_A(x,t), F_A(x,t): (x,t) \in X \times T)$$
such that $T_A(x,t_0) = \alpha$, $I_A(x,t_0) = \beta$, and $F_A(x,t_0) = \gamma$ for every $(x,t_0) \in X \times \{t_0\}$

ii. For each $A_1, A_2 \in \tau_{t_0}$, there exist a $F \in \tau_{t_0}$ such that $T_F(x,t_0) = T_{A_1 \cap_{t_0} A_2}(x,t_0)$ $I_F(x,t_0) = I_{A_1 \cap_{t_0} A_2}(x,t_0)$, $F_F(x,t_0) = F_{A_1 \cap_{t_0} A_2}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

iii. For any arbitrary family $\{A_i; i \in I\} \in \tau_{t_0}$, there exist a $D \in \tau_{t_0}$ such that $T_D(x,t_0) = T_{\cup_{i\in J}^{t_0} A_i}(x,t_0)$, $I_D(x,t_0) = I_{\cup_{i\in J}^{t_0} A_i}(x,t_0)$ and $F_D(x,t_0) = F_{\cup_{i\in J}^{t_0} A_i}(x,t_0)$ for each $(x,t_0) \in X \times \{t_0\}$.

The pair $\left((X,T),\tau_{t_0}\right)$ is called LFT-NT.

Definition 3.3.8

Let τ_{t_0} is a Fermatean temporal neutrosophic topological space in Chang's sense on non-empty set X and time moment set T and $A \in \tau_{t_0}$. Then Fermatean temporal neutrosophic interior and Fermatean temporal neutrosophic closure of A defined as follows:

$$int_{t_0}(A) = \cup \{G; G \in \tau_{t_0}, G \subseteq A\}$$

$$Cl_{t_0}(A) = \cap \{C; \overline{C} \in \tau_{t_0}, A \subseteq C\}$$

Definition 3.3.9

Let τ_{t_0} be a Fermatean temporal neutrosophic topological space in Chang's sense on non-empty set X with time moment. Then,

i. A is a FTN- closed set in $\tau_{t_0} \Leftrightarrow Cl_{t_0}(A) = A$

ii. A is a FTN- open set in $\tau_{t_0} \Leftrightarrow int_{t_0}(A) = A$

iii. $Cl_{t_0}(A) = \overline{int_{t_0}(A)}$ for any $A \in FTNS^{(X,T)}$

iv. $int_{t_0}(A) = \overline{Cl_{t_0}(A)}$ for any $A \in FTNS^{(X,T)}$

v. $int_{t_0}(A) \subseteq A$ for any $A \in FTNS^{(X,T)}$

vi. $A \subseteq Cl_{t_0}(A)$ for any $A \in FTNS^{(X,T)}$

vii. $A \subseteq B \Rightarrow int_{t_0}(A) \subseteq int_{t_0}(B)$ for any $A, B \in FTNS^{(X,T)}$

viii. $A \subseteq B \Rightarrow Cl_{t_0}(A) \subseteq Cl_{t_0}(B)$ for any $A, B \in FTNS^{(X,T)}$

ix. $Cl_{t_0}\left(Cl_{t_0}(A)\right) = Cl_{t_0}(A)$ for any $A \in FTNS^{(X,T)}$

x. $int_{t_0}\left(int_{t_0}(A)\right) = int_{t_0}(A)$ for any $A \in FTNS^{(X,T)}$

xi. $int_{t_0}(A \cap B) = int_{t_0}(A) \cap int_{t_0}(B)$ for any $A, B \in FTNS^{(X,T)}$

xii. $Cl_{t_0}(A \cup B) = Cl_{t_0}(A) \cup Cl_{t_0}(B)$ for any $A, B \in FTNS^{(X,T)}$

xiii. $int_{t_0}\left(1_N^{t_0}\right) = 1_N^{t_0}$

xiv. $Cl_{t_0}\left(0_N^{t_0}\right) = 0_N^{t_0}$

Proof:

The proof is straightforward

Definition 3.3.10

Let $\left((X,T'),\tau_{t_0}\right)$ and $\left((Y,T''),\phi_{t_0}\right)$ be two CFT-NTSs and let $f: X \to Y$ be a function. Then f known as Fermatean temporal neutrosophic continuous at time moment t_0 if and only if the Fermatean temporal preimage of each FTNS in ϕ_{t_0} is an FTNS in τ_{t_0}.

Definition 3.3.11

Let $\left((X,T'),\tau_{t_0}\right)$ and $\left((Y,T''),\phi_{t_0}\right)$ be two CFT-NTSs and let $f: X \to Y$ be a function. Then f said to be Fermatean temporal neutrosophic open at time moment t_0 if and only if the Fermatean temporal image of each FTNS in τ_{t_0} is an FTNS in ϕ_{t_0}.

