Certain Studies relating to Birnbaum-Saunders Distribution and Generalized Birnbaum-Saunders Distribution.

Deepika K

(17PMA005)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore - 641043

In Partial Fulfilment of the Requirements for the Degree of

Master of Science in Mathematics

April, 2019

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Signature of the	**Signature of the**
Head of the Department	**Supervisor**

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Abstract

1 Introduction

Reference

ABSTRACT

Abstract

This dissertation is devoted to the study of Birnbaum – Saunders Distribution and Generalized Birnbaum – Saunders Distribution .

The first chapter deals with basic concepts of Acceptance Sampling plan , Reliability , Theory of Estimation, Life time distributions, Notations and Review of literature.

Second chapter deals with the acceptance sampling plans for truncated life tests based on the Generalized Birnbaum-Saunders Distributions. The minimum sample size necessary to ensure the specified median life was obtained by assuming that the life times of the test units follow a generalized Birnbaum –Saunders Distribution. The operating characteristics values of the sampling plans as well as producer's risks were presented. Example is given to illustrate the developed procedure.

The third chapter deals with the shape of the hazard function of Birnbaum Saunders distribution. The hazard function of Birnbaum Saunders distribution which is an upside down function for all values of the shape parameter has been established. To determine the point at which the hazard function reaches its maximum, different estimators of that point have been proposed and evaluate their performance using Monte Carlo simulations. Then data are analyzed and all the inferential methods developed are illustrated.

The fourth chapter deals with the class of Generalized Birnbaum – Saunders Distribution , which is very flexible family of suitable for modelling lifetime data as it allows for different degrees of kurtosis and asymmetry and unimodality as well as bimodality. The theoretical developments on the model including properties, transformations and related distributions , lifetime analysis and shape analysis were described. The methods of inference based on uncensored and censored data , diagnostics methods , goodness –of –fit tests , and random number

generation algorithms for the Generalized Birnbaum – Saunders model have been discussed. The illustrated examples shows that the distribution fits the data.

Fifth chapter deals with the Birnbaum –Saunders model which is a fatigue life Distribution related to the normal one with appealing properties. It is non-negative transformation of N(0,1) random variable Z , and its extended version called the Generalized Birnbaum-Saunders model obtained replacing Z by any other symmetric continuous random variable . The Generalized Birnbaum-Saunders Distribution according to one of its properties and use the characterization to derive a graphical procedure to assess whether an observed data set follows Generalized Birnbaum-Saunders Distribution have been characterized . Goodness – of – fit test for the hypothesis , which the data follows Generalized Birnbaum-Saunders Distribution have been developed .

CHAPTER- 1

Chapter – 1

Introduction

Section – 1.1

Basic concepts of Acceptance sampling

1.1.1 Introduction

Acceptance sampling procedure is an essential tool in statistical quality control. It is a methodology that deals with quality contracting on product oreder between the producers and consumers and thus allowing the producers to take the decision to accept or reject the manufactured products based on the inspection of samples.

Acceptance sampling is necessary to limit the cost of inspection and is the only avaiable method to appraise the quality in destructive testing. Acceptance sampling itself does not improve quality, but whenever the lot is rejected it indicates the instability of the production process. Acceptance sampling is cost efficient and on admissible method of efficient tests with quick results.

1.1.2 Importance of Acceptance sampling

Acceptance sampling is one of the latest aspects of quality assurance and used primarily for the incoming and outgoing lot by lot quality assurance. The most effective use of acceptance sampling is not to " inspect quality into the product" , but rather as an audit tool to ensure that the output of a process confirms to requirements.

According to Duncan (1986), an acceptance sampling is likely to be implemented

- When the cost of inspection is high and the loss arising from the passing of a non confirming unit is not great; it is also possible in some cases that no inspection at all will be the cheapest plan.
- When 100 percent inspection is fatiguing, a carefully worked out sampling plan will produce good orbetter results. The 100 percent, may not mean

100percent perfect quality, and the percentage of non conforming items passed may be higher than under a scientifically designed sampling plan.

- When inspection is destructive i.e., a situation where inspection is not possible without destroying the article chemically or physically.
- When there are great quantities areas to be inspected.
- When it is desired to stimulate the buyer.

1.1.3 Major Areas of Acceptance Sampling

According to Dodge(1959), the major areas of acceptance sampling are

- Lot – by- lot sampling by the method of attributes, in which each unit in a sample is inspected on a go- not-go basis for one or more characteristics.
- Lot – by- lot sampling by the method of variables, in which each unit in a sample is measured for a single characteristics such as weight or strength .
- Continuous sampling of a flow of units by the method of attributes.
- Special purpose plan including chain sampling , skip-lot sampling, small sample plans,repetitve group sampling palns ,etc.,

1.1.4 Basic terminologies

Sampling plan

According to the American National Standards Institute / American Society for quality control (ANSI/ ASQC) Standard A2 (1987), an acceptance sampling plan is a specific plan that states the sampling rules to be used with the associated acceptance and non acceptance criteria.

Acceptance quality level (AQL)

The AQL is a percent defective that is the base line requirement for the quality of the producer's product. The produver would like to design a sampling plan such that there is ahigh probability of accepting a lot that has a defect level less than or equal to the AQL.

Single sampling plan

Sampling inspection in which the decision to accept a lot is based on the inspection of a single of sample siza 'n'.

Section - 1.2

Basic Concepts of Reliability

1.2.1 Introduction

The concepts of reliability has been known for a number of years, but it hasassumed greater significance and importance during the past decade, particularly due to the impact of automations, developments in complex missile and space programmes. The manufacture of highly complex equipment has served to focus greater attention on reliability. However, reliability is only one of the tools of management which must be supplemented by the other tools like quality control and design of experiments for the solution of problems of quality and cost. Reliability is the probability of a device performing its purpose adequately for the period of time intended under the operating conditions encountered. Reliability of a product is the measure of the ability of a product to fuction successfully, when required, in the specified environment.

Study of reliability is important because it is related to the quality of a product. Reliability of a product is more important because it is common for a person to think that , what is the use of buying a product that does not satisfy the customer needs and fails within a short period. Thus the effectiveness of a system is understood to mean the suitability of the system for the fulfillment of the intended tasks and the efficiency og utilizing the means put into it. The suitability of performing definite tasks is primarily determined by the quality of the system.

1.2.2 Definition of Reliability

Reliability of a unit is the probability that the unit performs its intended function adequately for a given period of time under the stated operating conditions or environment. By a unit we mean an element, a system or a part of a sysem. If T is the time till the failure of the unit (a random variable) occurs, then the probability that it will not fail in a given environment before time t (or its reliability) is $R(t) = P(T>t)$

Thus, the reliability is always a function of time. It also depends on environmental conditions which may or may not vary with time. Since it is a probability, its numerical value is always between 1 and 0, that is $R(0) = 1$, $R(\infty) = 0$ and $R(t)$ is a non increasing function between these limits.

1.2.3 Basic Elements of Reliability

The reliability definition stresses five element mainly

- Numerical value of probability
- Statement defining successful product performance.
- Statement defining the environment in which the equipment must operate.
- Statement of the required operating time.
- The type of distribution likely to be encountered in reliability measurement.

1.2.4 Design for Reliability

Reliability design is an iterative process that begins with the specification of reliability goals consistent with cost and objectives. This requires consideration of the life- cycle costs of the system and the effect that reliability has an overall costs and system effectiveness. Once these reliability goals have been establised, these goals must be translate into individual component, subcomponent, and part specifications. This not necessarily an easy task, and it generally requires reliability block analysis. After indivdual component and part requirements have been determined, various design methods can be applied in order to meet the goals. These methods include the proper selection of parts and materials, stress-strength analysis, simplification, identification of technologies and use of redundancy.

Following completion of a preliminary detailed design along with initial development and protyping, a failure analysis may be performed to determine whether the specifications are being met and also provide a systematic approach for identifying, ranking, and eliminating failure models. This requires the use of reliability testing, including, perhaps a formalizedreliability growth testins program. Once reliabilty goals have been achieved, verification that safety margins are also being met must be made. If either the reliability or safety goals not met, the design process must continue. This may require reallocating reliability goals among the components if it is not possible to achieve a desired component reliability. The effect of design changes should then be verified through continued use of failure analysis and reliability testing.

1.2.5 Achievement of Reliability

There are five effective areas for the achievement of reliability of the product. They are

- Design
- Production
- Measurement and testing
- Maintenance
- Field operation

Design is very important that the other four areas and a greater percentage of causes of unreliability can be traced out in this areas.

1.2.6 Failure Pattern

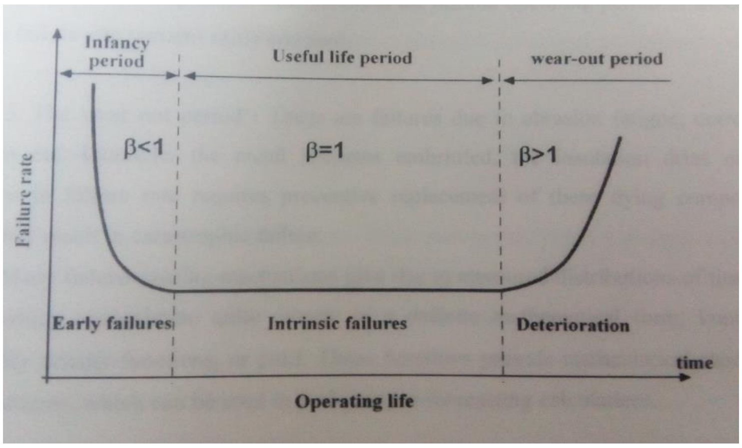


Figure 1.2.7.1 : Failure Pattern

The products often follow a familiar failure pattern of failure. When the failure rate (number of failures per unit time) is plotted against a continuous time scale, the resulting chart is known as " bath tub curve ". This curve exhibits three distinct zones. These zones differ from each other in frequency of failure and in the cause of failure pattern. These are as follows:

1.Infant Morality Period (or born in or the debugging period): This is characterized by high failure rates. It begins at the first point during manufacture that total equipment operation is possible and continues for such a period of time as permits (through maintenance and repairs), the elimination of marginal parts initially defective though not inoperative and unrecognizable as such until premature failure. Commonly, these are early failures resulting from defect in manufacturing, or other deficiencies which can be detected by debugging, running on or extended testing.

2.The constant failure rate period: Upon replacement of all prematurely failing items, the failure rate will have reached a lower value. From this point the failure rate remains fairly constant. These are chance failures which may result from the limitations inherent in the design plus accidents caused by usage or poor maintenance or hidden defects which escape inspection. This period is the normal operating period in which the average failure rate remains fairly constant.

3.The wear out period: These are failures due to abrasion fatigue, corrosion, vibration etc., Example, the metal becomes embrittled, the insulation dries out. A reduction in failure rate requires preventive replacement of these dying components before they result in catastrophic failure.

Mainly failure- causing mechanisms give rise to measured distributions of times- to- failure which approximate density functions, or probability density function. These functions provide mathematical models of failure pattern, which can be used in performance forecasting calculations.

1.2.7 Methods for improving design reliability

The following are some of the approaches used by the designers working jointly with reliability engineers to improve the design.

- Review the index selected to define product reliability to make sure that it reflects customer needs.
- Question the function of the unreliable parts with a view of eliminating them entirely if the function is found to be unnecessary.

- Review the selection of any parts which are relatively new.
- Conduct a research and development program to increase the reliability of the parts which are contributing most to the unreliability of the equipments.
- Specify corrective replacement items for unreliable parts and replace the parts before they fail.
- Select the parts which will be subjected to stress which are lower than the parts can normally withstand.
- Control the operating environment so that a part will be operating under conditions which yield a lower failure rate.
- Use redundancy so that if one unit fails a redundant unit will be available to do the job.
- Consider possible trade–offs of reliability with functional performance weight or other parameters.

1.2.8 Life testing

Reliability testing refers to the tests conducted to verify that a product will work satisfactorily for a given time period. Reliability testing therefore consists of functional test, environment test and life testing. A functional testing involves a test to determine if the product will function at time zone. An environmental test consists of determining the expected environmental levels and then carrying the functional test under the environments under which the product has to operate. The life of the component is the time period during which it retains its quality characteristic. Life tests are carried out to access the working life of a product, its capabilities and hence to form an idea of its quality level. The life test aims to measure the time or period during which the product will retain its desired quality characteristics. This may apply to either shelf life or life during use or both. Life tests are carried out in different manners under different conditions as follows :

- Tests under actual working conditions
- Tests under intensive conditions
- Tests under accelerated conditions

1.2.9 Acceptance sampling plans based on life tests

The sampling techniques and control charts are very important tools which analyses the data of life (destructive) tests. It is not necessary to subject all the sample pieces to destructive testing, the results in such case can be concluded from the time of first and middle failure. However, the potential capability of the product can be determined only through destructive testing. In recent times many authors have developed tables which show, for a life test, the relationship between the sample size, probability and percent of units which will fail before their shortest life for different parametric models.

Section 1.3

Theory of Estimation

1.3.1 Statistical inference

Statistical inference is the process of using data analysis to deduce properties of an underlying probability distribution. Inferential statistical analysis infers properties of a population, for example by testing hypotheses and deriving estimates. It is assumed that the observed data set is sampled from a larger population.

1.3.2 Theory of estimation

Estimation theory is a branch of inferential statistics that deals with estimating the values of parameters based on measured empirical data that has a random component. The parameters describe an underlying physical setting in such a way that their value affects the distribution of the measured data. An estimator attempts to approximate the unknown parameters using the measurements.

1.3.2.1 Commonly used methods of estimation:

- Maximum likelihood estimator.
- Bayes estimators.
- Method of moments estimators.
- Least squares.
- Minimum mean squared error (MMSE).
- Maximum a posteriori (MAP).

1.3.2.2 Maximum Likelihood Estimation (MLE)

In statistics, Maximum Likelihood estimation (MLE) is a method of estimating the parameters of a statistical model, given observations. The method obtains the parameter estimates by finding the parameter values that maximize the likelihood function. The estimates are called maximum likelihood estimates, which is also abbreviated as MLE.

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution that depends on one or more unknown parameters $\theta_1, \theta_2, \ldots, \theta_m$ with probability density (or mass) function $f(x_i ; \theta_1, \theta_2, \ldots, \theta_m)$. Suppose that $(\theta_1, \theta_2, \ldots, \theta_m)$ is restricted to a given parameter space Ω. Then:

(1) When regarded as a function of $\theta_1, \theta_2, \ldots, \theta_m$, the joint probability density (or mass) function of $X_1, X_2, \ldots, X_n$:

$L(\theta_1, \theta_2, \ldots, \theta_m) = \prod_{i=1}^{n} f(x_i ; \theta_1, \theta_2, \ldots, \theta_m)$

($(\theta_1, \theta_2, \ldots, \theta_m)$ in Ω) is called the likelihood function**.**

(2) If $[u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, x_2, \ldots, x_n)]$ is the m-tuple that the maximum likelihood function, then $\hat{\theta}_i = u_i(X_1, X_2, \ldots, X_n)$ is the maximum likelihood estimator of θ_i, for i = 1, 2, ..., m.

(3) The corresponding observed values of the statistics in (2), namely: $[u_1(x_1, x_2, \ldots, x_n), u_2(x_1, x_2, \ldots, x_n), \ldots, u_m(x_1, x_2, \ldots, x_n)]$ are called the maximum likelihood estimates of θ_i, for i = 1, 2, ..., m.

1.3.2.3 Methods of Moments

In statistical inference, the method of moments is a method of estimation of population parameters. It starts by expressing the population moments (i.e., the expected values of powers of the random variable under consideration) as functions of the parameters of interest. Those expressions are then set equal to the sample moments.

Let $x_1, x_2, \ldots, x_n$ be a random samples of n observations from a population having probability density function $f(x ; \theta_1, \theta_2, \ldots, \theta_k)$, where the parameters $\theta_1, \theta_2, \ldots, \theta_k$ are in the parametric space S. Let the r-th moment about zero of this distribution is $\mu'_r = E(X^r)$ and $\mu'_r = h(\theta_1, \theta_2, \ldots, \theta_k)$. Here $h(\theta_1, \theta_2, \ldots, \theta_k)$ is a function of parameters $\theta_1, \theta_2, \ldots, \theta_k$. The sample moment about zero for this population is $m'_r = \frac{1}{n} \sum_{i=1}^{n} x_i^r$ Since $\mu'_r = h(\theta_1, \theta_2, \ldots, \theta_k)$, we can write $\theta_r = g(\mu'_1, \mu'_2, \ldots, \mu'_r)$, when r = k.The method of moments states that m'_r is to be replaced by $\hat{\theta}_r = m'_r$.

Section 1.4

Life Time distributions

1.4.1 Introduction

The use of parametric distribution complements non-parametric techniques and provides the following advances :

- Parametric models can be described concisely with just few parameters, instead of having to report an entire curve.
- It is possible to use a parametric model to extrapolate (in time) to the lower or upper tail of distribution.
- Parametric models provide smooth estimates of failure – time distributions.

1.4.2 Birnbaum - Saunders distribution

The two-parameter Birnbaum-Saunders (BS) distribution was originally proposed by Birnbaum and Saunders (1969)[9,11] as a failure time distribution for fatigue failure caused under cyclic loading. The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable T is of the form

$$F(t;\alpha,\beta)=\phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}}-\left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right],\ 0<t<\infty,\alpha,\beta>0$$

where$\phi(.)$ is the standard normal cumulative distribution function. The parameters α and β are the shape and scale parameters, respectively.

The probability density function (PDF) of a two-parameter Birnbaum-Saunders random variable T corresponding to the cumulative distribution function is given by

$$f(t;\alpha,\beta)=\frac{1}{2\sqrt{2\pi\alpha\beta}}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}}+\left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{2}-2\right)\right],\ 0<t<\infty,\alpha,\beta>0$$

1.4.3 Generalized Birnbaum – Saunders Distribution

The Generalized Birnbaum Saunders Distribution is related to standard symmetrical distributions in R , also known as elliptically contoured univariate distributions. When a random variable Z follows a standard symmetrical distributions in R, the notation $Z \sim S(g)$ is used, where g is the kernel of the probability density function of Z. The probability density function and cumulative distribution function of Z $\sim$S(g) are denoted by f(.) and F(.), respectively, where f(z)= cg(z^2), with c being the normalization constant, such that

$\int_{-\infty}^{+\infty} g(z^2)dz = c^{-1}$.

The Cumulative Distribution Function (CDF) of T $\sim$GBS(α , β, g_X) is given by

$F_T(t) = G_X(\xi(t; \alpha , \beta)), t > 0$,

whereG_X ($\cdot$) is the cumulative distribution function of X and

$\xi(t; \alpha , \beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right), \quad t > 0$

The Probability Density Function is given by

$$f_T(t) = g_X\ (\xi(t ; \alpha , \beta))\xi'(t ; \alpha , \beta)\ , t > 0,$$

where$\xi'(t ; \alpha , \beta) = \frac{t+\beta}{2\alpha\sqrt{\beta}} t^{-3/2}$, t > 0 .

Section 1.5

Notations

$F_T(t)$ - cumulative distribution function of T .
$G_X(.)$ - cumulative distribution function of X random variable.
$f_T(t)$ - probability density function of T .
$\tilde{\beta}$ - Modified Moment estimator of β .
α - Shape parameter .
β - Scale parameter.
$t_{1:n}, t_{2:n}, \ldots, t_{n:n}$ - corresponding order statistics.
$T_{1:n}, T_{2:n}, \ldots, T_{n:n}$ - order statistics associated to random samples.
S - Sample arithmetic mean .
R - Harmonic Mean .
T,Z - random variables
g - the kernel of the probability density function of Z .
f(.) - probability density function of Z .
F(.) - cumulative density function of Z .
c - normalization constant
$h_T(t)$ - Generalized Birnbaum - Saunders hazard function .
q - number of parameters
$\hat{\Sigma}_{\hat{\theta}}$ - an estimate of the variance – covariance matrix of $\hat{\theta}$
$\dot{L}_{(i)}$ - the score vector dropping the ith case .
$\ddot{L}_{(i)}$ - Hessian matrix dropping the ithcase .
ϕ - Standard normal cumulative distribution function
$h(t, \alpha)$ - hazard function
E(T) - expected values
V(T) - variance
$\beta_1(T)$ - co-efficient of skewness
$\beta_2(T)$ - co- efficient of kurtosis

REVIEW OFLITERATURE

Review of the Literature

The reliability acceptance sampling plans are developed to obtain information concerning failures in order to quantify reliability and to improve product reliability. As required by the principles of statistical inference, it is necessary to specify the probability distribution of the variable quality characteristics under consideration. In the absence of such specifications, it is taken as the well known normal distribution. However, if the normal distribution is not good to fit to the data under consideration, the decision process constructed on this basis would be misleading. At the same time, an appeal to central limit theorem as a justification to normality assumption is not always valid as the sample size in quality control data is not large enough to adopt normality. In this backdrop, acceptance sampling plans based on truncated life tests for a variety of distributions was discussed by many members.Acceptance sampling based on truncated life tests were discussed by Epstein (1954) and Sobel and Tischendrof (1959) for the exponential model.

Attributes single sampling plan based on truncated life tests have been proposed for a variety of life time distribution by many authors. An extension of attributes single sampling plan tests work was carried out by Goode and Kao(1961) by considering the Weibull model which includes the exponential distribution as a particular case. Gupta and Groll (1961) and Gupta (1962) considered the gamma and log-normal distributions respectively. Seshadri (1964) and Saunders (1974) considered classes of absolutely continuous non-negative random variables closed under reciprocation and ways of generating such classes.

In the study of recovery from breast cancer, it has been observed by Langlands, Pockock, Kerr, and Gore(1979) that the maximum mortality occurs after about three year and then it decreases slowly over affixed period of time..

Cox and Oakes (1984) established that there are other ways to identify the suitable life distributions. A more general derivation of Birnbaum-Saunders distribution was provided by Desmond (1985) based on a biological considerations. Desmond (1985) also strengthened the physical justification for the use of Birnbaum-Saunders distribution by relaxing the assumptions made originally by Birnbaum and Saunders (1969).

Some recent work on the Birnbaum Saunders distribution are found in Chang and Tang (1993,1994), Rieck (1995,1999), Dupuis and Mills (1998),Ng ,Kundu, and Balakrishnan(2003,2006), From and Li(2006),Owen(2006), Balakrishnan, Leiva,Lopez (2007), Lemonte, Cribari-neto ,Vasconcellos (2007), and Xie and Wei (2007). Kantam (2001), Baklizi (2003), Baklizi and El Masri(2004) and Rosaiah and Kantam(2005) have all developed acceptance sampling plans for a variety of distributions.Díaz-García and Leiva (2002, 2005) proposed a highly flexible lifetime model that admits different degrees of kurtosis and asymmetry and possesses unimodality and bimodality was the generalized Birnbaum–Saunders (GBS) distribution. The generalized Birnbaum–Saunders distribution (GBS), introduced by Dıaz - Garcıa and Leiva (2005), is obtained replacing the normal generator in the Birnbaum–Saunders distribution by any symmetric absolutely continuous random variable. Diaz-Gracia and Leiva (2005,2007) generalized the Birnbaum-Saunders distribution and obtained a more flexible distribution.

To model survival times of patients with multiple myeloma by using prognostic variables with censored data used the Birnbaum– Saunders distribution by Leiva, Barros, Paula,Galea(2007). While a software program is operating, the development of intangible cumulative damage deteriorates the performance of this software and so they used the generalized Birnbaum– Saunders distribution for modelling have been argued by Balakrishnan, Leiva,Lopez (2007). Leiva Barros, Paula,Sanhueza (2008b) used the generalized Birnbaum– Saunders distribution to model air pollution data. Sanhueza,Leiva,Balakrishana(2008) for a discussion of its theory and applications, Balakrishnan, Leiva, Sanhueza,Vilca (2009) for the case generated by scale-mixtures of normal distributions and Leiva,Santos-Neto,Cysneiros(2014) for a family related to scale-mixture Birnbaum–Saunders distributions. Barro, Leiva, Ospina, Tsuyuguchi(2014) and Castro-Kuriss, Leiva, Athayde(2014) for an overview of available tests and graphical tools to assess Goodness – of – fit in non Location Scale distributions.

CHAPTER – 2

Chapter - 2

Acceptance Sampling Plans from Truncated Life Tests Based on the Generalized Birnbaum–Saunders Distribution

In this chapter , Acceptance Sampling Plans from Truncated Life Tests Based on the Generalized Birnbaum–Saunders Distribution by Balakrishnan , Victor Leiva, and Jorge Lopez (2007)[6] has been reviewed.

When the life test is truncated at a pre-fixed time , the acceptance sampling plans has been developed. The minimum sample size necessary to ensure the specified median life was obtained by assuming that the lifetimes of the test units follow a Generalized Birnbaum–Saunders distribution. The operating characteristic values of the sampling plans as well as producer's risk were presented.

The quality has become a differentiation tool between competitive enterprises. Two important tools for ensuring quality are the statistical quality control and the acceptance sampling. An acceptance sampling plan establishes the minimum sample size to be used and the associated acceptance and non-acceptance criteria for the lot. The consumer's and producer's risks are then the probabilities that a bad lot is accepted and a good lot is rejected.

