$\tilde{\Delta}$ - Closed sets in Topological Spaces

Kalaivani, R

(15PMA002)

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore – 641 043.

In Partial Fulfilment of the Requirement for the Degree of

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CONTENT

CONTENT

CHAPTER **TITLE**

INTRODUCTION

INTRODUCTION

Topology is a widely studied area of mathematics emerged through the works of the great Mathematician Henri Poincare in the 19th century. Topology developed as a field of study out of geometry and set theory, through analysis of such concepts as space, dimension and transformation. It is the study of continuity and connectivity. The topological structures are modeled suitably in the fields of computer graphics, pattern recognition, artificial intelligence, data mining, rough set theory, information systems, quantum physics etc.

The notion of open sets is the powerful tool for defining a topological space. Levine (1960) introduced and studied the concepts of semi-open sets in topological spaces, as a weaker form of open sets. It is found from literature that during recent years many topologists are interested in the study of generalized types of closed sets. The study of generalized closed sets was initiated by Levine (1970) in order to extend some important properties of closed sets to a larger family of sets. The productivity and fruitfulness of the notion of generalized closed sets motivated the mathematicians to introduce weaker and stronger forms of generalized closedness for the past four decades. With the aid of g-open sets, they introduced, investigated and modified continuous functions which are the core concept of topology. Detailed study in this regard by many investigators has enriched the field of generalized closed sets to a considerable extent.

In 1968, Velicko introduced δ - open sets which are stronger than open sets, in order to investigate the characterization of H - closed spaces in terms of arbitrary filter bases and showed that, the collection τ_δ of all δ - open sets, is a coarser topology on X where a set is δ - open if it is the union of regular open sets. The collection of all regular open sets forms a base for a coarser topology $\tau_s \subseteq \tau$ but $\tau_s = \tau_\delta$. The semi-regularization of τ is the family τ_s. In a semi-regular space $\tau = \tau_s = \tau_\delta$.

Through the semi-regularization of a given topology and the associated δ - closure operator, Dontchev et al (1996) considered a slightly stronger form of g - closedness namely δg - closedness, properly placed between δ - and g - closedness.

i.e., δ - closedness → δg - closedness → g - closedness

Thus, a separation axiom $T_{3/4}$ is introduced by Dontchev (1996) in which every δg - closed set is δ -closed. It is easily verified that

$$\mathbf{T_1 \rightarrow T_{3/4} \rightarrow T_{1/2}}$$

Chapter – 1 deals with definitions and results in topological spaces that are required for this study.

Chapter – 2 consists of 2 sections namely

2.1. δg - closed sets

2.2. δg* - closed sets

In **section 2.1,** the concept of δg - closed sets introduced by Dontchev (1996), is reviewed. The dependence, independence relationships of δg - closed sets with some already existing closed sets in topological spaces are studied. Further some interesting characterizations of high weightage are discussed.

In **section 2.2,** the notion δg* - closed sets, introduced by Sudha (1996), is taken for the study. Some relationships and characterizations of δg* - closed sets with some prominent sets in literature are discussed.

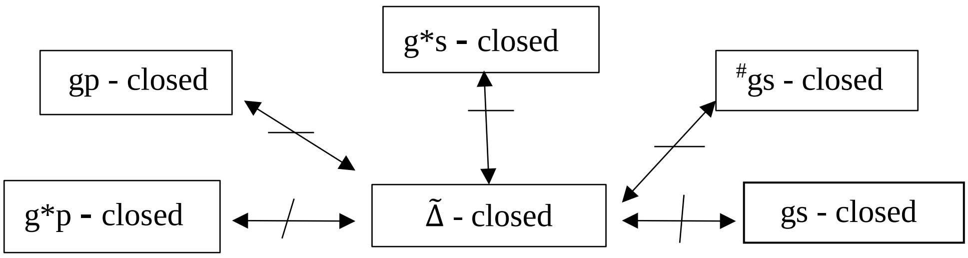
In **Chapter – 3,** a new class of sets is called $\tilde{\Delta}$ **-** closed sets is introduced. The collection of this new class contains the class of Δ* - closed sets and contained in the class of gδ - closed sets in topological spaces. Some characterizations and hereditary properties are obtained by proving some important theorems.

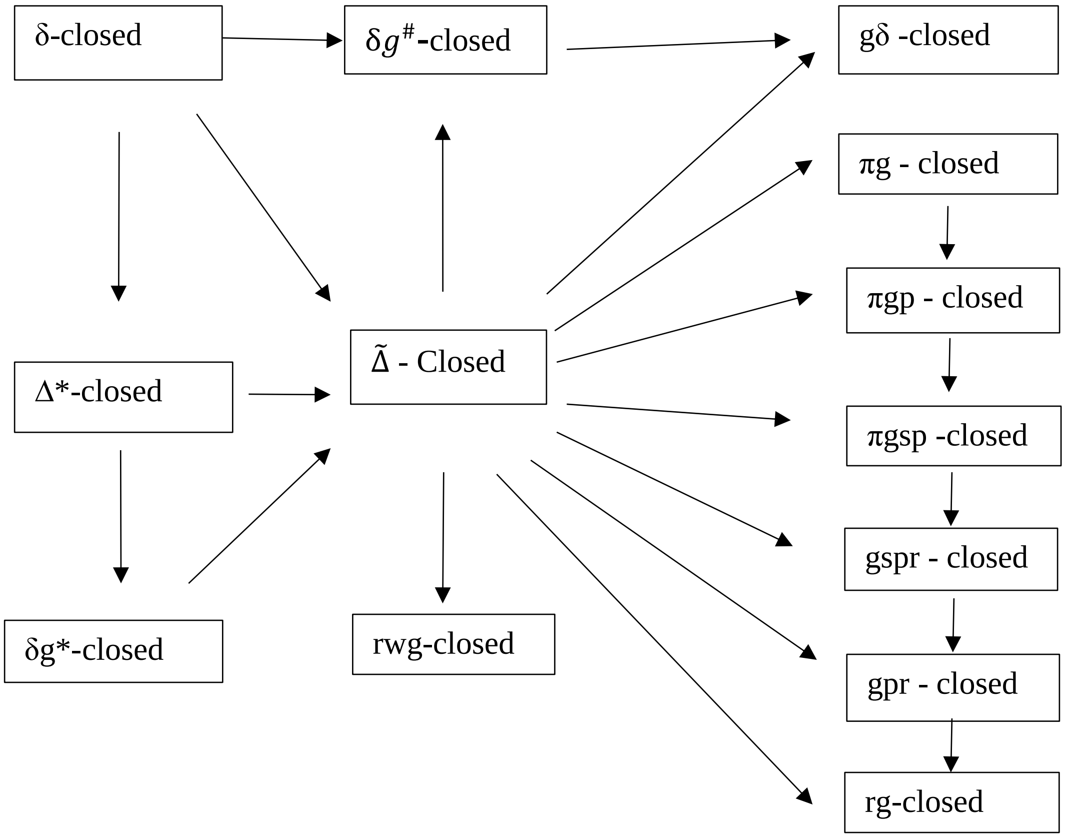
The important definitions and results proved in this chapter:

A subset A of a space (X, τ) is called $\tilde{\Delta}$ **- closed** if $cl_\delta(A) \subseteq U$, whenever $A \subseteq U$ and U is a δg* - open set in (X, τ).

The class of all $\tilde{\Delta}$ - closed sets in denoted by $\tilde{\Delta}C(X, \tau)$.

The following relations exist for $\tilde{\Delta}$ - closed sets





where A ⟶ B represents A implies B and A ⟷̸ B represents A and B are independent.

Chapter – 4 is devoted to applications of $\tilde{\Delta}$ - Closed sets namely $\tilde{\Delta}$ - separation axioms. In section 4.1, twelve new separation axioms using $\tilde{\Delta}$ - Closed sets are introduced and studied. This is followed by the notion of $\tilde{\Delta}$ - Continuity in section 4.2. As a progression, the dependent and independent relationships of $\tilde{\Delta}$ -Continuity with other already existing continuities are given. Further, composition on $\tilde{\Delta}$ - Continuity is discussed.

The twelve separation axioms are:

A space (X, τ) is said to be a

1. $\tilde{\Delta}\mathbf{T}\delta$ - space if every $\tilde{\Delta}$ - closed subset of (X, τ) is δ - closed in (X, τ).

2. $\tilde{\Delta}\mathbf{T}\delta g^*$ - space if every $\tilde{\Delta}$ - closed subset of (X, τ) is δg*- closed in (X, τ).

3. $\tilde{\Delta}\mathbf{T}\Delta^*$- space if every $\tilde{\Delta}$ - closed subset of (X, τ) is Δ*- closed in (X, τ).

4. $g\delta\mathbf{T}\tilde{\Delta}$ - space if every gδ - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

5. $\delta g^{\#}\mathbf{T}\tilde{\Delta}$ - space if every $\delta g^{\#}$ - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

6. $rg\mathbf{T}\tilde{\Delta}$ - space if every rg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

7. $rwg\mathbf{T}\tilde{\Delta}$ - space if every rwg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

8. $gpr\mathbf{T}\tilde{\Delta}$ - space if every gpr - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

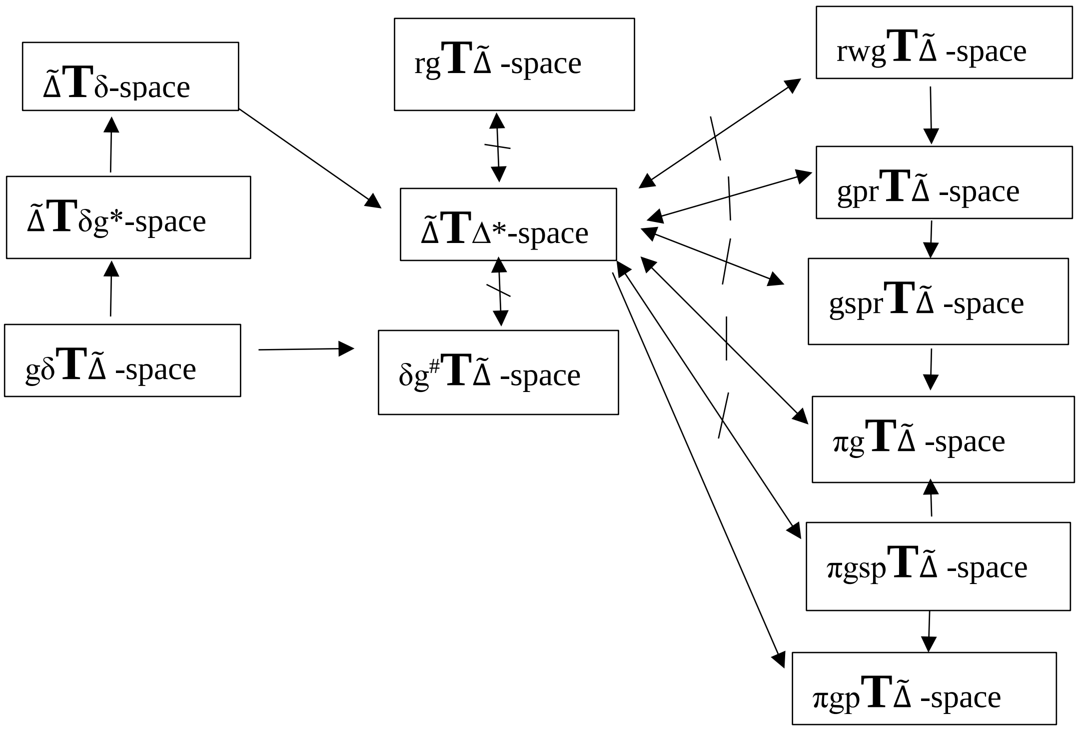
9. $gspr\mathbf{T}\tilde{\Delta}$ - space if every gspr - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

10. $\pi gp\mathbf{T}\tilde{\Delta}$ - space if every πgp - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

11. $\pi gsp\mathbf{T}\tilde{\Delta}$ - space if every πgsp - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

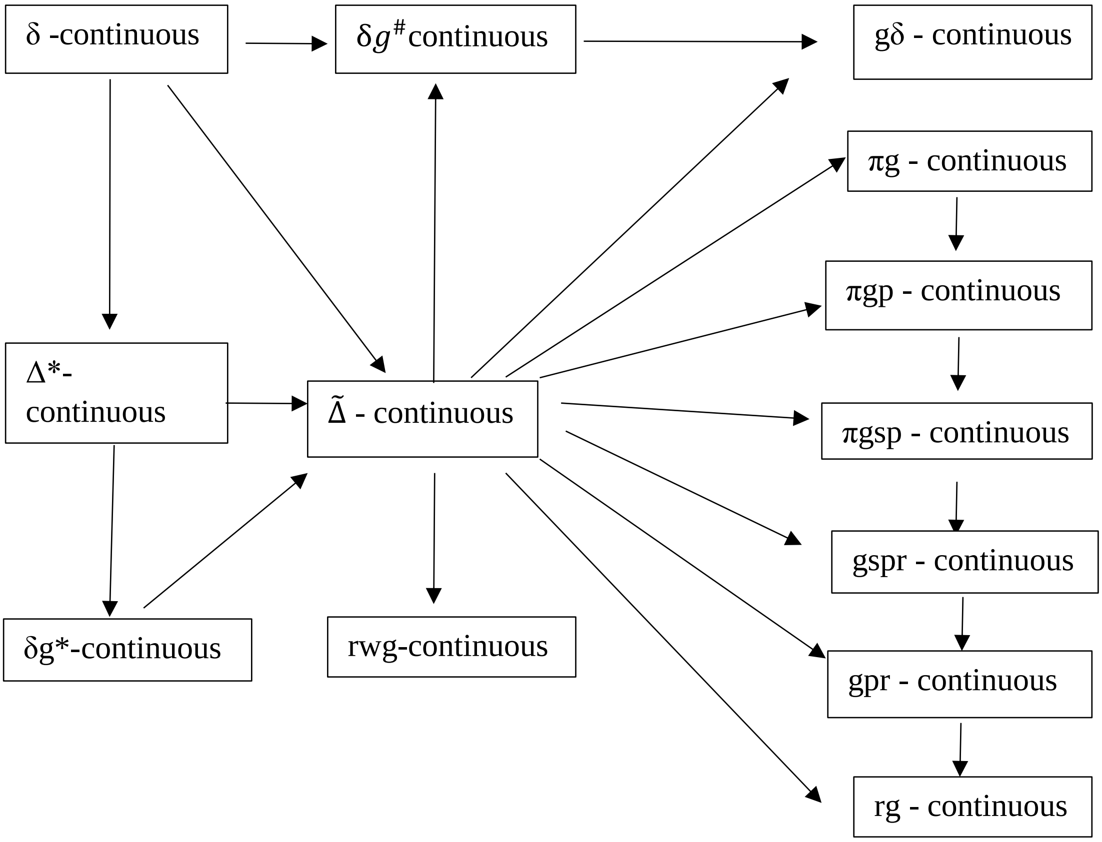
12. $\pi g\mathbf{T}\tilde{\Delta}$ - space if every πg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

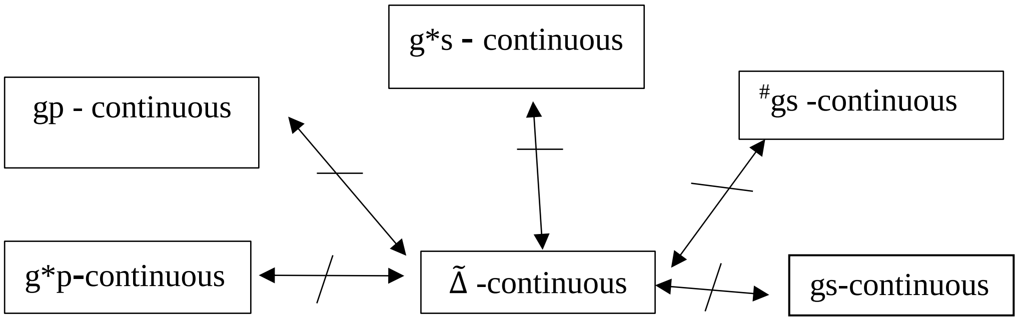
The following diagram shows the relationships between these spaces:



A function f: (X, τ) → (Y, σ) is said to be $\tilde{\Delta}$ **- continuous** if the inverse image of every closed set in (Y, σ) is $\tilde{\Delta}$ - closed in (X, τ).

The following two diagrams show the dependence and independence of $\tilde{\Delta}$ - continuous function and other already existing continuous functions respectively.





where A ⟶ B represents A implies B and A ⟷ B (with stroke) represents A and B are independent.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

Topology, the study of surfaces, is a branch of mathematics concerned with spatial properties preserved under bi-continuous deformation. It emerged through the development of concept from geometry and set theory. Topology plays an important role in pure and applied mathematics. Topological structures are suitable mathematical models for formulation of both qualitative and quantitative data.

Mashhour et al (1982) first studied the notion of pre open sets in topological spaces and obtained various properties. With the aid of pre open sets, they introduced and investigated modified continuous functions called pre continuous functions and weak pre continuous function. Dontchev et al (2000) introduced and investigated p - closed spaces.

Thivagar et al (2011) and their properties and applications were obtained, $\delta\hat{g}$ ($\delta\hat{g}$c) homeomorphism using $\delta\hat{g}$ - closed sets were introduced by Lellis Thivagar (2011) and their group structure was investigated. In a later paper, contra $\delta\hat{g}$ - continuous functions were introduced and several characterizations were obtained by Lellis Thivagar (2012). Furthermore, a stronger form of semi regularity called T_δ - spaces, was introduced and it is shown that it is equal to semi - regularity plus almost weak Hausdroffness. gδ - continuous and gδ - irresolute functions are considered as well.

K. Meena (2014) introduced a new class of generalised closed sets called Δ* - closed sets in topological spaces using δg - closed sets. Moreover we analyse the relations between Δ*- closed sets and already existing various closed sets are analysed. It is proved that Δ*- closed sets are independent of δg - closed sets and weaker than δg*- closed sets. The class of Δ*- closed sets is properly placed between various closed sets and a chain of relations is proved as follows.

regular - closed → π - closed → δ - closed→ δg* - closed → Δ* - closed → $\delta g^{\#}$ - closed → gδ - closed

Separation axioms is one of the most important and interesting concepts in topological spaces. One of the most well known low separation axioms is the one which requires that singletons are closed. i.e. T1 .

Dontchev introduced T_δ - space in which every gδ- closed set is δ - closed. He gave a characterization of T_δ - space using the concept of almost weekly hausdroff space and semi - regular space.

The separation axioms using Δ* - closed sets namely Δ^*T_δ - space, $\Delta^*T_{\delta g^*}$ - space, $g\delta T_{\Delta^*}$- space, $\delta g^{\#}T_{\Delta^*}$- space are introduced by K.Meena(2015) and their properties are discussed.

Continuity is an important concept in Mathematics and many forms of continuous functions have been introduced over the years. A significant theme in general topology concerns the variously modified forms of continuity by utilizing generalized closed sets.

K. Meena (2015) introduced some new class of functions called Δ* - continuous functions by using Δ* - closed sets. She investigated its implication and independence relationship with other types of continuous functions. Also she analysed the association of Δ* - continuous with various kinds of continuous functions via separation axioms. Furthermore she derived some of its properties under composition of mappings.

CHAPTER-I

CHAPTER 1

PRELIMINARIES

Definition 1.1 [37]

A subset A of a topological space (X,τ) is called **semi - open** if A ⊆ cl(int(A)) and **semi - closed** if int(cl(A)) ⊆A.

Definition 1.2 [76]

A subset A of a topological space (X,τ) is called **regular open** if A = int(cl(A)) and **regular closed** if cl(int(A)) ⊆ A.

Definition 1.3 [59]

A subset A of a topological space (X,τ) is called **α - open** if A ⊆ int(cl(int(A))) and **α - closed** if cl(int(cl(A))) ⊆ A.

Definition 1.4 [54]

A subset A of a topological space (X,τ) is called **pre - open** if A ⊆ int(cl(A)) and **pre - closed** if cl(int(A)) ⊆ A.