Proposition 3.3.12

$f:\left((X,T'),\tau_{t_0}\right) \to \left((Y,T''),\phi_{t_0}\right)$ is Fermatean temporal neutrosophic continuous iff the Fermatean temporal preimage of each FTN- closed set in ϕ_{t_0} is an FTN- closed set in τ_{t_0}.

Proposition 3.3.13

The following statements are equivalent to each other

a. $f:\left((X,T'),\tau_{t_0}\right) \to \left((Y,T''),\phi_{t_0}\right)$ is Fermatean temporal neutrosophic continuous,

b. $f^{-1}\left(int_{t_0}(B)\right) \subseteq int_{t_0} f^{-1}(B)$ for each $B \in FTNS^{(Y,T'')}$.

c. $Cl_{t_0}\left(f^{-1}(B)\right) \subseteq f^{-1}\left(Cl_{t_0}(B)\right)$ for each $B \in FTNS^{(Y,T'')}$

Proposition 3.3.14

Let τ_{t_0} be a Fermatean temporal neutrosophic topological space in Chang's sense on non-empty set X and time moment set t_0 and $\langle \alpha_{t_0}, \beta_{t_0}, \gamma_{t_0} \rangle$ is a FN-pair. Then, the FNT $\tau_{t_0}^{\langle \alpha_{t_0}, \beta_{t_0}, \gamma_{t_0} \rangle}$ which is defined as follows:

$$\tau_{t_0}{}^{\langle \alpha_{t_0},\beta_{t_0},\gamma_{t_0}\rangle}(A) = \begin{cases} \langle \alpha_{t_0},\beta_{t_0},\gamma_{t_0}\rangle & \text{if } A \in \tau_{t_0} - \{0_N^t, 0_N^t, 1_N^t\} \\ \langle 1,1,0\rangle & \text{if } A \in \tau\{0_N^t, 0_N^t, 1_N^t\} \\ \langle 0,0,1\rangle & \text{if } A \notin \tau_{t_0} - \{0_N^t, 0_N^t, 1_N^t\} \end{cases}$$

is a Fermatean temporal neutrosophic topology in Sostak's sense. This SFT-NTS is called as $\langle \alpha_{t_0},\beta_{t_0},\gamma_{t_0}\rangle$- SFT-NTS.

Definition 3.3.15

An Overall Fermatean neutrosophic topology in Chang's sense (Overall-CFT-NT) on a non-empty set X is a family τ_t of FTNSs satisfying the following axioms for each time moment t

i. $0_N^t \in \tau_t$ and $1_N^t \in \tau_t$.

ii. For each $A_1, A_2 \in \tau_t$, there exist a $F \in \tau_t$ such that $T_F(x,t) = T_{A_1 \cap A_2}(x,t)$, $I_F(x,t) = I_{A_1 \cap A_2}(x,t)$, $F_F(x,t) = F_{A_1 \cap A_2}(x,t)$ for each $(x,t) \in X \times T$.

iii. For any arbitrary family $\{A_i; i \in I\} \in \tau_t$, there exist a $D \in \tau_t$ such that $T_D(x,t) = T_{\bigcup_{i\in I} A_i}(x,t)$,$I_D(x,t) = I_{\bigcup_{i\in I} A_i}(x,t)$,$F_D(x,t) = F_{\bigcup_{i\in I} A_i}(x,t)$ for each $(x,t) \in X \times T$.

By this definition, it is understood that a CFT-NT can be obtained for each time moment in an Overall-CFT-NT.

Theorem 3.3.16

Let τ_t be an Overall-CFT-NT on non-empty set X and finite time moment set T. Then, family of FTNSs τ which defined as:

$$\bigcap_{A\in\tau_t} A = \left\{\left\langle x, \underset{t\in T}{\wedge} T_A(x,t), \underset{t\in T}{\wedge} I_A(x,t), \underset{t\in T}{\vee} F_A(x,t)\right\rangle\right\} \in \tau$$

is a Fermatean neutrosophic topological space in Chang's sense on FTNS.