In many situations, the quality of a product is measured through the lifetime (T), and the variance or the scale parameter of the distribution of T may serve as a quality parameter. If the units in the lot are classified as defective or non defective based on life testing experiment, then the acceptance sampling plan must consider a third element, i.e., the ratio $\frac{1}{\mu_0}$, where t is the pre-fixed test time and μ_0 is the specified mean or median lifetime. Thus, a random variable that represents the lifetime of the inspected unit is related to an acceptance sampling based on truncated life tests (ASTLT).

Most of the probabilistic models used to describe lifetime data are chosen for one or more of the following reasons:

(1) a physical or statistical theoretical argument for the mechanism of failure of the unit;

(2) a model that has previously been used successfully; and

(3) an appropriate model whose empirical fit is good to data. For example, an analysis of the probability density function is not always the best thing, and it may be preferable to analyze the hazard function (h(t)) or log[h(t)] versus t or log(t), or the cumulative hazard function (H(t)) or log-survival versus t or log(t).

Several products that must be put under inspection for acceptance sampling are exposed to cumulative degradation during their lifetimes. This degradation is recognized like “fatigue” and, in some cases, this development is untouchable. Therefore, it is necessary to consider methods of acceptance sampling from life models that account for this degradation and that are flexible the lifetime data is fitted. The main motivations for the use of the generalized Birnbaum–Saunders distribution are to make the kurtosis i.e. the measure of the tailedness of the probability distribution of a real valued random variable, more flexible and also to admit bimodality. Moreover, the Birnbaum–Saunders distribution is a particular case of the generalized Birnbaum–Saunders distribution.

In acceptance sampling based on truncated life tests, assume the following:

(1) the units are destructible or are degraded after the life test, and

(2) there are several distributions that model the product life reasonably well.

Thus, considering similar risk and operating conditions and the assumptions (1) and (2), the consumer will benefit with smaller number of units required to test. For this reason, one could use a distribution that gives the smallest sample size.

2.1 The Generalized Birnbaum–Saunders Distribution

The Birnbaum–Saunders distribution is defined in terms of the normal distribution by means of the random variate $T = \beta\left[\alpha Z/2 + \sqrt{(\alpha Z/2)^2 + 1}\right]^2$, where $Z = \frac{1}{\alpha}\left(\sqrt{T/\beta} - \sqrt{\beta/T}\right) \sim N(0,1), \alpha > 0$ is the shape parameter, and $\beta > 0$ is the scale parameter. This is denoted by $T \sim BS(\alpha, \beta)$. Recently, Díaz-García and Leiva (2005, 2007)[23,24] presented a generalization of the Birnbaum–Saunders distribution by assuming that Z follows a symmetrical distribution in $\mathbb{R}$(elliptically contoured distributions) with location parameter $\mu = 0$ and scale parameter $\sigma = 1$, i.e., $Z \sim EC(0, 1; f)$, where $f(\cdot)$ is the probability density function of Z. In this case, the notation $T \sim GBS(\alpha, \beta ; f)$ will be used.

2.1.1 Some characteristics of the Generalized Birnbaum–Saunders Distribution

The following results holds for the Generalized Birnbaum–Saunders Distribution.

If $T = \beta(\alpha Z/2 + \sqrt{(\alpha Z/2)^2 + 1})^2$ ~GBS(α, β ; f) , then :

(1) $Z = \frac{1}{\alpha}\left(\sqrt{T/\beta} - \sqrt{\beta/T}\right)$ ~EC(0, 1; f) ;

(2) The probability density function of T is

$f_T(t) = f\left(a_t(\alpha, \beta)\right)\frac{d}{dt}a_t(\alpha, \beta)$,t>0,$\alpha > 0$, $\beta > 0$,(2.1)

where $f(.)$ is the probability density function of Z ~EC (0, 1; f),

$$a_t(\alpha, \beta) = \frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right) \text{and} \frac{d}{dt}a_t(\alpha, \beta) = \frac{t^{-\frac{3}{2}}(t+\beta)}{2\alpha\sqrt{\beta}} = \frac{1}{2\alpha\beta}\left(\frac{t}{\beta} + 1\right)\left(\frac{t}{\beta}\right)^{-\frac{3}{2}}; \quad (2.2)$$

(3) cT~GBS(α, cβ ; f) , with c.0 ;

(4) T^{-1} ~GBS (α, β^{-1} ; f) ;

(5) For the random variable Z given in (1) , if $\mathbb{E}[Z^r]$ exists , then

$$\mathbb{E}[T^n] = \beta^n \sum_{s=0}^{k}\binom{k}{s}\mathbb{E}[Z^{2(n+s-k)}]\left(\frac{\alpha}{2}\right)^{2(n+s-k)}. \quad (2.3)$$

From Eq.(2.3), the mean and variance of T can be obtained as

$\mathbb{E}[T] = \frac{\beta}{2}(2 + \mathbb{E}[Z^2]\alpha^2)$and

$$\text{Var}[T] = \frac{\beta^2}{4}(2\alpha^4\mathbb{E}[Z^4] + 4\alpha^2\mathbb{E}[Z^2] - \alpha^4(\mathbb{E}[Z^2])^2), \quad (2.4)$$

where Z~EC (0, 1; f) ;

(6) The cumulative distribution function of t is

$$F_T(t) = \mathbb{P}(T \leq t) = F\left(a_t(\alpha, \beta)\right), t>0, \quad (2.5)$$

where F(.) is the cumulative distribution function of Z~EC (0, 1; f) and $a_t(\alpha, \beta)$ is as given in Eq.(2.2) .

(7)the p^{th}percentile of the Generalized Birnbaum Saunders distribution , $t_p = F_T^{-1}(p)$, isgiven by

$$t_p = \frac{\beta}{4}\left(\alpha z_p + \sqrt{\alpha^2 z_p^2 + 4}\right)^2, \quad (2.6)$$

wherez_p is the p^{th}percentile of Z~EC (0, 1; f) ; If p= 0.5 then $t_{0.5} = \beta$, and so $\beta$is the population median. This result is similar to that of Chang and Tang (1994)[15] for the Birnbaum Saunders distribution.

The probability density function and cumulative distribution function of the Birnbaum Saunders and GBS(t) distributions are presented in Table 2.1. The mean and variance of the Birnbaum Saunders and GBS(t) distributions are given in Table 2.2. From Table 2.2, one can note that if $v \to \infty$, then the mean and variance of $T \sim GBS(\alpha, \beta, t_v)$ converge to the mean and variance of the classical Birnbaum-Saunders Distribution .

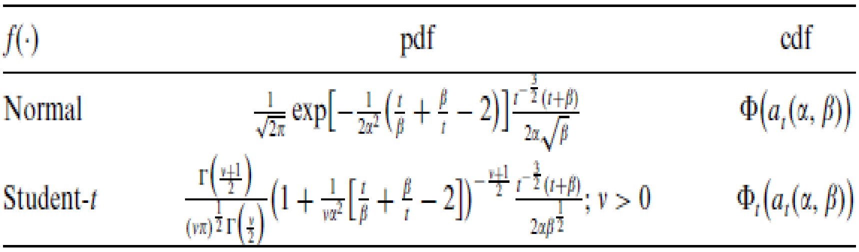
Here $\phi(.)$ and $\phi_t(.)$ are the cumulative distribution functions of the standard normal and Student ‘ t’ distributions , respectively , and $a_t(\alpha, \beta)$ is as given in Eq.(2.2) , where

$$\phi_t(z) = \frac{1}{2}\left[1 + I_{\frac{z^2}{z^2+v}}\left(\frac{1}{2}, \frac{1}{2}v\right)\right]$$

and$I_x(a, b)$ is the incomplete beta function ratio given by

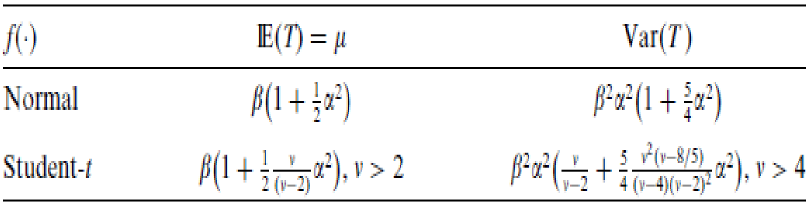
$$I_x(a, b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt}.$$

Table 2.1 : Probability Density function and cumulative distribution function of $T \sim GBS(\alpha, \beta)$, $\alpha, \beta > 0$, with f being the Probability Density function of $Z \sim EC(\ 0\ , 1\ ; f)$



$f(\cdot)$	pdf	cdf
Normal	$\frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right] \frac{t^{-\frac{3}{2}}(t+\beta)}{2\alpha\sqrt{\beta}}$	$\Phi\left(a_t(\alpha, \beta)\right)$
Student-t	$\frac{\Gamma\left(\frac{v+1}{2}\right)}{(v\pi)^{\frac{1}{2}}\Gamma\left(\frac{v}{2}\right)}\left(1 + \frac{1}{v\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-\frac{v+1}{2}} \frac{t^{-\frac{3}{2}}(t+\beta)}{2\alpha\beta^{\frac{1}{2}}}; v > 0$	$\Phi_t\left(a_t(\alpha, \beta)\right)$

Table 2.2 : Mean and variance of $T \sim BS(\alpha, \beta)$ and $T \sim GBS(\alpha, \beta; f), \alpha,$ $\beta > 0$with f being the probability density function of $Z \sim EC(\ 0\ , 1\ ; f)$



$f(\cdot)$	$\mathbb{E}(T) = \mu$	$\mathrm{Var}(T)$
Normal	$\beta\left(1 + \frac{1}{2}\alpha^2\right)$	$\beta^2\alpha^2\left(1 + \frac{5}{4}\alpha^2\right)$
Student-t	$\beta\left(1 + \frac{1}{2}\frac{v}{(v-2)}\alpha^2\right), v > 2$	$\beta^2\alpha^2\left(\frac{v}{v-2} + \frac{5}{4}\frac{v^2(v-8/5)}{(v-4)(v-2)^2}\alpha^2\right), v > 4$

2.1.2 The GBS Distribution as a Lifetime Model

An useful function in lifetime analysis is the failure rate or hazard function, defined by $h(t) = f(t)/(1 - F(t))$, where $f(\cdot)$ and $F(\cdot)$ are the probability density function and cumulative distribution function, respectively. The behaviour of h(t) allows one to characterize the aging of the units .If the failure rate is increasing (IFR distribution), then the units age with time. If h(t) is decreasing (DFR distribution), then the units improve in performance with time. Finally, if h(t) is constant, then the life distribution is necessarily exponential.

The normal distribution belongs to the Increasing Failure Rate family. However, the log normal model does not have a increasing failure rate, because it is initially increasing until its critical point and then it decreases to zero. For the hazard function of the classical Birnbaum Saunders distribution, one can have that:

(1) behaves similarly to the log normal model, but this decreases until it becomes stabilized in a positive constant (not at zero),

(2) its behaviour is similar to that of the inverse Gaussian distribution

(3) one cannot assume that this is always increasing while analysing a typical fatigue data; and

(4) its average is nearly non decreasing (≈IFRA distribution). The class of hazard functions which are non decreasing on the average has been studied by Birnbaum et al. (1968)[11].

If $T \sim BS(\alpha, \beta)$it is easy to verify that the behaviour of its hazard function,$h_T(t)$does not depend on the parameter β. On the other hand, for any α, we have: (1)$h_T(t)$ is unimodal, being increasing for $t < t_c$ and decreasing for $t > t_c$, where t_c is the critical point of $h_T(t)$; and (2) $h_T(t)$approach to $(2\alpha^2\beta)^{-1}$as $t \to \infty$.In addition, (3) when $\infty \to 0$, then $h_T(t)$tends to be increasing. Thus, it is interesting to know from which value of α the failure rate of T belongs to the Increasing Failure Rate class. Although it is not possible to obtain an analytical result in order to response to this question, a numerical study indicates that the Birnbaum Saunders distribution belongs to the Increasing Failure Rate family when $\alpha < 0.41$ and $0 < t < 8\beta$, which implies an increasing failure rate class. Also, Birnbaum and Saunders (1969)[9] showed by numerical calculation that the average failure rate of T decreases slowly.

For $t < 1.64$

If $T \sim GBS(\alpha, \beta; f)$, then the hazard function of T is given by

$$h_T(t) = \left(\frac{f(a_t(\alpha,\beta))}{F(-a_t(\alpha,\beta))}\right)\frac{d}{dt}a_t(\alpha,\beta), \qquad (2.7)$$

where f(.) and $a_t(\alpha, \beta)$are as given in (1), and F(.) is as in Eq.(2.5). Besides, $F(-a_t(\alpha,\beta)) = 1 - F(a_t(\alpha,\beta))$ due to the symmetry of F(.). The behaviour of $h_T(t)$given in Eq.(2.7) is similar to that of the classical Birnbaum Saunders model for all f(.) and in unique manner when the symmetric distribution in $\mathbb{R}$ is unimodal. Expressions for $h_T(t)$when $T \sim BS(\alpha, \beta)$ or $T \sim GBS(\alpha, \beta; t_\nu) \equiv GBS(\alpha, \beta; t_\nu)$, with f being the probability density function of $Z \sim t_\nu$, can be easily obtained from Eq.(2.7) and Table 2.1.

2.2. Acceptance Sampling Plans

It is assumed that the lifetime (T) of the product under study follows a GBS(t)distribution with parameters α and ν known (fixed). A common practice in lifeexperiments is to stop the life test at pre-fixed time (t) and register the number of failures. One object of these tests is to set a confidence (lower) limit on the mean or median life (or any other percentile of the distribution). It is then desired to establish a specified mean or median life with a probability of at least P^*(consumer's risk).Normally, the mean life is used if the life distribution is symmetric, and the median life is used if the life distribution is skewed. The decision to accept the specified mean or median life occurs if and only if the number of failures at the end of the pre-fixed time t does not exceed a given number c. Thus, the test is terminated at time t or at the $(c + 1)^{th}$ failure, whichever occurs first. For such a truncated life test and the associated decision rule, one can interested in obtaining sampling plans, i.e., to find the minimum sample size necessary to achieve our objective.

An acceptance sampling plan based on truncated life tests consists of: (1) the number of units on test (n); (2) the acceptance number (c), such that if at most c failures out of n occur at the end of the pre-fixed time t, the lot is accepted; and

(3) the ratio$\frac{t}{\mu_0}$, where μ_0 is the specified mean or median life and t is the maximum test duration. Thus, similar to the case of the log-normal distribution, for $T \sim GBS(\alpha, \beta; f)$, μ_0 will represent the median,that is, $\mu_0 = \beta_0$, where β is also the lot quality parameter, since β is the median life as well as the scale parameter of the Generalized Birnbaum Saunders distribution. An acceptance sampling plan based on truncated life tests for the Generalized Birnbaum Saunders distribution is $(n, c, \frac{t}{\beta_0})$.

2.2.1. Minimum Sample Size

One can fix the probability of accepting a bad lot (consumer's risk), i.e., the one for which the true median life β is below the specified β_0, not to exceed $1 - P^*$. We assume that the lot size (N) is large enough to be considered infinite, so that the binomial distribution can be used. Thus, the acceptance and non acceptance criteria for the lot are equivalent to the decisions of accepting or rejecting the hypothesis $\beta \geq \beta_0$. We want to find the minimum sample size (n) such that$\sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x} \leq 1 - P^*$. (2.8)

where $p = F_T(t)$given in Eq.(2.5) is monotonically increasing on $\frac{t}{\beta}$ and decreasing on β,for fixed t, which is easy to establish for the Generalized Birnbaum Saunders distribution. Thus, $p = F_T(t) = F_T(t; \beta) = F_T\left(\frac{t}{\beta}\right)$depends only on the ratio $\frac{t}{\beta}$, if we fix α and ν. Hence, it is sufficient to specify just this ratio. Therefore, if the number of observed failures is at most c, from Eq.(2.8) we can establish with probability P^*that $F_T\left(\frac{t}{\beta}\right) \leq F_T\left(\frac{t}{\beta_0}\right)$, which implies that $\beta \geq \beta_0$.

The minimum values of n satisfying Eq.(2.8) forP^*=0.75, 0.9, 0.95, 0.99 and $\frac{t}{\beta_0}$= 0.628,0.942,1.257,1.571, 2.356, 3.141, 3.927, 4.712 were determined are presented in Table 2.3, for fixed α and ν .

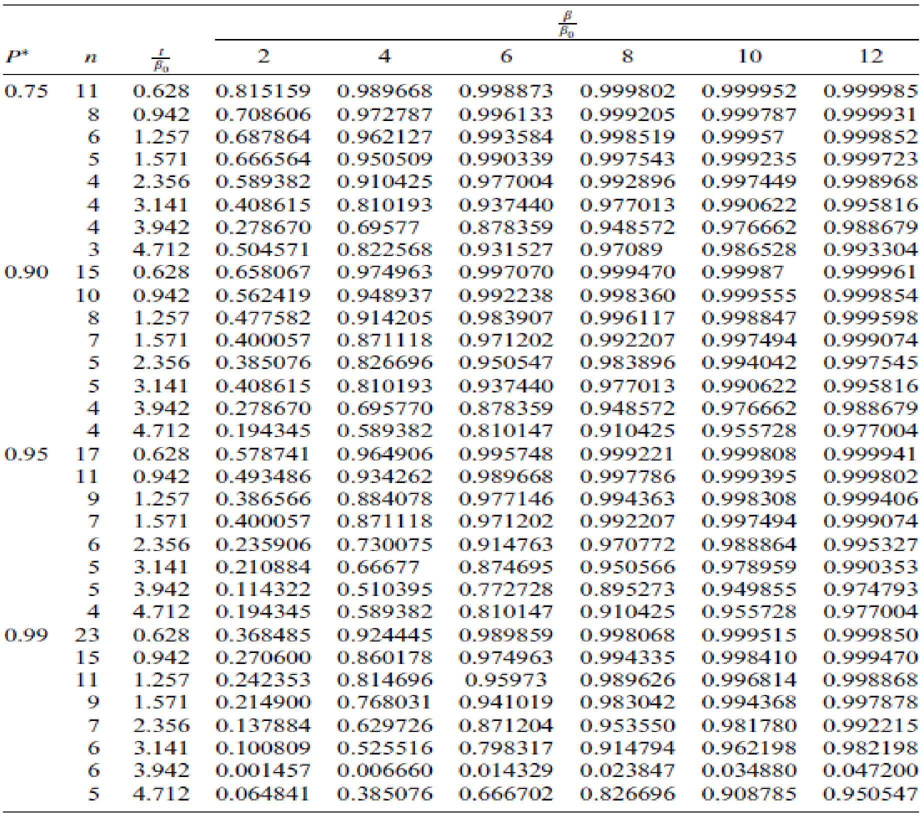
2.2.2. Operating Characteristic of the Sampling Plan (n,c , $\frac{t}{\beta_0}$)

The operating characteristic (OC) function of the acceptance sampling based on truncated life test plan (n ,c , $\frac{t}{\beta_0}$) gives the probability that the lot can be accepted.

$$L(p)=\sum_{x=0}^{c}\binom{n}{x}p^{x}(1-p)^{n-x} \qquad (2.9)$$

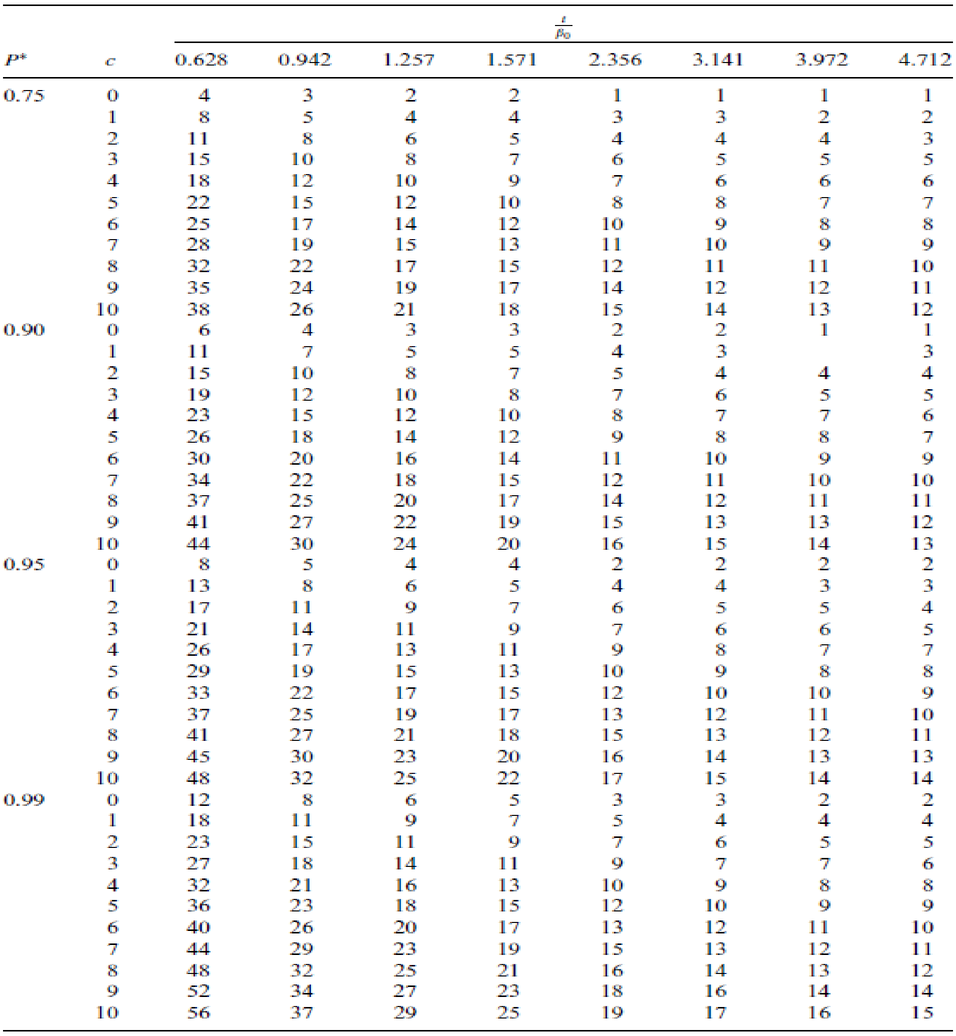
where p given in Eq.(2.8) is a monotonically decreasing function of $\beta \geq \beta_0$, for fixed t, while L(p) is decreasing in p. Based on Eq.(2.9), the operating characteristic values, as a function of $\frac{\beta}{\beta_0}$, for fixed α and ν, were determined and presented in Table 2.4 for the acceptance sampling based on truncated life test plan (n ,c , $\frac{t}{\beta_0}$) , for fixed c and different values of P^*.For given $P^*$and $\frac{t}{\beta_0}$, the choice of c and n can be made on the basis of the operating characteristic function.

Table 2.3 : Minimum sample size necessary to assert that the median life exceeds a given value, β_0, with probability $P^*$and the corresponding acceptance number, c, using binomial probabilities (when α= 1 and ν= 5)



			$\frac{\beta}{\beta_0}$					
P^*	n	$\frac{t}{\beta_0}$	2	4	6	8	10	12
0.75	11	0.628	0.815159	0.989668	0.998873	0.999802	0.999952	0.999985
	8	0.942	0.708606	0.972787	0.996133	0.999205	0.999787	0.999931
	6	1.257	0.687864	0.962127	0.993584	0.998519	0.99957	0.999852
	5	1.571	0.666564	0.950509	0.990339	0.997543	0.999235	0.999723
	4	2.356	0.589382	0.910425	0.977004	0.992896	0.997449	0.998968
	4	3.141	0.408615	0.810193	0.937440	0.977013	0.990622	0.995816
	4	3.942	0.278670	0.69577	0.878359	0.948572	0.976662	0.988679
	3	4.712	0.504571	0.822568	0.931527	0.97089	0.986528	0.993304
0.90	15	0.628	0.658067	0.974963	0.997070	0.999470	0.99987	0.999961
	10	0.942	0.562419	0.948937	0.992238	0.998360	0.999555	0.999854
	8	1.257	0.477582	0.914205	0.983907	0.996117	0.998847	0.999598
	7	1.571	0.400057	0.871118	0.971202	0.992207	0.997494	0.999074
	5	2.356	0.385076	0.826696	0.950547	0.983896	0.994042	0.997545
	5	3.141	0.408615	0.810193	0.937440	0.977013	0.990622	0.995816
	4	3.942	0.278670	0.695770	0.878359	0.948572	0.976662	0.988679
	4	4.712	0.194345	0.589382	0.810147	0.910425	0.955728	0.977004
0.95	17	0.628	0.578741	0.964906	0.995748	0.999221	0.999808	0.999941
	11	0.942	0.493486	0.934262	0.989668	0.997786	0.999395	0.999802
	9	1.257	0.386566	0.884078	0.977146	0.994363	0.998308	0.999406
	7	1.571	0.400057	0.871118	0.971202	0.992207	0.997494	0.999074
	6	2.356	0.235906	0.730075	0.914763	0.970772	0.988864	0.995327
	5	3.141	0.210884	0.66677	0.874695	0.950566	0.978959	0.990353
	5	3.942	0.114322	0.510395	0.772728	0.895273	0.949855	0.974793
	4	4.712	0.194345	0.589382	0.810147	0.910425	0.955728	0.977004
0.99	23	0.628	0.368485	0.924445	0.989859	0.998068	0.999515	0.999850
	15	0.942	0.270600	0.860178	0.974963	0.994335	0.998410	0.999470
	11	1.257	0.242353	0.814696	0.95973	0.989626	0.996814	0.998868
	9	1.571	0.214900	0.768031	0.941019	0.983042	0.994368	0.997878
	7	2.356	0.137884	0.629726	0.871204	0.953550	0.981780	0.992215
	6	3.141	0.100809	0.525516	0.798317	0.914794	0.962198	0.982198
	6	3.942	0.001457	0.006660	0.014329	0.023847	0.034880	0.047200
	5	4.712	0.064841	0.385076	0.666702	0.826696	0.908785	0.950547

Table 2.4: Operating characteristics for the acceptance sampling based on truncated life test plan $(n, c, \frac{t}{\beta_0})$ for a given P^* when $c = 2$, $\alpha = 1$ and $\nu = 5$.