Definition 1.5 [2]

A subset A of a topological space (X,τ) is called **semi - preopen** if A ⊆ cl(int(cl(A))) and **semi - pre closed** if int(cl(int(A))) ⊆ A.

Definition 1.6 [38]

A subset A of a topological space (X,τ) is called **generalized closed** (briefly g - closed) if cl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

Definition 1.7 [44]

A subset A of a topological space (X,τ) is called ***α* - generalized closed** (briefly *α*g - closed) if αcl(A) ⊆ U whenever A ⊆ U and U is open set in (X, τ).

Definition 1.8 [88]

A subset A of a topological space (X,τ) is called **δ - closed** if $cl_\delta(A) = A$ where $cl_\delta(A) = \{x \in X ; int(cl(U) \cap A) \neq \emptyset , U \in \tau$ and $x \in U\}$.

Definition 1.9 [69]

A subset A of a topological space (X,τ) is called **generalized semi- pre – regular - closed** (briefly gspr - closed) if spcl(A) ⊆ U whenever A ⊆ U and U is regular open in (X, τ).

Definition 1.10 [86]

A subset A of a topological space (X,τ) is called $\hat{g}$ - **closed** if cl(A) ⊆ U whenever A ⊆ U and U is semi - open in (X, τ).

Definition 1.11 [84]

A subset A of a topological space (X,τ) is called **g^* - closed** if cl(A) ⊆ U whenever A ⊆ U and U is g - open in (X, τ).

Definition 1.12 [89]

A subset A of a topological space (X,τ) is called **$^*g$ - closed** if cl(A) ⊆ U whenever A ⊆ U and U is $\hat{g}$ - open in (X, τ).

Definition 1.13 [88]

A subset A of a topological space (X,τ) is called **$^\#gs$ - closed** if scl(A) ⊆ U whenever A ⊆ U and U is $^*g$ - open in (X, τ).

Definition 1.14 [25]

A subset A of a topological space (X,τ) is called $\tilde{g}$ **- closed** if cl(A) ⊆ U whenever A ⊆ U and U is $^\#gs$ - open in (X, τ).

Definition 1.15 [67]

A subset A of a topological space (X,τ) is called **generalized star pre - closed** (briefly g*p - closed) if pcl(A) ⊆ U whenever A ⊆ U and U is g - open in (X, τ).

Definition 1.16 [65]

A subset A of a topological space (X,τ) is called **regular generalized - closed** (briefly rg - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open in (X, τ).

Definition 1.17 [12]

A subset A of a topological space (X,τ) is called **generalized δ - closed** (briefly gδ - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is δ - open in (X, τ).

Definition 1.18 [11]

A subset A of a topological space (X,τ) is called **δ - generalized closed (**briefly δ - closed) if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

Definition 1.19 [86]

A subset A of a topological space (X,τ) is called **$\delta g^{\#}$ - closed** if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is δ - open in (X, τ).

Definition 1.20 [83]

A subset A of a topological space (X,τ) is called **regular weekly generalized closed** (briefly rwg - closed) if $cl(int(A)) \subseteq U$ whenever $A \subseteq U$ and U is regular - open in (X, τ).

Definition 1.21 [14]

A subset A of a topological space (X,τ) is called **π - generalized closed** (briefly πg - closed) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is πg - open in (X, τ).

Definition 1.22 [27]

A subset A of a topological space (X,τ) is called **π - generalized pre closed** (briefly πgp - closed) if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is π - open in (X, τ).

Definition 1.23 [69]

A subset A of a topological space (X,τ) is called **π - generalized semi pre – closed** (briefly πgsp - closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is π - open in (X, τ).

Definition 1.24 [74]

A subset A of a topological space (X,τ) is called **δ - generalized star closed** (briefly δg* - closed) if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is g - open in (X, τ).

Definition 1.25 [75]

A subset A of a topological space (X,τ) is called **wδg*- closed** if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is g^* - open in (X, τ).

Definition 1.26 [72]

A subset A of a topological space (X,τ) is called **Δ* - closed** if $cl_{\delta}(A) \subseteq U$ whenever $A \subseteq U$ and U is δg - open in (X, τ).

Definition 1.27 [67]

A subset A of a topological space (X,τ) is called **generalized star s closed** (briefly g*s - closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs - open in (X, τ).

Definition 1.28 [5]

A subset A of a topological space (X,τ) is called **semi - generalized closed** (briefly sg - closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ).

Definition 1.29 [3]

A subset A of a topological space (X,τ) is called **generalized semi - closed** (briefly gs - closed) if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

Definition 1.30 [43]

A subset A of a topological space (X,τ) is called **generalized α - closed** (briefly gα - closed) if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α - open in (X, τ).

Definition 1.31 [1]

A subset A of a topological space (X,τ) is called **$\alpha\hat{g}$ - closed** if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$ - open in (X, τ).

Definition 1.32 [87]

A subset A of a topological space (X,τ) is called **$g^{\#}s$ - closed** if scl(A) ⊆ U whenever A ⊆ U and U is αg-open in (X, τ).

Definition 1.33 [52]

A subset A of a topological space (X,τ) is called **sαg* - closed** if αcl(A) ⊆ U whenever A ⊆ U and U is g* - open in (X,τ).

Definition 1.33 [26]

A subset A of a topological space (X,τ) is called **$\tilde{g}_{\alpha}$ - closed** if αcl(A) ⊆ U whenever A ⊆ U and U is $^{\#}$gs - open in (X, τ).

Definition 1.34 [79]

A subset A of a topological space (X,τ) is called **$\tilde{g}_{s}$ - closed** if scl(A) ⊆ U whenever A ⊆ U and U is $^{\#}$gs - open in (X, τ).

Definition 1.35 [21]

A subset A of a topological space (X,τ) is called **$\tilde{g}_{p}$ - closed** if pcl(A) ⊆ U whenever A ⊆ U and U is $^{\#}$gs - open in (X, τ).

Definition 1.36 [68]

A subset A of a topological space (X,τ) is called **$\ddot{g}$ - closed** if cl(A) ⊆ U whenever A ⊆ U and U is sg - open in (X,τ).

Definition 1.37 [66]

A subset A of a topological space (X,τ) is called **g''- closed** if cl(A) ⊆ U whenever A ⊆ U and U is gs - open in (X,τ).

Definition 1.38 [49]

A subset A of a topological space (X,τ) is called **gp - closed** if pcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

Definition 1.39 [85]

A subset A of a topological space (X,τ) is called **g*p - closed** if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is g - open in (X, τ).

Definition 1.40 [67]

A subset A of a topological space (X,τ) is called **g*s - closed** if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is gs - open in (X, τ).

Definition 1.41 [90]

The space (X,τ) is called **regular space** if $\tau_\theta = \tau$.

Definition 1.42 [90]

The family of all regular open sets is a base for a new topology denoted by τ_s on X is called the **semi regularization.**

Definition 1.43 [90]

A space (X,τ) is called **semi - regular space** if $\tau_s = \tau$.

Results 1.44 [90]

In general $\tau_s \subseteq \tau$.

Results 1.45 [90]

$\tau_s = \tau_\delta$.

Results 1.46 [74]

regular closed $\rightarrow$ π - closed $\rightarrow$ δ - closed $\rightarrow$ δg^* - closed $\rightarrow$ δg - closed $\rightarrow$ g - closed

Results 1.47 [70]

For every subset A of X, spcl $\subseteq$ $pcl(A) \subseteq cl_\delta(A)$.

Results 1.48

a. δ - closed $\rightarrow$ δg^* - closed $\rightarrow$ Δ^* - closed $\rightarrow$ $\delta g^\#$ - closed $\rightarrow$ $g\delta$ - closed.

b. πg - closed → πgp - closed → πgsp - closed → gspr - closed.

c. Closed → g – closed → αg - closed → gs - closed.

Remark 1.49

The result is obtained from the various g - closed set existing in the literature.

a. gpr - closed → gspr - closed.

b. gpr - closed → rwg - closed.

Definition 1.50 [23]

A subset A of a topological space (X,τ) is called **gpr - closed** if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

Definition 1.51 [38]

A topological space (X, τ) is said to be a **$T_{1/2}$ - space** if every g - closed subset of (X, τ) is closed in (X, τ).

Definition 1.52 [11]

A topological space(X, τ) is said to be a **$T_{3/4}$ - space** if every δg - closed subset of (X, τ) is δ - closed in (X, τ)

Definition 1.53 [10]

A topological space (X, τ) is said to be a **T_b - space** if every gs - closed subset of (X, τ) is closed in (X, τ).

Definition 1.54 [84]

A topological space (X, τ) is said to be a **T_c - space** if every gs - closed subset of (X, τ) is g*- closed in (X, τ).

Definition 1.55 [10]

A topological space (X, τ) is said to be a **T_d - space** if every gs - closed subset of (X, τ) is g - closed in (X, τ).

Definition 1.56 [46]

A topological space (X, τ) is said to be a **αTb - space** if every αg - closed subset of (X, τ) is closed in (X, τ).

Definition 1.57 [84]

A topological space (X, τ) is said to be a **αTc - space** if every αg - closed subset of (X, τ) is g*- closed in (X, τ).

Definition 1.58 [84]

A topological space (X, τ) is said to be a **$^*T_{1/2}$ - space** if every g - closed subset of (X, τ) is g* - closed in (X, τ).

Definition 1.59 [12]

A topological space (X, τ) is said to be a **Tδ - space** if every gδ - closed subset of (X, τ) is δ - closed in (X, τ).

Definition 1.60 [74]

A topological space (X, τ) is said to be a **δg*Tδ - space** if every δg* - closed subset of (X, τ) is δ - closed in (X, τ).

Definition 1.61 [74]

A topological space (X, τ) is said to be a **δgTδg* - space** if every δg - closed subset of (X, τ) is δg* - closed in (X, τ).

Definition 1.62 [74]

A topological space (X, τ) is said to be a **gsTδg*- space** if every gs - closed subset of (X, τ) is δg* - closed in (X, τ).

Definition 1.63 [74]

A topological space (X, τ) is said to be a **αgTδg*- space** if every αg - closed subset of (X, τ) is δg*- closed in (X, τ).

Definition 1.64 [74]

A topological space (X, τ) is said to be a **α$\hat{g}$Tδg* - space** if every α$\hat{g}$ - closed subset of (X, τ) is δg* - closed in (X, τ).

Definition 1.65 [74]

A topological space (X, τ) is said to be a **δ$\hat{g}$Tδg*- space** if every δ$\hat{g}$ - closed subset of (X, τ) is δg* - closed in (X, τ).

Definition 1.66 [73]

A topological space (X, τ) is said to be a **Δ*Tδ - space** if every Δ*- closed subset of (X, τ) is δ - closed in (X, τ).

Definition 1.67 [73]

A topological space (X, τ) is said to be a **Δ*Tδg* - space** if every Δ* - closed subset of (X, τ) is δg* - closed in (X, τ).

Definition 1.68 [73]

A topological space (X, τ) is said to be a g**δTΔ* - space** if every gδ - closed subset of (X, τ) is Δ* - closed in (X, τ).

Definition 1.69 [73]

A topological space (X, τ) is said to be a **$\delta g^{\#}$TΔ* - space** if every $\delta g^{\#}$ - closed subset of (X, τ) is Δ* - closed in (X, τ).

Definition 1.70 [38]

A function f: (X, τ) →(Y, σ) is said to be **continuous** if $f^{-1}(V)$ is closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.71 [61]

A function f: (X, τ) →(Y, σ) is said to be **super continuous** if $f^{-1}(V)$ is δ - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.72 [48]

A function f: (X, τ) →(Y, σ) is said to be **g - continuous** if $f^{-1}(V)$ is g - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.73 [11]

A function f: (X, τ) →(Y, σ) is said to be **δg - continuous** if $f^{-1}(V)$ is δg - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.74 [73]

A function f: (X, τ) →(Y, σ) is said to be **gδ - continuous** if $f^{-1}(V)$ is gδ - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.75 [65]

A function f: (X, τ) →(Y, σ) is said to be **rg - continuous** if $f^{-1}(V)$ is rg - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.76 [77]

A function f: (X, τ) →(Y, σ) is said to be **δg* - continuous** if $f^{-1}(V)$ is δg* - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.77 [67]

A function f: (X, τ) →(Y, σ) is said to be **g*s - continuous** if $f^{-1}(V)$ is g*s - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.78 [12]

A function f: (X, τ) →(Y, σ) is said to be **gp - continuous** if $f^{-1}(V)$ is gp - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.79 [24]

A function f: (X, τ) →(Y, σ) is said to be **gpr - continuous** if $f^{-1}(V)$ is gpr - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.80 [18]

A function f: (X, τ) →(Y, σ) is said to be **πg - continuous** if $f^{-1}(V)$ is πg - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.81 [69]

A function f: (X, τ) →(Y, σ) is said to be **πgp - continuous** if $f^{-1}(V)$ is πgp - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.82 [69]

A function f: (X, τ) →(Y, σ) is said to be **πgsp - continuous** if $f^{-1}(V)$ is πgsp - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.83 [69]

A function f: (X, τ) →(Y, σ) is said to be **gspr - continuous** if $f^{-1}(V)$ is gspr - closed in (X, τ) for every closed set V of (Y, σ).

Definition 1.84

A function f: (X, τ) →(Y, σ) is said to be **Δ* - continuous** if $f^{-1}(V)$ is Δ* - closed in (X, τ) for every closed set V of (Y, σ).

CHAPTER-II

CHAPTER 2

On δg - closed sets and δg* - closed sets in topological spaces

2.1 δg - closed sets

The initiation of the study of generalized closed sets was done by Levine [38] in 1970 as he considered sets whose closure belongs to every open superset. He called them generalized closed (= g - closed) and studied their most fundamental properties. The spaces in which the concepts of g-closed and closed sets coincide are called $T_{1/2}$ - spaces [38].

In this chapter the concept of semi – regularization from a different perspective is studied. Through the semi-regularization of a given topology and the associated δ - closure operator a slightly stronger form of g - closedness, properly placed between δ - closed and g - closedness, is considered. The sets are called δ - generalized closed. The basic properties are reviewed here.

Moreover, a new separation axiom $T_{3/4}$, the class of topological spaces where every δ - generalized closed set is δ - closed, i.e. closed in the semi-regularization topology, is considered for the study and the class of $T_{3/4}$ - spaces is shown to be properly placed between the classes of $T_{1/2}$ - spaces and T_1 - spaces.

Definition 2.1.1 [11]

A subset A of a topological space (X, τ) is called **δ - generalized closed** (briefly **δ-g - closed**) if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).

Throughout the sequel the family of all δ-generalized closed subsets of a topological space (X, τ) will be denoted by DGC(X, τ).

The class of δ - generalized closed sets is properly placed between the classes of δ - closed and g - closed sets.

Theorem 2.1.2.

Let (X, τ) be a space. Then

1. Every δ - closed set is δ - generalized closed.
2. Every δ - generalized closed set in (X, τ) is g - closed in (X, τ_s).

3. Every δ - generalized closed set in (X, τ) is g - closed in (X, τ) and hence αg - closed, gs - closed, gsp - closed and r-g - closed.

Proof:

The proof follows from the definitions.

The reverse claims in the above theorem are not usually true. First, we give an example of a δ - generalized closed set which is not δ - closed, not even semi - closed.

Example 2.1.3.

Let X = {a, b, c} and let τ = {∅, {a, b}, X}. Set A = {a, c}. Since the only open superset of A is X, A is clearly a δ - generalized closed. But it is easy to see that A is not δ - closed. In fact it is not even semi - closed.

Next we show that a closed set in a space (X, τ) need not be δ - generalized closed in (X, τ).

Example 2.1.4.

Let X be the real line and let τ be the point generated topology on X, i.e. the non - empty open sets are those containing a fixed point, say the zero point. Then the set P of all irrationals is closed in (X, τ) and thus g - closed. Since (X, τ_s) is the indiscrete space, P is g - closed in (X, τ_s) but clearly not δ - generalized closed in (X, τ).

We now observe that in semi - regular spaces the notions of δ - generalized closed and generalized closed sets coincide.

Theorem 2.1.5.

Let A be a subset of a semi - regular space (X, τ). Then,

1. A is δ - generalized closed if and only if A is g - closed.
2. If, in addition, (X, τ) is $T_{1/2}$, then A is δ - generalized closed if and only if A is closed.

The previous observation leads to the problem of finding the spaces (X, τ) in which the g - closed sets of (X, τ_s) are δ - generalized closed in (X, τ). Even though this problem is not completely resolved, a partial solution is offered. For that reason the spaces with $T_{1/2}$ semi-regularization is called almost weakly Hausdorff.

Definition 2.1.6.

A space is called weakly Hausdorff if its semi - regularization is T_1.

The point excluded topology on any infinite set gives an example of an almost weakly Hausdorff space, which is not weakly Hausdorff.

Theorem 2.1.7.

In an almost weakly Hausdorff space (X, τ) the g - closed sets of (X, τ_s) are δ - closed in (X, τ) and thus δ - generalized closed in (X, τ).

Proof.

Let $A \subseteq X$ be a g - closed subset of (X, τ_s). Let $x \in cl_\delta(A)$. If $\{x\}$ is δ - open, then $x \in A$. If not, then $X\backslash\{x\}$ is δ - open, since X is almost weakly Hausdorff. Assume that $x \notin A$. Since A is g - closed in (X, τ_s), then $cl_\delta(A) \subseteq X\backslash\{x\}$, i.e. $x \notin cl_\delta(A)$. By contradiction $x \in A$. Thus $cl_\delta(A) = A$ or equivalently A is δ - closed and hence δ - generalized closed in (X, τ).

Definition 2.1.8.

A topological space (X, τ) is called an R_1 - space if every two different points with distinct closures have disjoint neighbourhoods.

Theorem 2.1.9.

For a compact subset A of an R_1 topological space (X, τ) the following conditions are equivalent:

(a) A is a δ - generalized closed set.

(b) A is a generalized closed set.

Proof:

(a) $\Rightarrow$ (b)

The proof follows from the definitions.

(b) $\Rightarrow$ (a)

In R_1 – spaces, the concepts of closure and δ - closure coincide for compact sets [Theorem 3.6 from [30]]. Thus the rest of the proof is obvious.

Corollary 2.1.10.

In Hausdorff spaces, a finite set is g - closed if and only if it is δ-g - closed.

Theorem 2.1.11

Let A be a preopen subset of a topological space (X, τ). Then the following conditions are equivalent:

(a) A is δ-g - closed.

(b) A is g - closed.

(c) A is αg - closed.

Proof:

(a) ⇒ (b) and (b) ⇒ (c) are true for any subset A. If A is preopen in (X, τ) then, $cl(A) = cl_{\alpha}(A) = cl_{\delta}(A)$ [Janković [28]]and so (c) ⇒ (a).

Definition 2.1.12

A partition space [58] is a topological space where every open set is closed.

Corollary 2.1.13.

Let A be a subset of the partition space (X, τ). Then the following conditions are equivalent:

(a) A is δ-g - closed.

(b) A is g - closed.

(c) A is αg - closed.

Proof:

A topological space is a partition space if and only if every subset is preopen. Thus the claim follows straight from Theorem 2.1.11.

Concerning partition spaces, consider the following characterization via δ - generalized closed sets.

Theorem 2.1.14.

For a topological space (X, τ) the following conditions are equivalent:

(a) X is a partition space.

(b) Every subset of X is δ - generalized closed.

Proof:

(a) ⇒ (b)

Let $A \subseteq U$, where U is open and A is an arbitrary subset of X. Since X is a partition space, then U is clopen. Thus $cl_\delta(A) \subseteq cl_\delta(U) = U$.

(b) ⇒ (a)

If $U \subseteq X$ is open, then by (b) $U \subseteq U$ which implies $cl_\delta(U) \subseteq U$ or equivalently U is δ - closed and hence closed.

Theorem 2.1.15.

1. Finite union of δ-g - closed sets is always a δ-g - closed set.
2. Countable union of δ-g - closed sets need not be a δ-g - closed set.
3. Finite intersection of δ-g - closed sets may fail to be a δ-g - closed set.

Proof:

1. Let A, B $\subseteq$ X be δ-g - closed and let $A \cup B \subseteq U$ with U open. Then $cl_\delta(A \cup B) = cl_\delta(A) \cup cl_\delta(B)$, i.e. $A \cup B$ is δ-g - closed.