Proof:

In order to prove the theorem, it is necessary to show that the conditions of Fermatean neutrosophic topology in Chang's sense are satisfied.

i. Since $0_N^{t_0} \in \tau_t$ for each $t_0 \in T$, $\underset{t_0 \in T}{\wedge} 0_N^{t_0}(x,t) = 0$ and $\underset{t_0 \in T}{\vee} 0_N^{t_0}(x,t) = 1$ then $\underset{t_0 \in T}{\cap} 0_N^{t_0} = 0_N^{t}$. Then it is understood that $0_N^{t} \in \tau_t$. It can be shown by similar method $0_N^{t} \in \tau_t$.

ii. Let $\overset{\wedge}{A_i} = \cap_{A_i \in \tau_t} A_i = \left\{\left\langle x, \underset{t \in T}{\wedge} T_{A_i}(x,t), \underset{t \in T}{\wedge} I_{A_i}(x,t), \underset{t \in T}{\vee} F_{A_i}(x,t)\right\rangle\right\}$ and $\overset{\wedge}{A_j} = \cap_{A_j \in \tau_t} A_j = \left\{\left\langle x, \underset{t \in T}{\wedge} T_{A_j}(x,t), \underset{t \in T}{\wedge} I_{A_j}(x,t), \underset{t \in T}{\vee} F_{A_j}(x,t)\right\rangle\right\}$. Then

$$\overset{\wedge}{A_i} \cap \overset{\wedge}{A_j} = \left(\bigcap_{A_i \in \tau_t} A_i\right) \cap \left(\bigcap_{A_j \in \tau_t} A_j\right)$$

$$= \left\{\left\langle \begin{array}{c} x, \left(\underset{t \in T}{\wedge} T_{A_i}(x,t)\right) \wedge \left(\underset{t \in T}{\wedge} T_{A_j}(x,t)\right), \\ \left(\underset{t \in T}{\wedge} I_{A_i}(x,t)\right) \wedge \left(\underset{t \in T}{\wedge} I_{A_j}(x,t)\right), \\ \left(\underset{t \in T}{\vee} F_{A_i}(x,t)\right) \vee \left(\underset{t \in T}{\vee} F_{A_j}(x,t)\right) \end{array} \right\rangle\right\}$$

$$= \left\{\left\langle \begin{array}{c} x, \left(\underset{t \in T}{\wedge} \left(T_{A_i}(x,t) \wedge T_{A_j}(x,t)\right)\right), \\ \left(\underset{t \in T}{\wedge} \left(I_{A_i}(x,t) \wedge I_{A_j}(x,t)\right)\right), \\ \left(\underset{t \in T}{\vee} \left(F_{A_i}(x,t) \vee F_{A_j}(x,t)\right)\right) \end{array} \right\rangle\right\}$$

$= \cap_{t \in T} A_i \cap A_j$. Since τ_t is Overall-CFT-NTS. It is understood that $\overset{\wedge}{A_i} \cap \overset{\wedge}{A_j} \in \tau$.

iii. Let $\overset{\wedge}{A_i} = \cup_{A_i \in \tau_t} A_i = \left\{\left\langle x, \underset{t \in T}{\vee} T_{A_i}(x,t), \underset{t \in T}{\vee} I_{A_i}(x,t), \underset{t \in T}{\wedge} F_{A_i}(x,t)\right\rangle\right\}$ and $\overset{\wedge}{A_j} = \cup_{A_j \in \tau_t} A_j = \left\{\left\langle x, \underset{t \in T}{\vee} T_{A_j}(x,t), \underset{t \in T}{\vee} I_{A_j}(x,t), \underset{t \in T}{\wedge} F_{A_j}(x,t)\right\rangle\right\}$.

Then

$$\overset{\wedge}{A_i} \cup \overset{\wedge}{A_j} = \left(\bigcup_{A_i \in \tau_t} A_i\right) \cup \left(\bigcup_{A_i \in \tau_t} A_j\right)$$

$$= \left\{ \left\langle \begin{array}{c} x, \left(\bigvee_{t \in T} T_{A_i}(x,t) \right) \vee \left(\bigvee_{t \in T} T_{A_j}(x,t) \right), \\ \left(\bigvee_{t \in T} I_{A_i}(x,t) \right) \vee \left(\bigvee_{t \in T} I_{A_j}(x,t) \right), \\ \left(\bigwedge_{t \in T} F_{A_i}(x,t) \right) \wedge \left(\bigwedge_{t \in T} F_{A_j}(x,t) \right) \end{array} \right\rangle \right\} = \left\{ \left\langle \begin{array}{c} x, \left(\bigvee_{t \in T} \left(T_{A_i}(x,t) \vee T_{A_j}(x,t) \right) \right), \\ \left(\bigvee_{t \in T} \left(I_{A_i}(x,t) \vee I_{A_j}(x,t) \right) \right), \\ \left(\bigwedge_{t \in T} \left(F_{A_i}(x,t) \wedge F_{A_j}(x,t) \right) \right) \end{array} \right\rangle \right\}$$

$= \bigcup_{t \in T} A_i \cap A_j$. Since τ_t is Overall-CFT-NTS. It is understood that $\overset{\wedge}{A_i} \cup \overset{\wedge}{A_j} \in \tau$.