		$\frac{t}{\beta_0}$							
P^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.972	4.712
0.75	0	4	3	2	2	1	1	1	1
	1	8	5	4	4	3	3	2	2
	2	11	8	6	5	4	4	4	3
	3	15	10	8	7	6	5	5	5
	4	18	12	10	9	7	6	6	6
	5	22	15	12	10	8	8	7	7
	6	25	17	14	12	10	9	8	8
	7	28	19	15	13	11	10	9	9
	8	32	22	17	15	12	11	11	10
	9	35	24	19	17	14	12	12	11
	10	38	26	21	18	15	14	13	12
0.90	0	6	4	3	3	2	2	1	1
	1	11	7	5	5	4	3		3
	2	15	10	8	7	5	4	4	4
	3	19	12	10	8	7	6	5	5
	4	23	15	12	10	8	7	7	6
	5	26	18	14	12	9	8	8	7
	6	30	20	16	14	11	10	9	9
	7	34	22	18	15	12	11	10	10
	8	37	25	20	17	14	12	11	11
	9	41	27	22	19	15	13	13	12
	10	44	30	24	20	16	15	14	13
0.95	0	8	5	4	4	2	2	2	2
	1	13	8	6	5	4	4	3	3
	2	17	11	9	7	6	5	5	4
	3	21	14	11	9	7	6	6	5
	4	26	17	13	11	9	8	7	7
	5	29	19	15	13	10	9	8	8
	6	33	22	17	15	12	10	10	9
	7	37	25	19	17	13	12	11	10
	8	41	27	21	18	15	13	12	11
	9	45	30	23	20	16	14	13	13
	10	48	32	25	22	17	15	14	14
0.99	0	12	8	6	5	3	3	2	2
	1	18	11	9	7	5	4	4	4
	2	23	15	11	9	7	6	5	5
	3	27	18	14	11	9	7	7	6
	4	32	21	16	13	10	9	8	8
	5	36	23	18	15	12	10	9	9
	6	40	26	20	17	13	12	11	10
	7	44	29	23	19	15	13	12	11
	8	48	32	25	21	16	14	13	12
	9	52	34	27	23	18	16	14	14
	10	56	37	29	25	19	17	16	15

2.2.3. Producer's Risk

The producer's risk is defined as the probability of rejecting of the lot when $\beta \geq \beta_0$.For the sampling plan under consideration and a given value for the producer's risk, say γ, one may be interested in knowing the value of $\frac{\beta}{\beta_0}$ that will ensure the producer's risk to be at most γ. One can note that Eq.(2.2) can be written as

$$a_t(\alpha, \beta) = \frac{1}{\alpha}\left(\frac{\sqrt{\frac{t}{\beta_0}}}{\sqrt{\frac{\beta}{\beta_0}}} - \frac{\sqrt{\frac{\beta}{\beta_0}}}{\sqrt{\frac{t}{\beta_0}}}\right), (2.10)$$

which will denoted by $a_{t,\alpha}\left(\frac{\beta}{\beta_0}\right)$, for fixed α. Based on Eq.(2.10) , the probability p =F $\left(a_t(\alpha,\beta)\right)$, with F(.) as in Eq.(2.5) , may be obtained as function of $\frac{\beta}{\beta_0}$, that is , p = F $\left(a_{t,\alpha}\left(\frac{\beta}{\beta_0}\right)\right)$. Then ,$\frac{\beta}{\beta_0}$ is the smallest positive number for which p = F $\left(a_{t,\alpha}\left(\frac{\beta}{\beta_0}\right)\right)$ satisfies the inequality

$$\sum_{x=0}^{c}\binom{n}{x}p^x(1-p)^{n-x} \geq 1-\gamma \tag{2.11}$$

For a given acceptance sampling based on truncated life test plan (n, c, $\frac{t}{\beta_0}$), at a specified confidence level P^* , the minimum values of $\frac{\beta}{\beta_0}$ satisfying Eq.(2.8) were determined and are presented in Table 2.5 , for fixed $\alpha = 1$ and ν = 5 .

2.2.4. Extensions and Approximations

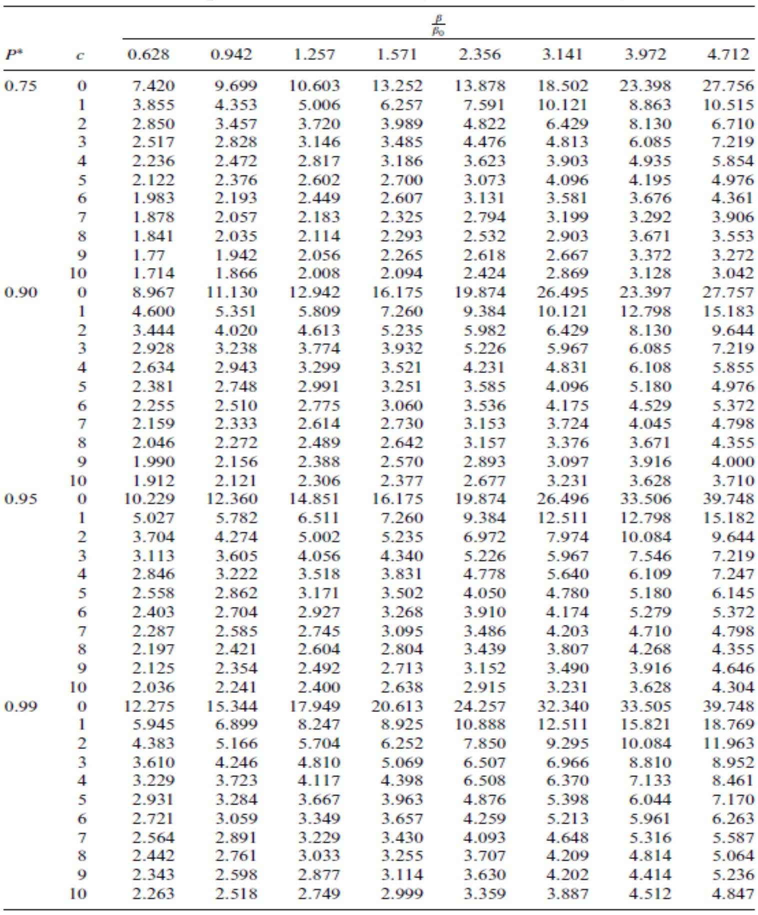
For T ~ GBS(α, β ; t_ν) and fixed α and ν , if p = $F_T(t;\beta)$is small and n is large, the binomial probability can be approximated by the Poisson probability with parameter λ= np. So, Eq.(2.8) can be approximated as

$$\sum_{x=0}^{c}\frac{\exp(-\lambda)\lambda^x}{x!} \leq 1-P^*. \tag{2.12}$$

Thus, a acceptance sampling based on truncated life test plan (n, c, $\frac{t}{\beta_0}$) can be obtained based on this approximation. Also, following a similar procedure as in Gupta (1962)[31], it is possible to obtain an approximate formula for n from Eq.(2.12).

Instead of the median, some other quantiles can also be used. The tables presented in this chapter are directly usable in order to carry out acceptance sampling based on truncated life test plan for mean or other quantiles of the Generalized Birnbaum Saunders distribution or log- Generalized Birnbaum Saunders distribution.

Table 2. 5 :Minimum ratio of true median life to specified median life for the acceptance of a lot with producer's risk of 0.05 (when α= 1 and ν= 5)



		$\frac{\beta}{\beta_0}$							
P^*	c	0.628	0.942	1.257	1.571	2.356	3.141	3.972	4.712
0.75	0	7.420	9.699	10.603	13.252	13.878	18.502	23.398	27.756
	1	3.855	4.353	5.006	6.257	7.591	10.121	8.863	10.515
	2	2.850	3.457	3.720	3.989	4.822	6.429	8.130	6.710
	3	2.517	2.828	3.146	3.485	4.476	4.813	6.085	7.219
	4	2.236	2.472	2.817	3.186	3.623	3.903	4.935	5.854
	5	2.122	2.376	2.602	2.700	3.073	4.096	4.195	4.976
	6	1.983	2.193	2.449	2.607	3.131	3.581	3.676	4.361
	7	1.878	2.057	2.183	2.325	2.794	3.199	3.292	3.906
	8	1.841	2.035	2.114	2.293	2.532	2.903	3.671	3.553
	9	1.77	1.942	2.056	2.265	2.618	2.667	3.372	3.272
	10	1.714	1.866	2.008	2.094	2.424	2.869	3.128	3.042
0.90	0	8.967	11.130	12.942	16.175	19.874	26.495	23.397	27.757
	1	4.600	5.351	5.809	7.260	9.384	10.121	12.798	15.183
	2	3.444	4.020	4.613	5.235	5.982	6.429	8.130	9.644
	3	2.928	3.238	3.774	3.932	5.226	5.967	6.085	7.219
	4	2.634	2.943	3.299	3.521	4.231	4.831	6.108	5.855
	5	2.381	2.748	2.991	3.251	3.585	4.096	5.180	4.976
	6	2.255	2.510	2.775	3.060	3.536	4.175	4.529	5.372
	7	2.159	2.333	2.614	2.730	3.153	3.724	4.045	4.798
	8	2.046	2.272	2.489	2.642	3.157	3.376	3.671	4.355
	9	1.990	2.156	2.388	2.570	2.893	3.097	3.916	4.000
	10	1.912	2.121	2.306	2.377	2.677	3.231	3.628	3.710
0.95	0	10.229	12.360	14.851	16.175	19.874	26.496	33.506	39.748
	1	5.027	5.782	6.511	7.260	9.384	12.511	12.798	15.182
	2	3.704	4.274	5.002	5.235	6.972	7.974	10.084	9.644
	3	3.113	3.605	4.056	4.340	5.226	5.967	7.546	7.219
	4	2.846	3.222	3.518	3.831	4.778	5.640	6.109	7.247
	5	2.558	2.862	3.171	3.502	4.050	4.780	5.180	6.145
	6	2.403	2.704	2.927	3.268	3.910	4.174	5.279	5.372
	7	2.287	2.585	2.745	3.095	3.486	4.203	4.710	4.798
	8	2.197	2.421	2.604	2.804	3.439	3.807	4.268	4.355
	9	2.125	2.354	2.492	2.713	3.152	3.490	3.916	4.646
	10	2.036	2.241	2.400	2.638	2.915	3.231	3.628	4.304
0.99	0	12.275	15.344	17.949	20.613	24.257	32.340	33.505	39.748
	1	5.945	6.899	8.247	8.925	10.888	12.511	15.821	18.769
	2	4.383	5.166	5.704	6.252	7.850	9.295	10.084	11.963
	3	3.610	4.246	4.810	5.069	6.507	6.966	8.810	8.952
	4	3.229	3.723	4.117	4.398	6.508	6.370	7.133	8.461
	5	2.931	3.284	3.667	3.963	4.876	5.398	6.044	7.170
	6	2.721	3.059	3.349	3.657	4.259	5.213	5.961	6.263
	7	2.564	2.891	3.229	3.430	4.093	4.648	5.316	5.587
	8	2.442	2.761	3.033	3.255	3.707	4.209	4.814	5.064
	9	2.343	2.598	2.877	3.114	3.630	4.202	4.414	5.236
	10	2.263	2.518	2.749	2.999	3.359	3.887	4.512	4.847

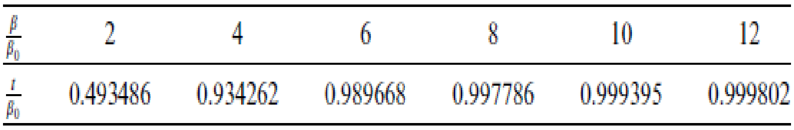
2.3 Description of Tables and Illustrative Examples

The numerical results for $T \sim GBS(\alpha, \beta\,; t_\nu)$ and fixed $\alpha = 1$ and $\nu = 5$, are presented in Tables 2.3–2.5. Table 2.3 presents the minimum sample size necessary to assert that the median life exceeds a given value, β_0, with probability $P^*$and the corresponding acceptance number, c, using binomial probabilities. Table 2.4 presents the operating characteristic values for the sampling plan $(n\ ,c\ ,\ \frac{t}{\beta_0})$ for a given $P^*$when c = 2. Finally, Table 2.5 presents the minimum ratios of true median life to specified median life for the acceptance of a lot with producer's risk of 0.05.

Example 2.3.1

Suppose that the lifetime (T) of a product follows a generalized Birnbaum–Saunders distribution. Specifically, let $T \sim GBS(\alpha, \beta\,; t_\nu)$, with α= 1 (which is close to increasing failure rate model) and $\nu = 5$. An experimenter wants to establish that the true unknown median life is at least 1,000 hours (β_0) with confidence $P^* = 0.95$, and that the life test will be terminated at $t = 942$ hours. Then, for an acceptance number $c = 2$, the required n is found from Table 2.3 to be 11. If during 942 hours, no more than 2 failures out of 11 are observed, then the experimenter can assert with confidence 0.95 that the median life is at least 1,000 hours. Instead of the binomial formula, if the Poisson approximation is used, the value of n would instead be 14. The values of n presented in Table 2.3 are indeed less than the corresponding values of n tabulated in of Gupta and Groll (1961)[32] for gamma distribution with shape parameter 2 (which is an increasing failure rate model), Kantam et al. (2001)[37] for the log-logistic distribution, and Baklizi and El Masri (2004)[3] for the Birnbaum Saunders distribution for $\alpha = 1$ (which is approximately an increasing failure rate model).

For the sampling plan ($n = 11$, $c = 2$, $\frac{t}{\beta_0} = 0.942$), the operating characteristic values from Table 2.4 are as follows:



$\frac{\beta}{\beta_0}$	2	4	6	8	10	12
$\frac{t}{\beta_0}$	0.493486	0.934262	0.989668	0.997786	0.999395	0.999802

This simply means that if the true median life is twice the specified median life $\left(\frac{\beta}{\beta_0} = 2\right)$, then the producer's risk is about 0.507, while it is about 0.066 when the true median life is 4 times the specified median life. Table 2.4 can be used to get the value of $\frac{\beta}{\beta_0}$ for various choices of $\left(c, \frac{t}{\beta_0}\right)$ such that the producer's risk may not exceed 0.05. For example, the value of $\frac{\beta}{\beta_0}$ is 4.274 for $c = 2$,$\frac{t}{\beta_0} = 0.942$ and $P^* = 0.95$. This means that the product should have a median life of 4. 274 times the specified median life in order for the lot to be accepted with probability 0. 95.

It is concluded that under similar conditions, in order to ensure a specified median life with a given confidence level, the GBS(t_ν) model results in smaller sample sizes than some other models used in acceptance sampling . The GBS(t_ν) distribution fits the data better than the classical Birnbaum–Saunders and inverse Rayleigh models.

CHAPTER - 3

Chapter -3

On the Hazard Function of Birnbaum-Saunders Distribution and Associated Inference

In this chapter, On the hazard function of Birnbaum –Saunders distribution and Associated Inference by Debasis Kundu, Nandini Kannan and Balakrishnan [2008][39] has been reviewed.

The two-parameter Birnbaum-Saunders (BS) distribution was originally proposed by Birnbaum and Saunders (1969)[9,10] as a failure time distribution for fatigue failure caused under cyclic loading. The cumulative distribution function (CDF) of a two-parameter Birnbaum-Saunders random variable T is of the form

$$F(t;\alpha,\beta)= \phi\left[\frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}}-\left(\frac{\beta}{t}\right)^{\frac{1}{2}}\right\}\right],\ 0< t <\infty\ , \alpha,\beta > 0 \qquad (3.1)$$

where $\phi(.)$ is the standard normal cumulative distribution function. The parameters α and β in Eq.(3.1) are the shape and scale parameters, respectively. Although the Birnbaum -Saunders distribution was originally proposed as a failure time distribution for fatigue failure under the assumption that the failure is due to development and growth of a dominant crack, a more general derivation was provided by Desmond (1985)[25] based on a biological model. Desmond (1985)[25] also strengthened the physical justification for the use of this distribution by relaxing the assumptions made originally by Birnbaum and Saunders (1969)[10].

The density function of the Birnbaum -Saunders distribution is unimodal. Mann et al. (1974)[46] mentioned that the hazard function of the Birnbaum – Saunders distribution is not an increasing function of t, although they did not provide a formal proof for it. In this chapter, the author prove that the hazard function of the Birnbaum - Saunders distribution is indeed an upside down function of $t > 0$ for all values of the shape parameter α and scale parameter β .

It is not uncommon to model survival and failure time data by distributions which have monotone hazard function. But in many practical situations, the hazard function is not monotone and in fact it increases up to a point and then decreases.

In this chapter, the change point of the hazard function of the Birnbaum - Saunders distribution is investigated and some methods of estimation for that change point are discussed. Since the change point is a function of the two parameters of the Birnbaum - Saunders distribution, it would be logical to estimate the change point by replacing the parameters by their estimates. Several estimation methods have been discussed by Ng et al. (2003)[47] who have observed that the modified moment estimators (MMEs) and the bias-corrected modified moment estimators (BCMMEs), in particular, are quite efficient and are also easy to implement computationally. For this reason, the change point is estimated by replacing the two unknown parameters by the corresponding modified moment estimators and the bias-corrected modified moment estimators. The asymptotic distributions of these estimators of the change point are obtained. To obtain confidence intervals for the change point, non-parametric bootstrap method is used. The performance of these estimators are evaluated in terms of bias and variance by means of Monte Carlo simulations for small, moderate and large sample sizes. Finally, areal data set was analysed and all the inferential methods discussed here are illustrated.

3.1 Birnbaum –Saunders distribution

The probability density function (PDF) of a two-parameter Birnbaum-Saunders random variable T corresponding to the cumulative distribution function in Eq.(2.1) is given by

$$f(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi\alpha\beta}}\left[\left(\frac{\beta}{t}\right)^{\frac{1}{2}} + \left(\frac{\beta}{t}\right)^{\frac{3}{2}}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{2}-2\right)\right],\ 0 < t < \infty,\ \alpha,\beta > 0 \tag{3.2}$$

The monotone transformation

$$X=\frac{1}{2}\left[\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{\frac{-1}{2}}\right] \tag{3.3}$$

$$T=\beta\left\{1 + 2X2 + 2X(1 + X^2)^{\frac{1}{2}}\right\} \tag{3.4}$$

Then from Eq.(3.1), it readily follows that X is distributed as normal with mean zero and variance ($\alpha^2/4$). The transformation in Eq.(3.4) is a very useful transformation as it enables the determination of the moments of T through known results on expectations of functions of X. For example,

$$E(T)=\beta\left(1+\frac{1}{2}\alpha^2\right), \tag{3.5}$$

$$V(T)=(\alpha\beta)^2\left(1+\frac{5}{4}\alpha^2\right), \tag{3.6}$$

$$\beta_1(T)=\frac{16\alpha^2(11\alpha^2+6)}{(5\alpha^2+4)^3}, \tag{3.7}$$

$$\beta_2(T)=3+\frac{6\alpha^2(93\alpha^2+41)}{(5\alpha^2+4)^2}, \tag{3.8}$$

where E(T), V (T), $\beta_1(T)$ and $\beta_2(T)$ are the expected value, variance, coefficient of skewness, and coefficient of kurtosis, respectively.

3.2 Shape of the Hazard

To examine the shape of the hazard function, let us assume that the scale parameter $\beta = 1$, without loss of any generality. Consider the function

$$\epsilon(t) = t^{\frac{1}{2}} - t^{-\frac{1}{2}} \tag{3.9}$$

for which

$$\left.\begin{aligned} \epsilon'(t) &= \frac{d}{dt}\epsilon(t) = \frac{1}{2}\left(t^{\frac{-1}{2}} + t^{-\frac{3}{2}}\right) = \frac{1}{2t}\left(t^{\frac{1}{2}} + t^{-\frac{1}{2}}\right), \\ \epsilon''(t) &= \frac{d}{dt}\epsilon'(t) = -\frac{1}{4t^2}\left(t^{\frac{1}{2}} + 3t^{-\frac{1}{2}}\right) \end{aligned}\right\} \tag{3.10}$$

and also

$$\epsilon^2(t) = t + \frac{1}{t} - 2 \tag{3.11}$$

The density function of the Birnbaum - Saunders distribution in Eq.(3.2)
(for $\beta = 1$) is then

$$f(t;\alpha)=\frac{1}{\sqrt{2\pi\alpha}}\epsilon'(t)e^{\frac{-1}{2\alpha^2}\epsilon^2(t)} \tag{3.12}$$

which, in conjunction with the expression of the distribution function in Eq.(3.1), gives the hazard function as

$$h(t;\alpha)=\frac{f(t;\alpha)}{1-F(t;\alpha.1)} = \frac{\frac{1}{\sqrt{2\pi\alpha}}\epsilon'(t)e^{\frac{-1}{2\alpha^2}\epsilon^2(t)}}{\phi\left(\frac{-\epsilon(t)}{\alpha}\right)} \tag{3.13}$$

From Eq . (3.13), the shape of h(t; α) is not at all clear. The lemmas for establishing the main result regarding the shape of the hazard function h(t; α) in Eq.(3.13) are as follows.

Lemma 3.1: Suppose f(t), for t> 0, is the density function of a positive real-valued continuous random variable,$f'(t)$ is the derivative of f(t), and $\eta(t) = \frac{-f'(t)}{f(t)}$. Then, if there exists at_0 such that $\eta'(t) > 0\ \forall t \in (0,t_0), \eta'(t_0) = 0$ and $\eta'(t) < 0\ \forall t \in (t_0,\infty)$, the hazard function corresponding to f(t) is either an upside down or a decreasing function of t.

Lemma 3.2: The hazard function of Birnbaum-Saunders distribution is either an upside down or a decreasing function of t > 0, for all values of the shape parameter α.

Lemma 3.3: For α > 0, the hazard function of the Birnbaum Saunders distribution is indeed an upside down function.

Theorem 3.1: The hazard function of the Birnbaum Saunders distribution is an upside down function for all values of the shape parameter α .

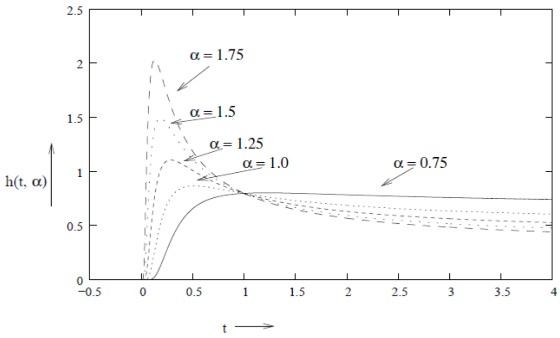


Figure 3.1: Hazard functions of the Birnbaum-Saunders distribution for different values of α, when β = 1.

3.3 Change Point of the Hazard

Since β is the scale parameter, it is evident that if $c_{\alpha,\beta}$ is the change point of $h(t;\alpha,\beta)$, then$c_{\alpha,\beta} = \beta c_{\alpha,1}$ for notational simplicity is denoted by$c_{\alpha,1}$ byc_α. In Figure 1, plots of the hazard function of the Birnbaum Saunders distribution for different values ofαare presented. The change point $c_\alpha$is a decreasing function ofα, which can be obtained as the solution of the following non- linear equation;

$$\phi\left(-\frac{\epsilon(t)}{\alpha}\right)\{-(\epsilon'(t))^2\epsilon(t)+\alpha^2\epsilon''(t)\}+\alpha\phi\left(-\frac{1}{\alpha}\epsilon(t)\right)(\epsilon'(t))^2=0 \qquad (3.15)$$

Table 3.1 : values of c_α for different choices of α

α	c_α	α	c_α	α	c_α	α	c_α	α	c_α
0.6	2.5364	1.0	0.5149	1.4	0.2178	1.8	0.1221	2.2	0.0787
0.7	1.5233	1.1	0.3981	1.5	0.1850	1.9	0.1083	2.3	0.0716
0.8	0.9923	1.2	0.3180	1.6	0.1594	2.0	0.0968	2.4	0.0654
0.9	0.6950	1.3	0.2605	1.7	0.1388	2.1	0.0870	2.5	0.0599

There is no explicit solution for Eq. (3.15), and so it needs to be determined by numerical methods. In Table 1, the values of c_α for different choices of α are presented. The calculation of c_α from Eq.(3.15) is not difficult for large values of α. Standard root solving technique like Newton-Raphson method works very well. But for smaller values of α, finding is extremely difficult. As α approaches zero, c_α approaches infinity. Particularly, it is observed that for $\alpha < 0.5$, the numerical solution of Eq.(3.15) is very unstable, and so extreme care is needed to calculate c_α in this case. The standard root solving methods do not work.