2. Let X be the real line with the usual topology. Since X is semi - regular, then by Theorem 2.1.5 every singleton in X is δ-g - closed. Let N be the set of all positive integers. Set $A = \cup_{n\epsilon N} \{1/n\}$. Clearly A is a countable union of δ - generalized closed sets but A is not δ - generalized closed, since $A \in (0, \infty)$ but $0 \in cl_\delta(A)$.

3. Let X = {a, b, c, d, e} and let τ = {∅,{a, b},{c},{a, b, c},X}. Set A = {a, c, d} and B = {b, c, e}. Clearly A and B are δ - generalized closed sets, since X is their only open superset. But C = {c} = A ∩ B is not δ - generalized closed, since $C \subseteq \{c\} \in \tau$ and $cl_\delta(C) = \{c, d, e\} \subseteq \{c\}$.

Theorem 2.1.16.

The intersection of a δ - generalized closed set and a δ - closed set is always δ - generalized closed.

Proof:

Let A be δ - generalized closed and let F be δ - closed. If U is an open set with $A \cap F \subseteq U$, then $A \subseteq U \cup (X\backslash F)$ and so $cl_\delta(A) \subseteq U \cup (X\backslash F)$. Now $cl_\delta(A \cap F) \subseteq cl_\delta(A) \cap F \subseteq U$, and so $A \cap F$ is δ - generalized closed.

Theorem 2.1.17.

For a subset A of a topological space (X, τ) the following conditions are equivalent:

(a) A is clopen.

(b) A is δ - generalized closed, preopen and semi - closed

(c) A is δ - generalized closed and (regular) open.

(d) A is αg - closed and (regular) open.

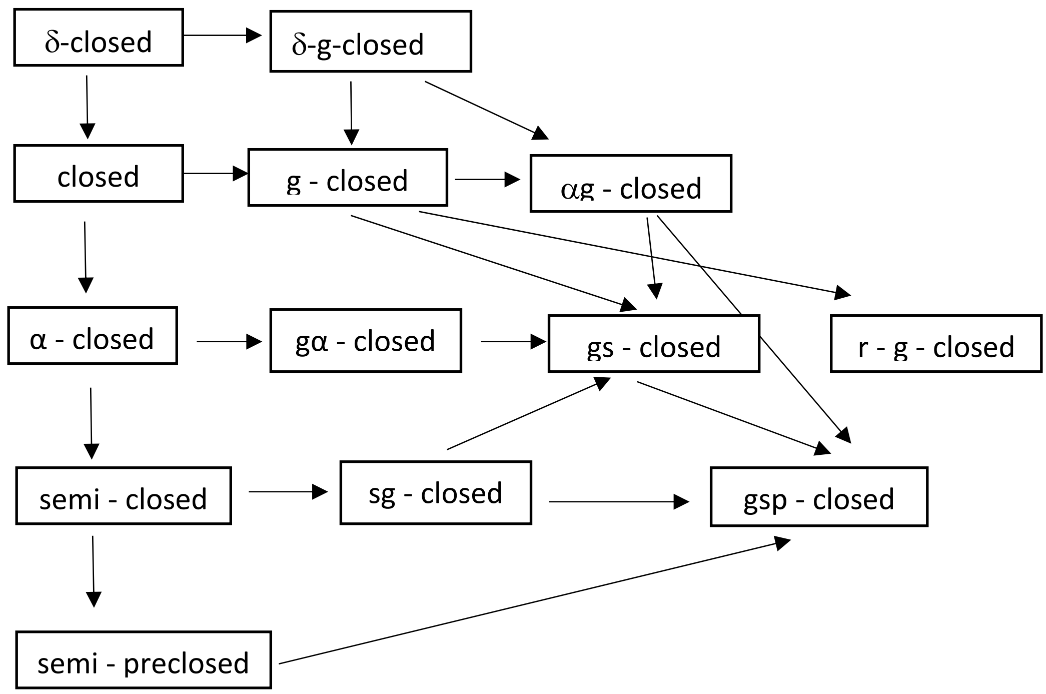
Proof:

(a) ⇒ (b), (b) ⇒ (c) and (c) ⇒ (d) are obvious.

(d) ⇒ (a) Since A is αg-closed, then $cl_\alpha(A) \subseteq$ A and thus A is α- closed. Since α - closed (semi-pre) - open sets are (regular) closed, then (a) is clear.

Remark 2.1.18.

The following diagram shows the relationships between δ - generalized closed sets with other existing closed sets.



2.2 δg* - closed sets

The concept of generalized closed sets plays a significant role in topology. There are many research papers which deal with different types of generalized closed sets. Levine [38] introduced generalized closed (briefly g - closed) sets and studied their basic properties. Bhattacharya and Lahiri [5], Arya and Nour [3], Maki et al [43,44], Dontchev and Ganster [11], Maragathavalli et al [52] and VeeraKumar [87] introduced semi-generalized closed sets, generalized semi - closed sets, generalized α - closed sets, α - generalized closed sets, δ - generalized closed sets, sαg*- closed sets and $g^{\#}s$ - closed sets respectively. VeeraKumar [89] introduced g* - closed sets in topological spaces.

In this chapter we study about the concept of a new class of closed sets called δg* - closed sets and also the basic properties of δg* - closed sets in topological spaces. Applying this set, we obtain the new spaces which are called $T_{\delta g*}$ - spaces and $^{\#}T_{\delta g*}$ - spaces.

`

Definition 2.2.1.

A subset A of a space X is called **δg *- closed** if $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is a g - open set in X.

Proposition 2.2.2

Every δ - closed set is δg* - closed

Proof:

Let A be an δ - closed set and U be any g - open set containing A. Since A is δ - closed, $cl_\delta(A) = A$. Therefore $cl_\delta(A) = A \subseteq U$ and hence A is δg*- closed.

Remark 2.2.3

The converse of the above theorem is not true as shown in the following example.

Example 2.2.4.

Let X = {a, b, c}, τ = {X, ∅, {a}, {b}, {a, b}, {a, c}}. Here, g - open sets with respect to τ are open sets. Then the set {b, c} is δg* - closed but not δ - closed, since the only non-trivial δ - closed sets are {a, c} and {b}.

Proposition 2.2.5.

Every δg *- closed set is g - closed.

Proof:

Let A be a δg* - closed set and U be an any open set containing A in X. Since every open set is g - open and A is δg* - closed, $cl_\delta(A) \subseteq U$. Since $cl(A) \subseteq cl_\delta(A) \subseteq U$, we have $cl(A) \subseteq U$ and hence A is g - closed.

Remark 2.2.6.

A g - closed set need not be δg*- closed as shown in the following example.

Example 2.2.7.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. Then the set {b} is g - closed but not δg*- closed.

Proposition 2.2.8.

Every δg*- closed set is g*- closed.

Proof:

Let A be a δg* - closed set and U be an any g - open set containing A in X. Since A is δg * - closed, $cl_\delta(A) \subseteq U$. But $cl(A) \subseteq cl_\delta(A) \subseteq U$, we have $cl(A) \subseteq U$ and hence A is g* - closed.

Remark 2.2.9.

A g* - closed set need not be δg* - closed as shown in the following example.

Example 2.2.10.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. Then the set {b} is g* - closed but not δg* - closed.

Proposition 2.2.11.

Every δg* - closed set is gs - closed.

Proof:

Let A be δg*- closed and U be any open set containing A in X. Since every open set is g - open, $cl_\delta(A) \subseteq U$ for every subset A of X. Since $scl(A) \subseteq cl_\delta(A) = U$, we have scl(A) ⊆ U and hence A is gs - closed.

Remark 2.2.12.

A gs - closed set need not be δg*- closed as shown in the following example.

Example 2.2.13.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}}. Then the set {b} is gs - closed but not δg* - closed.

Proposition 2.2.14.

Every δg* - closed set is αg - closed.

Proof:

It is true from the fact that αcl(A) ⊆ $cl_δ$(A) for every subset A of X.

Remark 2.2.15.

A αg - closed set need not be δg* - closed as shown in the following example.

Example 2.2.16.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. Then the set {c} is αg - closed but not δg* - closed.

Proposition 2.2.17.

Every δg* - closed set is sαg* - closed.

Proof:

Let A be δg* - closed and U be any g* - open set containing A in X. Since every g* - open set is g-open and A is δg*- closed, $cl_δ$(A) ⊆ U, for every subset A of X. Since αcl(A) ⊆ $cl_δ$(A) ⊆ U, we have αcl(A) ⊆ U and hence A is sαg*- closed.

Remark 2.2.18.

A sαg* - closed set need not be δg* - closed as shown in the following example.

Example 2.2.19.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. Then the set {c} is sαg* - closed but not δg*-closed.

Proposition 2.2.20.

Every δg* - closed set is δ$\hat{g}$ - closed.

Proof:

Let A be δg* - closed and U be any $\hat{g}$ - open set containing A in X. Since every $\hat{g}$ - open set is g - open and A is δg* - closed, $cl_\delta(A) \subseteq U$. Hence A is δ$\hat{g}$ - closed.

Remark 2.2.21.

A δ$\hat{g}$ - closed set need not be δg* - closed as shown in the following example.

Example 2.2.22.

Let X = {a, b, c}, τ = {X, ∅, {a}}. Then the set {a, b} is δ$\hat{g}$ - closed but not δg* - closed.

Proposition 2.2.23.

Every δg* - closed set is α$\hat{g}$ - closed.

Proof:

Let A be δg* - closed and U be any $\hat{g}$ - open set containing A in X. Since every $\hat{g}$ - open set is g - open and A is δg*- closed, $cl_\delta(A) \subseteq U$. Since $\alpha cl(A) \subseteq cl_\delta(A) \subseteq A$, we have $\alpha cl(A) \subseteq U$ and hence A is α$\hat{g}$ - closed.

Remark 2.2.24.

α$\hat{g}$ - closed set need not be δg*- closed as shown in the following example.

Example 2.2.25.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}}. Then the set {b} is α$\hat{g}$ - closed but not δg* - closed.

Proposition 2.2.26.

Every δg* - closed set is wδg * - closed.

Proof:

Let A be δg*- closed and U be any g*- open set containing A in X. Since every g*- open set is g - open and A is δg*- closed, $cl_\delta(A) \subseteq U$, for every subset A of X. Hence A is wδg*- closed.

Remark 2.2.27.

A wδg * - closed set need not be δg*- closed as shown in the following example.

Example 2.2.28.

Let X = {a, b, c}, τ = {X, ∅, {a}, {b, c}}. Then the set {a, c} is wδg*- closed but not δg* - closed.

Proposition 2.2.29.

Every δg* - closed set is δg - closed.

Proof:

Let A be δg* - closed and U be any open set containing A in X. Since every open set is g - open and A is δg*- closed, $cl_\delta(A) \subseteq U$, for every subset A of X. Hence A is δg - closed.

Remark 2.2.30.

A δg - closed set need not be δg*- closed as shown in the following example.

Example 2.2.31.

Let X = {a, b, c}, τ = {X, ∅, {a}}. Then the set {c} is δg - closed but not δg* - closed.

Proposition 2.2.32.

Every δg*- closed set is $^{\#}$gs - closed.

Proof:

Let A be δg* - closed and U be any *g - open set containing A in X. Since every *g - open set is g - open and A is δg* - closed, $cl_\delta(A) \subseteq U$, for every subset A of X. But $scl(A) \subseteq cl_\delta(A) \subseteq A$, we have $scl(A) \subseteq U$ Hence A is $^{\#}$gs - closed.

Remark 2.2.33.

A $^{\#}$gs - closed set need not be δg* - closed as shown in the following example.

Example 2.2.34.

Let X = {a, b, c}, τ = {X, ∅, {a}, {b, c}}. Then the set {a, b} is $^{\#}$gs - closed but not δg* - closed.

Proposition 2.2.35.

Every δg*- closed set is *g - closed.

Proof:

Let A be δg* - closed and U be any $\hat{g}$ - open set containing A in X. Since every $\hat{g}$ - open set is g - open and A is δg*- closed, $cl_\delta(A) \subseteq$ U for every subset A of X. since cl(A) ⊆ cl (A) ⊆ U, we have cl(A) ⊆ U and hence A is * g - closed.

Remark 2.2.36.

A *g - closed set need not be δg * - closed as shown in the following example.

Example 2.2.37.

Let X = {a, b, c}, τ = {X, ∅, {a}}. Then the set {a, c} is *g - closed but not δg* - closed.

Proposition 2.2.38.

Every δg*- closed set is g*p - closed.

Proof:

Let A be δg* - closed and U be any g - open set containing A in X. Since A is δg* - closed, $cl_\delta(A) \subseteq$ U for every subset A of X. Since pcl(A) ⊆ $cl_\delta(A) \subseteq$ U, we have pcl(A) ⊆ U and hence A is g*p - closed.

Remark 2.2.39.

A g*p - closed set need not be δg* - closed as shown in the following example.

Example 2.2.40.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}}. Then the set {b} is g*p - closed but not δg* - closed.

Proposition 2.2.41.

Every g* - closed set is gp - closed.

Proof:

It follows from the fact that every open set is g - open.

Remark 2.2.42.

A gp - closed set need not be g * - closed as shown in the following example.

Example 2.2.43.

Let X = {a, b, c}, τ = {X, ∅, {a, b}}. Then the set {a} is gp - closed but not δg* - closed.

Remark 2.2.44.

The following examples show that δg* - closedness is independent from $\tilde{g}$ - closedness, $\tilde{g}_\alpha$ - closedness and $\tilde{g}_s$ - closedness.

Example 2.2.45.

Let X = {a, b, c}, τ = {X, ∅, {a, b}}. In this topology the set {b, c} is δg* - closed but not $\tilde{g}$ - closed, $\tilde{g}_\alpha$ - closed and $\tilde{g}_s$ - closed.

Example 2.2.46.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. In this topology the set {b} is $\tilde{g}$ - closed, $\tilde{g}_\alpha$ - closed and $\tilde{g}_s$ - closed but not δg*-closed.

Remark 2.2.47.

The following examples show that δg* - closeness is independent from gα - closedness, $g^{\#}s$ - closedness, g*s - closedness and α - closedness.

Example 2.2.48.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}}. In this topology the set {b} is g - closed, $g^{\#}s$ - closed, g*s - closed and α - closed but not δg* - closed. The set {a, c} is δg*- closed but not g - closed, $g^{\#}s$ - closed, g*s - closed and α - closed.

Remark 2.2.49.

The following examples show that δg*- closedness is independent from αg* - closedness.

Example 2.2.50.

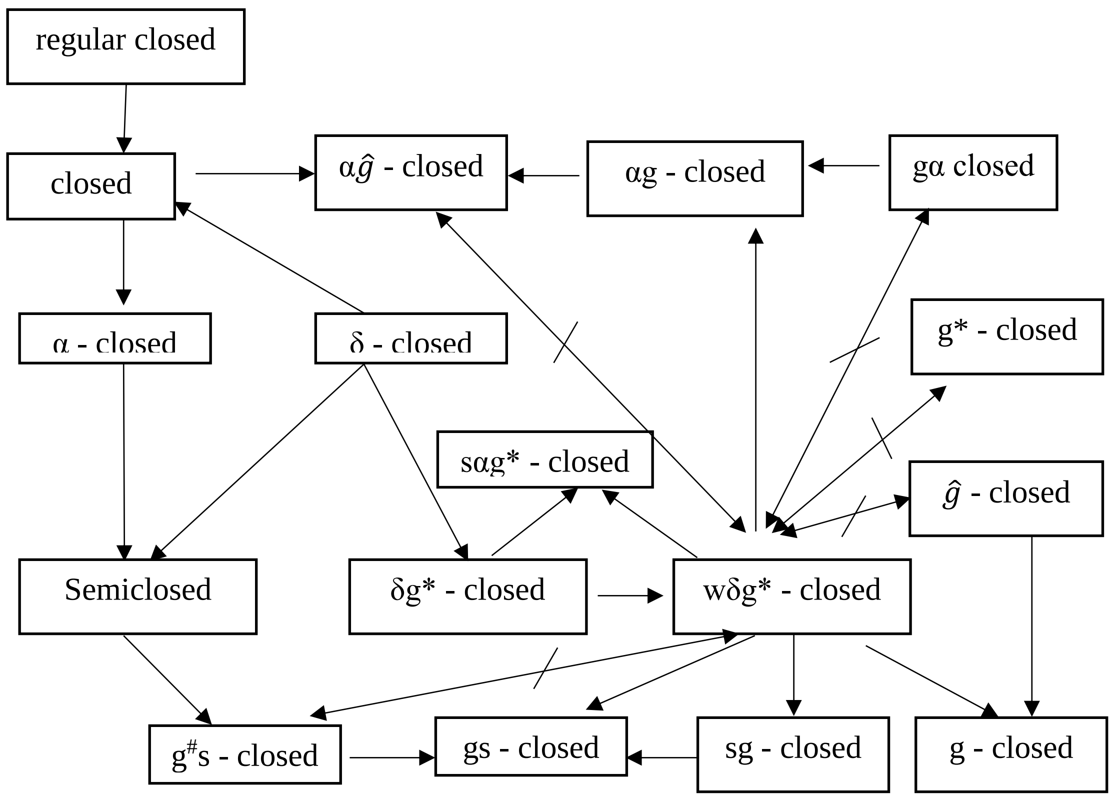
Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}, {a, c}}. In this topology the set {b} is αg* - closed but not δg* - closed.

Example 2.2.51.

Let X = {a, b, c}, τ = {X, ∅, {a}, {a, b}}. In this topology the set {a, c} is αg* - closed but not δg* - closed.

Remark 2.2.52.

The following diagram has shown the relationship of δg * -closed sets with other known existing sets. A⟶B represents A implies B but not conversely and A ↚↛ B, represents A and B are independent to each other.



2.3 Properties of δg* - closed sets

Theorem 2.3.1.

The finite union of δg* - closed sets is δg* - closed.

Proof:

Let {Ai / i= 1, 2, 3,.., n } be a finite class of δg* - closed subsets of X. Let A = $\bigcup_{i=1}^{n} A_i$. Let V be a g - open set containing A which implies $\bigcup_{i=1}^{n} A_i \subseteq V$. This implies $A_i \subseteq V$, for every i. By assumption $cl_\delta(A_i) \subseteq V$, for every i. Which implies $\bigcup_{i=1}^{n} cl_\delta(A_i) \subseteq V$. Then $cl_\delta(\bigcup_{i=1}^{n} A_i) \subseteq V$. Thus $cl_\delta(A) \subseteq V$. Hence finite union of δg*- closed sets is δg* - closed.

Remark 2.3.2.

The following example shows that intersection of any two δg* - closed sets in X need not be δg* - closed.

Example 2.3.3.

Let X = {a, b, c}, τ = {X, ∅, {a}}. Then the set {a, b} and {a, c} are δg* - closed but their intersection {a} is not δg* - closed.

Theorem 2.3.4.

Let A be a δg* - closed set of X. Then $cl_\delta(A)$ - A does not contain a non empty g - closed set.

Proof:

Suppose that A is δg* - closed, let F be a g - closed set contained in $cl_\delta(A)$ - A. Now F^c is a g - open set in X such that $A \subseteq F^c$. Since A is a δg*- closed set of X, then $cl_\delta(A) \subseteq F^c$. Thus $F \subseteq (cl(A))^c$. Also $F \subseteq cl_\delta(A) - A$. Therefore $F \subseteq (cl\,(A))^c \cap cl_\delta(A) = \emptyset$. Hence F = ∅ .

Proposition 2.3.5.

If A is a g - open set and δg* - closed subset of X then A is an δ - closed subset of X.

Proof:

Since A is g-open and δg* - closed, $cl_\delta(A) \subseteq A$. Hence A is δ - closed.

Theorem 2.3.6.

The intersection of a δg* - closed set and a δ - closed set is always δg*- closed.

Proof:

Let A be δg* - closed and F be δ - closed. Let V = A ∩ F. Let U be g - open such that V ⊆ U implies A ∩ F ⊆ U which implies A ⊆ U ∩ F^c. Here F^c is δ - open, so F^c is open. Thus F^c is g - open. Hence U ∪ F^c is g - open and by assumption A ⊆ U ∩ F^c ⇒ $cl_\delta(A)$ ⊆ U ∩ F^c, $cl_\delta(V) = cl_\delta(A \cap F) \subseteq cl_\delta(A) \cap cl_\delta(F) = cl_\delta(A) \cap F$ which is contained in U. Therefore $cl_\delta(V)$ ⊆ V. Hence A ∩ F is δg*- closed.

Theorem 2.3.7.

In a $T_{3/4}$ - space every δg* - closed set is δ - closed.

Proof:

Let X be a $T_{3/4}$ - space. Let A be a δg*- closed set of x. We know that every δg* - closed set is g - closed. Since X is $T_{3/4}$ - space, A is δ - closed.

Proposition 2.3.8.

If A is an δg* - closed set in a space X and A ⊆ B ⊆ $cl_\delta(A)$, then B is also a δg* - closed.

Proof:

Let U be a g - open set of X such that B ⊆ U. Then A ⊆ U. Since A is δg* - closed set, $cl_\delta(A)$ ⊆ U. Also since B ⊆ $cl_\delta(A)$, $cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$. Hence $cl_\delta(B)$ ⊆ U. Therefore B is also a δg* - closed set.

Theorem 2.3.9.

Let A be a δg* - closed set of X. Then A is δ - closed iff $cl_\delta(A)$ - A is g - closed.

Proof: (Necessity)

Let A be a δ - closed subset of X. Then $cl_\delta(A)$ = A and so $cl_\delta(A)$ - A = ∅, which is g - closed.

Sufficiency:

Let $cl_\delta(A)$ - A be g - closed. Since A is δg*- closed, by Proposition 2.3.4., $cl_\delta(A)$ - A does not contain a non-empty g - closed set which implies $cl_\delta(A)$ - A = ∅. That is $cl_\delta(A)$ = A. Hence A is δ - closed.

Theorem 2.3.10.

For each $a \in X$ either $\{a\}$ is g - closed or $\{a\}^c$ is δg^*- closed in X.

Proof:

Suppose that $\{a\}$ is not g - closed in X, then $\{a\}^c$ is not g - open and the only g - open set containing $\{a\}^c$ is the space X itself. That is $\{a\}^c \subseteq X$. Therefore $cl_\delta(\{a\}^c) \subseteq X$ and so $\{a\}^c$ is δg^* - closed.

Definition 2.3.11.

The intersection of all g - open subsets of X containing A is called the g - kernel of A and is denoted by g - ker(A).

Lemma 2.3.12.

A subset A of X is g * - closed if and only if $cl_\delta(A) \subseteq$ g - ker(A).

Proof:

Suppose that A is δg^*- closed in X, then $cl_\delta(A) \subseteq U$, whenever $A \subseteq U$ and U is g - open on X. Let $x \in cl_\delta(A)$. If $x \notin$ g - ker(A), then there is a g - open set U such that $x \notin U$. Since U is a g - open set containing A, $x \notin cl_\delta(A)$ a contradiction.

Conversely let $cl_\delta(A) \subseteq$ g - ker(A). If U is any g - open set containing A, then $cl_\delta(A) \subseteq$ g - ker(A) $\subseteq U$. Then A is δg^* - closed.

CHAPTER-III

CHAPTER 3

$\tilde{\Delta}$ - Closed sets in topological spaces

The concept of generalised closed (briefly, g - closed) sets were introduced by Norman Levine [38] in 1970. Velicko [90] introduced δ - open sets which are stronger than open sets in 1968. By combining the concepts of δ - closedness and g - closedness, Julian Dontchev [11] proposed a class of generalised closed sets called δg - closed set in 1996. Sudha. R [62] introduced and investigated a new concept of generalised closed sets namely δg* - closed sets in 2012. K. Meena [72] introduced a new class of generalised closed sets called Δ* - closed sets.

In this chapter we introduce a new class of sets called $\tilde{\Delta}$ - closed sets which contains the class of Δ* - closed sets and is contained in the class of gδ - closed sets in topological spaces.

3.1 $\tilde{\Delta}$ - closed sets

Definition 3.1.1.

A subset A of a space (X, τ) is called $\tilde{\Delta}$ - **closed** if $cl_{\delta}(A) \subseteq U$, whenever $A \subseteq U$ and U is a δg* - open set in (X, τ).

Proposition 3.1.2.

Every δ - closed set is $\tilde{\Delta}$ - closed but not conversely.

Proof:

Let A be a δ - closed set and U be any δg* - open set containing A. Since A is δ - closed, $cl_{\delta}(A) = A$. Therefore $cl_{\delta}(A) = A \subseteq U$ and hence A is $\tilde{\Delta}$ - closed.

Example 3.1.3.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {b} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Proposition 3.1.4.

Every δg* - closed set is $\tilde{\Delta}$ - closed but not conversely.

Proof:

Let A be a δg^* - closed set and U be any δg^* - open set containing A. By Proposition 2.2.5, every δg^* - open is g - open and by hypothesis, A is δg^* - closed. Therefore $cl_\delta(A) \subseteq U$ and hence A is $\tilde{\Delta}$ - closed.

Example 3.1.5.

Let X = {a, b, c}, $\tau = \{\emptyset, X, \{a\}\}$.Then the subset {a, c} is $\tilde{\Delta}$ - closed but not δg^* - closed in (X, τ).

Proposition 3.1.6.

Every $\tilde{\Delta}$ - closed set is πg - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any π - open set containing A. By Remark 1.46, every π - open is δg^* - open and by hypothesis A is $\tilde{\Delta}$ - closed. Therefore $cl_\delta(A) \subseteq U$. For every subset A of X, we have $cl(A) \subseteq cl_\delta(A)$ and so $cl(A) \subseteq U$ and hence A is πg - closed.

Example 3.1.7.

Let X = {a, b, c}, $\tau = \{\emptyset, X, \{a\}\}$. Then the subset {a} is πg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.8.

Every $\tilde{\Delta}$ - closed set is rg - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any regular open containing A. By Remark 1.46, every regular open is δg^* - open and by hypothesis A is $\tilde{\Delta}$ - closed. Therefore $cl_\delta(A) \subseteq U$. For every subset A of X, we have $cl(A) \subseteq cl_\delta(A)$ and so $cl(A) \subseteq U$ and hence A is rg - closed.

Example 3.1.9.

Let X = {a, b, c}, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then the subset {a, b} is rg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.10.

Every $\tilde{\Delta}$ - closed set is gpr - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any regular open containing A. By Remark 1.46, every regular open is δg*- open and by hypothesis A is $\tilde{\Delta}$ - closed, which gives $cl_\delta(A) \subseteq U$. For every subset A of X, we have $pcl(A) \subseteq cl_\delta(A)$ which implies $pcl(A) \subseteq U$ and hence A is gpr - closed.

Example 3.1.11.

Let X = {a, b, c}, τ = {∅, X, {a, b}}. Then the subset {a} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.12.

Every $\tilde{\Delta}$ - closed set is πgp - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any π - open containing A. By Remark 1.46, every π - open is δg*- open and by hypothesis A is $\tilde{\Delta}$ - closed, which gives $cl_\delta(A) \subseteq U$. For every subset A of X, we have $pcl(A) \subseteq cl_\delta(A)$ which implies $pcl(A) \subseteq U$ and hence A is πgp - closed.

Example 3.1.13.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {a} is πgp - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.14.

Every $\tilde{\Delta}$ - closed set is gδ - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any δ - open containing A. By Remark 1.46, every δ - open is δg* - open and by hypothesis A is $\tilde{\Delta}$ - closed, which gives $cl_\delta(A) \subseteq U$. For every subset A of X, we have $cl(A) \subseteq cl_\delta(A)$ which implies $cl(A) \subseteq U$ and hence A is gδ - closed.

Example 3.1.15.

Let X = {a, b, c}, τ = {∅, X, {a, b}}. Then the subset {b} is gδ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.16.

Every Δ* - closed set is $\tilde{\Delta}$ - closed but not conversely.

Proof:

Let A be a Δ* - closed set and U be any δg* - open set containing A. By Proposition 2.2.29, every δg* - open is δg - open and by hypothesis A is Δ* - closed, which implies $cl_\delta(A) \subseteq U$. Therefore A is $\tilde{\Delta}$ - closed.

Example 3.1.17.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {a, b} is $\tilde{\Delta}$ - closed but not Δ* - closed in (X, τ).

Proposition 3.1.18.

Every $\tilde{\Delta}$ - closed set is rwg - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any regular open containing A. By Remark 1.46, every regular open is δg* - open and by hypothesis A is $\tilde{\Delta}$ - closed, which implies $cl_\delta(A) \subseteq U$. We know that $int(A) \subseteq A$ which implies $cl(int(A)) \subseteq cl(A)$. For every subset A of X, we have $cl(A) \subseteq cl_\delta(A)$ and hence $cl(int(A)) \subseteq U$. Therefore A is rwg - closed.

Example 3.1.19.

Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a, b}}. Then the subset {a, b} is rwg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.20.

Every $\tilde{\Delta}$ - closed set is gspr - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any regular open containing A. By Remark 1.46, every regular open is δg* - open and by hypothesis A is $\tilde{\Delta}$ - closed, which implies

$cl_\delta(A) \subseteq U$. For every subset A of X , we have $spcl(A) \subseteq pcl(A) \subseteq cl_\delta(A)$ which implies $spcl(A) \subseteq U$ and hence A is gspr - closed.

Example 3.1.21.

Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a, b}}. Then the subset {a, b} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.22.

Every $\tilde{\Delta}$ - closed set is πgsp - closed but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed set and U be any π - open containing A. By Remark 1.46, every π - open is δg* - open and by hypothesis A is $\tilde{\Delta}$ - closed, which implies $cl_\delta(A) \subseteq U$. For every subset A of X , we have $spcl(A) \subseteq pcl(A) \subseteq cl_\delta(A)$ which gives $spcl(A) \subseteq U$ and hence A is πgsp - closed.

Example 3.1.23.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {a} is πgsp - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 3.1.24.

Every $\tilde{\Delta}$ - closed set is $\delta g^{\#}$ - closed but not conversely.

Proof:

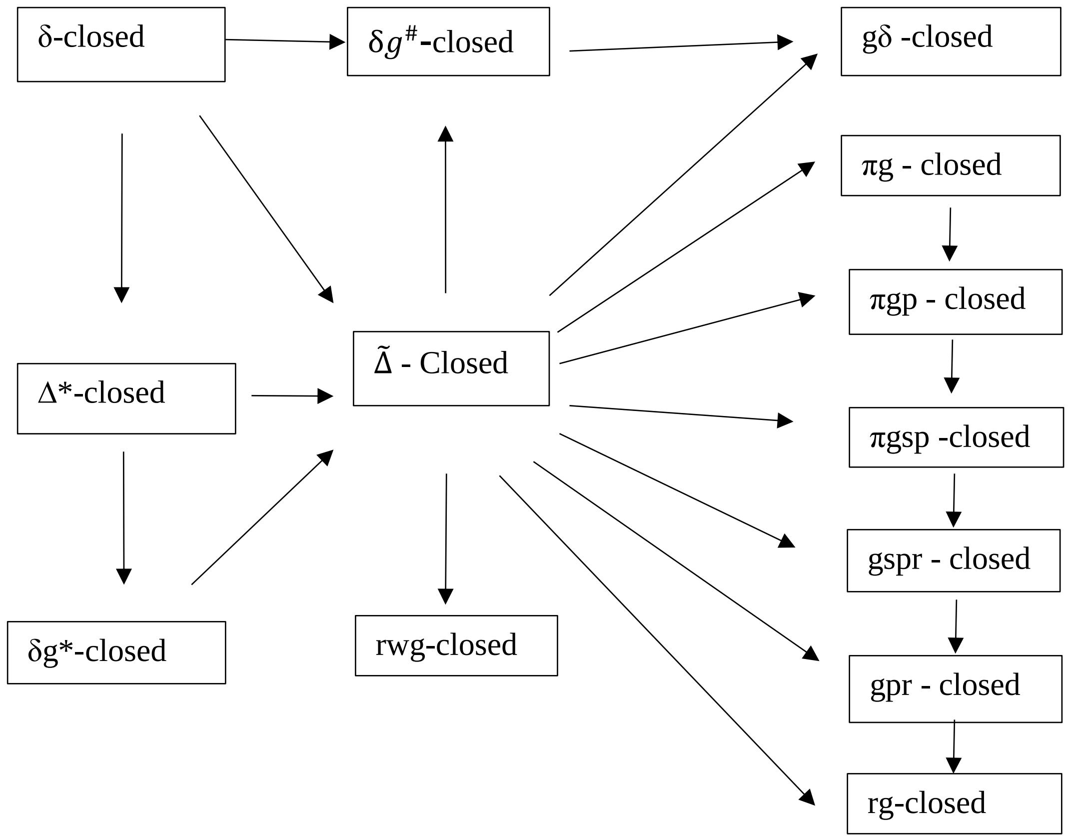
Let A be a $\tilde{\Delta}$ - closed set and U be any δ - open containing A. By Remark 1.46, every δ - open is δg* - open and by hypothesis, A is $\tilde{\Delta}$ - closed, which gives $cl_\delta(A) \subseteq U$. Hence A is $\delta g^{\#}$ - closed.

Example 3.1.25.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {a} is $\delta g^{\#}$- closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 3.1.26.

The following figure gives the dependence of $\tilde{\Delta}$ - closed set with twelve other already existing closed sets.



Remark 3.1.27.

The following counter examples show that Δ̃ - closedness is independent of gp - closedness.

Example 3.1.28.

Let X = {a, b, c}, τ = {∅, X, {a, b}}.Then the subset {a} is gp - closed but not Δ̃ - closed in (X, τ).

Example 3.1.29.

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}, {a, c}}. Then the subset {a, b} is Δ̃ - closed but not gp - closed in (X, τ).

Remark 3.1.30.

The following counter examples show that Δ̃ - closedness is independent of g*p - closedness.

Example 3.1.31.

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}, {a, c}}. Then the subset {a, b} is $\tilde{\Delta}$ - closed but not g*p - closed in (X, τ).

Example 3.1.32.

Let X = {a, b, c}, τ = {∅, X, {a, b}}. Then the subset {a} is g*p - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 3.1.33.

The following counter examples show that $\tilde{\Delta}$ - closedness is independent of gs - closedness, #gs - closedness.

Example 3.1.34.

Let X = {a, b, c}, τ = {∅, X, {a},{a, b},{a, c}}. Then the subset {a, b} is $\tilde{\Delta}$ - closed but not gs - closed, #gs - closed in (X, τ).

Example 3.1.35.

Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a, b}}. Then the subset {a} is gs - closed, #gs - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 3.1.36.

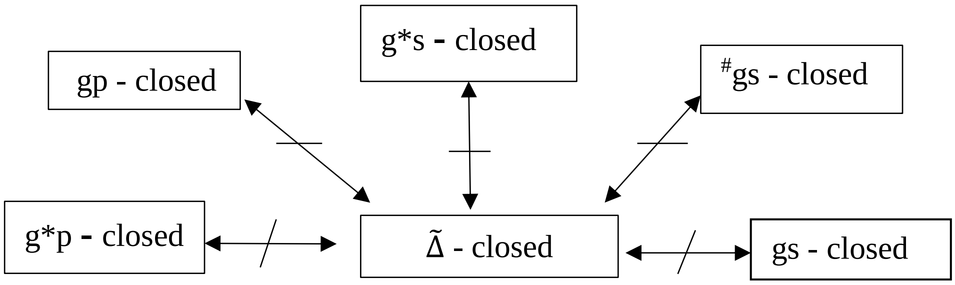
The following counter examples show that $\tilde{\Delta}$ - closedness is independent of g*s - closedness.

Example 3.1.37.

Let X = {a, b, c}, τ = {∅, X, {a},{b},{a, b}}. Then the subset {b} is g*s - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 3.1.38.

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}, {a, c}}.Then the subset {a, c} is $\tilde{\Delta}$ - closed but not g*s - closed in (X, τ).



Theorem 3.1.39.

Union of two $\tilde{\Delta}$ - closed sets is $\tilde{\Delta}$ - closed.

Proof:

Let A and B be the $\tilde{\Delta}$ - closed sets in X. Let U be a δg^*- open set in X, such that A ⊆ U and B ⊆ U. Then A ∪ B ⊆ U. Since A and B are $\tilde{\Delta}$ - closed sets, $cl_\delta(A) \subseteq U$ and $cl_\delta(B) \subseteq U$. This implies $cl_\delta(A \cup B) = cl_\delta(A) \cup cl_\delta(B) \subseteq U$. Therefore A∪B is a $\tilde{\Delta}$ - closed set.

Remark 3.1.40.

The following example shows that intersection of any two $\tilde{\Delta}$ - closed sets in X need not be $\tilde{\Delta}$ - closed set.

Example 3.1.41.

Let X = {a, b, c}, τ = {∅, X, {a}}. Then the subset {a, c} and {a, b} are $\tilde{\Delta}$ - closed but their intersection {a} is not $\tilde{\Delta}$ - closed.

Proposition 3.1.42.

Let A be a $\tilde{\Delta}$ - closed set of (X, τ). Then $cl_\delta(A)$ - A does not contain any non-empty δg^* - closed set.

Proof:

Suppose F be a non-empty δg^* - closed set contained in $cl_\delta(A)$-A. Now $F \subseteq cl_\delta(A)$-A then $F \subseteq cl_\delta(A) \cap A^c$ which implies $F \subseteq cl_\delta(A)$ and $F \subseteq A^c$ which implies $A \subseteq F^c$. Then F^c, δg^*- open and A, $\tilde{\Delta}$ - closed implies $cl_\delta(A) \subseteq F^c$. Thus $F \subseteq (cl_\delta(A))^c$. Also $F \subseteq cl_\delta(A)$ - A. Therefore $((cl_\delta (A))^c \cap cl_\delta(A)) = \emptyset$. Hence F = ∅ which is a contradiction.

Proposition 3.1.43.

If A is a δg^*- open and a $\tilde{\Delta}$ - closed subset of (X, τ). Then A is a δ - closed subset of (X, τ).

Proof:

Let A is δg^* - open and $\tilde{\Delta}$ - closed then $cl_\delta(A) \subseteq A$. We know that, $A \subseteq cl(A) \subseteq cl_\delta(A)$ which implies $A \subseteq cl_\delta(A)$. Therefore $A = cl_\delta(A)$ which implies A is δ - closed.

Theorem 3.1.44.

If A is a $\tilde{\Delta}$ - closed set and $A \subseteq B \subseteq cl_\delta(A)$, then B is $\tilde{\Delta}$ - closed.

Proof:

Let U be an δg^*-open set of X such that $B \subseteq U$ then $A \subseteq B \subseteq U$ which implies $A \subseteq U$. Since A is $\tilde{\Delta}$ - closed, then $cl_\delta(A) \subseteq U$. Given that $B \subseteq cl_\delta(A)$ and $cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A)$. Thus $cl_\delta(B) \subseteq U$. Hence B is $\tilde{\Delta}$ - closed in X.

Proposition 3.1.45.

For each $a \in X$ either {a} is δg^*-closed (or) $\{a\}^c$ is $\tilde{\Delta}$ - closed in X.

Proof:

Suppose {a} is not δg^*-closed in X then $\{a\}^c$ is not δg^*-open. The only δg^*- open set containing $\{a\}^c$ is X. Thus $\{a\}^c \subseteq X$ which implies $cl_\delta \{a\}^c \subseteq cl_\delta(X) = X$. Therefore $cl_\delta\{a\}^c \subseteq X$ and so $\{a\}^c$ is $\tilde{\Delta}$ - closed in X.

Theorem 3.1.46.

Let A be a $\tilde{\Delta}$ - closed set of (X, τ). Then A is δ - closed if and if only $cl_\delta(A)$-A is a δg^*- closed set.

Proof: **(Necessity)**

Let A be a δ - closed subset of X. Then $cl_\delta(A) = A$ which implies $cl_\delta(A) \cap A^c = A \cap A^c = \emptyset$ Thus $cl_\delta(A) \cap A^c = \emptyset$. And so $cl_\delta(A) - A = \emptyset$. Therefore $cl_\delta(A)$ - A is a δg^*- closed set.