Since c_α does not have a closed-form expression, we seek to obtain a functional approximation of c_α as a function of α .Using Box-Cox type transformation, it was observed that $c_\alpha^{\frac{-1}{2}}$ approximately a linear function of α and so a reasonably good approximation of c_α, say $\widetilde{c_\alpha}$ as a function of α is given by

$$\widetilde{c_\alpha} = \frac{1}{(-0.4604+1.8417\alpha)^2} , \qquad (3.16)$$

For $\alpha > \frac{0.4604}{1.8417} = 0.25$.

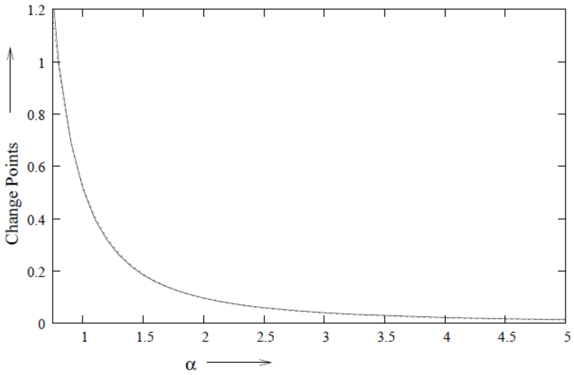


Figure 3.2: The change points and the approximate change points of the hazard functions of the Birnbaum-Saunders distribution for different values of α, when β= 1.

It is observed that for $\alpha > 0.6$, the approximation in Eq.(3.16) work very well. Since the computation of $c_\alpha$itself is very difficult for $0.25 < \alpha < 0.5$, it makes it difficult to find a reasonable approximation of c_α in that range. Plots of $c_\alpha$and $\widetilde{c_\alpha}$ against $\alpha$are presented in Fig. 3.2 where from it is observed that the approximation in Eq.(3.16) is quite accurate whenever α is not very small.

3.5 Estimation of the Change Point

The different estimators for the change point $c_{\alpha,\beta}$of the hazard function $h(t,\alpha,\beta)$ of the BirnbaumSaunders distribution, where

$h(t,\alpha,\beta) = \frac{f(t;\alpha,\beta)}{1-F(t;\alpha,\beta)} = \frac{\frac{1}{\sqrt{2\pi\alpha}}\epsilon'\left(\frac{t}{\beta}\right)e^{-\frac{1}{2\alpha^2}\epsilon^2\left(\frac{t}{\beta}\right)}}{\Phi\left(-\frac{\epsilon\left(\frac{t}{\beta}\right)}{\alpha}\right)}$ are presented.

For this purpose, we use the modified moment estimators (MMEs) and the bias-corrected modified moment estimators (BCMMEs) of αandβ discussed by Ng et al. (2003)[47].These authors observed that the modified moment estimators are very simple to use and that they behave very much like maximum likelihood estimators (MLEs). They are also slightly biased, like the maximum likelihood estimators, in the case of small sample sizes and for this reason the bias-corrected estimators were proposed. The bias- corrected modified moment estimators were shown to be good estimators in terms of both bias and variance. More importantly, as aptly mentioned by Ng et al. (2003)[47], both these estimators are very simple to implement as they do not require any non-linear root solving procedure as needed by the maximum likelihood estimators.

Let $t_1, \ldots, t_n$ be a random sample of size n from the Birnbaum Saunders distribution with probability density function as in Eq.(3.2).

Let , $s = \frac{1}{n}\sum_{i=1}^{n} t_i$ and $r = \left[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{t_i}\right]^{-1}$

denote the sample arithmetic and harmonic means, respectively. Then, the modified moment estimators of $\alpha$and β are given by

$$\hat{\alpha} = \left(2\left[\left(\frac{s}{r}\right)^{\frac{1}{2}} - 1\right]\right)^{\frac{1}{2}} \text{ and } \hat{\beta} = (sr)^{\frac{1}{2}},$$

and the bias- corrected modified moment estimators are given by

$\tilde{\alpha} = \left(\frac{n}{n-1}\right)\hat{\alpha}$ and $\tilde{\beta} = \left(1+\frac{\tilde{\alpha}^2}{4n}\right)^{-1}\hat{\beta}$;

Therefore, one can propose to estimate$c_{\alpha,\beta}$by $c_{\hat{\alpha},\hat{\beta}}$or $c_{\tilde{\alpha},\tilde{\beta}}$. Denote $c_{\hat{\alpha},\hat{\beta}}$ and $c_{\tilde{\alpha},\tilde{\beta}}$ as the modified moment estimators and bias- corrected modified moment estimators of $c_{\alpha,\beta}$respectively. Ina similar vein, $c_{\tilde{\alpha}}$in Eq.(3.16) can be used in conjunction with modified moment estimators and bias- corrected modified moment estimators of α andβ to produce approximate modified moment estimators (AMME) and approximate bias- corrected modified moment estimators (ABCMME) of$c_{\alpha,\beta}$as $\hat{\beta}\tilde{c}_{\hat{\alpha}}$and $\tilde{\beta}\tilde{c}_{\tilde{\alpha}}$respectively.The asymptotic distributions of all these estimators can be derived by using delta method.

3.6 Simulation Results

Monte Carlo simulations are carried out in order to compare the performance of all the estimators proposed .

The sample sizes considered were n = 10, 15, 25, 50 and the shape parameter was taken asα = 0.75, 1.00, 1.50 and 2.00. In all cases, set the scale parameter β = 1. Used 1,000 replications to estimate the change point c_{α} by using the modified moment estimators, bias- corrected modified moment estimators, approximate modified moment estimators and approximate bias- corrected modified moment estimators methods. The values of the average and variance of all estimates were computed and these are reported in Table 2.2, wherein the true value of c_{α} [determined numerically from Eq.(3.15)] are also presented for comparison purposes.

From the values in Table3.2, it is clear that both bias and variance of all estimators decrease when the sample size increases, as one would expect. When the sample size is very small, say n = 10, then none of the methods work satisfactorily. While all the methods overestimate c_{α} generally, the bias- corrected modified moment estimators, and approximate bias- corrected modified moment estimators seem to possess smaller bias and variance than the modified moment estimators and approximate modified moment estimators, respectively. Since the performance of the estimators bias- corrected modified moment estimators and approximate bias- corrected modified moment estimators are quite similar and that they can be computed explicitly without requiring the use of any non-linear equation solver, the use of approximate bias- corrected modified moment estimators is recommended for estimating the change point of the hazard function of the Birnbaum Saunders distribution as it shows a slightly better performance.

Table 3.2: **Values of average and variance of different estimators of c_α .**

n	α	c_α	MME	BCMME	AMME	ABCMME
10	0.75	1.2162	1.5980(0.7204)	1.3381(0.6534)	1.8891(0.8148)	1.4141(0.6570)
	1.00	0.5149	0.8315(0.2475)	0.6000(0.1486)	0.8385(0.2041)	0.6082(0.1021)
	1.50	0.1850	0.3097(0.0559)	0.2273(0.0275)	0.3148(0.0434)	0.2303(0.0233)
	2.00	0.0968	0.1551(0.0091)	0.1515(0.0180)	0.1595(0.0112)	0.1345(0.0204)
15	0.75	1.2162	1.5284(0.5704)	1.2801(0.3953)	1.6194(0.5043)	1.2898(0.2997)
	1.00	0.5149	0.7615(0.1959)	0.5920(0.1207)	0.7230(0.1287)	0.5873(0.0823)
	1.50	0.1850	0.2679(0.0333)	0.2154(0.0190)	0.2703(0.0290)	0.2068(0.0099)
	2.00	0.0968	0.1389(0.0059)	0.1265(0.0079)	0.1376(0.0077)	0.1164(0.0119)
25	0.75	1.2162	1.4690(0.4384)	1.2518(0.3236)	1.4390(0.3296)	1.2601(0.2432)
	1.00	0.5149	0.6596(0.1127)	0.5683(0.0793)	0.6479(0.0926)	0.5736(0.0712)
	1.50	0.1850	0.2290(0.0146)	0.2022(0.0104)	0.2292(0.0107)	0.2031(0.0085)
	2.00	0.0968	0.1198(0.0033)	0.1091(0.0031)	0.1194(0.0043)	0.1047(0.0034)
50	0.75	1.2162	1.3714(0.2381)	1.2376(0.0796)	1.2969(0.1199)	1.2159(0.1036)
	1.00	0.5149	0.5773(0.0410)	0.5402(0.0328)	0.5837(0.0414)	0.5442(0.0269)
	1.50	0.1850	0.2037(0.0048)	0.1920(0.0042)	0.2057(0.0036)	0.1937(0.0032)
	2.00	0.0968	0.1061(0.0013)	0.1000(0.0012)	0.1059(0.0014)	0.0992(0.0013)

For a given n, against each α, the first figure represents the average value of the estimate and the figure inside the parentheses is the corresponding variance.

Example 3.1 :

The data represent the survival times of guinea pigs injected with different doses of tubercle bacilli. It is known that guinea pigs have high susceptibility to human tuberculosis and that is why they were used in this study. Here, we are primarily concerned with the animals in the same cage that were under the same regimen. The regimen number is the common logarithm of the number of bacillary units in 0.5 ml. of challenge solution; i.e., regimen 6.6 corresponds to 4.0×10^6 bacillary units per 0.5 ml. (log ($4.0\ 10^6$) = 6.6).

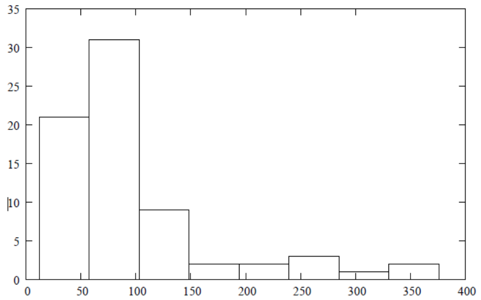


Figure 3.3: **Histogram of the survival times of guinea pigs in Regimen 6.6 data**

Corresponding to regimen 6.6, there were 72 observations listed: 12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

The mean, standard deviation and the coefficient of skewness are calculated as 99.82, 80.55 and 1.80 respectively. The skewness measure indicates that the data are positively skewed. The histogram plot, presented in Fig. 3.3, also supports this point. Moreover, Gupta et al. (1997)[33] observed that the empirical hazard function calculated from these data is unimodal. Therefore, use the Birnbaum Saunders distribution to analyze these data. The modified moment estimators of$\alpha$and β are determined as 0.7707 and 77.2931, respectively, and the bias-corrected modified moment estimators as 0.7600 and 77.4525. The plots of empirical survival function and the fitted survival function are presented in Fig.3.4. They are nearly identical. Furthermore, the Kolmogorov-Smirnov (KS) distance between the empirical and fitted survival functions is found to be 0.1044 with the corresponding p-value being 0.4125, which clearly suggests that the Birnbaum Saunders distribution provides a good fit to the Guinea pig data. The plots of the estimated hazard functions based on the modified moment estimators and bias- corrected modified moment estimators are presented in Figure 3.5. The modified moment estimators and the approximate modified moment estimators of the change point $c_\alpha$are 90.30 and 87.80, respectively, and the corresponding 95% bootstrap confidence intervals to be (59.37 ,137.55) and (59.31, 131,44). bias-corrected modified moment estimators and approximate bias- corrected modified moment estimators to be 86.20 and 84.05, respectively, and the corresponding

95% bootstrap confidence intervals to be (57.84,134.96) and (57.92, 128.61). One can observe that the results obtained by the two methods to be quite close.

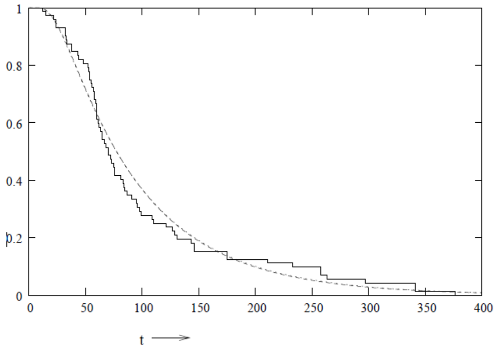


Figure 3.4: The empirical survival function and the fitted survival functions of the guinea pigs in Regimen 6.6 data.

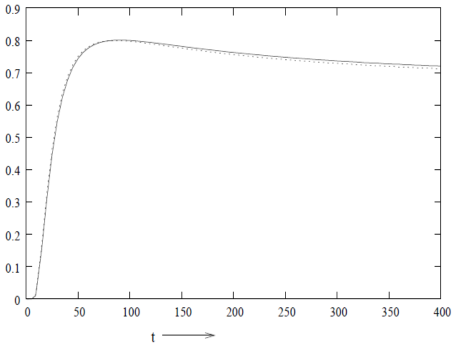


Figure 3.5: The estimated hazard functions based on modified moment estimators and bias- corrected modified moment estimators.

In this chapter it is proved that the hazard function of the two-parameter Birnbaum Saunders distribution is unimodal (of upside down form). The change point in the hazard function can be obtained as a solution of a non-linear equation. An approximation to this change point is provided and have shown that the approximation works very well whenever the shape parameter is not too small. Different methods for estimating the change point are proposed and have compared their performance through Monte Carlo simulations. Both bias-corrected modified moment estimators and approximate bias- corrected modified moment estimators perform well, and since approximate bias- corrected modified moment estimators is a simple explicit estimator, it can be used for the estimation of change point. The asymptotic distribution of all the estimators are derived and because of their complexity, bootstrap confidence intervals for data analysis has used.

CHAPTER - 4

Chapter – 4

The Generalized Birnbaum–Saunders Distribution and Its Theory, Methodology, And Application

In this chapter , the Generalized Birnbaum –Saunders Distribution and Its Theory, Methodology, and Application by Antonio Sanhueza , Victor Leiva and Balakrishnan (2008)[54] has been reviewed.

In parametric analysis of data, the identification of the most suitable distribution is an important problem. When data are modelled, several probability distributions fit in the central part, but poorly in the tails due to lack of observations. This poses special problems in situations when the primary interest of the study is at the extremes of the distributions as establishing warranties, maintenance problems or lethal doses. Therefore identification of a suitable model, therefore, becomes a very important issue.

Lifetime distributions can be identify by examining the hazard function (h.f.) or its logarithm against the lifetime or its logarithm or the cumulative hazard function or the logarithm of the survival function against the lifetime or its logarithm. While analyzing the hazard function the points to remember: If the hazard function. is increasing (IHF), the survival probability will decrease throughout time suitable for modelling lifetimes when wear-out or aging is present. Hazard functions with decreasing behaviour (DHF class) are suitable for modelling survival data after successful surgery when an initial risk exists possibly due to infection or haemorrhaging, followed by a decreasing risk as the patient recovers, since the survival probability increases throughout time. If the risk decreases and becomes stabilized during a period and then increases, a U-shaped hazard function must be assumed, which is generally present in human mortality data. Finally, if the hazard function is constant, then the lifetime model is exponential. Hence, to consider several ways of selecting a lifetime distribution is the best and the chosen must be logical model and supported by goodness-of-fit.

Problems related to the medical sciences, such as chronic cardiac diseases and different types of cancer, a cumulative damage caused by several risk factors produce degradation that leads to a fatigue process and the corresponding lifetimes can then be modeled suitably by the Generalized Birnbaum Saunders distribution.

4.1 The Generalized Birnbaum Saunders Distribution:

The Generalized Birnbaum Saunders Distribution is related to standard symmetrical distributions in R , also known as elliptically contoured univariate distributions. When a random variable Z follows a standard symmetrical distributions in R, the notation $Z \sim S(g)$ is used, where g is the kernel of the probability density function (pdf) of Z. The probability density function and cumulative distribution function (cdf) of $Z \sim S(g)$are denoted by f(.) and F(.), respectively, where $f(z)= cg(z^2)$, with c being the normalization constant, such that $\int_{-\infty}^{+\infty} g(z^2)dz = c^{-1}$.

4.1.2 Theoretical Properties :

If a random variable T follows a Generalized Birnbaum Saunders Distribution with shape parameter $\alpha > 0$, scale parameter $\beta > 0$ and associated kernel g, then the notation $T \sim$Generalized Birnbaum Saunders Distribution$(\alpha, \beta; g)$ is used. Now with

$$T = \beta\left(\frac{\alpha Z}{2} + \left[\left(\frac{\alpha Z}{2}\right)^2 + 1\right]^{\frac{1}{2}}\right)^2 \sim GBS(\alpha, \beta; g), \tag{4.1}$$

the following properties hold :

(A1). $Z = \frac{1}{\alpha}\left\{\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{\frac{-1}{2}}\right\}$,

(A2). $U = Z^2 = \frac{1}{\alpha^2}\left(\frac{T}{\beta} + \frac{T}{\beta} - 2\right)$ follows a generalized chi-square (G_{χ^2}) distribution with one degree of freedom, which is denoted by $U \sim G_{\chi^2}(f)$. The probability density function of U is given by $f_U(u) = cg(u)u^{-\frac{1}{2}}$, with $u > 0$, where cg(.) is the pdf of $Z \sim S(g)$;

(A3). The pdf of T is $f_T(t) = cg(\kappa_t)A_t = f(a_t)A_t$, with $t > 0$, $\alpha > 0$ and $\beta > 0$, where

$$a_t = a_t(\alpha, \beta) = \frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{2}} - \left(\frac{t}{\beta}\right)^{\frac{-1}{2}}\right\},$$

$$A_t = \frac{d}{dt} a_t = \frac{1}{2\alpha\beta}\left\{\left(\frac{t}{\beta}\right)^{-\frac{1}{2}} + \left(\frac{t}{\beta}\right)^{\frac{-3}{2}}\right\} \text{ and } \kappa_t = a_t^2 \;;$$

(A4). The mode(s) of T , denoted by t_m , is (are) given by the solution(s) of

$$\omega_g = \left(\kappa_{t_m}\right) = \frac{\alpha^2 \beta t_m (t_m + 3\beta)}{2(t_m - \beta)(t_m + \beta)^2},$$

where $\omega_g(u) = g'(u)/g(u)$ with u>0 , $g'(.)$ is derivative of g(.) and κ_t is as in (A3).

The Generalized Birnbaum Saunders Distribution can have multiple modes;

(A5).cT~ GBS $(\alpha, c\beta; g)$, with c >0 , i.e., the Generalized Birnbaum Saunders Distribution is closed under scale transformations;

(A6). $T^{-1} \sim$ GBS $(\alpha, \beta^{-1}; g)$, i.e., the Generalized Birnbaum Saunders Distribution is closed under reciprocation;

(A7). The cumulative distribution function of T is $F_T(t) = F(a_t)$, with t >0 , and a_t is ans in (A3).

(A8). The quantile function of T is t(p) = $F_T^{-1}(p) = \frac{\beta}{4}\left(\alpha z_p + \left[\alpha^2 z_p^2 + 4\right]^{\frac{1}{2}}\right)^2$,

where z_p is the pth percentile of $Z \sim S(g)$. If p = 0.5 , then t(0.5) = β and so β is the median ;

(A9) For the variates Z and U = Z^2given as in (A1) and (A2), respectively, if $\mathbb{E}\,(Z^k)$ exists, then

$$\mathbb{E}(T^k)=\beta^k \sum_{i=0}^{k} \binom{2k}{2i} \sum_{j=0}^{i} \binom{i}{j} \mathbb{E}\,(U^{k+j-i} \left(\frac{\alpha}{2}\right)^{2(k+j-i)}, \qquad (4.2)$$

From Properties (A5) and (A6), the negative moments of T can be obtained by using the fact that βT^{-1} and $\beta^{-1}T$ have the same distribution. Hence, $\mathbb{E}\,(T^{-k}) = \beta^{-2k}\,\mathbb{E}(T^k)$. An alternative way of computing the kth moment of the Generalized Birnbaum Saunders Distribution, instead of Eq.(4.2) , is

$$\mathbb{E}(T^k)=\frac{\alpha^{2k}\beta^k}{2} u_k - \frac{1}{2}\left\{\sum_{i=1}^{k} \binom{2k}{i} \frac{\mathbb{E}\,(T^{k-i})}{\beta^{-i}(-1)^i} + \sum_{i=k+1}^{2k-1} \binom{2k}{i} \frac{\mathbb{E}\,(T^{i-k})}{\beta^{i-2k}(-1)^i}\right\},$$

$$k = 1,2,... \qquad (4.3)$$

where $u_k = \mathbb{E}\,(U^k)$ and $U \sim G_{\chi^2}$ (g) . From Eqs.(4.2) and (4.3) , the first four moments of T can be obtained as

(A9.1) $\mu_1' = \mathbb{E}(T) = \frac{\beta}{2} \quad (2 + u_1\alpha^2)$;

(A9.2) $\mu_2' = \mathbb{E}(T^2) = \frac{\beta^2}{2} \quad (2 + 4u_1\alpha^2 + u_2\alpha^4)$;

(A9.3) $\mu_3' = \mathbb{E}(T^3) = \frac{\beta^3}{2} \quad (2 + 9u_1\alpha^2 + 6u_2\alpha^4 + u_3\alpha^6)$; and

(A9.4) $\mu_4' = \mathbb{E}(T^4) = \frac{\beta^4}{2} \quad (2 + 16u_1\alpha^2 + 20u_2\alpha^4 + 8u_3\alpha^6 + u_4\alpha^8)$.

From (A9.1)–(A9.4) , the first four central moments of the Generalized Birnbaum Saunders distribution , defined as $\mu_k = \mathbb{E}\left[(T - \mathbb{E}\{T\})^k\right], \mu_1 = 0,$ are obtained to be

(A9.5) $\mu_2 = \text{Var}\,(T) = \frac{\beta^2\alpha^2}{4}[4u_1 + (2u_2 - u_1^2)\alpha^2]$;

(A9.6) $\mu_3 = \frac{\beta^3\alpha^4}{4}[6(u_2 - u_1^2) + (u_1^3 - 3u_1u_2 + 2u_3)\alpha^2]$; and

(A9.7) $\mu_4 = \frac{\beta^4\alpha^4}{16}\begin{bmatrix} 16u_2 + (24u_1^3 - 48u_1u_2 + 32\,u_3)\alpha^2 \\ -(3u_1^4 - 12u_1^2u_2 + 16u_1u_3 8u_4)\alpha^4 \end{bmatrix}$

(A10) The mean and variance of T are as given in (A9.1) and (A9.5) respectively, while the coefficients of variation (CV), skewness (CS), and kurtosis (CK) of T are given, respectively, by

$$\gamma(T) = \frac{\alpha[4u_1+(2u_2-u_1^2)\alpha^2]^{\frac{1}{2}}}{(2+u_1\alpha^2)},$$

$$\alpha_3(T) = \frac{4\alpha[(3u_2-3u_1^2)+\frac{1}{2}(2u_3-3u_1u_2+u_1^3)\alpha^2]}{[4u_1+(2u_2-u_1^2)\alpha^2]^{\frac{3}{2}}} \text{ and}$$

$$\alpha_4(T) = \frac{16u_2+(32u_3-48u_1u_2+24u_1^3)\alpha^2+(8u_4-16u_1u_3+12u_1^2-3u_1^4)\alpha^4}{[4u_1+(2u_2-u_1^2)\alpha^2]^2} .$$

The dimensionless ratios $\gamma(T)$, $\alpha_3(T)$, and $\alpha_4(T)$ are functionally independent of the scale parameter β, with the skewness and kurtosis being mainly controlled by the shape parameter α.