Sufficiency:

Let $cl_\delta(A)$ - A is δg^*- closed and A is $\tilde{\Delta}$ -closed then $cl_\delta(A)$ - A does not contain a non - empty δg^*- closed set. Therefore $cl_\delta(A) - A = \emptyset$. That is $cl_\delta(A) = A$. Hence A is a δ - closed subset of X.

Theorem 3.1.47.

The intersection of a $\tilde{\Delta}$ - closed set and a δ - closed set is $\tilde{\Delta}$ - closed.

Proof:

Let A be a $\tilde{\Delta}$ - closed and F be a δ - closed set. Let $A \subseteq U$, U is a δg^*- open set and A is a $\tilde{\Delta}$ - closed set, $cl_\delta(A) \subseteq U$. To prove $A \cap F$ is $\tilde{\Delta}$ - closed, it is enough to show that $cl_\delta(A \cap F) \subseteq U$, whenever $A \cap A^c \subseteq U$, where U is an δg^*- open set.

Now $cl_\delta(A \cap F) \subseteq cl_\delta(A) \cap cl_\delta(F) \subseteq cl_\delta(A) \cap F \subseteq U \cap F \subseteq U$. And so $cl_\delta(A \cap F) \subseteq U$. Therefore $A \cap F \subseteq U$, which implies $A \cap F$ is $\tilde{\Delta}$ - closed.

3.2 $\tilde{\Delta}$ - open sets

Definition 3.2.1.

A subset of A of a topological space (X, τ) is called $\tilde{\Delta}$ **- open** if its complement is called $\tilde{\Delta}$ - closed set in (X, τ).

Proposition 3.2.2.

If a subset A of a topological space (X, τ) is δ - open then it is $\tilde{\Delta}$ - open in X.

Proof:

Let A be a δ - open in a topological space (X, τ). Then A^c is δ - closed. From Proposition 3.1.2 we know that every δ - closed set is $\tilde{\Delta}$ - closed set. Therefore A^c is $\tilde{\Delta}$ - closed set which implies A is $\tilde{\Delta}$ - open in X.

Remark 3.2.3.

The converse of the above theorem need not be true as seen from the following example.

Example 3.2.4.

Let X = {a, b, c}, τ = {∅, X,{a},{b},{a, b},{a, c}}.Then the subset {a} is $\tilde{\Delta}$ - open but not δ - open in (X, τ).

Proposition 3.2.5.

In a topological space (X, τ), we have the following

(a) Every δg* - open set is $\tilde{\Delta}$ - open.

(b) Every $\tilde{\Delta}$ - open set is πg - open.

(c) Every $\tilde{\Delta}$ - open set is rg - open.

(d) Every $\tilde{\Delta}$ - open set is gpr - open.

(e) Every $\tilde{\Delta}$ - open set is πgp - open.

(f) Every $\tilde{\Delta}$ - open set is gδ - open.

(g) Every Δ* - open set is $\tilde{\Delta}$ - open.

(h) Every $\tilde{\Delta}$ - open set is rwg - open.

(i) Every $\tilde{\Delta}$ - open set is gspr - open.

(j) Every $\tilde{\Delta}$ - open set is πgsp - open.

(k) Every $\tilde{\Delta}$ - open set is $\delta g^{\#}$ - open.

Proof:

Similar to Proposition 3.2.2 using corresponding definition.

Remark 3.2.6.

The converse of the above proposition is not true in general. The easy justification through examples is omitted as it follows from the examples in section 3.1.

Lemma 3.2.7.

For a subset A of X, $cl_\delta(X - A) = X - int_\delta(A)$.

Theorem 3.2.8.

A subset A of a topological space (X, τ) is $\tilde{\Delta}$-open if and only if $G \subseteq int_\delta(A)$ whenever $G \subseteq A$ and G is δg^*- closed.

Proof:

Assume that A is $\tilde{\Delta}$ - open. Then A^c is $\tilde{\Delta}$ - closed set. Let G be a δg^*- closed set of (X, τ) contained in A. Then G^c is a δg^*- open and $A \subseteq G^c$. Since is $\tilde{\Delta}$ - closed, $cl_\delta(A^c) \subseteq G^c$ which implies $G \subseteq int_\delta(A)$, by lemma 3.2.7.

Conversely, Assume that $G \subseteq int_\delta(A)$ whenever $G \subseteq A$ and G is a δg^*- closed in (X, τ). Let $A^c \subseteq F$, where F is δg^*- open in (X, τ) which implies $F^c \subseteq A$. Then by hypothesis $F^c \subseteq int_\delta(A)$. Thus $cl_\delta(A^c) \subseteq F$, which implies A^c is $\tilde{\Delta}$ - closed. Therefore A is $\tilde{\Delta}$ - open.

Theorem 3.2.9.

If $int_\delta(A) \subseteq B \subseteq A$ and A is $\tilde{\Delta}$ - open, then B is $\tilde{\Delta}$ - open in (X, τ).

Proof:

Let $int_\delta(A) \subseteq B \subseteq A$ and A is $\tilde{\Delta}$ - open in (X, τ). Then $A^c \subseteq B^c \subseteq (int_\delta(A))^c$. By Lemma 3.2.7, $A^c \subseteq B^c \subseteq cl_\delta(A^c)$. Since A is $\tilde{\Delta}$ - open, then A^c is $\tilde{\Delta}$ - closed. By Theorem 3.1.44, B^c is $\tilde{\Delta}$ - closed and hence B is $\tilde{\Delta}$ - open in (X, τ).

Theorem 3.2.10

If A and B are $\tilde{\Delta}$ - open set in a space (X, τ). Then $A \cap B$ is also a $\tilde{\Delta}$ - open set in X.

Proof:

Let A and B are $\tilde{\Delta}$ - open set in a space (X, τ).Then A^c and B^c are $\tilde{\Delta}$ - closed in (X, τ). By Theorem 3.1.39, $A^c \cup B^c$ again $\tilde{\Delta}$ - closed in (X, τ). Thus $A^c \cup B^c = (A \cap B)^c$ is $\tilde{\Delta}$ - closed. Therefore $A \cap B$ is $\tilde{\Delta}$ - open in X.

Theorem 3.2.11

If a subset A of a topological space X is both $\tilde{\Delta}$ - open and δg^*- closed then it is δ - open.

Proof:

Let A be a δg^* - closed and $\tilde{\Delta}$ - open set in X. Then A^c is δg^*- open and $\tilde{\Delta}$ - closed set in X. By Proposition 3.1.43, A^c be δ - closed, which implies A is δ - open.

Theorem 3.2.12

A set A is $\tilde{\Delta}$ - open in (X, τ) iff G = X whenever G is δg^*- open and $int_\delta(A) \cup A^c \subseteq G$

Proof: **(Necessity)**

Suppose that A is $\tilde{\Delta}$ - open in X. Let G is δg^*- open and $int_\delta(A) \cup A^c \subseteq G$. Thus $G^c \subseteq (int_\delta(A) \cup A^c)^c = (int_\delta(A))^c \cap A = (int_\delta(A))^c - A^c = cl_\delta(A^c) - A^c$. Since A^c is $\tilde{\Delta}$ - closed and G^c is δg^*- closed by Theorem 3.1.46, $G^c = \emptyset$. Hence G = X.

Sufficiency:

Suppose that F is δg^* - closed and F ⊆ A. By theorem 3.2.8, it suffices to show that F ⊆ int_δ(A).Now int_δ(A) ∪ A^c ⊆ int_δ(A) ∪ F^c and hence int_δ(A) ∪ F^c = X, it follows that F ⊆ int_δ(A).

Theorem 3.2.13.

Intersection of two $\tilde{\Delta}$ - open sets is $\tilde{\Delta}$ - open.

Proof:

The proof follows from the theorem 3.1.39.

Now the notation of $\tilde{\Delta}$ - closure of a set is defined and some of its properties are studied.

Definition 3.2.14.

The closure operator of $\tilde{\Delta}$ - closed set is defined as $\tilde{\Delta}$cl(A) = ∩{F ⊆ X:A⊆F and F is $\tilde{\Delta}$ - closed in (X, τ)}.

Theorem 3.2.15.

Let A be a subset of a topological space X. Then x ∈ $\tilde{\Delta}$cl(A) if and only if for any $\tilde{\Delta}$ - open set containing x, A∩U ≠ ∅.

Proof:

Let x ∈ $\tilde{\Delta}$ cl(A) and suppose that, there is a $\tilde{\Delta}$ - open set U containing x ∋ x ∈ U and A∩U = ∅, which implies A ⊆ U^c, $\tilde{\Delta}$ cl(A) ⊆ $\tilde{\Delta}$ cl(U^c) = U^c. Since x ∈ U, x ∉ U^c, which implies x ∉ $\tilde{\Delta}$ cl(A),which is a contradiction. Therefore A∩U ≠ ∅.

Conversely, let V be a $\tilde{\Delta}$ - open set containing x and A∩U ≠ ∅. To prove x ∈ $\tilde{\Delta}$ cl(A). Suppose that x ∉ $\tilde{\Delta}$ cl(A), then there is a $\tilde{\Delta}$ - closed set F in X ∋ x ∉ F and A ⊆ F. If x ∉ F then x ∈ F^c, which is $\tilde{\Delta}$ - open set. Since A ⊆ F which implies A∩F^c = ∅, which is a contradiction. Therefore x ∈ $\tilde{\Delta}$ cl(A).

CHAPTER-IV

CHAPTER 4

$\tilde{\Delta}$ - Separation Axioms and $\tilde{\Delta}$ - Continuity

4.1 $\tilde{\Delta}$ - Separations Axiom

As an application of $\tilde{\Delta}$ - closed sets twelve new spaces namely $\tilde{\Delta}T\delta$ - space, $\tilde{\Delta}T\delta g^*$ - space, $\tilde{\Delta}T\Delta^*$- space, $g\delta T\tilde{\Delta}$ - space, $\delta g^{\#}T\tilde{\Delta}$ - space, $rgT\tilde{\Delta}$ - space, $rwgT\tilde{\Delta}$ - space, $gprT\tilde{\Delta}$ - space, $gsprT\tilde{\Delta}$ - space, $\pi gpT\tilde{\Delta}$ - space, $\pi gspT\tilde{\Delta}$ - space and $\pi gT\tilde{\Delta}$ - space are introduced and some properties are studied.

Definition 4.1.1

A space (X, τ) is said to be a

1. $\tilde{\Delta}T\delta$ - space if every $\tilde{\Delta}$ - closed subset of (X, τ) is δ - closed in (X, τ).
2. $\tilde{\Delta}T\delta g^*$ - space if every $\tilde{\Delta}$ - closed subset of (X, τ) is δg^*- closed in (X, τ).
3. $\tilde{\Delta}T\Delta^*$- space if every $\tilde{\Delta}$ - closed subset of (X, τ) is Δ^*- closed in (X, τ).
4. $g\delta T\tilde{\Delta}$ - space if every $g\delta$ - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
5. $\delta g^{\#}T\tilde{\Delta}$ - space if every $\delta g^{\#}$ - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
6. $rgT\tilde{\Delta}$ - space if every rg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
7. $rwgT\tilde{\Delta}$ - space if every rwg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
8. $gprT\tilde{\Delta}$ - space if every gpr - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
9. $gsprT\tilde{\Delta}$ - space if every gspr - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
10. $\pi gpT\tilde{\Delta}$ - space if every πgp - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
11. $\pi gspT\tilde{\Delta}$ - space if every πgsp - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).
12. $\pi gT\tilde{\Delta}$ - space if every πg - closed subset of (X, τ) is $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.2.

For a space (X, τ) the following conditions are equivalent

i. (X, τ) is a $\tilde{\Delta}T\delta$ - space

ii. For a each x ∈ X, {x} is either δ - open or δg* - closed.

Proof:

i ⟹ ii,

Let x ∈ X and suppose {x} is not δg* - closed and X-{x} is not δg* - open, the only δg*- open set containing X-{x} is X. So X-{x} is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is $\tilde{\Delta}T\delta$ - space, X-{x} is δ - closed set in (X, τ). Hence {x} is δ - open in (X, τ).

ii ⟹ i,

Let A be a $\tilde{\Delta}$ - closed in (X, τ) and x ∈ cl_δ(A). We show that x ∈ A for the following two cases

Case (i):

Assume that {x} is δ - open. Then X-{x} is δ - closed set. If x ∉ A then A ⊆ X-{x}. Since x ∈ cl_δ(A), we have x ∈ X-{x}, which is a contradiction. Hence x ∈ A.

Case (ii):

Assume that {x} is δg* - closed. Assume that x ∉ A, then we would have x ∈ cl_δ(A)-A, which contradicts that the Proposition 3.1.41. Therefore x ∈ A.

Proposition 4.1.3.

If (X, τ) is a $\tilde{\Delta}T\delta$ - space then $\tilde{\Delta}$cl(B) = cl_δ(B), for each subset B of X.

Proof:

Let (X, τ) be a $\tilde{\Delta}T\delta$ - space. We know that every δ - closed is $\tilde{\Delta}$ - closed set. By definition 1 of $\tilde{\Delta}T\delta$-space, every $\tilde{\Delta}$ - closed subset of (X, τ) is δ - closed in (X, τ). Therefore $\tilde{\Delta}C(X,\tau)$ = cl_δ(X, τ). Hence by definition of δ - closure and $\tilde{\Delta}$ - closure, $\tilde{\Delta}$cl(B) = cl_δ(B) for each subset B of X.

Corollary 4.1.4.

If (X, τ) is a $\tilde{\Delta}T\delta$ - space then for every subset A of X, $\tilde{\Delta}cl(A)$ is δ - closed in (X, τ).

Proof:

Since (X, τ) is a $\tilde{\Delta}T\delta$ - space. By Proposition 4.1.3, $\tilde{\Delta}cl(A) = cl_{\delta}(A)$. Therefore $\tilde{\Delta}cl(A)$ is a δ - closed set in (X, τ).

Proposition 4.1.5.

Every $\tilde{\Delta}T\delta$ - space is a $\tilde{\Delta}T\delta g^*$- space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed and (X, τ) be a $\tilde{\Delta}T\delta$ - space. Then A is δ - closed in (X, τ). Since every δ - closed is δg*- closed [Proposition 2.2.2], A is δg*- closed in (X, τ). Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

Example 4.1.6.

Let X = {a, b, c}, τ = {∅, X, {a, b}} then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not a $\tilde{\Delta}T\delta$ - space, since the subset {c} is $\tilde{\Delta}$-closed but not δ-closed in (X, τ).

Proposition 4.1.7.

Every $\tilde{\Delta}T\delta$ - space is a $\tilde{\Delta}T\Delta^*$- space but not conversely.

Proof:

Let A be $\tilde{\Delta}$-closed and (X, τ) be a $\tilde{\Delta}T\delta$ - space. Then A is δ - closed in (X, τ). Since every δ - closed set is Δ*- closed [Proposition 3.2 of [72]], A is Δ*- closed in (X, τ). Hence (X, τ) is a $\tilde{\Delta}T\Delta^*$- space.

Example 4.1.8.

Let X and τ be defined in Example 4.1.6 then (X, τ) is a $\tilde{\Delta}T\Delta^*$- space but not a $\tilde{\Delta}T\delta$ - space, since the subset {c} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Proposition 4.1.9.

Every $\tilde{\Delta}T_\delta$ - space is a δg^*T_δ - space but not conversely.

Proof:

Let A be δg*- closed and (X, τ) be a $\tilde{\Delta}T_\delta$ - space. Since every δg*- closed is $\tilde{\Delta}$ - closed [Proposition 3.1.4], A is $\tilde{\Delta}$ - closed in (X, τ). Therefore A is δ - closed, as (X, τ) is $\tilde{\Delta}T_\delta$ - space. Hence (X, τ) is a δg^*T_δ - space.

Example 4.1.10.

Let X = {a, b, c}, τ = {∅, X, {a}, {b, c}} then (X, τ) is a δg^*T_δ - space but not a $\tilde{\Delta}T_\delta$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Remark 4.1.11.

T_b - space and $\tilde{\Delta}T_\delta$ - space are independent of each other as seen from the following examples.

Example 4.1.12.

Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a, b}} then (X, τ) is a $\tilde{\Delta}T_\delta$ - space but not a T_b - space, since the subset {a} is gs - closed but not closed in (X, τ).

Example 4.1.13.

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}, {a, c}} then (X, τ) is a T_b - space but not a $\tilde{\Delta}T_\delta$ - space. Since the subset {b} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Remark 4.1.14.

T_c - space and T_d - spaces are independent of $\tilde{\Delta}T_\delta$ - space as seen from the following examples.

Example 4.1.15.

Let X and τ be defined as in Example 4.1.6 then (X, τ) is a T_c - space as well as T_d - space but not a $\tilde{\Delta}T_\delta$ - space. Since the subset {c} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Example 4.1.16.

Let X and τ be defined as in example 4.1.12 then (X, τ) is a $\tilde{\Delta}T_\delta$ - space but neither T_c - space nor T_d - space, since the subset {a} is gs - closed but neither g* - closed nor g - closed in (X, τ).

Remark 4.1.17.

${}_{gspr}T_{\tilde{\Delta}}$ - space is independent of $\tilde{\Delta}T_\delta$ - space as seen from the following examples.

Example 4.1.18.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T_\delta$ - space but not a ${}_{gspr}T_{\tilde{\Delta}}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.19.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a ${}_{gspr}T_{\tilde{\Delta}}$ - space but not a $\tilde{\Delta}T_\delta$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Remark 4.1.20.

${}_{gpr}T_{\tilde{\Delta}}$ - space is independent of $\tilde{\Delta}T_\delta$ - space as seen from the following examples.

Example 4.1.21.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T_\delta$ - space but not a ${}_{gpr}T\tilde{\Delta}$ - space, since the subset {a, b} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.22.

Let X = {a, b, c} and τ = {∅, X, {a}} then (X, τ) is a ${}_{gpr}T\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\delta$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Remark 4.1.23.

${}_{rg}T\tilde{\Delta}$ - space and ${}_{rwg}T\tilde{\Delta}$ - space are independent of $\tilde{\Delta}T\delta$ - space as seen from the following examples.

Example 4.1.24.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\delta$ - space but neither a ${}_{rg}T\tilde{\Delta}$ - space nor a ${}_{rwg}T\tilde{\Delta}$ - space, since the subset {a, b} is rg - closed as well as rwg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.25.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a ${}_{rg}T\tilde{\Delta}$ - space as well as ${}_{rwg}T\tilde{\Delta}$ - space but not $\tilde{\Delta}T\delta$ - space, as the subset {c} is $\tilde{\Delta}$ - closed but not δ - closed in (X, τ).

Theorem 4.1.26.

If (X, τ) is a $\delta gT\delta g^*$ - space as well as a $\tilde{\Delta}T\delta$ - space then it is a $T_{3/4}$ - space.

Proof:

Let A be a δg - closed set in (X, τ). Since (X, τ) is a $\delta gT\delta g^*$ - space, A is δg* - closed in (X, τ). As every δg* - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.4] and (X, τ) is a $\tilde{\Delta}T\delta$ - space, A is δ - closed in (X, τ). Hence (X, τ) is a $T_{3/4}$ - space.

Theorem 4.1.27.

If (X, τ) is a $\tilde{\Delta}T\delta$ - space as well as gs$T\delta g^*$ - space then it is a $\tilde{\Delta}T\delta g^*$ - space.

Proof:

Let A be a $\tilde{\Delta}$ - closed set in (X, τ). Since (X, τ) is a $\tilde{\Delta}T\delta$ - space in (X, τ), A is δ - closed in (X, τ). As every δ - closed set is gs - closed [Remark 1.46 and 1.48c] and (X, τ) is a gs$T\delta g^*$ - space, A is δg* - closed in (X, τ). Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

Theorem 4.1.28.

1. If (X, τ) is a $\tilde{\Delta}T\delta$ - space and g$\delta T\delta g^*$ - space then it is a $\tilde{\Delta}T\delta g^*$ - space.
2. If (X, τ) is a $\tilde{\Delta}T\delta$ - space and $\alpha g T\delta g^*$ - space then it is a $\tilde{\Delta}T\delta g^*$ - space.
3. If (X, τ) is a $\tilde{\Delta}T\delta$ - space and $\delta g T\delta g^*$ - space then it is a $\tilde{\Delta}T\delta g^*$ - space.

Proof:

Let A be $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}T\delta$ - space in (X, τ), A is δ - closed. Then the proof of 1, 2, 3 follows from the fact that every δ - closed set is gδ - closed [corollary 3.3[12]], αg - closed [Remark 1.46 and 1.48c], δg - closed [Theorem 2.1.2] and (X, τ) is g$\delta T\delta g^*$ - space, $\alpha g T\delta g^*$ - space and $\delta g T\delta g^*$ - space which implies A is δg*- closed. Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

$\tilde{\Delta}T\delta g^*$- space

Proposition 4.1.29.

Every $\tilde{\Delta}T\delta g^*$ - space is a $\tilde{\Delta}T\Delta^*$ - space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed and (X, τ) be a $\tilde{\Delta}T\delta g^*$ - space. Therefore A is δg* - closed in (X, τ). As every δg*- closed is Δ* - closed [Proposition 3.4 [72]], A is Δ*- closed in (X, τ). Hence (X, τ) is a $\tilde{\Delta}T\Delta^*$- space.

Example 4.1.30.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space but not a $\tilde{\Delta}T\delta g^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg* - closed.

Proposition 4.1.31.

Every $g\delta T\delta g^*$ - space is a $\tilde{\Delta}T\delta g^*$- space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed and (X, τ) be a $g\delta T\delta g^*$- space. As every $\tilde{\Delta}$ - closed is gδ - closed [Proposition 3.1.14], A is gδ - closed in (X, τ). Since (X, τ) is $g\delta T\delta g^*$ - space, A is δg* - closed. Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

Example 4.1.32.

Let X = {a, b, c}, τ = {∅, X, {a}, {a, b}} then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not a $g\delta T\delta g^*$ - space, as the subset {a} is gδ - closed but not δg*- closed.

Proposition 4.1.33.

Every $T\delta$ - space is a $\tilde{\Delta}T\delta g^*$ - space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed. Since every $\tilde{\Delta}$ - closed is gδ - closed, A is a gδ - closed in (X, τ). As (X, τ) is a $T\delta$ - space, A is δ - closed in (X, τ). We know that every δ - closed is δg* - closed [Proposition 2.2.2] Therefore A is δg* - closed. Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

Example 4.1.34.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not a $T\delta$ - space, as the subset {a} is gδ - closed but not δ - closed.

Remark 4.1.35.

$T_{1/2}$ - space is independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.36.

Let X and τ be defined as in example 4.1.13 then (X, τ) is a $T_{1/2}$ - space but not a $\tilde{\Delta}T\delta g^*$ - space, as the subset {b} is $\tilde{\Delta}$ - closed but not δg*- closed in (X, τ).

Example 4.1.37.

Let X = {a, b, c}, τ = {∅, X, {b, c}} then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not a $T_{1/2}$ - space since the subset {a, b} is g - closed but not closed in (X, τ).

Remark 4.1.38.

rg$T\tilde{\Delta}$ - space and rwg$T\tilde{\Delta}$ - space are independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.39.

Let X and τ be defined as in example 4.1.10 then (X, τ) is a rg$T\tilde{\Delta}$ - space and rwg$T\tilde{\Delta}$ - space but not $\tilde{\Delta}T\delta g^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg* - closed in (X, τ).

Example 4.1.40.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but neither a rg$T\tilde{\Delta}$ - space nor a rwg$T\tilde{\Delta}$ - space, since the subset {a, b} is rg - closed and rwg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.41.

gpr$T\tilde{\Delta}$ - space is independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.42.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a gpr$T\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\delta g^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg*- closed in (X, τ).

Example 4.1.43.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\delta g^*$- space but not a gpr$T\tilde{\Delta}$ - space, as the subset {a, b} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.44.

gspr$T\tilde{\Delta}$ - space is independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.45.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a gspr$T\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\delta g^*$- space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg*- closed in (X, τ).

Example 4.1.46.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not a gspr$T\tilde{\Delta}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.47.

πgp$T\tilde{\Delta}$ - space, πg$T\tilde{\Delta}$- space and πgsp$T\tilde{\Delta}$ - space are independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.48.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a πgp$T\tilde{\Delta}$ - space,

$\pi gT\tilde{\Delta}$ - space and $\pi gspT\tilde{\Delta}$ - space but not $\tilde{\Delta}T\delta g^*$ - space, Since the subset {b} is $\tilde{\Delta}$ - closed but not δg^* - closed in (X, τ).

Example 4.1.49.

Let X = {a, b, c}, τ = {∅, X, {a}, {b}, {a, b}, {a, c}} then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but a not $\pi gpT\tilde{\Delta}$ - space, $\pi gT\tilde{\Delta}$ - space and $\pi gspT\tilde{\Delta}$ - space, since the subset {a} is πgp - closed, πg - closed and πgsp - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.50.

T_c - space, T_b - space and T_d - space are independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.51.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a T_c - space, T_b - space and T_d - space but not a $\tilde{\Delta}T\delta g^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg^*- closed in (X, τ).

Example 4.1.52.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but not T_c - space, T_b - space and T_d - space, since the subset {a} is gs - closed, but not closed, g* - closed, g - closed in (X, τ).

Remark 4.1.53.

$g\delta T\tilde{\Delta}$ - space and $\delta g^{\#}T\tilde{\Delta}$ - space are independent of $\tilde{\Delta}T\delta g^*$ - space as seen from the following examples.

Example 4.1.54.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space as well as $g\delta T\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\delta g^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not δg^* - closed in (X, τ).

Example 4.1.55.

Let X and τ be defined as in Example 4.1.49 then (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space but neither $\delta g^{\#}T\tilde{\Delta}$ - space nor a $g\delta T\tilde{\Delta}$ - space, since the subset {a} is $\delta g^{\#}$ - closed as well as gδ - closed but not a $\tilde{\Delta}$ - closed in (X, τ).

Theorem 4.1.56.

If (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space and $T_{3/4}$ - space then it is a $\delta g^*T\delta$ - space.

Proof:

Let A be a δg^* - closed in (X, τ). Since (X, τ) be a $\tilde{\Delta}T\delta g^*$ - space, A is δg^* - closed. Since every δg^* - closed is gδ – closed [Theorem 4.1 [12]] and (X, τ) is a $T_{3/4}$ - space, A is δ - closed. Hence (X, τ) is a $\delta g^*T\delta$ - space.

$g\delta T\tilde{\Delta}$ - space

Proposition 4.1.57.

Every $g\delta T\tilde{\Delta}$ - space is a $\delta g^{\#}T\tilde{\Delta}$ - space but not conversely.

Proof:

Let A be $\delta g^{\#}$ - closed and (X, τ) be a $g\delta T\tilde{\Delta}$ - space. Since every $\delta g^{\#}$ - closed set is gδ – closed [Remark 1.48a], A is gδ - closed. Since (X, τ) is a $g\delta T\tilde{\Delta}$ - space A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space.

Example 4.1.58.

Let X and τ be defined as in Example 4.1.49 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not $g\delta T\tilde{\Delta}$ - space. Since the subset {a} is gδ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.59.

Every $g\delta T_{\Delta *}$ - space is a $g\delta T_{\tilde{\Delta}}$ - space but not conversely.

Proof:

Let A be gδ - closed and (X, τ) be a $g\delta T_{\Delta *}$ - space. Then A is $\Delta *$ - closed. Since every $\Delta *$ - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.16], A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - sspace.

Example 4.1.60.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space but not a $g\delta T$- space, since the subset {b} is gδ - closed but not $\Delta *$ - closed in (X, τ).

Proposition 4.1.61.

Every T_δ - space is a $g\delta T_{\tilde{\Delta}}$ - space but not conversely.

Proof:

Let A be gδ - closed and (X, τ) be a T_δ - space. Then, A is δ - closed. We know that, every δ - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.2] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space.

Example 4.1.62.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space but not T_δ - space. Since the subset {c} is gδ - closed but not δ - closed.

Remark 4.1.63.

$^{*}T_{1/2}$ - space is independent of $g\delta T_{\tilde{\Delta}}$ - space as seen from the following examples.

Example 4.1.64.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space but not $^{*}T_{1/2}$ - space, since the subset {b} is g - closed but not g*- closed in (X, τ).

Example 4.1.65.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a $^{*}T_{1/2}$-space but not a $g\delta T_{\tilde{\Delta}}$ - space. Since the subset {a} is gδ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.66.

$gprT_{\tilde{\Delta}}$ - space is independent of $g\delta T_{\tilde{\Delta}}$ - space as seen from the following examples.

Example 4.1.67.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a $gprT_{\tilde{\Delta}}$ - space but not a $g\delta T_{\tilde{\Delta}}$ - space, since the subset {a} is gδ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.68.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space but not a $gprT_{\tilde{\Delta}}$ - space, since the subset {a, b} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.69.

T_c - space and T_b - space are independent of $g\delta T_{\tilde{\Delta}}$ - space as seen from the following examples.

Example 4.1.70.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T_{\tilde{\Delta}}$ - space but not T_c - space as well as T_b - space. Since the subset {a, b} is gs - closed but not g* - closed and closed in (X, τ).

Example 4.1.71.

Let X and τ be defined as in Example 4.1.49 then (X, τ) is a T_c - space and T_b - space but not a $g\delta T\tilde{\Delta}$ - space, since the subset {a} is $g\delta$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.72.

T_d - space is independent of $g\delta T\tilde{\Delta}$ - space as seen from the following example.

Example 4.1.73.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $g\delta T\tilde{\Delta}$ - space but not a T_d - space, since the subset {a} is gs - closed but not g - closed in (X, τ).

Example 4.1.74.

Let X and τ be defined as in Example 4.1.6 then (X, τ) is a T_d - space but not a $g\delta T\tilde{\Delta}$ - space, since the subset {a} is $g\delta$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.75.

$T_{1/2}$ - space is independent of $g\delta T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.76.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a $T_{1/2}$ - space but not a $g\delta T\tilde{\Delta}$ - space, since the subset {a} is $g\delta$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.77.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T\tilde{\Delta}$ - space but not a $T_{1/2}$ - space, since the subset {b} is g - closed but not closed in (X, τ).

Theorem 4.1.78.

If (X, τ) is a $g\delta T\tilde{\Delta}$ - space as well as a $\tilde{\Delta}T\delta$ - space then it is a $T\delta$ - space.

Proof:

Let A be a gδ - closed in (X, τ). Since (X, τ) is a $g\delta T\tilde{\Delta}$ - space, A is $\tilde{\Delta}$ - closed. Also since (X, τ) is a $\tilde{\Delta}T\delta$ - space, A is δ - closed. Hence (X, τ) is a $T\delta$ - space.

Theorem 4.1.79.

If (X, τ) is a $g\delta T\tilde{\Delta}$-space as well as a $\tilde{\Delta}T\delta g^*$-space then it is a $g\delta T\delta g^*$-space.

Proof:

Let A be a gδ - closed in (X, τ). Since (X, τ) is a $g\delta T\tilde{\Delta}$ - space, A is $\tilde{\Delta}$ - closed. Also since (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space, A is δg* - closed. Hence (X, τ) is a $g\delta T\delta g^*$ - space.

$\delta g^{\#}T\tilde{\Delta}$ - space

Proposition 4.1.80.

Every $\delta g^{\#}T_{\Delta^*}$ - space is a $\delta g^{\#}T\tilde{\Delta}$ - space but not conversely.

Proof:

Let A be $\delta g^{\#}$ - closed and (X, τ) be a $\delta g^{\#}T_{\Delta^*}$ - space. Since every Δ^* - closed is $\tilde{\Delta}$ - closed, A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space.

Example 4.1.81.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not a $\delta g^{\#}T_{\Delta^*}$ - space, since the subset {b} is $\delta g^{\#}$ - closed but not Δ^* - closed in (X, τ).

Proposition 4.1.82.

Every $T\delta$ - space is a $\delta g^{\#}T\tilde{\Delta}$ - space but not conversely.

Proof:

Let A be $\delta g^{\#}$ - closed and (X, τ) be a T_δ - space. Since every $\delta g^{\#}$ - closed is gδ – closed [Remark 1.48a]. A is gδ - closed. In (X, τ), A is δ - closed as (X, τ) be a T_δ - space. We know that every δ - closed is $\tilde{\Delta}$ - closed and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\delta g^{\#}T_{\tilde{\Delta}}$ - space.

Example 4.1.83.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\delta g^{\#}T_{\tilde{\Delta}}$ - space but not a T_δ - space, since the subset {b} is gδ - closed but not δ - closed in (X, τ).

Remark 4.1.84.

T_b-space is independent of $\delta g^{\#}T_{\tilde{\Delta}}$ - space as seen from the following examples.

Example 4.1.85.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\delta g^{\#}T_{\tilde{\Delta}}$ - space but not T_b - space. Since the subset {a} is gs - closed but not closed in (X, τ).

Example 4.1.86.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a T_b - space but not a $\delta g^{\#}T_{\tilde{\Delta}}$ - space, since the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.87.

$T_{1/2}$ - space is independent of $\delta g^{\#}T_{\tilde{\Delta}}$ - space as seen from the following examples.

Example 4.1.88.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a $T_{1/2}$ - space but not a $\delta g^{\#}T\tilde{\Delta}$ - space, as the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.89.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not a $T_{1/2}$ - space, since the subset {b} is g - closed but not closed in (X, τ).

Remark 4.1.90.

$gprT\tilde{\Delta}$ - space is independent with $\delta g^{\#}T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.91.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a $gprT\tilde{\Delta}$ - space but not a $\delta g^{\#}T\tilde{\Delta}$ - space, since the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.92.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not $gprT\tilde{\Delta}$ - space, since the subset {a, b} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.93.

T_c - space, T_b - space and T_d - space are independent of $\delta g^{\#}T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.94.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a T_c - space, T_b - space and T_d - space but not a $\delta g^{\#}T\tilde{\Delta}$- space. Since the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.95.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not a T_c - space, T_b - space and T_d - space. Since the subset {a} is gs - closed but not g* - closed, closed and g - closed in (X, τ).

Remark 4.1.96.

$T_{1/2}$ - space is independent of $\delta g^{\#}T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.97.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a $T_{1/2}$ - space but not a $\delta g^{\#}T\tilde{\Delta}$ - space, since the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.98.

Let X and τ be defined as in Example 4.1.6 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not $T_{1/2}$ – space, since the subset {b, c} is g - closed but not closed in (X, τ).

Theorem 4.1.99.

If (X, τ) is a $T\delta$ - space and $\delta g^{\#}T\tilde{\Delta}$ - space then it is a $g\delta T\tilde{\Delta}$ - space.

Proof:

Let A be a gδ-closed in (X, τ). Since (X, τ) be a $T\delta$ - space, A is δ - closed. Since every δ-closed is $\delta g^{\#}$ - closed [Remark 1.48a] and (X, τ) is also a $\delta g^{\#}T\tilde{\Delta}$ - space, A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $g\delta T\tilde{\Delta}$ - space.

$\tilde{\Delta}T_{\Delta^*}$ - space

Proposition 4.1.100.

Every $g\delta T_{\Delta^*}$ - space is a $\tilde{\Delta}T_{\Delta^*}$ - space but not conversely.

Proof:

Let A be a $\tilde{\Delta}$ - closed and (X, τ) be a $g\delta T_{\Delta^*}$ - space. Since every $\tilde{\Delta}$ - closed is gδ - closed [Proposition 3.1.14]. Therefore A is Δ* - closed. Hence (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space.

Example 4.1.101.

Let X and τ be defined as in Example 4.1.6 then (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space but not a $g\delta T_{\Delta^*}$ - space, since the subset {a} is gδ - closed but not Δ* - closed in (X, τ).

Proposition 4.1.102.

Every $\delta g^{\#}T_{\Delta^*}$ - space is a $\tilde{\Delta}T_{\Delta^*}$ - space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed and (X, τ) be a $\delta g^{\#}T_{\Delta^*}$- space. Since every $\tilde{\Delta}$ - closed is $\delta g^{\#}$ - closed and (X, τ) is a $\delta g^{\#}T_{\Delta^*}$ - space. A is Δ* - closed. Hence (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space.

Example 4.1.103.

Let X and τ be defined as in Example 4.1.13 then (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space but not a $\delta g^{\#}T_{\Delta^*}$ - space, since the subset {a} is $\delta g^{\#}$ - closed but not Δ* - closed in (X, τ).

Proposition 4.1.104.

Every T_δ - space is a $\tilde{\Delta}T_{\Delta^*}$- space but not conversely.

Proof:

Let A be $\tilde{\Delta}$ - closed and (X, τ) be a $T\delta$ - space. Since every $\tilde{\Delta}$ - closed is gδ - closed [Proposition 3.1.14] and (X, τ) is a $T\delta$ - space, A is δ - closed. We know that every δ - closed is Δ* - closed. Hence (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space.

Example 4.1.105.

Let X and τ be defined as in Example 4.1.49 then (X, τ) is a $\tilde{\Delta}T\Delta^*$- space but not a $T\delta$ - space, since the subset {a} is gδ - closed but not δ - closed in (X, τ).

Remark 4.1.106.

$\tilde{\Delta}T\Delta^*$ - space is independent of $g\delta T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.107.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space but not a $g\delta T\tilde{\Delta}$ - space, since the subset {a} is gδ-closed but not $\tilde{\Delta}$-closed in (X, τ).

Example 4.1.108.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $g\delta T\tilde{\Delta}$ - space but not $\tilde{\Delta}T\Delta^*$ - space. Since the subset {a} is $\tilde{\Delta}$ - closed but not Δ* - closed in (X, τ).

Remark 4.1.109.

$\tilde{\Delta}T\Delta^*$ - space is independent of $\delta g^{\#}T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.110.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space but not a $\delta g^{\#}T\tilde{\Delta}$ - space, since the subset {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.111.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\delta g^{\#}T\tilde{\Delta}$ - space but not $\tilde{\Delta}T\Delta^{*}$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not Δ^{*} - closed in (X, τ).

Remark 4.1.112.

$\tilde{\Delta}T\Delta^{*}$ - space is independent of $\pi gspT\tilde{\Delta}$ - space as well as $\pi gT\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.113.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $\pi gspT\tilde{\Delta}$ - space and a $\pi gT\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\Delta^{*}$- space, since the subset {b} is $\tilde{\Delta}$ - closed but not Δ^{*}-closed in (X, τ).

Example 4.1.114.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T\Delta^{*}$ - space but neither $\pi gspT\tilde{\Delta}$ - space nor $\pi gT\tilde{\Delta}$ - space, since the subset {a} is πgsp - closed as well as πg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.115.

$\tilde{\Delta}T\Delta^{*}$- space is independent of $rwgT\tilde{\Delta}$ - space and $rgT\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.116.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a $rwgT\tilde{\Delta}$ - space as well as $rgT\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\Delta^{*}$- space, since the subset {b} is $\tilde{\Delta}$ - closed but not Δ^{*} - closed in (X, τ).

Example 4.1.117.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space but not rwg$T\tilde{\Delta}$ - space and rg$T\tilde{\Delta}$ - space. Since the subset {a, b} is rwg - closed and rg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.118.

$\tilde{\Delta}T\Delta^*$ - space is independent of gspr$T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.119.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space but not a gspr$T\tilde{\Delta}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.120.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a gspr$T\tilde{\Delta}$ - space but not $\tilde{\Delta}T\Delta^*$ - space, since the subset {c} is $\tilde{\Delta}$ - closed but not Δ^*- closed in (X, τ).

Remark 4.1.121.

$\tilde{\Delta}T\Delta^*$- space is independent of gpr$T\tilde{\Delta}$ - space as seen from the following examples.

Example 4.1.122.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\tilde{\Delta}T\Delta^*$- space but not a gpr$T\tilde{\Delta}$ - space, since the subset {a, b} is gpr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Example 4.1.123.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a gpr$T\tilde{\Delta}$ - space but not a $\tilde{\Delta}T\Delta^*$ - space, since the subset {b} is $\tilde{\Delta}$ - closed but not Δ^*- closed in (X, τ).