4.1.2 Transformations and Related Distributions

Some transformations of T ~GBS(α, β ;g), in addition to those given in (A1), (A2), (A5), and (A6) are the logarithm of T and the absolute value and exponentiation of Z in (A1). Thus, we have that :

(B1)The probability density function of Y = log (T) is given by

$f_Y(y) = \frac{1}{\alpha}\cosh\left(\frac{y-\gamma}{2}\right) f\left(\frac{2}{\alpha}\sinh\left(\frac{y-\gamma}{2}\right)\right)$, with $y \in \mathbb{R}$, $\alpha > 0$, $\gamma \in \mathbb{R}$;

(B2) the probability density function of H = | Z| = $\left|\frac{1}{\alpha}\left\{\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}}\right\}\right|$ is given by $f_H(h) = 2f(h)$, with h>0 ;

(B3) the probability density function of L = exp (Z) = $\exp\left(\frac{1}{\alpha}\left\{\left(\frac{T}{\beta}\right)^{\frac{1}{2}} - \left(\frac{T}{\beta}\right)^{-\frac{1}{2}}\right\}\right)$ is given by$f_L(l) = \frac{f(\log(l))}{l}$, with l>0;

The following additional properties are readily obtained from (A1), (A2), (A5), (A6), (B1), (B2), and (B3):

(B4) O = $\frac{\alpha^2}{\beta}$ T $\sim$GBS$(\alpha, \alpha^2; g)$, i.e., O follows a one-parameter Generalized Birnbaum Saunders distribution;

(B5) C = $\beta^2 T^{-1}$$\sim$GBS$(\alpha, \beta; g)$, i.e., C is the complementary reciprocal of T ;

(B6) Y = log (T) follows the log- Generalized Birnbaum Saunders distribution, denoted by Y $\sim$log-GBS$(\alpha, \gamma; g)$, where γ = log (β) ;

(B7) H given in (B2) follows the half-symmetrical distribution;

(B8) L given in (B3) follows the log-symmetrical model;

(B9) An important distribution that is related to the Generalized Birnbaum Saunders distribution is the sinh-spherical (SHS) law;

The following results hold for the sinh-spherical model:

(B9.1) For a random variable Y following the sinh-spherical distribution, the notation Y $\sim$SHS$(\alpha, \gamma, \sigma; g)$ is used. Thus, if

$Z = \frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right) \sim S(g)$, then $Y = \gamma + \sigma \operatorname{arcsinh}\left(\frac{\alpha Z}{2}\right) \sim SHS(\alpha, \gamma, \sigma; g)$.

The pdf of Y is

$f_Y(y) = f\left(\frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right)\right)\frac{2}{\alpha\sigma}\cosh\left(\frac{Y-\gamma}{\sigma}\right)$, $y \in \mathbb{R}, \alpha > 0, \gamma \in \mathbb{R}, \sigma > 0$.

If Y $\sim$SHS$(\alpha, \gamma = \ln(\beta), \sigma = 2; g)$, then the random variable

T = exp(Y) $\sim$GBS$(\alpha, \beta; g)$ as in (B6);

(B9.2) If the GBS distribution is generated through its associated sinh-spherical model for any value of σ, then the three-parameter Generalized Birnbaum Saunders distribution is obtained; Thus, if Y $\sim$SHS$(\alpha, \gamma, \sigma; g)$, then the pdf of the random variable T = exp(Y) $\sim$GBS$(\alpha, \beta, \sigma; g)$ is given by $f_T(t)$= $f(b_t)B_t$, with t > 0, α> 0,β> 0, and σ> 0, where

$b_t = b_t(\alpha\,,\beta) = \frac{1}{\alpha}\left\{\left(\frac{t}{\beta}\right)^{\frac{1}{\sigma}} - \left(\frac{t}{\beta}\right)^{-\frac{1}{\sigma}}\right\}$ and

$B_t = \frac{d}{dt} b_t = \frac{1}{\alpha\beta\sigma}\left\{\left(\frac{t}{\beta}\right)^{-\frac{1}{\sigma}} + \left(\frac{t}{\beta}\right)^{-\left(\frac{1+\sigma}{\sigma}\right)}\right\}$;

(B10) Another important probability law related to the Generalized Birnbaum Saunders model is the inverse Gaussian type (IGT) distribution; see Sanhueza et al. (2008)[54]. If X follows a inverse Gaussian type distribution, denoted by X ~IGT(μ , λ ; g) then its pdf is given by

$f_X(x) = f\left(\frac{\lambda(x-\mu)}{\mu\sqrt{x}}\right)\frac{\sqrt{\lambda}}{\sqrt{x^3}}$, with x > 0,μ> 0 and λ> 0. If T ~GBS(α , β ; g), after some algebraicmanipulations, the pdf in (A3) can be expressed as

$$f_T(t) = \frac{1}{2}\, f(a_t)\sqrt{\frac{\beta}{\alpha^2 t^3}} + \frac{1}{2} f(a_t)\sqrt{\frac{1}{\alpha^2\beta t\beta}} = \frac{1}{2} f_X(t) + \frac{1}{2}\frac{x f_X(t)}{\beta}\ , t > 0\,,$$

$$\alpha > 0, \beta > 0\,.$$

Therefore, if $\alpha = \frac{\mu}{\lambda}$ and $\beta = \mu$, the Generalized Birnbaum Saunders model is an equally weighted mixture of a inverse Gaussian type distribution and its complementary reciprocal (also known as length-biased version). Thus, the Generalized Birnbaum Saunders model is a particular case of the mixture inverse Gaussian type distribution .

(B11) An important aspect related to the Generalized Birnbaum Saunders distribution is its generation by means of skew-spherical distributions; see Kotz and Vicari (2005)[37]. Then, we have the following:

(B11.1) If the Generalized Birnbaum Saunders law is obtained through a skew-spherical model, then a parameter of asymmetry (λ) is incorporated and a new version of the Birnbaum Saunders distribution is reached, which called as the skew Birnbaum–Saunders (SBS) distribution and denote by T ~SBS(α , β, λ; g); In this case, the pdf of T is given by $f_T(t)$= 2 $f(a_t)F(\lambda a_t)A_t$, with t > 0 ,λ ∈ℝ.

(B11.2) The sinh - spherical distribution can be obtained from a skew-spherical model, which is called the sinh-skew-spherical (SSS) distribution, denoted by Y ~SSS(α , γ , σ, λ ; g)and its pdf is given by

$f_Y(y) = 2f\left(\frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right)\right) F\left(\lambda\left\{\frac{2}{\alpha}\sinh\left(\frac{Y-\gamma}{\sigma}\right)\right\}\right)\frac{2}{\alpha\sigma}\cosh\left(\frac{Y-\gamma}{\sigma}\right)$, y∈ ℝ .

The Generalized Birnbaum Saunders distribution is related to the sinh-skew-spherical distribution and so we can generate an extended version of the Generalized Birnbaum Saunders distribution, which is denoted by
$T \sim EGBS(\alpha, \beta, \sigma, \lambda; g)$and has pdf as by $f_T(t) = 2\, f(b_t)F(\lambda b_t)B_t$,
with $t > 0$,$\lambda \in \mathbb{R}$.

4.1.3 Lifetime Analysis

Some useful functions in lifetime analysis are the hazard function, the survival function (s.f.), and the mean residual life. Leiva et al. (2008a)[42] carried out a lifetime analysis of the Generalized Birnbaum Saunders distribution. A summary of their results are as follows. Let $T \sim GBS(\alpha, \beta; g)$. Then, we have that :

(C1) The hazard function of T is$h_T(t) = \frac{f_T(t)}{1-F_T(t)} = \frac{f(a_t)}{F(-a_t)} A_t$, with $t > 0$ and $0 < F_T(.) < 1$. In addition, the behaviour of the Generalized Birnbaum Saunders hazard function does not depend on the scale parameter β ;

(C2) The hazard function average (HFA) of T is $HFA(t) = \frac{H_T(t)}{t} = \frac{\int_0^t h_T(t)\,dt}{t}$,
with $t > 0$,where $H_T(t)$ is the cumulative hazard function;

(C3) The survival function and conditional survival function of T are, respectively, $S_T(t) = F(-a_t)$ and$S_T(t \setminus x) = \frac{F(-a_{t+x})}{F(-a_x)}$, with $t > 0$, $x > 0$,
and $0 < S_T(.) < 1$;

(C4) The mean residual life of T is given by
$\mu_x = \mu = \mathbb{E}(T) = \int_0^\infty F(-a_t)dt = \frac{\beta}{2}(2 + u_1\alpha^2)$,when $x = 0$, as in (A9.1) ;

(C5) The critical point (t_c) of the hazard function of T is obtained as the solution of theEquation$F(-a_t) = \frac{\alpha\beta^{\frac{1}{2}}t_c^{\frac{1}{2}}(t_c+\beta)^2\, f(a_t)}{2\omega\kappa_{t_c}(t_c-\beta)(t_c+\beta)^2 + (t_c+3\beta)\alpha^2\beta t_c}$;

(C6) If $\lim_{t\to\infty} \omega_g(\kappa_t)$ exists , then $\lim_{t\to\infty} h_T(t) = -\lim_{t\to\infty}\left\{\frac{d}{dt}\log\big(g(\kappa_t)\big)\right\}$
$$= -\frac{1}{\alpha^2\beta}\lim_{t\to\infty}\omega_g(\kappa_t) ;$$

(C7) Systems of components connected in series, parallel or combinations of these, have lifetimes related to the minimum and maximum of the component lifetimes of these systems, i.e., associated with order statistics; Thus, if
$T_1, \ldots, T_n$ are independent and identically distributed random variable's, where

$T_i \sim GBS(\alpha, \beta; g)$, with i = 1 ,… , n, then the indicated random variable has probability density function given by:

(i) $T_{(1)}$ (the minimum of $T_1, \ldots, T_n$): $f_{T_{(1)}}(t) = nf(a_t)A_t[F(-a_t)]^{n-1}$;

(ii)$T_{(n)}$ (the maximum of $T_1, \ldots, T_n$): $f_{T_{(n)}}(t) = nf(a_t)A_t[F(a_t)]^{n-1}$;

(iii) $T_{(j)}$(the jth order statistic of$T_1, \ldots, T_n$); $f_{T_{(j)}}(t) = \frac{n!}{(j-1)!(n-j)!} f(a_t)A_t[F(a_t)]^{j-1}[F(-a_t)]^{n-j}$;

(C8) It is also possible to model the Generalized Birnbaum Saunders hazard function. in a flexible way. We consider T $\sim GBS(\alpha, \beta, \sigma; g)$, the three-parameter Generalized Birnbaum Saunders distribution, and analyze the limiting behaviour of its hazard function. Depending on the value of σ, we note that this extended version of the Generalized Birnbaum Saunders model admits, besides the classical ∩-shaped hazard function, the increasing hazard function class. Thus, if $h_T(t)$is the hazard function of T, then as t→∞,

$$h_T(t) \rightarrow \begin{cases} (2\alpha^2\beta)^{-1} \lim\limits_{t\to\infty} \omega_g(\kappa_t) & , \text{if } \sigma = 2 ; \\ \infty & , \text{if} \sigma < 2 ; \\ 0 & , \text{if } \sigma > 2 ; \end{cases} \tag{4.4}$$

4.1.4. Shape Analysis

A graphical analysis of some special cases of Generalized Birnbaum Saunders distributions are presented. Models presenting different degrees of kurtosis and asymmetry, modality as well as bimodality and those with absence of moments are considered. Also, models with ∩-shaped, increasing and bimodal hazard functions are presented. We give the densities of these Generalized Birnbaum Saunders models and Figs. 4.1 and 4. 2 show plots of some densities and hazard functions.

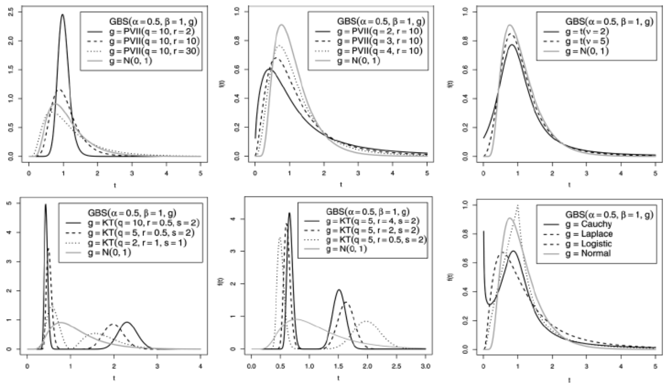


Figure 4.1.Plots of Generalized Birnbaum Saunders densities for the indicated α , β and kernel.

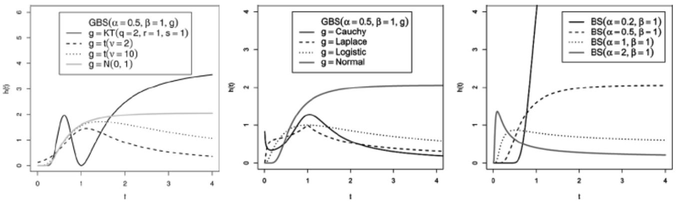


Figure 4.2.Plots of Generalized Birnbaum Saunders hazard functions for the indicated α , β and kernel.

Example 4.1 Let T ∼GBS(α , β ; g) . Then, for the specified kernel inside the brackets, the probability density function of T following a two-parameter Generalized Birnbaum Saunders distribution is given by

$$f_T(t) = \frac{1}{2\pi\alpha\beta}\left(1 + \frac{1}{\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-1}\left(\left[\frac{t}{\beta}\right]^{-\frac{1}{2}} + \left[\frac{t}{\beta}\right]^{-\frac{3}{2}}\right), \quad t > 0 \text{ ; [Cauchy]}$$

$$f_T(t) = \frac{1}{4\alpha\beta}\exp\left(-\frac{1}{\alpha}\left|\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right|\right)\left(\left[\frac{t}{\beta}\right]^{-\frac{1}{2}} + \left[\frac{t}{\beta}\right]^{-\frac{3}{2}}\right), \quad t > 0 \text{ ; [Laplace]}$$

$$f_T(t) = \frac{1}{2\alpha\beta}\frac{\exp\left(\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right)}{\left\{1 + \exp\left(\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right)\right\}^2}\left(\left[\frac{t}{\beta}\right]^{-\frac{1}{2}} + \left[\frac{t}{\beta}\right]^{-\frac{3}{2}}\right), \quad t > 0 \text{ ; [Normal]}$$

where$I_x(a, b) = \frac{\int_0^x t^{a-1}(1-t)^{b-1}dt}{\int_0^1 t^{a-1}(1-t)^{b-1}dt}$ is the incomplete beta ratio function ;

4.2.Inference,Diagnostics , Simulation and Goodness – of –fit

Some results related to inferential aspects, diagnostics tools, simulation algorithms, and goodness of fit methods for the generalized Birnbaum Saunders distribution are provided.

4.2.1 Inference

Díaz-García and Domínguez-Molina (2007)[22], Leiva et al. (2008a,b)[42,43] have all discussed estimation of the parameters α and β of the Generalized Birnbaum Saunders distribution. Some of the results concerning to the maximum likelihood estimators (MLEs) and modified moment estimators (MMEs) for the parameters of the Generalized Birnbaum Saunders distribution are described. The parameter estimates are obtained when censored data are presented.

4.2.1.2 Maximum Likelihood Estimation Method

Uncensored data: The log-likelihood function based on an uncensored sample $T_1, \ldots, T_n$ from the r.v. T ~GBS$(\alpha, \beta\,; g)$ is given by

$\ell(\theta) = \sum_{i=1}^{n} \ell_i(\theta)$,

with $\ell_i(\theta) \propto -\log(\alpha) - \frac{1}{2}\log(\beta) + \log(t_i + \beta) + \log\left(g\left(\kappa_{t_i}\right)\right)$, (4.5)

where $\theta = (\alpha, \beta)^T$ and g(.) κ_t are as given in (A3). In order to maximize $\ell(\theta)$, we need the score vector with first derivatives $\dot{L} = \left(\dot{L}_\alpha, \dot{L}_\beta\right)^T$, where

$\dot{L}_\alpha = \frac{\partial}{\partial\alpha}\ell(\alpha, \beta)$ and $\dot{L}_\beta = \frac{\partial}{\partial\beta}\ell(\alpha, \beta)$, which are given by

$$\dot{L}_\alpha = \sum_{i=1}^{n} \frac{v_i}{\alpha^3}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] - \frac{n}{\alpha} \text{ and } \dot{L}_\beta = \frac{1}{t_i+\beta} - \sum_{i=1}^{n} \frac{v_i}{2\alpha^2}\left[\frac{1}{t_i} - \frac{t_i}{\beta^2}\right] - \frac{n}{2\beta}, \qquad (4.6)$$

Where $v_i = v_i(\alpha, \beta) = -2\omega_g\left(\kappa_{t_i}\right)$ and $\omega_g(.)$ is a sin (A4). Thus, solving the equations $\dot{L}_\alpha = 0$ and $\dot{L}_\beta = 0$, we get

$$\alpha = \left(\frac{1}{n}\sum_{i=1}^{n} v_i \left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{\frac{1}{2}} \text{and } \beta = \left(\frac{\frac{1}{2\alpha^2}\sum_{i=1}^{n} v_{i t_i}}{\frac{1}{2\alpha^2}\sum_{i=1}^{n}\frac{v_i}{t_i} - \sum_{i=1}^{n}\frac{1}{t_i+\beta} + \frac{n}{2\beta}}\right)^{-\frac{1}{2}}. \qquad (4.7)$$

Censored data: The log-likelihood function based on a type-II censored sample $T_1, \ldots, T_r$ (with the censored largest n − r observations) from T ~GBS$(\alpha, \beta\ ; g)$ is given by

$$\ell(\theta) \propto -r\log(\alpha) - \frac{r}{2}\log(\beta) + \sum_{i=1}^{r}\log(t_i + \beta) + \sum_{i=1}^{r}\log\left(g(\kappa_{t_i})\right) + (n - r)\log(1 - F(a_t)), \quad (4.8)$$

where g(.) and κ_t are as in (A3) . in this case , the score vector $\dot{L} = \left(\dot{L}_\alpha, \dot{L}_\beta\right)^T$ has elements given by

$$\dot{L}_\alpha = -\frac{r}{\alpha} + \frac{1}{\alpha^3}\sum_{i=1}^{r} v_i\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] + \frac{2(n-r)}{\alpha}\frac{(t_r - \beta)}{(t_r + \beta)t_r}H_T(t_r) \text{and}$$

$$\dot{L}_\beta = -\frac{r}{2\beta} + \sum_{i=1}^{r}(t_i + \beta)^{-1} - \frac{1}{2\alpha^2}\sum_{i=1}^{r} v_i\left[\frac{1}{t_i} - \frac{t_i}{\beta^2}\right] + \frac{(n-r)t_r}{\beta}H_T(t_r), \quad (4.9)$$

where $H_T(.)$ is the Generalized Birnbaum Saunders hazard function. Thus, by solving the equations get

$\dot{L}_\alpha = 0$ and $\dot{L}_\beta = 0$, we get

$$\alpha = \left(\frac{1}{r}\sum_{i=1}^{r} v_i\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] + \frac{2(n-r)\alpha^2}{r}\frac{(t_r - \beta)}{(t_r + \beta)}t_r H_T(t_r)\right)^{\frac{1}{2}} \text{and} \quad (4.10)$$

$$\beta = \left(\frac{\frac{1}{2\alpha^2}\sum_{i=1}^{r} v_i t_i}{\frac{1}{2\alpha^2}\sum_{i=1}^{r}\frac{v_i}{t_i} - \sum_{i=1}^{r}\frac{1}{t_i+\beta} + \frac{r}{2\beta} - (n-r)\frac{t_r}{\beta}H_T(t_r)}\right)^{-\frac{1}{2}}, \quad (4.11)$$

which once again requires the use of an iterative process.

Asymptotic inference :

Based on the asymptotic normality of the Maximum Likelihood Estimators, $\hat{\theta} = \left(\hat{\alpha}, \hat{\beta}\right)^T$, one can construct hypothesis tests and confidence regions for α and β by using that $\hat{\theta} \dot{\sim} N_2(\theta, \sum_{\hat{\theta}})$N2, where $\sum_{\hat{\theta}}$ is the variance–covariance matrix of $\hat{\theta}$. Thismatrix can be approximated by $-\ddot{L}^{-1}$, with - $\ddot{L}$being the observed informationmatrix evaluated at $\hat{\theta}$and obtained from $\ddot{L}$.For the Birnbaum Saunders distribution, it is known that $\mathbb{E}(\ddot{L}_{\alpha\beta})$= 0; see Ng et al. (2003).[47] . However, when the Generalized Birnbaum Saunders model is generated from symmetrical distributions different than the normal one, then $\mathbb{E}(\ddot{L}_{\alpha\beta}) \neq 0$. Thus, in order to construct a confidenceregion for θ, the fact that $(\hat{\theta} - \theta)^{T} \Sigma^{-1} (\hat{\theta} - \theta) \dot{\sim} \chi^{2}(2)$ can be used. Then, an approximate $100(1 - \gamma)\%$ confidence region, with $0 < \gamma < 1$, for θis given by$\mathcal{R} = \left\{\theta \in \mathbb{R}^{2} : (\hat{\theta} - \theta)^{T} \Sigma_{\hat{\theta}}{}^{-1} (\hat{\theta} - \theta) \leq \chi^{2}{}_{1-\gamma}(2)\right\}$,where ${}^{2}{}_{1-\gamma}(2)$ denotes the$(1 - \gamma)^{th}$ percentile of the $\chi^{2}(2)$ distribution.

4.2.1.2 Modified Moment Estimation

As is well-known, the moment estimators may not be unique and also may not always exist. This occurs in the case of the Generalized Birnbaum Saunders model as well. An alternative way for estimating the parameters of the Generalized Birnbaum Saunders model is the modified moment estimation method, which estimates always exists uniquely; We now describe the Modified Moment Estimators of the parameters α and β for the Generalized Birnbaum Saunders distribution and their asymptotic distributions, which can be used to construct confidence intervals and hypothesis tests for the parameters α and β. Thus, if $T_1, \ldots, T_n$is asample from T ~GBS$(\alpha, \beta; g)$, then

(D1) The modified moment estimators ofα and β , denoted by $\tilde{\alpha}$ and $\tilde{\beta}$, are given by $\tilde{\alpha} = \left\{\frac{2}{u_1}\left[(S/R)^{\frac{1}{2}} - 1\right]\right\}^{\frac{1}{2}}and\tilde{\beta} = (S/R)^{\frac{1}{2}}$, where $u_1 = \mathbb{E}(U)$ with

$U \sim G\chi^{2}(g)$, $S = \frac{1}{n}\sum_{i=1}^{n} T_i$ and $R = \left[\frac{1}{n}\sum_{i=1}^{n} T_i{}^{-1}\right]^{-1}$. The shape parameter estimate $(\tilde{\alpha})$ is only affected by the considered kernel through the value of u_1. However, the scale parameter estimate $(\tilde{\beta})$ is independent of the considered kernel.

(D2) The asymptotic joint distribution of the Modified Moment Estimators$\tilde{\alpha}$ and $\tilde{\beta}$ is bivariatenormal, given by$\left[\begin{pmatrix}\tilde{\alpha}\\ \tilde{\beta}\end{pmatrix} - \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right]\sqrt{n} \dot{\sim} N_2(0, \Sigma)$,as $n \to \infty$,

$$\text{where } \Sigma = \begin{pmatrix} \frac{\alpha^{2}}{4}\left(\frac{u_1}{u_2} - 1\right) & 0 \\ 0 & (\alpha\beta)^{2}\frac{(4u_1 + \alpha^{2}u_2)}{(2 + \alpha^{2}u_1)^{2}} \end{pmatrix}, \tag{4.12}$$

and$u_k = \mathbb{E}(U^k)$,with $U \sim G\chi^2(g)$ and $k = 1,2$. From Eq.(4.12), we obtain the marginal asymptotic distributions of $\tilde{\alpha}$and $\tilde{\beta}$, which can be used to construct confidenceintervals for α and β.

4.2.2. Influence of Atypical Cases on the Estimates

The detection of atypical cases is an important step when parameters are estimated. Case deletion is a way to assess the effect of an observation on the estimates. This is a global influence analysis, since the effect of the case is evaluated by dropping it from the data. Alternatively, local influence is based on geometric differentiation rather than the elimination of cases. A differential comparison of estimates is used before and after perturbing the data or model. In order to evaluate possible a typical cases in the data, the local influence method was implemented. As in Cook (1986)[18], one can use the likelihood displacement to evaluate the local influence. Some global and local influence techniques are described.

4.2.2.1.Global Influence.

An important diagnostics technique of the global influence method is the Cook's distance; see Cook and Weisberg (1982)[18]. A generalization of Cook's distance is given by$D_i = \frac{1}{q}(\hat{\theta} - \hat{\theta}_{(i)})^T \hat{\Sigma}_{\hat{\theta}}^{-1}(\hat{\theta} - \hat{\theta}_{(i)})$, where q is the number of parameters and $\hat{\Sigma}_{\hat{\theta}}$ is an estimate of the variance–covariance matrix of $\hat{\theta}$, which can be approximated by -$\ddot{L}^{-1}$, so that we have

$D_i = \frac{1}{q}(\hat{\theta} - \hat{\theta}_{(i)})^T[-\ddot{L}](\hat{\theta} - \hat{\theta}_{(i)})$. Then, byusing an approximation of the first order, we obtain$\hat{\theta} - \hat{\theta}_{(i)} \approx [\ddot{L}_{(i)}]^{-1}\dot{L}_{(i)}$, where$\dot{L}_{(i)}$and $\ddot{L}_{(i)}$are respectively the score vector and Hessian matrix dropping the ith case.

Thus,$D_i \approx \frac{1}{q}\left\{[\dot{L}_{(i)}]^T[\ddot{L}_{(i)}]^{-1}[-\ddot{L}][\ddot{L}]^{-1}\dot{L}_{(i)}\right\}$ (4.13)

where high values of Di indicate cases with high impact on the Maximum Likelihood Estimators of θ .

4.2.2.2. Local Influence.

The contributions $\ell_i(\theta)$are equally weighted. A perturbed log-likelihood function—allowing different weights for different cases—can be defined as $\ell\ (\theta|\omega)= \sum_{i=1}^{n} \omega_i \ell_i(\theta)$, where$\omega = (\omega_1 , \ldots , \omega_n)^T$is a vector of weights for the contributions from each caseto the likelihood and $\omega_0 = 1_n = (1 , \ldots ,1)^T$ is the non- perturbed point, that is, $\ell\ (\theta|\omega)$=$\ell\ (\theta)$. This scheme of perturbation is intended to evaluate whether the contribution of the observations with different weights affects the Maximum Likelihood Estimators of θ .Let $\hat{\theta}_\omega$ be the Maximum Likelihood Estimators of θobtained from the perturbed likelihood function. The influence of the perturbation ωon the Maximum likelihood estimators can be evaluated by the likelihood displacement defined by LD (ω)=2$\left[\ell(\hat{\theta}) - \ell(\hat{\theta}_\omega)\right]$. Cook (1986) proposes to studythe local behaviour of LD (ω) around of ω_0, using the normal curvature $C_l$of LD(ω) in the no-perturbation vector in the direction of some unitary vector **l**. He showed that $C_l = 2\left|l^T \Delta^T \ddot{L}^{-1} \Delta l\right|$, with ||l||= 1, where $\ddot{L}$is defined by

$$\ddot{L} = \begin{bmatrix} \ddot{L}_{\alpha\alpha} & \ddot{L}_{\alpha\beta} \\ \ddot{L}_{\beta\alpha} & \ddot{L}_{\beta\beta} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2}{\partial\alpha^2} \ell(\alpha , \beta) & \frac{\partial^2}{\partial\alpha\partial\beta} \ell(\alpha , \beta) \\ \frac{\partial^2}{\partial\alpha\partial\beta} \ell(\alpha , \beta) & \frac{\partial^2}{\partial\beta^2} \ell(\alpha , \beta) \end{bmatrix}$$ and $\Delta$is a 2× n matrix

given byΔ= $[\Delta_1(\theta) , \ldots , \Delta_n(\theta)]$, both evaluated at θ=$\hat{\theta}$and ω_0.The elements

of the perturbation matrix, Δ, are given by
$\Delta_i(\theta)$= $[\Delta_i(\alpha), \Delta_i(\beta)]^T$ = $\left[\frac{\partial^2 \ell(\alpha|\omega)}{\partial\alpha\partial\omega_i} , \frac{\partial^2 \ell(\beta|\omega)}{\partial\beta\partial\omega_i}\right]^T$ with i = 1, …, n, where

$$\Delta_i(\alpha) = \frac{v_i}{\alpha^3}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right] - \frac{1}{\alpha} \text{and} \Delta_i(\beta) = \frac{1}{t_i+\beta} - \frac{v_i}{2\alpha^2}\left[\frac{1}{t_i} - \frac{t_i}{\beta^2}\right] - \frac{1}{2\beta} \quad ,$$

for the uncensored case. Let $\boldsymbol{l_{max}}$ be the direction of maximum normal curvature, which is the perturbation that produces the greatest local change in $\hat{\theta}$. The most influential elements of the data can be identified by the large components of the vector $\boldsymbol{l_{max}}$. Moreover, $\boldsymbol{l_{max}}$ is the Eigen vector corresponding to the largest Eigen value of B= $\Delta^T \ddot{L}^{-1} \Delta$Another important direction is l = e_{in}, which corresponds to the ith unitary vector of $\mathbb{R}^n$. In this case, the normal curvature called total local influence of the ith case — is given by $C_i = 2|b_{ii}|$, where b_{ii} is the ith diagonal element of B. We can use l_{max} and C_i as diagnostics for local influence. We consider as cut-off point those cases such that $C_i > 2\bar{C}$, where $\bar{C} = \frac{1}{n}\sum_{i=1}^{n} C_i$;

4.2.3. Simulation

Several simulation studies have been carried out for the Generalized Birnbaum Saunders distribution. Leiva et al. (2008c)[43] considered different ways to generate random numbers from this distribution. Leiva et al. (2008a)[41] discussed a lifetime analysis and maximum likelihood estimation for the Generalized Birnbaum Saunders distribution. They presented a Monte Carlo simulation study for evaluating their estimation method. Díaz-García and Domínguez-Molina(2007)[22] also discussed the Maximum Likelihood Estimators of the parameters for the Generalized Birnbaum Saunders distribution by considering samples that are independent as well as dependent. However, the estimation process is different from the one discussed by Leiva et al. (2008a)[42]. Díaz-García and Domínguez-Molina (2007)[22] simulated a set of lifetime data for the dependent case and then determined the Maximum Likelihood Estimators the parameters under different distributions. Díaz-García and Domínguez-Molina (2007)[22] presented an optimization algorithm based on the simulated annealing method for the Generalized Birnbaum Saunders model with independent as well as dependent data by likelihood methods and mentioned that this algorithm is very efficient for optimization purposes and does not require any manipulation of the log-likelihood.

4.2.3.1 Generation of Random Numbers

Three random number (r.n.) generators based on the relationship between the Generalized Birnbaum Saunders distribution and the spherical, sinh – spherical and IGT models are considered.

Generator based on standard symmetrical distributions in $\mathbb{R}$ (G1). The steps of the algorithm are:

(G1.1) Generate a random number z from $Z \sim S(g)$;

(G1.2)Generate a random number v from $V \sim U(0,1)$;

(G1.3)Fix values for α and β, and then produce the random number $t = t_1$ or $t = t_2$ from $T \sim GBS(\alpha, \beta; g)$according to the following criterion: if $v < 1/2$, then $t_1 = \left[\beta(2 + \alpha^2 z^2) - \sqrt{\beta^2(2 + \alpha^2 z^2)^2 - 4\beta^2}\right]/2$ and if $v<1/2$, then $t_2 = \left[\beta(2 + \alpha^2 z^2) + \sqrt{\beta^2(2 + \alpha^2 z^2)^2 - 4\beta^2}\right] / 2$.

Generator based on the Sinh- spherical distribution (G2) :

The steps of the algorithm are:

(G2.1) Generate a random number z from $Z \sim S(g)$;

(G2.2) Fix values of α, γ, and $\sigma = 2$; and then generate a random number y from $y = \gamma + \sigma \ \operatorname{arcsinh}\left(\frac{\alpha z}{2}\right)$ where $Y \sim SHS(\alpha, \gamma, \sigma; g)$and produce the random number $t = \exp(y)$, which corresponds to a random variable $T \sim GBS(\alpha, \beta; g)$.

Generator based on the mixture Inverse Gaussian Type distribution (G3).

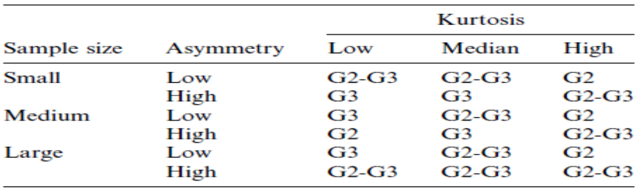
The steps of the algorithm are:

(G3.1) Fix values of α and β and then generate a random number x_1 from $X_1 \sim IGT(\beta, \alpha^{-2}\beta; g)$ and x_2 from $X_2 \sim IGT(\beta^{-1}, \alpha^{-2}\beta^{-1}; g)$ and compute the reciprocal ofx_2 and obtain a random number $s = {x_2}^{-1}$;

(G3.2) Generate a random number w from W according to $\mathbb{P}(W = 0) = 1/2$ and $\mathbb{P}(W = 1) = 1/2$; and produce the random number t from $T \sim GBS(\alpha, \beta; g)$ by $T = WX_1 + (1 - W)S$, where $S = {X_2}^{-1}$.

Table 4.1 shows a summary of the results that are related to the most appropriate generation method for different configurations, i.e., for several sample size (small, medium and large) and kurtosis (low, median, and high) and asymmetry (low and high) levels.

Table 4.1: Selected generator method for the indicated configurations



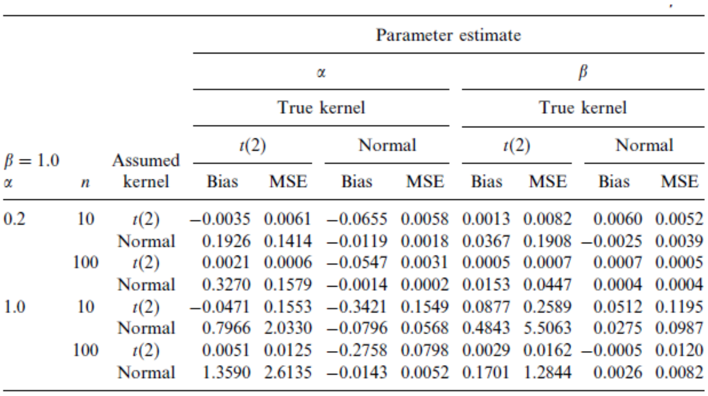
		Kurtosis		
Sample size	Asymmetry	Low	Median	High
Small	Low	G2-G3	G2-G3	G2
	High	G3	G3	G2-G3
Medium	Low	G3	G2-G3	G2
	High	G2	G3	G2-G3
Large	Low	G3	G2-G3	G2
	High	G2-G3	G2-G3	G2-G3

4.2.3.2 Empirical Studies

A Monte Carlo simulation study was conducted for verifying the quality of the estimation method .The bias and mean square error (MSE) of the Maximum

Likelihood Estimators were computed. The samples were generated from the Generalized Birnbaum Saunders model with a specific kernel, called “true kernel” and the estimates of the parameters computed from samples obtained by using the same or other kernels, called “assumed kernel.” The normal and t with$v$ degrees of freedom (denoted by t_v) kernels were considered here. The empirical bias and mean square error values are the average of the corresponding values determined from simulated samples for each combination of n [sample size: small (n = 10), medium (n = 25) and large (n=100)], α[skewness: low (α= 0.5) and high (α= 1.0)] and kernel [kurtosis: high (v= 2), medium (v= 8) and low ($v = \infty$)]. A summary of this study reveals that the bias gets smaller when the sample size increases, the bias gets larger when the asymmetry increases and the bias gets smaller when the kurtosis increases. We also found larger bias in the estimate of α than the estimate of β. The mean square error gets smaller when the sample size gets larger, when the asymmetry decreases or when the kurtosis increases. It was also seen that the mean square error of the estimate of β is generally larger than that of the estimate of α. Table 4.2 presents some representative results of this simulation study.

Table 4.2: Biases and Mean Square Estimators based on Monte Carlo simulations for estimates of $\alpha$and .



			Parameter estimate							
			α				β			
			True kernel				True kernel			
$\beta = 1.0$		Assumed	$t(2)$		Normal		$t(2)$		Normal	
α	n	kernel	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.2	10	$t(2)$	−0.0035	0.0061	−0.0655	0.0058	0.0013	0.0082	0.0060	0.0052
		Normal	0.1926	0.1414	−0.0119	0.0018	0.0367	0.1908	−0.0025	0.0039
	100	$t(2)$	0.0021	0.0006	−0.0547	0.0031	0.0005	0.0007	0.0007	0.0005
		Normal	0.3270	0.1579	−0.0014	0.0002	0.0153	0.0447	0.0004	0.0004
1.0	10	$t(2)$	−0.0471	0.1553	−0.3421	0.1549	0.0877	0.2589	0.0512	0.1195
		Normal	0.7966	2.0330	−0.0796	0.0568	0.4843	5.5063	0.0275	0.0987
	100	$t(2)$	0.0051	0.0125	−0.2758	0.0798	0.0029	0.0162	−0.0005	0.0120
		Normal	1.3590	2.6135	−0.0143	0.0052	0.1701	1.2844	0.0026	0.0082

4.2.4. Goodness-of-Fit

Some goodness-of-fit methods for the Generalized Birnbaum Saunders distribution are described. These methods can be considered jointly with the classical techniques, such as the quantile versus quantile (QQ) or probability versus probability (PP) plots, Kolmogorov- Smirnov (KS) test and model selection criteria such as the Schwartz information criterion (SIC).

Based on graphical plots:

Chang and Tang (1994)[15] developed a simple graphical technique analogous to the PP plots for the Birnbaum Saunders distribution. This method can be useful as a goodness-of-fit test and can also be used for estimating the parameters of the Birnbaum Saunders distribution or at least for finding starting values for iterative estimation procedures.

Leiva et al. (2008b,c)[43,44] applied this technique to the Generalized Birnbaum Saunders distribution. This consists of transforming the data, which results in obtaining pairs of values that have a linear relationship. By using a simple linear regression method, the slope and intercept of the line are then estimated. The line is used for goodness-of-fit, like PP-plots. Thus, if we consider the cdf given in (A7), we have

$$t = +\alpha\sqrt{\beta}\sqrt{t}F^{-1}\big(F_T(t)\big) ,$$

where $F^{-1}(.)$ is the inverse cdf of the standard symmetrical distribution in $\mathbb{R}$ and $F_T(.)$ is the Generalized Birnbaum Saunders cdf. However, it is difficult to derive a linear function on t, which is fundamental for probability plotting. By considering p = $\sqrt{t}F^{-1}\big(F_T(t)\big)$, we obtain the linear function y ≈ a + bx, where the X-axis is x = p, Y -axis is y = t, intercept is a =$\beta$and slope is b = $\alpha\sqrt{\beta}$. Now, suppose that we have n observations $t_1 , \dots , t_n$.Then, plotting $t_1$against $\hat{p}_i$, where $\hat{p}_i = \sqrt{t_i}F^{-1}\left(\hat{F}_T(t_i)\right)$ and $\hat{F}_T(t_i)$ is estimated by the mean rank given as $\hat{F}_T(t_i)$= (i– 0 .3)/(n + 0.4), the result is approximately a straight line of the data from the Generalized Birnbaum Saunders distribution. Goodness-of-fit can be evaluated either visually or analytically by using the coefficient of determination or R-square, which is obtained when we fit ($\hat{p}_i , t_i$), with i = 1, …., n. The parameters α and β can be estimated by the least-squares method, where $\hat{\beta}$= $\hat{a}$and $\hat{\alpha}$= $\hat{b}$/ $\sqrt{\hat{a}}$.

Based on moment:

The moments can be used to compare distributional families and identify the most suitable model.

Also, graphical methods of goodness-of-fit based on moments consist of developing rectangular charts whose coordinate axes are (β_1, β_2), where $\beta_1 = \left(\alpha_3(T)\right)^2$ and $\beta_2 = \alpha_4(T)$ according to the theoretical properties. This diagram, known asβ_1–β_2 chart is given by

$$\beta_2 = \frac{\mu_2\mu_4}{(\mu_3)^2}\beta_1 , \tag{4.14}$$

where μ_2, μ_3, and μ_4 are the second, third, and fourth central moments, respectively. The β_1–β_2 chart, however, is not always appropriate because the first four moments lack sufficient accuracy when used on small data sets. For this reason, we can replace the β_1–β_2 chart by a curve on the plane $\gamma - \gamma_3$, where

$$\gamma(T) = \gamma = \frac{\sqrt{\mu_2}}{\mu}, \alpha_3(T) = \gamma_3 = \frac{\mu_3}{(\sqrt{\mu_2})^3}, \quad \text{and } \gamma_3 = \frac{\mu\mu_3}{(\sqrt{\mu_2})^4}\gamma , \tag{4.15}$$

where μ is the mean and $\sqrt{\mu_2}$ is the standard deviation. A large value of $\gamma_3$indicates a long tail in the distribution. Leiva et al. (2008b)[43] discussed the β_1–β_2 and $\gamma - \gamma_3$diagrams for the Generalized Birnbaum Saunders distribution, which are based on the results in theoretical properties.

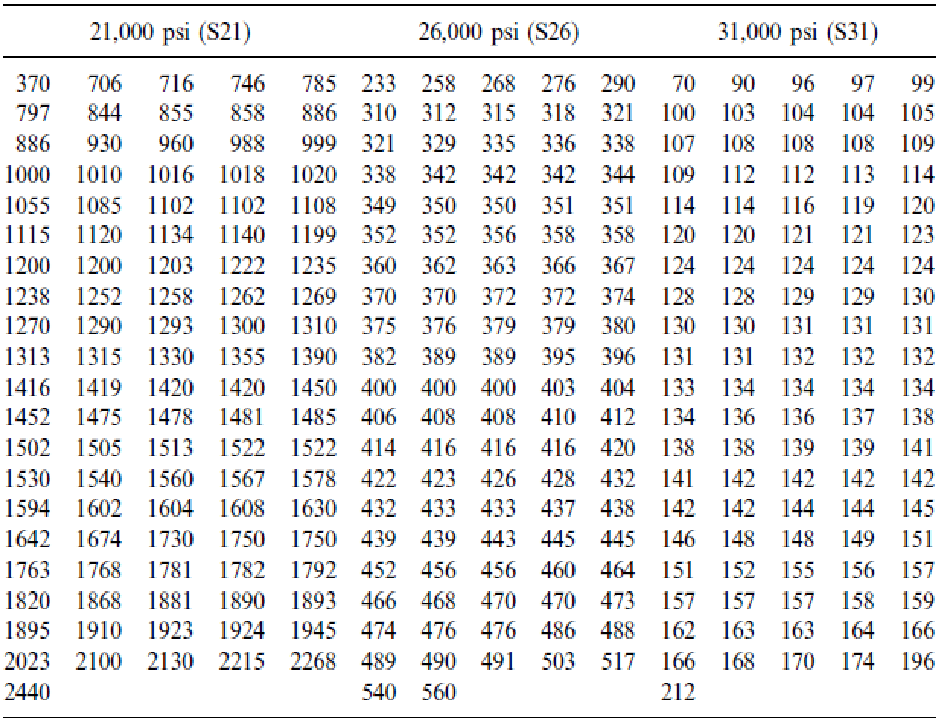
Choosing the kernel :

Some heuristic manner in how to choose the kernel for the Generalized Birnbaum Saunders distribution is as follows. Firstly, one use the $\gamma - \gamma_3$ chart for considering or discarding kernels for the This is an initial method of goodness-of-fit. Secondly, if more than one kernel is selected by using this chart, we then use the classical techniques, i.e., PP, KS, and SIC, for obtaining the final kernel associated with the Generalized Birnbaum Saunders distribution. Finally, diagnostics tools are considered in order to validate the chosen kernel.

4.3. Numerical Examples

In this section, for the purpose of illustration, we analyze the data of Birnbaum and Saunders (1969)[12]. The lifetime data in this case correspond to the cycles $(\times 10^{-3})$ of aluminum specimens of type 6061-T6. These specimens were cut in a parallel angle to the direction of rotation and oscillating at 18 cycles per seconds. They were exposed to a pressure with maximum stress of 21,000, 26,000, and 31,000 psi, for n = 101,102 and 101 specimens, respectively. All specimens were tested until failure. Table 4.3 presents the data from where we first produce a complete analysis for uncensored data and then an analysis for censored data, mainly to illustrate the estimation procedure for type-II censored data .

Table 4.3: Lifetimes (in cycles) of aluminium specimens exposed to the indicated stress levels



21,000 psi (S21)					26,000 psi (S26)					31,000 psi (S31)				
370	706	716	746	785	233	258	268	276	290	70	90	96	97	99
797	844	855	858	886	310	312	315	318	321	100	103	104	104	105
886	930	960	988	999	321	329	335	336	338	107	108	108	108	109
1000	1010	1016	1018	1020	338	342	342	342	344	109	112	112	113	114
1055	1085	1102	1102	1108	349	350	350	351	351	114	114	116	119	120
1115	1120	1134	1140	1199	352	352	356	358	358	120	120	121	121	123
1200	1200	1203	1222	1235	360	362	363	366	367	124	124	124	124	124
1238	1252	1258	1262	1269	370	370	372	372	374	128	128	129	129	130
1270	1290	1293	1300	1310	375	376	379	379	380	130	130	131	131	131
1313	1315	1330	1355	1390	382	389	389	395	396	131	131	132	132	132
1416	1419	1420	1420	1450	400	400	400	403	404	133	134	134	134	134
1452	1475	1478	1481	1485	406	408	408	410	412	134	136	136	137	138
1502	1505	1513	1522	1522	414	416	416	416	420	138	138	139	139	141
1530	1540	1560	1567	1578	422	423	426	428	432	141	142	142	142	142
1594	1602	1604	1608	1630	432	433	433	437	438	142	142	144	144	145
1642	1674	1730	1750	1750	439	439	443	445	445	146	148	148	149	151
1763	1768	1781	1782	1792	452	456	456	460	464	151	152	155	156	157
1820	1868	1881	1890	1893	466	468	470	470	473	157	157	157	158	159
1895	1910	1923	1924	1945	474	476	476	486	488	162	163	163	164	166
2023	2100	2130	2215	2268	489	490	491	503	517	166	168	170	174	196
2440					540	560				212				

4.3.1. Uncensored Data

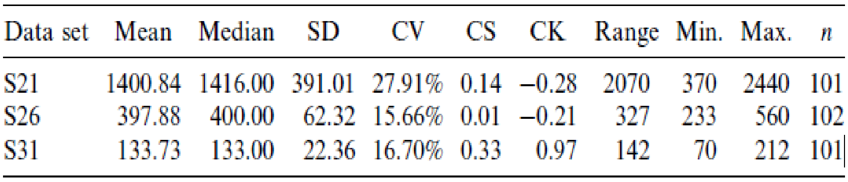
By using goodness-of-fit methods and model selection criteria, It is showed that the Generalized Birnbaum Saunders distribution fits these data better than the classical Birnbaum Saunders distribution. Once the “best” model is selected, we then develop inference on the lifetime characteristics of the corresponding Generalized Birnbaum Saunders distribution.

4.3.1.1. Exploratory Data Analysis:

One can present an exploratory data analysis. Table 4.4 presents a descriptive summary while Fig. 4.2 shows the histograms for the three data sets (S21, S26, and S31).

A careful look at Table 4.4 reveals slightly positively skewed distributions with moderate kurtosis and Fig.4.4 (first panel) shows some influential potentially observations on the parameter estimates. The Generalized Birnbaum Saunders distribution accounts well the degrees of skewness and kurtosis present in the data sets and also enables the estimation of the parameters of the model in a robust way when outliers are present. The Generalized Birnbaum Saunders distribution is , a suitable distribution for modelling these data. γ_3 chart and postulate that the logistic, normal and t distributions are good choices for the kernel associated with the Generalized Birnbaum Saunders distribution.

Table 4.4: Descriptive statistics for lifetimes of aluminium specimens for the indicated data set



Data set	Mean	Median	SD	CV	CS	CK	Range	Min.	Max.	n
S21	1400.84	1416.00	391.01	27.91%	0.14	−0.28	2070	370	2440	101
S26	397.88	400.00	62.32	15.66%	0.01	−0.21	327	233	560	102
S31	133.73	133.00	22.36	16.70%	0.33	0.97	142	70	212	101

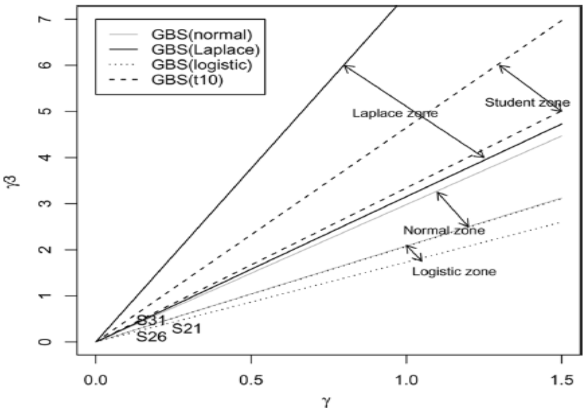


Figure 4.3 :$\gamma - \gamma_3$ chart with estimated γ and γ_3 for the indicated data sets.

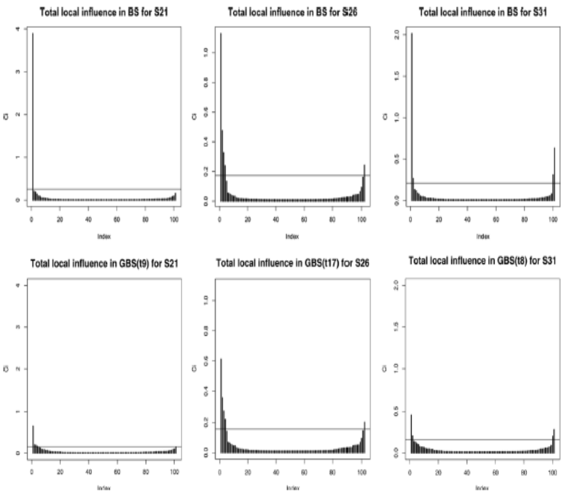


Figure 4.4. Influence index plots for the indicated stress levels and models.

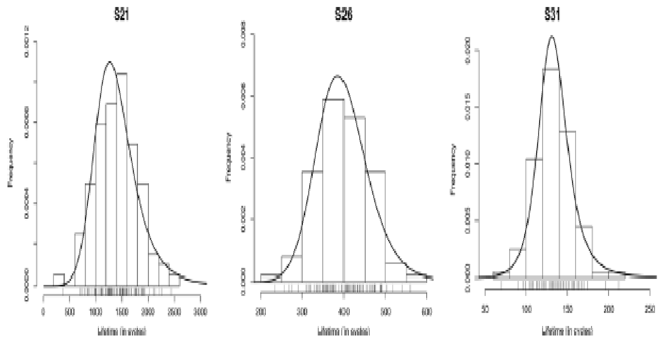


Figure 4.5: Histograms for the indicated data sets with estimated pdf of the GBS(t_v) model.