Remark 4.1.124.

$\tilde{\Delta}T_{\Delta^*}$ - space is independent of T_d - space as seen from the following examples.

Example 4.1.125.

Let X and τ be defined as in Example 4.1.32 then (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space but not a T_d - space, since the subset {b} is gs - closed but not g - closed in (X, τ).

Example 4.1.126.

Let X and τ be defined as in Example 4.1.10 then (X, τ) is a T_d - space but not a $\tilde{\Delta}T_{\Delta^*}$- space, since the subset {b} is $\tilde{\Delta}$ - closed but not Δ* - closed in (X, τ).

Theorem 4.1.127.

If (X, τ) is a T_δ - space as well as $\tilde{\Delta}T_{\Delta^*}$ - space then it is a $g\delta T_{\Delta^*}$ - space.

Proof:

Let A be a gδ - closed in (X, τ). Since (X, τ) be a T_δ - space, A is δ - closed. Since every δ - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.2] and (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space, A is Δ* - closed. Hence (X, τ) is a $g\delta T_{\Delta^*}$ - space.

Theorem 4.1.128.

If (X, τ) is a $\tilde{\Delta}T_{\Delta^*}$ - space as well as a Δ^*T_δ - space then it is a $\tilde{\Delta}T_\delta$ - space.

Proof:

Let A be a $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) be a $\tilde{\Delta}T_{\Delta^*}$ - space, A is Δ* - closed. Since every Δ* - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.16] and (X, τ) is a $\tilde{\Delta}T_\delta$ - space, A is δ - closed. Hence (X, τ) is a $\tilde{\Delta}T_\delta$ - space.

Theorem 4.1.129.

If (X, τ) is a $\tilde{\Delta}T\Delta^*$ - space as well as a $g\delta T\delta g^*$ - space then it is a $\tilde{\Delta}T\delta g^*$ - space.

Proof:

Let A be a $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) be a $\tilde{\Delta}T\Delta^*$ - space, A is Δ^*- closed. Since every Δ^* - closed is gδ - closed [Proposition 3.6[72]] and (X, τ) is a $g\delta T\delta g^*$ - space, A is δg*- closed. Hence (X, τ) is a $\tilde{\Delta}T\delta g^*$ - space.

$\pi gspT\tilde{\Delta}$ - space

Proposition 4.1.130.

Every $\pi gspT\tilde{\Delta}$ - space is a $\pi gT\tilde{\Delta}$ - space but not conversely.

Proof:

Let A be πg - closed and (X, τ) be a $\pi gspT_{\Delta^*}$ - space. Since every πg - closed is πgsp - closed [Remark 1.48b] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\pi gT\tilde{\Delta}$ - space.

Example 4.1.131.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\pi gT\tilde{\Delta}$ - space but not a $\pi gspT\tilde{\Delta}$ - space, since the subset {a} is πgsp - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.132.

Every $\pi gspT\tilde{\Delta}$ - space is a $\pi gpT\tilde{\Delta}$ - space but not conversely.

Proof:

Let A be πgp - closed and (X, τ) be a $\pi gspT\tilde{\Delta}$ - space. Since every πgp - closed is πgsp - closed [Remark 1.48b] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\pi gpT\tilde{\Delta}$ - space.

Example 4.1.133.

Let X and τ be defined as in Example 4.1.12 then (X, τ) is a $\pi gpT_{\tilde{\Delta}}$ - space but not a $\pi gspT_{\tilde{\Delta}}$ - space, since the subset {a} is πgsp - closed but not $\tilde{\Delta}$ - closed in (X, τ).

$gsprT_{\tilde{\Delta}}$ - space

Proposition 4.1.134.

Every $gsprT_{\tilde{\Delta}}$ - space is a $\pi gpT_{\tilde{\Delta}}$ - space but not conversely.

Proof:

Let A be πgp - closed and (X, τ) be a $gsprT_{\tilde{\Delta}}$ - space. Since every πgp - closed is gspr - closed [Remark 1.48b] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\pi gpT_{\tilde{\Delta}}$ - space.

Example 4.1.135.

Let X = {a, b, c}, τ = {∅, X, {b}, {c}, {b, c}} then (X, τ) is a $\pi gpT_{\tilde{\Delta}}$ - space but not a $gsprT_{\tilde{\Delta}}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.136.

Every $gsprT_{\tilde{\Delta}}$ - space is a $\pi gT_{\tilde{\Delta}}$ - space but not conversely.

Proof:

Let A be πg - closed and (X, τ) be a $gsprT_{\tilde{\Delta}}$ - space. Since every πg - closed is gspr - closed [Remark 1.48b] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a $\pi gT_{\tilde{\Delta}}$ - space.

Example 4.1.137.

Let X and τ be defined as in Example 4.1.135 then (X, τ) is a $\pi gT_{\tilde{\Delta}}$ - space but not $gsprT_{\tilde{\Delta}}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.138.

Every gspr$T_{\tilde{\Delta}}$ - space is a gpr$T_{\tilde{\Delta}}$ - space but not conversely.

Proof:

Let A be gpr - closed and (X, τ) be a gspr$T_{\tilde{\Delta}}$ - space. Since every gpr - closed is gspr - closed [Remark 1.49a] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a gpr$T_{\tilde{\Delta}}$ - space.

Example 4.1.139.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a gpr$T_{\tilde{\Delta}}$ - space but not gspr$T_{\tilde{\Delta}}$ - space, since the subset {a} is gspr - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Proposition 4.1.140.

Every rwg$T_{\tilde{\Delta}}$ - space is a gpr$T_{\tilde{\Delta}}$ - space but not conversely.

Proof:

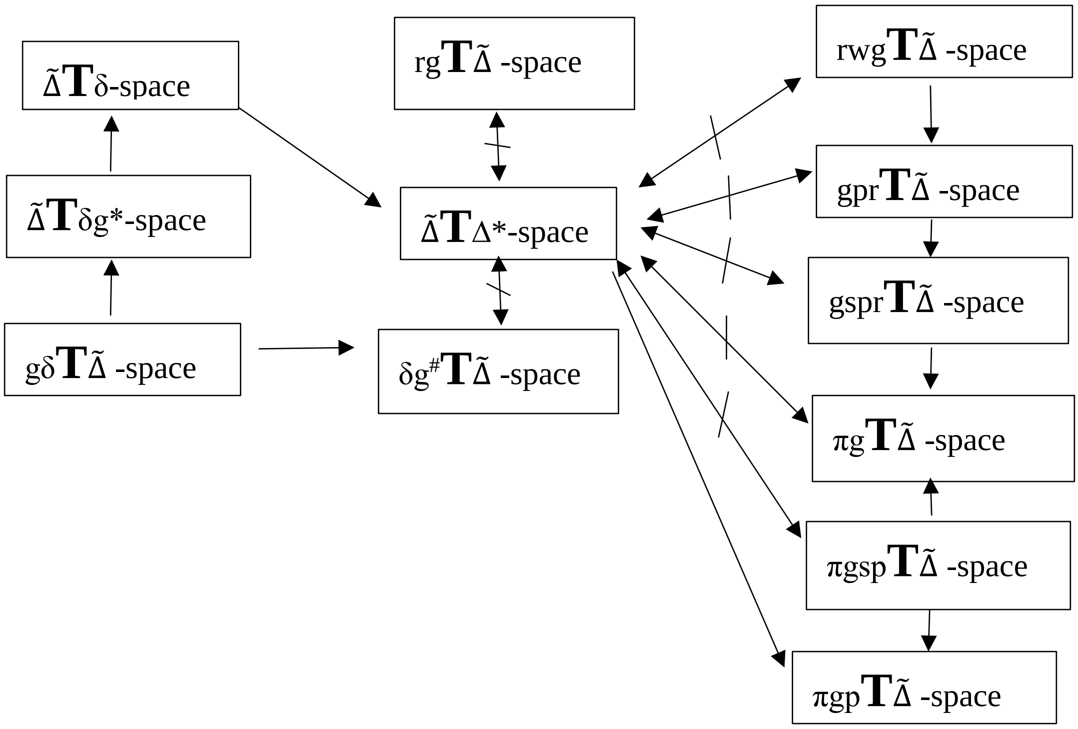
Let A be gpr - closed and (X, τ) is a rwg$T_{\tilde{\Delta}}$ - space. Every gpr - closed is rwg - closed [Remark 1.49b] and therefore A is $\tilde{\Delta}$ - closed. Hence (X, τ) is a gpr$T_{\tilde{\Delta}}$ - space.

Example 4.1.141.

Let X and τ be defined as in Example 4.1.22 then (X, τ) is a gpr$T_{\tilde{\Delta}}$ - space but not rwg$T_{\tilde{\Delta}}$ - space. Since the subset {a} is rwg - closed but not $\tilde{\Delta}$ - closed in (X, τ).

Remark 4.1.142.

The below figure represents the dependence and independence of different newly defined spaces.



4.2 Δ̃ - continuous function

Continuity is a most important concept in mathematics and over the year many types of continuous functions have been analyzed. In this chapter Δ̃ - continuous functions in topological spaces and study their properties are introduced.

Definition 4.2.1

A function f: (X, τ) → (Y, σ) is said to be Δ̃ - continuous if the inverse image of every closed set in (Y, σ) is Δ̃ - closed in (X, τ).

Theorem 4.2.2.

A mapping f: (X, τ) → (Y, σ) is Δ̃ - continuous if and only if the inverse image of every open set in (Y, σ) is Δ̃ - open in (X, τ).

Proof: (Necessity)

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous and U be an open set in (Y, σ) then Y - U is closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous, f^{-1}(Y - U) = X - f^{-1}(U) is closed in (X, τ) and hence f^{-1}(U) is $\tilde{\Delta}$ - open in (X, τ).

Sufficiency:

Assume that f^{-1}(U) is $\tilde{\Delta}$ - open set in (X, τ) for each open set U in (Y, σ). Let U be closed set in (Y, σ) then Y - U is open in (Y, σ). By assumption f^{-1}(Y - U) = X - f^{-1}(U) is $\tilde{\Delta}$ - open in (X, τ). Then f^{-1}(U) is $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Proposition 4.2.3.

Every super continuous function is $\tilde{\Delta}$ - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be super - continuous map. Let V be any closed set in (Y, σ). Since f is super continuous function, f^{-1}(V) is a δ - closed in (X, τ). By Proposition 3.1.2, every δ - closed is a $\tilde{\Delta}$ - closed set which implies f^{-1}(V) is a $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Example 4.2.4.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous, but not super - continuous because inverse image of the closed set {c} in (Y, σ) and f^{-1}{c} = {c} which is $\tilde{\Delta}$ - closed in (X, τ) but not δ-closed in (X, τ).

Proposition 4.2.5.

Every δg*- continuous function is $\tilde{\Delta}$ - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a δg*-continuous map. Let V be any closed set in (Y, σ). Since f is δg* - continuous function, f^{-1}(V) is a δg* - closed in (X, τ). By Proposition 3.1.4, every δg* - closed is a $\tilde{\Delta}$ - closed set which implies f^{-1}(V) is a $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Example 4.2.6.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}. {a, c}} , σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous, but not δg^* - continuous because inverse image of the closed set {b} in (Y, σ) and f^{-1}{b} = {b} is $\tilde{\Delta}$ - closed but not δg^* - closed.

Proposition 4.2.7.

Every Δ^* - continuous function is $\tilde{\Delta}$ - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a Δ^* - continuous map. Let V be any closed set in (Y, σ). Since f is Δ^* - continuous function, $f^{-1}(V)$ is a Δ^* - closed in (X, τ). By Proposition 3.1.16, every Δ^* - closed is a $\tilde{\Delta}$ - closed set which implies $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Example 4.2.8.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {b, c}}, σ = {∅, Y, {a, b}}. Let f: (X, τ) → (Y, σ) be a map such that f(a) = (a), f(b) = (c), f(c) = (b). Then f is $\tilde{\Delta}$ - continuous but not Δ^* - continuous because inverse image of the closed set {c} in (Y, σ) and f^{-1}(c) = {b} is $\tilde{\Delta}$ - closed but not Δ^* - closed.

Proposition 4.2.9.

Every $\tilde{\Delta}$ - continuous function is $\delta g^{\#}$ - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By proposition 3.1.24, every $\tilde{\Delta}$ - closed is a $\delta g^{\#}$ - closed set which implies $f^{-1}(V)$ is a $\delta g^{\#}$ - closed in (X, τ). Hence f is $\delta g^{\#}$ - continuous.

Example 4.2.10.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\delta g^{\#}$ - continuous but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {a} in (Y, σ) and f^{-1}{a} = {a} is $\delta g^{\#}$ - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.11.

Every $\tilde{\Delta}$ - continuous function is πgsp - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.22, every $\tilde{\Delta}$ - closed is a πgsp - closed set which implies $f^{-1}(V)$ is a πgsp - closed in (X, τ). Hence f is πgsp - continuous.

Example 4.2.12.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is πgsp - continuous but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {a} in (Y, σ) and f^{-1}{a} = {a} is πgsp - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.13.

Every $\tilde{\Delta}$ - continuous function is gspr - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.20, every $\tilde{\Delta}$ - closed is a gspr - closed set which implies $f^{-1}(V)$ is a gspr - closed in (X, τ). Hence f is gspr - continuous.

Example 4.2.14.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {b, c}}. Let f: (X, τ) → (Y, σ) be a map such that f(a) = (a), f(b) = (c), f(c) = (b). Then f is gspr - continuous, but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {c} in (Y, σ) and f^{-1}{c} = {b} is gspr - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.15.

Every $\tilde{\Delta}$ - continuous function is gδ - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.14, every $\tilde{\Delta}$ - closed is gδ - closed set which implies f $^{-1}$(V) is a gδ - closed in (X, τ). Hence f is gδ - continuous.

Example 4.2.16.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}}, σ = {∅, Y, {a}, {a, b}}. Let f: (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b. Then f is gδ - continuous, but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {c} in (Y, σ) and f $^{-1}${c} = {b} is gδ - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.17.

Every $\tilde{\Delta}$ - continuous function is πgp - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, f $^{-1}$(V) is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.12, every $\tilde{\Delta}$ - closed is a πgp - closed set which implies f $^{-1}$(V) is a πgp - closed in (X, τ). Hence f is πgp - continuous.

Example 4.2.18.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}}, σ = {∅, Y, {a}, {b}, {a,c}, {a, b}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is πgp - continuous, but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {b} in (Y, σ) and f $^{-1}${b} = {b} is πgp - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.19.

Every $\tilde{\Delta}$ - continuous function is rg - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, f $^{-1}$(V) is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.8, every $\tilde{\Delta}$ - closed is a rg - closed set which implies f $^{-1}$(V) is a rg - closed in (X, τ). Hence f is rg - continuous.

Example 4.2.20.

Let X = {a, b, c} = Y, τ = {∅, X, {b, c}, σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is rg - continuous, but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {b} in (Y, σ) and f^{-1}{b} = {b} is rg - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.21.

Every $\tilde{\Delta}$ - continuous function is rwg - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.18, every $\tilde{\Delta}$ - closed is a rwg - closed set which implies $f^{-1}(V)$ is a rwg - closed in (X, τ). Hence f is rwg - continuous.

Example 4.2.22.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}} σ = {∅, Y, {a}, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is rwg - continuous, but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {a} in (Y, σ) and f^{-1}{a} = {a} is rwg - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.23.

Every $\tilde{\Delta}$ - continuous function is πg - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a πg - closed in (X, τ). By Proposition 3.1.6, every $\tilde{\Delta}$ - closed is a πg - closed set which implies $f^{-1}(V)$ is a πg - closed in (X, τ). Hence f is πg - continuous.

Example 4.2.24.

Let X = {a, b, c} = Y, τ = {∅, X, {b, c}}, σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is πg - continuous, but not $\tilde{\Delta}$ - continuous, because inverse image of the closed set {a} in (Y, σ) and f^{-1}{b} = {b} is πg - closed but not $\tilde{\Delta}$ - closed.

Proposition 4.2.25.

Every $\tilde{\Delta}$ - continuous function is gpr - continuous but not conversely.

Proof:

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map. Let V be any closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous function, $f^{-1}(V)$ is a $\tilde{\Delta}$ - closed in (X, τ). By Proposition 3.1.10, every $\tilde{\Delta}$-closed is a gpr - closed set which implies $f^{-1}(V)$ is a gpr - closed in (X, τ). Hence f is gpr - continuous.

Example 4.2.26.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}}, σ = {∅, Y, {a}, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is gpr - continuous but not $\tilde{\Delta}$ - continuous, because inverse image of the closed set {a} in (Y, σ) and f^{-1}{a} = {a} is gpr - closed but not $\tilde{\Delta}$ - closed.

Remark 4.2.27.

The following examples show that g*s - continuous function is independent of $\tilde{\Delta}$ - continuous function.

Example 4.2.28.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {b}, {a, b}}, σ = {∅, Y, {b}, {c}, {b, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is g*s - continuous but not $\tilde{\Delta}$ - continuous, because the closed set {a} in (Y, σ) and f^{-1}{a} = {a} is g*s - closed but not $\tilde{\Delta}$ - closed.

Example 4.2.29.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {b, c}}, σ = {∅, Y, {a}, {a, b}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous but not g*s - continuous because inverse image of the closed set {c} in (Y, σ) and f^{-1}{c} = {c} is $\tilde{\Delta}$ - closed but not g*s - closed.

Remark 4.2.30.

The following examples show that g*p - continuous function and gp - continuous are independent of $\tilde{\Delta}$ - continuous.

Example 4.2.31.

Let X = {a, b, c} = Y, τ = {∅, X, {a, b}}, σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is gp - continuous and g*p - continuous but not $\tilde{\Delta}$ - continuous because the closed set {b} in (Y, σ) and f^{-1}{b} = {b} is g*p - closed as well as gp - closed but not $\tilde{\Delta}$ - closed.

Example 4.2.32.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {a}, {b}, {a, b}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous but not gp - continuous and g*p - continuous because inverse image of the closed set {a, c} in (Y, σ) is f^{-1}{a, c} = {a, c} is $\tilde{\Delta}$ - closed but neither g*p - closed nor gp - closed.

Remark 4.2.33.

The following examples show that gs - continuous is independent of $\tilde{\Delta}$ - continuous.

Example 4.2.34.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {a}, {b}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous but not gs - continuous, because the closed set {a, c} in (Y, σ) is f^{-1}{a, c} = {a, c} is $\tilde{\Delta}$ - closed but not gs - closed.

Example 4.2.35.

Let X = {a, b, c} = Y, τ = {∅, X, {b}, {c}, {b, c}}, σ = {∅, Y, {a}, {b}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is gs - continuous but not $\tilde{\Delta}$ - continuous because inverse image of the closed set {b} in (Y, σ) and f^{-1}{b} = {b} is gs - closed but not $\tilde{\Delta}$ - closed.

Remark 4.2.36.

The following examples show that $^{\#}$gs - continuous is independent of $\tilde{\Delta}$ - continuous.

Example 4.2.37.