4.3.1.2. Inference and Diagnostics

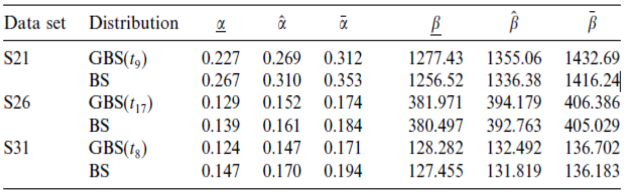
One can find the Maximum Likelihood Estimators of the parameters $\alpha\ and\ \beta$ of the Generalized Birnbaum Saunders distribution based on the t kernel, which will be denoted by GBS(t_v).The influence diagnostics is also carried out.

In order to estimate the parameters$\alpha and \beta$ of the GBS(t_v) model, different fixed values for v and use the Maximum Likelihood Estimation method We use the Maximum Likelihood Estimators of $\alpha\ and\ \beta$ for the Birnbaum Saunders model as starting values in the numerical procedure .Then choose the value of v that maximizes the likelihood function over several values of v∈[2, 100]. The value selected for v in this manner is 9 for S21, 17 for S26, and 8 for S31, which produce high levels of kurtosis and robust parameter estimates in presence of outliers.

Next, we give estimates and 95% approximate confidence intervals (ACI) of the form $ACI_{95\%}(\theta_j)$= $[\underline{\theta}_j; \bar{\theta}_j]$ for the parameter of interest θ_j , with j = 1, 2. Consider the normal and t cases as generator kernels of the Generalized Birnbaum Saunders distribution. These results are summarized in Table 4.5.

In Fig.4.4, we can observe the inherent robustness of the estimation procedure for the GBS(t_v) distribution, with v= 9 ,17 and 8 for S21, S26, and S31 respectively. Values of C_i, for i = 1 ,…, n—total local influence for the ith case—show potentially influential observations only for the classical Birnbaum Saunders model.

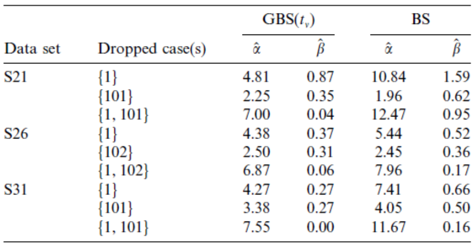
Table 4.5 :Estimates and 95% ACIs of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ for the indicated data set and distributions .



Data set	Distribution	$\underline{\alpha}$	$\hat{\alpha}$	$\bar{\alpha}$	$\underline{\beta}$	$\hat{\beta}$	$\bar{\beta}$
S21	GBS(t_9)	0.227	0.269	0.312	1277.43	1355.06	1432.69
	BS	0.267	0.310	0.353	1256.52	1336.38	1416.24
S26	GBS(t_{17})	0.129	0.152	0.174	381.971	394.179	406.386
	BS	0.139	0.161	0.184	380.497	392.763	405.029
S31	GBS(t_8)	0.124	0.147	0.171	128.282	132.492	136.702
	BS	0.147	0.170	0.194	127.455	131.819	136.183

Table 4.6 presents the relative changes (RC), in percentage, of each parameter estimate, defined by$RC_{\theta_j} = \left|\left(\hat{\theta}_j - \hat{\theta}_{j(I)}/\hat{\theta}_j\right)\right| \times 100\ \%$ where $\hat{\theta}_{j(I)}$ denotes the Maximum Likelihood Estimator of $\hat{\theta}_j$ after the set I of cases has been removed. From Table 4.9, we note that the relative changes are greater for the classical Birnbaum Saunders model than for the GBS(t_v) model, with the latter being negligible (<5%). Thus, those specimens will be considered in the analysis because they do not affect the Maximum Likelihood Estimators under the GBS(t_8) model. This reveals that the GBS(t_8) distribution presents robust parameter estimates in the presence of outliers.

Table 4.6: RC (in %) for the indicated parameters, models and data sets



		GBS(t_v)		BS	
Data set	Dropped case(s)	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\beta}$
S21	{1}	4.81	0.87	10.84	1.59
	{101}	2.25	0.35	1.96	0.62
	{1, 101}	7.00	0.04	12.47	0.95
S26	{1}	4.38	0.37	5.44	0.52
	{102}	2.50	0.31	2.45	0.36
	{1, 102}	6.87	0.06	7.96	0.17
S31	{1}	4.27	0.27	7.41	0.66
	{101}	3.38	0.27	4.05	0.50
	{1, 101}	7.55	0.00	11.67	0.16

4.3.1.3. Goodness-of-Fit

In order to select the best model, we consider three different criteria: Quantile versus Quantile plot, Kolmogorov Smirnov test and Schwartz Information Criterion. Table 10 presents the coefficients of determination (CD in %) of the Quantile versus Quantile -plots and the Kolmogorov Smirnov and Schwartz Information Criterion values for the considered data sets under the Birnbaum Saunders and BS(t_v)distributions. The model that provides the best fit to the data in all the stress levels is the GBS(t_8) distribution .

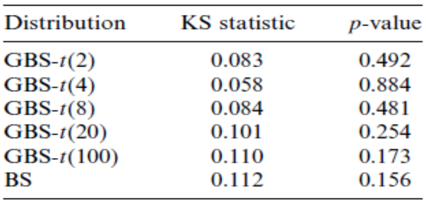
4.3.1.4. Lifetime Analysis

One can estimate the hazard function and its critical point based on the GBS(t_v) distribution, which was indeed selected as the best model as in previous case. Hazard function. Let T ~GBS$(\alpha, \beta\ , t_8)$. Then, based on (C1), the hazard function is given by

$$h_T(t) = \frac{1024\Gamma\left(\frac{9}{2}\right)}{3\sqrt{\pi}} \frac{\alpha^8 \beta^4 t^3 (t+\beta)}{(8\alpha^2\beta t + t^2 + \beta^2 - 2\beta t)^{\frac{9}{2}} \left(1 + I_{\frac{t^2+\beta^2-2\beta t}{t^2+\beta^2-2\beta t+8\alpha^2\beta t}}\left(\frac{1}{2}, 4\right)\right)} \ , t > 0 \ , \qquad (4.16)$$

whereI_x(.,.) is as in Table 4.1. An analogous result is obtained for v= 9 ,17. Now, based on Table 4.8, we get $\hat{\alpha}$ and $\hat{\beta}$and then by the invariance property of the Maximum Likelihood Estimators, we obtain the Maximum Likelihood Estimator of $h_T(t)$, $\hat{h}_T(t)$. By using the Maximum Likelihood Estimator of $h_T(t)$, it is possible to find the Maximum Likelihood Estimators of $S_T(t)$, $S_T(t \backslash x)$, HRA(t) and μ_x according to (C2)–(C4). Figure 4.6 presents the estimated curves of the survival function and hazard function corresponding to the GBS(t_v) distribution for the indicated stress levels.

Table 4.7:Values of KS, SIC and CD-QQ for the indicated GBS distributions and data sets.



Distribution	KS statistic	p-value
GBS-$t(2)$	0.083	0.492
GBS-$t(4)$	0.058	0.884
GBS-$t(8)$	0.084	0.481
GBS-$t(20)$	0.101	0.254
GBS-$t(100)$	0.110	0.173
BS	0.112	0.156

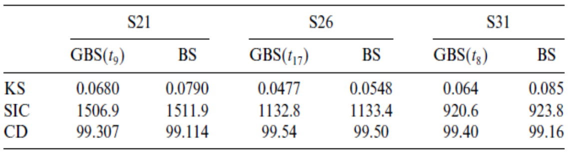
4.3.2. Censored Data

One can find the Maximum Likelihood Estimators of the parameters α and β of the Generalized Birnbaum Saunders distribution based on the data set S31 with right censoring of 20% in these data. Once again, the normal and t kernels are considered in the analysis. The Kolmogorov-Smirnov test for censored data given by Barr and Davidson (1973)[8] is used for the model validity.

4.3.2.1. Estimation.

Table 4.8 presents the Maximum Likelihood Estimators of the parameters α and $\beta$of the Birnbaum Saunders and GBS(t_v) distributions based on the right type-II censored data at 20%.

Table 4.8 :Maximum Likelihood Estimators α and $\beta$for the indicated distributions



	S21		S26		S31	
	GBS(t_9)	BS	GBS(t_{17})	BS	GBS(t_8)	BS
KS	0.0680	0.0790	0.0477	0.0548	0.064	0.085
SIC	1506.9	1511.9	1132.8	1133.4	920.6	923.8
CD	99.307	99.114	99.54	99.50	99.40	99.16

4.3.2.2. Goodness-of-Fit

In order to select the best model based on these censored data, we used the Kolmogorov-Smirnov test. Table 4.9 presents the values of the Kolmogorov-Smirnov statistics and the corresponding p-values for the Birnbaum Saunders and

GBS(t_v) distributions. The model that provides the best fit to these censored data is now the GBS(t_4) distribution .

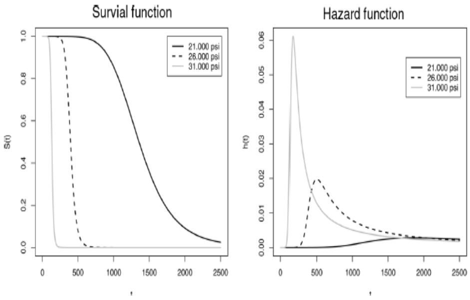
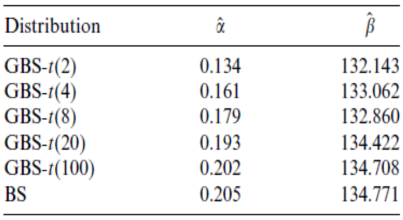


Figure 4.6: Estimated lifetime functions of the GBS(t_8) model for the indicated stress level.

Table 4.9 :Kolmogorov-Smirnov statistics and p-values for the indicated distributions



Distribution	$\hat{\alpha}$	$\hat{\beta}$
GBS-t(2)	0.134	132.143
GBS-t(4)	0.161	133.062
GBS-t(8)	0.179	132.860
GBS-t(20)	0.193	134.422
GBS-t(100)	0.202	134.708
BS	0.205	134.771

One can conclude the Generalized Birnbaum- Saunders distribution fits the data better than the classical Birnbaum- Saunders model.

CHAPTER - 5

Chapter -5

A Characterization of the Generalized Birnbaum–Saunders Distribution

In this chapter A Characterization of the Generalized Birnbaum–Saunders distribution by Emilla Athayde [2017] [26] has been reviewed.

The Birnbaum–Saunders distribution is positively skewed allowing for different degrees of kurtosis and its hazard or failure rate (FR) has an inverse bathtub shape .The Birnbaum–Saunders distribution is closed under scale transformations and under reciprocation.

Common random variables satisfying the “reciprocal property”, meaning that the random variable and its own reciprocal are identically distributed, are the Fisher–Snedecor $F_{n,n}$and thelognormal with appropriate mean, as well as the quotient of two independent and identically distributed (IID) non-negative random variable.

The generalized Birnbaum–Saunders distribution (GBS), introduced by Diaz-Garcia and Leiva (2005)[21], is obtained replacing the normal generator in the Birnbaum–Saunders distribution by any symmetric absolutely continuous random variable. It is a highly flexible class of positively skewed distributions allowing for a wide range of kurtosis .The probability density functions include unimodal and bimodal cases and failure rate can be monotone, inverse bathtub or have more than one change-point. Heavy tails are also allowed depending on the tails of the generating random variable. The generalized Birnbaum–Saunders distribution is closed under scale transformations and under reciprocation. Any generalized Birnbaum–Saunders distributed random variable, suitably scaled, satisfies the reciprocal property.

The three-parameter model of power transformations of Birnbaum–Saunders distributed random variables introduced by Owen(2006)[49], and the four-parameter extension based on the Johnson system has been point out. The Birnbaum–Saunders distribution also belongs to the family of cumulative damage distributions. Truncated Birnbaum–Saunders and shifted Birnbaum–Saunders distributions have been considered. The case generated by non-symmetric random variables has also been addressed. In this case the resulting random variable does not satisfy the reciprocal property for any scale transformation.

Considering the problem of fitting a distribution to univariate data comes from a non-negative random variable, the generalized Birnbaum–Saunders distribution is a good candidate to model the data, and an appropriate way to find the test for goodness-of-fit (GOF) has been assessed. The Birnbaum–Saunders and the generalized Birnbaum–Saunders distributions are not in the location-scale (LS) family. In the case of parametric distributions with unknown parameters, Goodness of fit techniques have also been addressed for non location-scale models, including graphical techniques and graphical tools to assess goodness-of-fit in non location-scale distributions. These techniques can be applied to the case of a Birnbaum Saunders distribution or a generalized Birnbaum–Saunders generated by a parameterized Distribution.

The converse is also true, for any Generalized Birnbaum-Saunders distributed random variable, suitably scaled and reciprocal property. To find an alternative estimator for the scale parameter, to consider an empirical graphical technique that requires no estimation of the scale parameter and to test whether the data come from a Generalized Birnbaum–Saunders distribution using symmetry tests about an unknown constant have been enabled.

The Birnbaum–Saunders distribution is a transformation T of a standard normal random variable given by

$$T = \beta\left(\frac{\alpha Z}{2} + \sqrt{\left(\frac{\alpha Z}{2}\right)^2 + 1}\right)^2 , \qquad (5.1)$$

where Z ~N(0, 1), α (α> 0) is a shape parameter and β (β> 0) is a scale parameter (and also the median), denoted here by T ~BS(α, β), with inverse transformation given by

$$Z = \frac{1}{\alpha}\left(\sqrt{\frac{T}{\beta}} - \sqrt{\frac{\beta}{T}}\right) \sim N(0,1) \ . \qquad (5.2)$$

The distribution of T given by Eq.(5.1) is positively skewed allowing for different degrees of kurtosis (greater than 3) and its failure rate has an inverse bathtub shape. Among the properties of this distribution, if T ~BS$(\alpha\,,\beta)$ then
(i) cT ~BS$(\alpha\,,c\beta)$ with c > 0 and (ii) T^{-1}~BS$(\alpha\,,\beta^{-1})$, i.e., the Birnbaum–Saunders distribution is closed under scale transformations and under

reciprocation. Thus, denoting Y = T/β , Y and 1/Y are identically distributed, i.e., Y has the reciprocal property. Analogously T $\sim$BS(α , β), suitably scaled, satisfies the reciprocalproperty.

The Generalized Birnbaum–Saunders distribution, introduced by Díaz-García and Leiva(2005) [21], is obtained replacing Z in Eq.(5.1) by any symmetric absolutely continuous random variable X, thus leading to

$$T = \beta\left(\frac{\alpha X}{2} + \sqrt{\left(\frac{\alpha X}{2}\right)^2 + 1}\right)^2, \tag{5.3}$$

whereα (α> 0) is a shape parameter andβ (β> 0) is a scale parameter (and also the median). T given by Eq.(5.3) is generated by X and denote it by
T $\sim$GBS(α , β, g_X), where $g_X(\cdot)$ is the probability density function (PDF) of X.

The cumulative distribution function (CDF) of T $\sim$GBS(α , β, g_X) is given by F_T(t) = $G_X(\xi$(t; α , β)), t > 0, where G_X $(\cdot)$ is the cumulative distribution function of X and ξ(t; α , β) $=\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)$, t > 0, and the Probability Density Function is given by f_T(t) = g_X (ξ(t ; α , β))ξ'(t ; α , β) , t > 0,
where ξ'(t ; α , β) = $\frac{t+\beta}{2\alpha\sqrt{\beta}} t^{-3/2}$, t > 0 .

Let T $\sim$GBS (α , β, g_X), then T, suitably scaled, satisfies the reciprocal property considering like before Y = T/β. In formula Eq.(5.3) one may assume that α = 1 without loss of generality since letting X_α= α X, the distributions GBS(α , β, g_X) and GBS(1, β, g_X)are clearly the same.

As mentioned before, the Generalized Birnbaum–Saunders distribution given by Eq.(5.3), where X is a symmetric absolutely continuous random variable , is such that T/β and β /T are equally distributed. The converse is also true, and consequently that this remarkable property characterizes the class of Generalized Birnbaum–Saunders distributions.

Theorem 5 .1. Let T be a non-negative absolutely continuous random variable. Then T $\sim$GBS(α , β, g_X) , if and only if T, suitably scaled, satisfies the reciprocal property.

Corollary 5.1.1. Let T be a non-negative absolutely continuous random variable. Then T ~GBS(α,β, g_X) if and only if log(T) − log(β) is a symmetric random variable.

Corollary 5.1.2. Let T and U be two independent Generalized Birnbaum–Saunders distributed random variables and a ≠0. Then T_a, TU and T/U are also Generalized Birnbaum–Saunders distributed random variables.

Corollary 5.1.3. Any non-negative random variable that is written as a quotient of two identically distributed random variables is Generalized Birnbaum–Saunders distributed.

Corollary 5.1.4. The modified moment estimator of β is Generalized Birnbaum–Saunders distributed.

Theorem 5.2. For a random sample from the GBS(α, β , g_X) distribution and assuming E(X^4) < +∞, the modified moment estimator of β is asymptotically BS($n^{-1/2}\alpha\theta , \beta$) distributed, where

$$\theta^2 = \frac{u_1+\frac{1}{4}\alpha^2 u_2}{1+\frac{1}{2}\alpha^2 u_1}, \text{ and } u_i = \mathrm{E}(X^{2i}),\ i = 1, 2.$$

Corollary 5.2.5. For a random sample of the GBS(α, β , g_X) distribution, the modified moment estimator of β is asymptotically N($\beta, n^{-1/2}\alpha \,.\, \theta\beta$) .

5.1.Goodness - of – fit

When dealing with a univariate lifetime random sample, t = $(t_1, t_2, \ldots, t_n)$ from a random variable T, a natural question to consider is whether a member of the Generalized Birnbaum–Saunders class is suitable to model these data. Let Y = log(T), and let t^{-1} and y denote the transformed samples $\left(\frac{1}{t_1}, \frac{1}{t_2}, \ldots, \frac{1}{t_n}\right)$ and (log(t1t_1), log(t_2), ..., log(t_n)), respectively. Theorem 5.1 and Corollary 5.1 leads us to tackle this problem (i) testing for equal distributions of T/β and β/T with unknown β, or (ii) testing Y for symmetry about an unknown constant. Both these procedures rely on estimating β or log(β), and the same applies from an empirical point of view using a graphical approach such as a quantile-quantile plot (QQ-plot) for the two samples, $\beta^{-1}t$ and βt^{-1}. A graphical procedure known as the total time on test (TTT) plot can also be used and this plot requires no estimation of β.

To test for equal distributions for T/β and β/T, β may be estimated by minimizing some "distance" between the two samples, $\beta^{-1}t$ and βt^{-1}. Two possible distances are:

- The square of the difference between the sample means of $\beta^{-1}t$ and βt^{-1}. This leads to the usual modified moment estimator of β, given by Eq.(5.4) with S $=\bar{T}$and R^{-1}= T^{-1} as before. This is not surprising since the modified moment estimator of β in the GBS(α, β, f_X) model does not depend on either $f_X$or α.
- A Kolmogorov–Smirnov (KS) type distance between the empirical cumulative distribution function (ECDF) of the samples, T_1 / β..., T_n / β and β/T_1, ..., β/T_n, given by

$$D_{KS} = \sup_x |F_1(x) - F_2(x)| ,$$

where $F_1(x)$ and $F_2(x)$are the empirical cumulative distribution functions of the two samples. Notice that these two samples are not independent.

To test Y = log(T) for symmetry about unknown location, one can have two tests with asymptotically distribution-free test statistics, namely (i) a classical test based on the sample skewness coefficient b_1 and (ii) the triples test .In the first case, the test statistic $\frac{\sqrt{n}b_1}{\tau}$, where $b_1 = \frac{m_3}{m_2^{3/2}}$, $\tau = \frac{m_6 - 6m_2m_4 + 9m_2^3}{m_2^3}$ and m_i is the central moment of order i, i ∈ ℕ, is asymptotically N(0, 1) under the null hypothesis of symmetry, provided μ_6 = E(Y^6) exists. The second test is based on the difference D between the number of "right triples" and the number of "left triples" in the sample, where each triple (Y_i, Y_j , Y_k), 1 ≤ i< j < k ≤ n, is defined as a "right triple" if the middle ordered observation in (Y_i, Y_j , Y_k) is closer to the smallest than to the largest of the three observations, and as a "left triple" if the middle ordered observation is closer to the largest than to the smallest of the three observations .The test statistic, V = D/$\hat{\sigma}$, is asymptotically N(0, 1) under the null hypothesis of symmetry. Notice further that these two tests are unsensitive to power transformations in T, as well as to scale changes.

5.2 A graphical procedure based on the TTT plot

The failure rate is an important indicator in lifetime analysis. Some particular outstanding failure rate shapes include increasing (IFR), decreasing (DFR), bathtub (BT) and inverse bathtub (IBT) ones. For a random variable T with finite expectation, it is possible to identify the shape of its failure rate by the scaled TTT curve , given by

$$W_T(y) = \frac{\int_0^{F_T^{-1}(y)}[1-F_T(t)]dt}{\int_0^{F_T^{-1}(1)}[1-F_T(t)]dt},\ 0\leq y \leq 1. \tag{5.4}$$

This function can be empirically approximated by

$$W_n(k/n)=\frac{\sum_{i=1}^{k} T_{1:n}+[n-k]T_{k:n}}{\sum_{i=1}^{n} T_{1:n}},\ \text{k = 0,…,n} \tag{5.5}$$

where $T_{1:n}$, $T_{2:n}$, ..., $T_{n:n}$ denote the order statistics associated to a random sample T_1, T_2, ...,$T_n$which is given in Fig 5.1 .Thus, the plot of $[k/n\,, W_n(k/n)]$, , where the consecutive points are connected by straight lines, gives us information about the underlying failure rate.

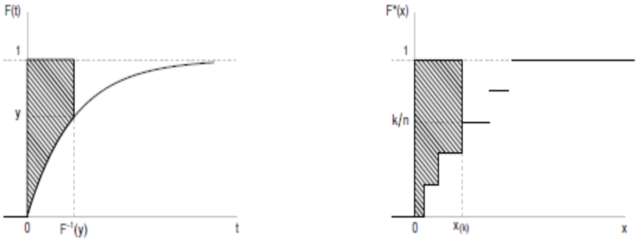


Figure 5.1 : Shaded areas corresponding to $\int_0^{F_T^{-1}(1)}[1-F_T(t)]dt$ (left) and $\frac{1}{n}\left[\sum_{i=1}^{k} T_{1:n}+[n-k]T_{k:n}\right]$ (right) in equations (5) and (6).

The scaled TTT plot is a straight line in the case of the exponential distribution, a concave (convex) function in the case of an increasing (decreasing) failure rate, first concave (convex) and then convex (concave) in the case of an inverse bathtub (bathtub) failure rate, thus providing a useful tool in identifying the shape of the failure rate in Fig 5.2 . Further, the scaled TTT is invariant under change of scale, and so in the case of a Generalized Birnbaum–Saunders distribution no estimation of β is required for the plot. The only drawback is that it requires T to have a finite expectation.

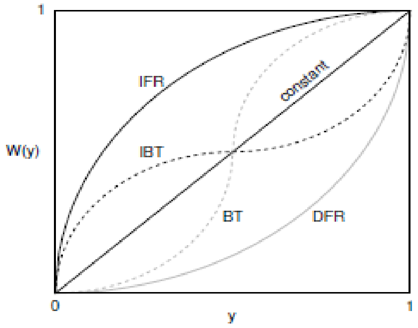


Figure 5.2: **Scaled TTT plot for indicated shape of failure rate-bathtub(BT),decreasing (DFR), inverse bathtub (IBT), increasing (IFR).**

Once again, it follows from Theorem (5.1) that the TTT curves are the same for T/β and β/T if and only if T ~GBS(1, β , g_X) for some symmetric X. Based on this result, we propose to assess the fit to the Generalized Birnbaum–Saunders distribution by comparing the empirical scaled TTTs of the samples t and t^{-1}. If the data do follow a Generalized Birnbaum–Saunders distribution, these two plots should look alike, regardless of β. Denoted by D_{TTT} the maximum vertical distance between these two scaled TTT plots.

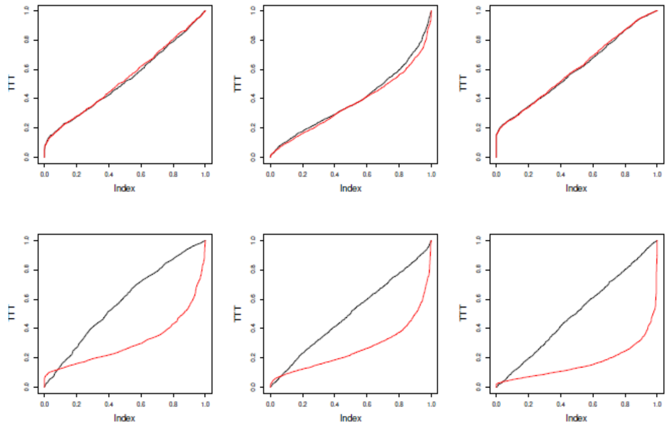


Figure 5.3: Scaled TTT plot for some Generalized Birnbaum–Saunders (top) and non- Generalized Birnbaum–Saunders (bottom) simulated samples and reciprocals, with n= 10^3.

For plots of the empirical scaled TTT for simulated random samples (n = 10^3) for some Generalized Birnbaum–Saundersdistributions (namely, a BS(1, 1), a

BS-t_3 generated by the Student t with 3 degrees of freedom, and the Generalized Birnbaum–Saunders distribution with cumulative distribution function $H_a(.;\theta)$ for a = θ = 0.2, mentioned in Remark 1) and non- Generalized Birnbaum–Saunders distributions (half-normal, half-Student t3 and exponential). The behaviour of the statistic D_{TTT} is under investigation.

5.3 Testing for the Birnbaum Saunders model

For the case of an absolutely continuous lifetime random variable T, to test the null hypothesis H_0 that the cumulative distribution function of T is F(·; θ), based on a random sample ($t_1, t_2, \ldots, t_n$), we consider the Cramer–von Mises (CM) statistic given by

$$W_n^2 = n\int_0^{+\infty}(F_n^*(t) - F(t;\theta))^2\, dF(t;\theta)\ , \qquad (5.6)$$

where$F_n^*(.)$ is the Empirical cumulative distribution function associated to the sample. This reduces to

$$W_n^2 = \frac{1}{12n} + \sum_{j=1}^{n}\left(\frac{2j-1}{2n} - F(t_{j:n};\theta)\right)^2 ,$$

where $t_{1:n}, t_{2:n}, \ldots, t_{n:n}$denote the corresponding order statistics. If θ is known,W_n^2 is distribution-free, in the sense that its distribution depends only on n but not on the true F(·;θ), since F(T) is uniformly distributed in [0, 1] under H_0.The asymptotic distributions were derived by Anderson and Darling (1952)[1].

As is well known, the Empirical cumulative distribution function statistics, such as W_n^2, for the case of unknown parameters usually depend on the cumulative distribution function F(·;θ) in H_0as well as on n. However,in the case of a location-scale family, these statistics depend only on the family itself and n but not on the true values of the location and scale parameters, as long as an appropriate estimation method is provided. In some cases of a shape parameter, such as in the Gamma family, the dependence of the asymptotic and finite sample Empirical cumulative distribution function statistics on the shape parameter is slight, and tables of asymptotic percentage points were provided for different values of the parameter, to be used with the estimated values. This method uses a randomly chosen half of the original sample to compute the parameter estimates, say θ^*, by asymptotically efficient methods, such as maximum likelihood

(ML).Then the Empirical cumulative distribution function statistics are computed with F(·;θ^*) using the whole sample.The remarkable result is that asymptotically these Empirical cumulative distribution function statistics will behave like the ones for the case of known parameters. However, besides the dependence of the test conclusion on the choice of the half-sample, a considerable loss in power has been reported, namely in the case of testing for a normal or exponential distribution .

For a random sample $T_1, T_2, \dots, T_n$n from T ~BS(α, β), let θ = (α, β) and $\hat{\theta}$and $\tilde{\theta}$ denote respectively the maximum likelihood and modified moment estimators of θ, and θ^*denote the maximum likelihood estimator based on a randomly chosen half-sample. One can carry out a study of the asymptotic distribution of W_n^2in Eq.(5.6) for the case of unknown θ, using these three statistics. Thus let

$$C_n^2 = \text{n}\int_0^{+\infty} \left(F_n(t) - F\left(t;\hat{\theta}\right)\right)^2 dF(t;\hat{\theta}) \tag{5.7}$$

instead of Eq.(5.7) , as in Darling (1955)[20] , or alternatively

$$C_n'^2 = \text{n} \int_0^{+\infty} \left(F_n(t) - F\left(t;\tilde{\theta}\right)\right)^2 dF(t;\tilde{\theta}) \tag{5.8}$$

or

$$C_n^{*2} = = \text{n} \int_0^{+\infty} \left(F_n(t) - F(t;\theta^*)\right)^2 dF(t;\theta^*) \tag{5.9}$$

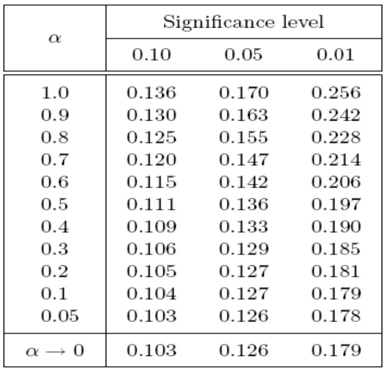
For the BS(α, β) distribution, using the asymptotic distributionsof $\hat{\beta}$ and$\tilde{\beta}$, we have var($\tilde{\beta}$) ~var($\hat{\beta}$) as $\alpha \rightarrow 0$, so the relative efficiency of these two estimators tends to 1 as αdecreases. Moreover, quoting Birnbaum & Saunders(1969b)[10] “under this condition [α< 1/2], which we shall later empirically verify, $\tilde{\beta}$ is virtually the maximum likelihood estimator whose optimal properties are well known”, we then expect to have similar asymptotic distributions (as n → ∞) in Eqs.(5.7) and (5.8) when using either$\hat{\beta}$ or $\tilde{\beta}$, at least for small values of α.

The asymptotic percentage points are computed for C_n^2 for testing

H_0 : T ~BS(α, β) with unknown parameters, based on 10^5 simulations, by the method described in Stephens (1986)[59], for significance levels 0.10, 0.05

and 0.01.This was achieved, for fixed α (α = 0.05, 0.1, 0.2, ..., 1.0), by plotting the points obtained with simulated samples of size n (n = 30, 40, ..., 120) against m = 1/n and extra polating to m = 0. Then, the values obtained for each fixed significance level were plotted against α to extrapolate toα= 0 by means of a polynomialfit from Table 5.1. These values for $\alpha \rightarrow 0$ are almost exactly the same as for the case of a normal distribution with unknown parameters , as expected, due to the asymptotic normality of the BS(α, β)distribution as $\alpha \rightarrow 0$. One can also report that, for the range of α values considered, the dependence of the percentage points on n (n ≥ 30) is slight, being negligible as α decreases and as the significance level increases. Table 5.1 is to be used with estimated α from the data, as mentioned before. In general, the well known data that have been fitted to a Birnbaum Saunders model have $\hat{\alpha}$< 1, for example the lifetime data sets psi31, psi26 and psi21 in Birnbaum & Saunders (1969b)[10] or the survival data set in Kundu et al. (2008)[39].

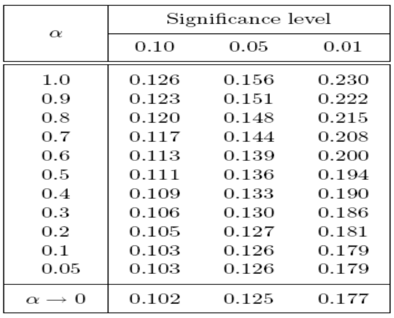
Table 5.1: Asymptotic upper-tail percentage points for $C_n^2$for testing H_0 : T ~BS(α, β),bothparameters unknown, based on 10^5 simulations.



α	Significance level		
	0.10	0.05	0.01
1.0	0.136	0.170	0.256
0.9	0.130	0.163	0.242
0.8	0.125	0.155	0.228
0.7	0.120	0.147	0.214
0.6	0.115	0.142	0.206
0.5	0.111	0.136	0.197
0.4	0.109	0.133	0.190
0.3	0.106	0.129	0.185
0.2	0.105	0.127	0.181
0.1	0.104	0.127	0.179
0.05	0.103	0.126	0.178
$\alpha \rightarrow 0$	0.103	0.126	0.179

One then repeated this procedure using modified moment instead of Maximum likelihood estimates of both parameters, and obtained the asymptotic percentage points for $C_n'^2$. Theresults, shown in Table 5.2, are similar to the former ones for small values and the similarity is stronger as α decreases to 0, as expected.

Table 5.2: Asymptotic upper-tail percentage points for $C_n'^2$ for testing H_0 : T ~BS(α, β),both parameters unknown, based on 10^5 simulations.



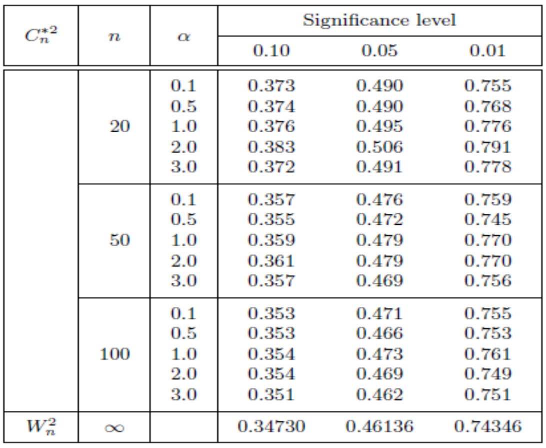
α	Significance level		
	0.10	0.05	0.01
1.0	0.126	0.156	0.230
0.9	0.123	0.151	0.222
0.8	0.120	0.148	0.215
0.7	0.117	0.144	0.208
0.6	0.113	0.139	0.200
0.5	0.111	0.136	0.194
0.4	0.109	0.133	0.190
0.3	0.106	0.130	0.186
0.2	0.105	0.127	0.181
0.1	0.103	0.126	0.179
0.05	0.103	0.126	0.179
$\alpha \rightarrow 0$	0.102	0.125	0.177

In the case of the Generalized Birnbaum Saunders family, the percentage points for C_n^2 strongly depend on the true shape parameter $\alpha$for a fixed generator X. However, if the parameters $\alpha\ and\ \beta$ are estimated by Maximum Likelihood via the split-sample method , then similar results to the ones reported for testing normality and exponentiality based on Carmer – von Mises statistic as in Tables 5.1 and 5.2 were obtained. Illustrate this feature for the Birnbaum Saunders case with Table 3. This table shows the percentage points for C_n^{*2} for α = 0.1, 0.5, 1.0, 2.0, 3.0 and n = 20, 50, 100, each one computed from 105 simulated samples at significance levels 0.10, 0.05 and 0.01, for unknown parameters estimated by the split-sample method.

The asymptotic percentage points for W_n^2 are shown in the last row of the Table 5.3. can observe that the dependence of upper-percentage points on αvalues is no longer strong, that it decreases as n increases and that the upper-percentage points are fairly close to the asymptotic ones for W_n^2 **.**

It is realized that a drawback of these methods is the dependence of the critical points on the unknown parameter and that there are other possible goodness of-fit tests that can be useful in such cases;

Table 5.3: Upper-tail percentage points for C_n^{*2} for testing H_0 : T ~BS(α,β), both parameters unknown, and upper-tail asymptotic percentage points for W_n^2 for testing H_0 : T ~BS(α,β)both parameters known.



C_n^{*2}	n	α	Significance level		
			0.10	0.05	0.01
	20	0.1	0.373	0.490	0.755
		0.5	0.374	0.490	0.768
		1.0	0.376	0.495	0.776
		2.0	0.383	0.506	0.791
		3.0	0.372	0.491	0.778
	50	0.1	0.357	0.476	0.759
		0.5	0.355	0.472	0.745
		1.0	0.359	0.479	0.770
		2.0	0.361	0.479	0.770
		3.0	0.357	0.469	0.756
	100	0.1	0.353	0.471	0.755
		0.5	0.353	0.466	0.753
		1.0	0.354	0.473	0.761
		2.0	0.354	0.469	0.749
		3.0	0.351	0.462	0.751
W_n^2	∞		0.34730	0.46136	0.74346

5.4 Some Applications with Data

5.4.1 The data sets

The three data sets under analysis are (i) the survival times of 72 guinea pigs infected with tubercle bacilli in regimen 6.6 (corresponding to 4.0×10^{6} bacillary units per 0.5ml), analyzed by Kundu et al.(2008)[38], denoted by survpig, (ii) the data set of lifetimes in cycles of aluminium coupons (maximum stress per cycle 31,000 psi) analyzed by Birnbaum & Saunders (1969 b)[10] and other authors (e.g., Ng et al.(2003)[46], Sanhueza et al.(2008)[54] and Balakrishnan et al. (2009)[6]), denoted by psi31 and (iii) the data set of daily ozone concentrations collected in New York during May–September 1973, analyzed by Ferreira et al.(2012)[27], denoted by ozone. The sample dimensions are respectively n = 72, n = 101 and n = 116.

Example 5.4.1.

The estimation procedure based on the KS-type distance D_{KS} is illustrated here by means of the data set survpig. For these data, all β values in the interval [72.35, 72.92] minimize D_{KS}, so we took the center of this interval as its estimate, say β_{KS} = 72.635. This corresponds to a distance D_{KS} = 6/72 = 0.0833.

See Fig. 5.4.

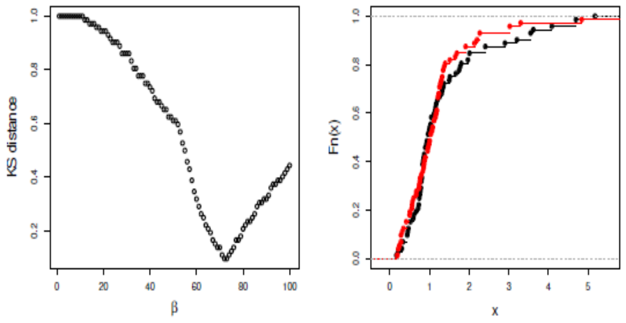
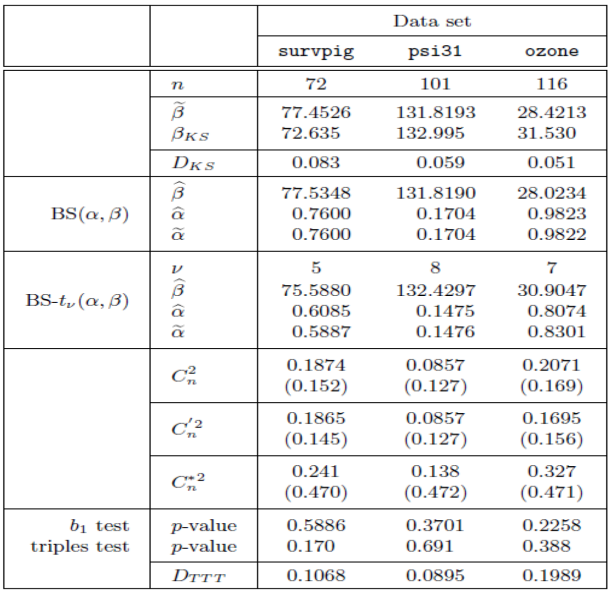


Figure 5.4: D_{KS} (KS-type distance) as a function of β (left) for survpig data and empirical cumulative distribution function for the β -scaled sample and its reciprocal, with β estimated by minimizing D_{KS} (right).

5.4.3. Analyzing the data

For each of the samples, say t = $(t_1, t_2, \ldots, t_n)$, the different procedures are applied and the results are summarized in Table 5.4. The scaled TTT plots for t and $t^{-1}$are provided in Figure 5.5 and the QQ-plots for x = t/$\hat{\beta}$and y = $\hat{\beta}$/t are given in Figure 5.6. Estimates $\tilde{\beta}$and β_{KS}(for β_{KS}we took the center of the interval of β values corresponding to a minimum distance D_{KS}, as explained before) were computed, as well as Maximum Likelihood estimates of α and β for the parametric models BS(α , β) and BS-t_ν(α , β), with ν estimated as in Azevedo et al. (2012)[2]. The CM-type statistics C_n^2, $C_n'^2$ and C_n^{*2} have also been computed and critical values for these statistics at significance level 5% are shown in parentheses. These values were obtained by interpolation, using Tables 5.1 and 5.2, in the first two cases, and from 10^5 simulated samples for each n (n = 72, 101, 116) and α (α = 0.76, 0.17, 0.98), respectively. The classical test for symmetry about unknown location based on b1 and the triples test were also applied to the transformed sample y = log(t).

Table 5.4: Results for samples survpig, psi31 and ozone.

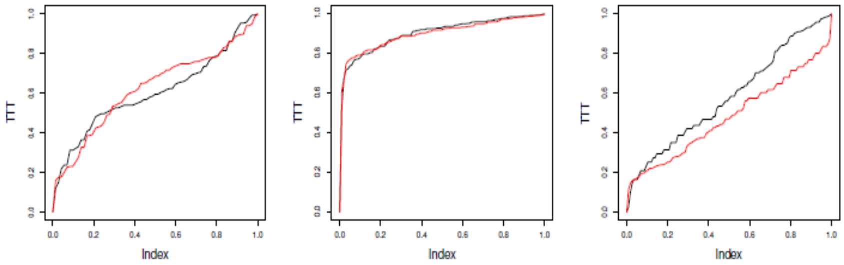


		Data set		
		survpig	psi31	ozone
	n	72	101	116
	$\widetilde{\beta}$	77.4526	131.8193	28.4213
	β_{KS}	72.635	132.995	31.530
	D_{KS}	0.083	0.059	0.051
BS(α,β)	$\widehat{\beta}$	77.5348	131.8190	28.0234
	$\widehat{\alpha}$	0.7600	0.1704	0.9823
	$\widetilde{\alpha}$	0.7600	0.1704	0.9822
BS-$t_\nu(\alpha,\beta)$	ν	5	8	7
	$\widehat{\beta}$	75.5880	132.4297	30.9047
	$\widehat{\alpha}$	0.6085	0.1475	0.8074
	$\widetilde{\alpha}$	0.5887	0.1476	0.8301
	C_n^2	0.1874 (0.152)	0.0857 (0.127)	0.2071 (0.169)
	$C_n'^2$	0.1865 (0.145)	0.0857 (0.127)	0.1695 (0.156)
	C_n^{*2}	0.241 (0.470)	0.138 (0.472)	0.327 (0.471)
b_1 test	p-value	0.5886	0.3701	0.2258
triples test	p-value	0.170	0.691	0.388
	D_{TTT}	0.1068	0.0895	0.1989

The Cramer- von Mises -type tests based on $C_n^2$and $C_n'^2$ both reject the Birnbaum Saunders model for samples survpig and ozone, but not for psi31. The symmetry tests do not reject a Generalized Birnbaum Saunders model for any of these samples.

Finally, the distance D_{TTT}has been computed for each sample. The upper 5% percentage points for the distance D_{TTT}in the BS (α, β) and BS-$t_\nu(\alpha, \beta)$ models for each n (72, 101 and 116, respectively) with $\alpha = \widehat{\alpha}$ and $\beta = \widehat{\beta}$ in each case (ν =5, 8 and 7, respectively) with 10^4simulations (see Table 5.5) are simulated. This rules out these two particular models for ozone. The graphical analysis (Figures 5.5 and 5.6) also indicates that a Generalized Birnbaum Saunders model seems reasonable for survpig, excellent for psi31 and not adequate for ozone.

Figure 5.5: TTT for samples t and $t^{-1}$for survpig (left), psi31 (center) and ozone (right).



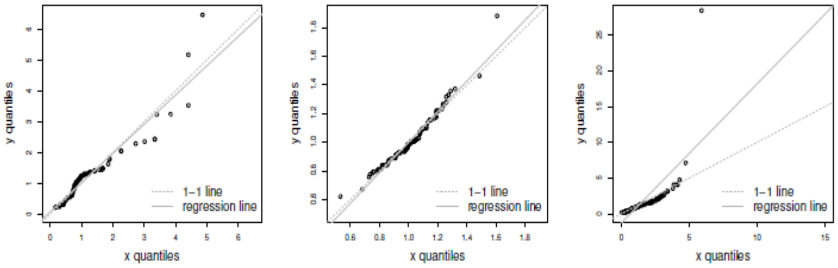


Figure 5.6: QQ-plots for $x = t/\tilde{\beta}$ and $y = \tilde{\beta}t^{-1}$, for survpig (left), psi31 (center) and ozone (right).

Table 5.5: Simulated upper-tail percentage points (at significance level 5%) for D_{TTT} assuming $T \sim BS(\hat{\alpha}, \hat{\beta})$ or $T \sim BS\text{-}t_\nu(\hat{\alpha}, \hat{\beta})$ ($\hat{\alpha}$ and $\hat{\beta}$ estimated from the three samples).

	sample		
	survpig	psi31	ozone
BS	0.156	0.122	0.157
BS-t_ν	0.218 ($\nu = 5$)	0.197 ($\nu = 8$)	0.184 ($\nu = 7$)

On the other side, as the classical symmetry b_1 test does not reject a Generalized Birnbaum Saunders model for ozone, we have carried out a brief simulation study on the power of this test against several alternatives, for n = 116, including the extreme value Birnbaum–Saunders model generated by the Gumbel distribution for minima, denoted by $\text{EVBS}^*(\alpha, \beta, 0)$. This model was proposed by Ferreira et al.(2012)[28] as the best among several other models, including the Birnbaum Saunders one. The power of the test, based on 10^5simulations, was estimated as 0.762 supposing the true model is $\text{EVBS}^*(\hat{\alpha},\hat{\beta},0)$.

In this chapter a characterization of the Generalized Birnbaum Saunders class related to the reciprocal property and analyzed some of its consequences has been derived. Some graphical procedures to assess the fit of the Generalized Birnbaum Saunders model to observed data are discussed , the asymptotic percentage points are tabulated for a test of the null hypothesis that the data come from a Birnbaum Saunders distribution with unknown parameters, and finally the results are applied to three well-known data sets. The case of tests for other Generalized Birnbaum Saunders distributions, such as the ones generated by the Student $t_\nu$or logistic distributions, is under investigation.

SUMMARY AND CONCLUSION

Summary and Conclusion

Reliability of a product (or a system) is the probability that a given product will successfully perform a required function without break or failure under specified environmental conditions, for a specified period of time. Reliability characteristics, such as probability of survival, mean time to failure, availability, mean down time and frequency of failures are some measures of system effectiveness. Acceptance sampling is the middle ground between 100% inspection and no inspection. The important thing in acceptance sampling is to minimize the cost and time required for quality control or reliability test for the decision about the acceptance or rejection of the submitted lot of the products.

The Birnbaum-Saunders distribution also known as the fatigue life distribution, is a probability distribution used extensively in reliability applications to model failure times. Birnbaum-Saunders distributions are used to study of recovery from breast cancer.

The generalized Birnbaum-Saunders distribution is a new class of positively skewed model with lighter and heavier tails than the traditional Birnbaum-Saunders distribution which is largely applied to study lifetimes. The theoretical argument and the properties of the distribution are useful model for describing pollution data and deriving its positive and negative moments.

The first chapter deals with basic concepts of Acceptance Sampling Plan, Theory of Estimations, Lifetime distributions, Notations and Review of literature.

In second chapter , when the life test is truncated at a pre-fixed time and the life time of the test units follow a Student-t generalized Birnbaum-Saunders distribution the acceptance sampling plans have been developed. Under similar conditions , in order to ensure a specified median life with a given confidence level. The generalized Birnbaum-Saunders (t_ν) model results in smaller sample sizes than some other models using in acceptance sampling. The generalized Birnbaum-Saunders (t_ν) distributions fits the data better than the classical Birnbaum-Saunders and inverse Rayleigh model have been demonstrated for a real data.

The third chapter deals with the shape of the hazard function of Birnbaum Saunders distribution. It is proved that the hazard function of the two-parameter Birnbaum Saunders distribution is unimodal (of upside down form). The change point in the hazard function is obtained as a solution of a non-linear equation. An approximation to this change point is provided and have shown that the approximation works very well whenever the shape parameter is not too small. Different methods for estimating the change point are proposed and have compared their performance through Monte Carlo simulations. Both bias-corrected modified moment estimators and approximate bias- corrected modified moment estimators perform well, and since approximate bias- corrected modified moment estimators is a simple explicit estimator, it can be used for the estimation of change point. The asymptotic distribution of all the estimators are derived and because of their complexity, bootstrap confidence intervals for data analysis has used.

In fourth chapter, the class of Generalized Saunders Distribution, which is very flexible family of suitable for modelling lifetime data as it slows for different degree of kurtosis and asymmetry and unimodality as well as bimodality. The methods of inference based on uncensored and censored data, diagnostics methods, goodness of fit tests, and random number generation algorithms for the generalized Birnbaum Saunders model have been discussed. Generalized Birnbaum-Saunders distribution fits the data better than the classical Birnbaum-Saunders model.

In fifth chapter, a characterization of the Generalized Birnbaum Saunders class related to the reciprocal property and analyzed some of its consequences has been derived. Some graphical procedures to assess the fit of the Generalized Birnbaum Saunders model to observed data are discussed, the asymptotic percentage points are tabulated for a test of the null hypothesis that the data come from a Birnbaum Saunders distribution with unknown parameters, and finally the results are applied to three well-known data sets.

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