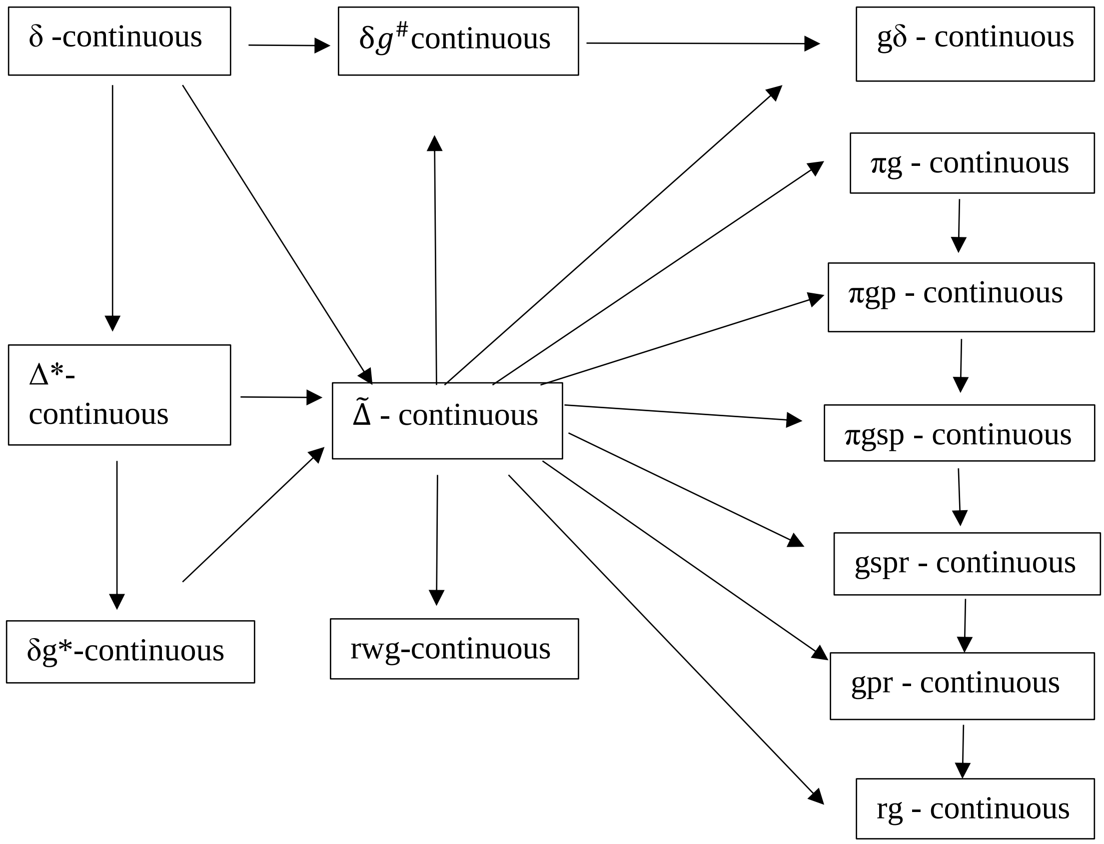
Let X = {a, b, c} = Y, τ = {∅, X, {a}, {a, b}, {a, c}}, σ = {∅, Y, {a}, {b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $\tilde{\Delta}$ - continuous but not $^{\#}$gs - continuous because inverse image of the closed set {a, c} in (Y, σ) and f^{-1}{a, c} = {a, c} is $\tilde{\Delta}$ - closed but not $^{\#}$gs - closed.

Example 4.2.38.

Let X = {a, b, c} = Y, τ = {∅, X, {a}, {b}, {a, b}}, σ = {∅, Y, {a}, {a, b}, {a, c}}. Let f: (X, τ) → (Y, σ) be an identity map. Then f is $^{\#}$gs - continuous but not $\tilde{\Delta}$ - continuous, because inverse image of the closed set {b} in (Y, σ) and f^{-1}{b} = {b} is #gs-closed but not $\tilde{\Delta}$ - closed.

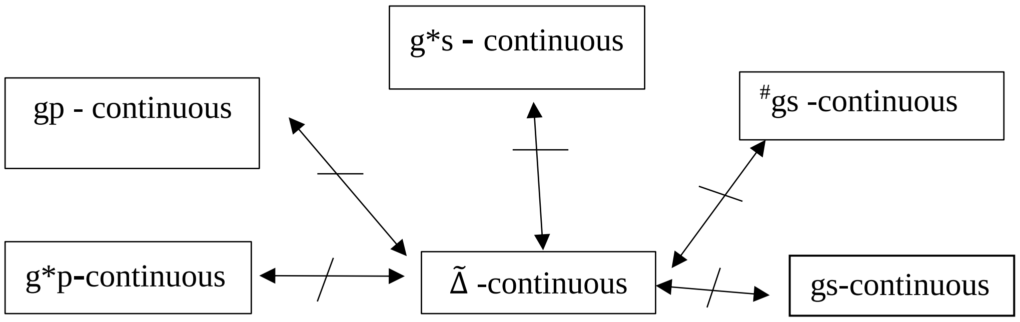
Remark 4.2.39.

The following diagram shows the dependence of $\tilde{\Delta}$ - continuous function on other already existing continuous functions.



Remark 4.2.40.

The following diagram shows the independence of $\tilde{\Delta}$ - continuous function with other already existing continuous functions.



Proposition 4.2.41.

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous map and (X, τ) be a $\tilde{\Delta}\mathbf{T}\delta$ - space. Then f is super - continuous function.

Proof:

Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta$ - space, we get $f^{-1}(V)$ is δ - closed in (X, τ). Hence f is super - continuous.

Proposition 4.2.42.

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous and (X, τ) be a $\tilde{\Delta}\mathbf{T}\delta g^*$ - space. Then f is δg* - continuous function.

Proof:

Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$ - space, we get $f^{-1}(V)$ is δg* - closed in (X, τ). Hence f is δg*- continuous.

Proposition 4.2.43.

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$-continuous and (X, τ) be a $\tilde{\Delta}\mathbf{T}\Delta^*$ - space. Then f is Δ^* - continuous function.

Proof:

Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}\mathbf{T}\Delta^*$ - space, we get $f^{-1}(V)$ is Δ^* - closed in (X, τ). Hence f is Δ^* - continuous.

Proposition 4.2.44.

Let f: (X, τ) → (Y, σ) be a gδ - continuous and (X, τ) be a $g\delta\mathbf{T}\tilde{\Delta}$ - space. Then f is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is gδ - closed in (X, τ). Since (X, τ) is a $g\delta\mathbf{T}\tilde{\Delta}$ - space, we get $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Proposition 4.2.45.

Let f: (X, τ) → (Y, σ) be a $\delta g^{\#}$-continuous and (X, τ) be a $\delta g^{\#}\mathbf{T}\tilde{\Delta}$ - space. Then f is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be a closed set in (Y, σ). Then $f^{-1}(V)$ is $\delta g^{\#}$ - closed in (X, τ). Since (X, τ) is a $\delta g^{\#}\mathbf{T}\tilde{\Delta}$ - space, we get $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Hence f is $\tilde{\Delta}$ - continuous.

Proposition 4.2.46.

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous, where (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$ - space and $\mathbf{T}_{3/4}$ - space then f is super - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous, $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). As (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$- space, $f^{-1}(V)$ is δg*- closed. Since every δg*- closed is δg - closed [Proposition 2.2.26] and (X, τ) is a $\mathbf{T}_{3/4}$ - space, $f^{-1}(V)$ is δ - closed. Therefore f is super - continuous.

Proposition 4.2.47.

Let f: (X, τ) → (Y, σ) be gδ – continuous, where (X, τ) is $g\delta\mathbf{T}\tilde{\Delta}$ - space as well as $\tilde{\Delta}\mathbf{T}\delta g^*$- space then f is δg* - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is gδ - continuous, $f^{-1}(V)$ is gδ - closed in (X, τ). Since (X, τ) is a $g\delta\mathbf{T}\tilde{\Delta}$ - space, $f^{-1}(V)$ is $\tilde{\Delta}$ - closed. As (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$ - space, $f^{-1}(V)$ is δg* - closed. Therefore f is δg* - continuous.

Proposition 4.2.48.

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous, where (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$- space as well as $\tilde{\Delta}\mathbf{T}\Delta^*$ - space then f is δg* - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous, $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$- space, $f^{-1}(V)$ is δg*- closed. Since every δg* - closed is $\tilde{\Delta}$ - closed [Proposition 3.1.4] and (X, τ) is a $\tilde{\Delta}\mathbf{T}\Delta^*$ - space, $f^{-1}(V)$ is Δ^*-closed. Therefore f is Δ^* - continuous.

Proposition 4.2.49.

Let f: (X, τ) → (Y, σ) be gδ - continuous, where (X, τ) is a $\mathbf{T}_\delta$ - space as well as $\delta g^\#\mathbf{T}\tilde{\Delta}$ - space then f is $\tilde{\Delta}$ - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is gδ - continuous, $f^{-1}(V)$ is gδ - closed in (X, τ). Since (X, τ) is a $\mathbf{T}_\delta$ - space, $f^{-1}(V)$ is δ - closed. Since every δ - closed is $\delta g^\#$ - closed [Remark 1.48a] and (X, τ) is a $\delta g^\#\mathbf{T}\tilde{\Delta}$ - space, $f^{-1}(V)$ is $\tilde{\Delta}$ - closed. Therefore f is $\tilde{\Delta}$ - continuous.

Proposition 4.2.50.

Let f: (X, τ) → (Y, σ) be rg - continuous where (X, τ) is a $rg\mathbf{T}\tilde{\Delta}$ - space as well as $\mathbf{T}_{\delta}$ - space then f is super - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is rg - continuous, $f^{-1}(V)$ is rg - closed in (X, τ). Since (X, τ) is a $rg\mathbf{T}\tilde{\Delta}$ - space, $f^{-1}(V)$ is $\tilde{\Delta}$-closed. Since every $\tilde{\Delta}$ - closed is gδ - closed [Proposition 3.1.14] and (X, τ) is a $\mathbf{T}_{\delta}$ - space, $f^{-1}(V)$ is δ - closed. Therefore f is super - continuous.

Proposition 4.2.51.

Let f: (X, τ) → (Y, σ) be $\tilde{\Delta}$ - continuous, where (X, τ) is a $\tilde{\Delta}\mathbf{T}_{\delta}$ - space as well as $\widehat{\delta g}\mathbf{T}_{\delta g*}$ - space then f is δg* - continuous.

Proof:

Let V be a closed set in (Y, σ). Since f is $\tilde{\Delta}$ - continuous, $f^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Since (X, τ) is a $\tilde{\Delta}\mathbf{T}_{\delta}$ - space, $f^{-1}(V)$ is δ - closed. Since every δ - closed is $\widehat{\delta g}$ - closed [Proposition 3.2 [36]] and (X, τ) is $\widehat{\delta g}\mathbf{T}_{\delta g*}$ - space, $f^{-1}(V)$ is δg* - closed. Therefore f is δg* - continuous.

Theorem 4.2.52.

Let f: (X, τ) → (Y, σ) be a $\tilde{\Delta}$ - continuous. Then for every subset V of (X, τ), $f(\tilde{\Delta}\,cl(V)) \subseteq cl(f(V))$.

Proof:

Let V be any subset of (X, τ). Then cl(f(v)) is closed in (Y, σ). Since f is $\tilde{\Delta}$ - continuous $f^{-1}(cl(f(V)))$ is $\tilde{\Delta}$ - closed in (X, τ). Since $f(V) \subseteq cl(f(V))$ which implies $V \subseteq f^{-1}(cl(f(V)))$. Hence $f^{-1}(cl(f(V)))$ is $\tilde{\Delta}$ - closed set containing V. By definition of $\tilde{\Delta}$ - closure [Definition 3.3.1], we have $\tilde{\Delta}cl(V) \subseteq f^{-1}(cl(f(V)))$ which implies $f(\tilde{\Delta}cl(V)) \subseteq cl(f(V))$.

Theorem 4.2.53.

Let f: (X, τ) → (Y, σ) be a function from a topological space X to topological space Y. Then the following statements are equivalent

(a) for each point x in X and each open set V in Y with $f(x) \in V$, there is a $\tilde{\Delta}$ - open set U in X ∋ $x \in U$ and $f(U) \subseteq V$.

(b) for each subset A of X, $f(\tilde{\Delta}\ cl(A)) \subseteq cl(f(A))$.

(c) for each subset B of Y, $\tilde{\Delta}\ cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$

Proof:

(a) ⇒ (b)

Let $y \in f(\tilde{\Delta}\ cl(A))$, then there exist a open set V, which is neighbourhood of y. If $y \in f(\tilde{\Delta}\ cl(A))$ there is a $x \in \tilde{\Delta}\ cl(A)$ then $y = f(x)$. Since $f(x) \in V$ by (i) there is a $\tilde{\Delta}$ - open set U in X ∋ $x \in U$ and $f(U) \subseteq V$. By Theorem 3.2.15, we know that $A \cap U \neq \emptyset$. Then $f(A \cap U) \neq \emptyset$, which implies $f(A) \cap f(U) \neq \emptyset$, $f(A) \cap V \neq \emptyset$. Then by definition of neighbourhood $y = f(x) \subseteq cl(f(A))$. Therefore $f(\tilde{\Delta}\ cl(A)) \subseteq cl(f(A))$.

(b) ⇒ (a)

Let $x \in X$ and V be any open set in (Y, σ) containing f(x). Let $A = f^{-1}(V^c)$ then $x \notin A$. By (ii), $f(\tilde{\Delta}\ cl(A)) \subseteq cl(f(A)) \subseteq cl(f(f^{-1}(V^c))) \subseteq cl(V^c) = V^c$. Therefore $f^{-1}(f(\tilde{\Delta}\ cl(A))) \subseteq f^{-1}(V^c)$ which implies $\tilde{\Delta}\ cl(A) \subseteq f^{-1}(V^c) = A$. Since $x \notin A$ then $x \notin \tilde{\Delta}\ cl(A)$. Therefore there exist a open set U containing x ∋ $U \cap A = \emptyset$, which implies $U \subseteq A^c$, $f(U) \subseteq f(A^c) \subseteq V$. Therefore $f(U) \subseteq V$.

(b) ⇒ (c)

Suppose that (ii) holds and let B be any subset of Y. Replacing A by $f^{-1}(B)$ from (b).$f(\tilde{\Delta}cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$. Therefore $\tilde{\Delta}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

(c) ⇒ (b)

Suppose that (iii) holds and let B = f(A), where A is subset of X. Then $\tilde{\Delta}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$, $f(\tilde{\Delta}cl(f^{-1}(B))) \subseteq cl(B) \subseteq cl(f(A))$. Hence proved.

4.3 Composition of $\tilde{\Delta}$ - continuous functions.

Remark 4.3.1.

The composition of two $\tilde{\Delta}$ - continuous functions need not be $\tilde{\Delta}$ - continuous as seen from the following example.

Example 4.3.2.

Let X = {a, b, c} = y = z with τ = {∅, X, {a}, {a, b}}, σ = {∅, X, {a, b}}, η = {∅, X, {a}}. Let f: (X, τ) → (Y, σ) be an identity map. Let g: (Y, σ) → (Z, η) be a map such that g(a) = (b), g(b) =(c), g(c)= (a), then the composition mapping (g·f): (X, τ) → (Z, η) is defined by (g·f)(a) = (b), (g·f)(b) = (c), (g·f)(c) = (a), (g·f)(a, b) = (b, c), (g·f)(b, c) = (a, c), (g·f)(a, c) = (a, b). Here both f and g are $\tilde{\Delta}$ - continuous but the composition mapping (g·f) is not $\tilde{\Delta}$ - continuous since for the closed set {b, c} in (Z, η), $(g\cdot f)^{-1}$ {b, c} = {a, b} is not $\tilde{\Delta}$ - closed in (X, τ).

Theorem 4.3.3.

If f: (X, τ) → (Y, σ) is $\tilde{\Delta}$ - continuous and g: (Y, σ) → (Z, η) is super - continuous then (g·f): (X, τ) → (Z, η) is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be closed set in (Z, η). Since g is δ - continuous, $g^{-1}(V)$ be δ - closed in (Y, σ). We know that δ - closed set is closed. Therefore $g^{-1}(V)$ is closed in (Y, σ). Since f is $\tilde{\Delta}$ - continuous which implies that $f^{-1}(g^{-1}(V)) = (g\cdot f)^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Hence (g·f) is $\tilde{\Delta}$ - continuous.

Theorem 4.3.4.

If f: (X, τ) → (Y, σ) is $\tilde{\Delta}$ - continuous in which (Y, σ) is a $\tilde{\Delta}\mathbf{T}\delta g^*$ - space. Let g: (Y, σ) → (Z, η) is super - continuous then (g·f): (X, τ) → (Z, η) is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be any closed set in (Z, η). Since g is $\tilde{\Delta}$ - continuous, therefore $g^{-1}(V)$ be $\tilde{\Delta}$ -closed in (Y, σ) and also $\tilde{\Delta}\mathbf{T}\delta g^*$ - space which implies $g^{-1}(V)$ is δg^*- closed. Then every δg^* - closed set is $\tilde{\Delta}$ - closed [Proposition 3.1.4] in (Y, σ). Also f is $\tilde{\Delta}$ - continuous therefore $f^{-1}(g^{-1}(V)) = (g\cdot f)^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Hence (g·f) is $\tilde{\Delta}$ - continuous.

Theorem 4.3.5.

If f: (X, τ) → (Y, σ) is gδ - continuous in which (Y, σ) is a $g\delta\mathbf{T}\tilde{\Delta}$ - space. Let g: (Y, σ) → (Z, η) is gδ - continuous then (g·f): (X, τ) → (Z, η) is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be any closed set in (Z, η). Then $g^{-1}(V)$ is gδ - closed in (Y, σ). Since f is gδ - continuous, $f^{-1}(g^{-1}(V)) = (g\cdot f)^{-1}(V)$ is gδ - closed and (X, τ) is gδ$T_{\tilde{\Delta}}$ - space, $(g\cdot f)^{-1}(V)$ is $\tilde{\Delta}$ - closed in (X, τ). Hence (g·f) is $\tilde{\Delta}$ - continuous.

Theorem 4.3.6.

Let f: (X, τ) → (Y, σ) is gδ - continuous map in which (Y, σ) is a T_δ - space and $\delta g^{\#}T_{\tilde{\Delta}}$ - space. Let g: (Y, σ) → (Z, η) is gδ - continuous then (g·f): (X, τ) → (Z, η) is $\tilde{\Delta}$ - continuous function.

Proof:

Let v be any closed set in (Z, η). Then $g^{-1}(V)$ is gδ - closed in (Y, σ). Since f is gδ - continuous, $f^{-1}(g^{-1}(V)) = (g\cdot f)^{-1}(V)$ is gδ - closed. Since (X, τ) is T_δ - space, $f^{-1}(g^{-1}(V))$ becomes δ - closed set in (X, τ). Moreover (X, τ) is $\delta g^{\#}T_{\tilde{\Delta}}$ - space, since every δ - closed set is $\delta g^{\#}$ - closed set [Remark 1.48]. Therefore $f^{-1}(g^{-1}(V))$ is $\tilde{\Delta}$ - closed in (X, τ). Hence (g·f) is $\tilde{\Delta}$ - continuous.

Theorem 4.3.7.

If f: (X, τ) → (Y, σ) is δg*- irresolute and g: (Y, σ) → (Z, η) is super - continuous then (g·f): (X, τ) → (Z, η) is $\tilde{\Delta}$ - continuous function.

Proof:

Let V be any closed set in (Z, η). Since g is super - continuous, $g^{-1}(V)$ is δ - closed in (Y, σ). Since every δ - closed is δg*- closed [Proposition 2.2.2]. Here f is δg*- irresolute, $f^{-1}(g^{-1}(V))$ is δg*- closed. Since every δg*- closed is $\tilde{\Delta}$ - closed [Proposition 3.1.4]. Therefore $f^{-1}(g^{-1}(V))$ is $\tilde{\Delta}$ - closed in (X, τ). Hence (g·f) is $\tilde{\Delta}$ - continuous.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

This dissertation work consists of a study of $\tilde{\Delta}$ - closed sets and its properties.

In **chapter 1**, Preliminary and definitions which are needed for the course of the work are listed.

In **chapter 2**, δ - genaralized closed sets, δ generalized star closed sets and Properties of δg* - closed sets in topological spaces are reviewed and discussed. As applications, certain spaces and their interrelations are studied.

In **chapter 3**, $\tilde{\Delta}$ - closed sets, $\tilde{\Delta}$ - open sets, $\tilde{\Delta}$ - closure of subsets in topological spaces are introduced and their properties are discussed.

In **chapter 4,** $\tilde{\Delta}$ - separation axioms and $\tilde{\Delta}$ - continuous functions in topological spaces are discussed. As application certain new spaces are introduced and their interrelations are discussed. In $\tilde{\Delta}$ - continuous functions, properties and characterizations are analysed.

As further study, the new concept of $\tilde{\Delta}$ - closed sets can be extended to bitopological spaces, fuzzy topological spaces and digital topological spaces.

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