

Separation Axioms for Bitopological Spaces

BY

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INTRODUCTION

The separation properties play an important role in the study of topological spaces because the separation properties enable us to state with precision that whether a given topological space has enough open sets. Open sets possessed by a topological space is intimately related to continuous functions on the space and continuous functions are of central importance in the study of topology. If the only open sets in a topological space are the empty set and full space, then the only continuous functions are the constant functions. In the case of discrete spaces every function is continuous. But however only a few important spaces are discrete and so this goes a bit far off. The separation properties make it possible for us to ensure that our spaces have enough continuous functions without committing ourselves to the excess of discrete spaces.

The concept of Bitopological spaces was introduced by J.C. Kelly in 1963. A set X together with two topologies J_1, J_2 is called a bitopological space. The separation axioms are generalised to Bitopological spaces and they are pairwise T_0 , pairwise $T_{\frac{1}{2}}$, pairwise T_1 , pairwise rT_1 , pairwise semi T_1 , pairwise semi rT_1 , pairwise semi T_2 , Pairwise rT_2 etc...

In this dissertation the author wants to discuss separation axioms for biotopological spaces. The material presented here are mainly taken from the following article.

(i) On biotopological separation properties by Ivan L. Reilly [5].

(ii) Separation axioms for bitopological spaces by S.P. Arya [1].

Section 1

In this section the author studies T_0 , T_1 , T_2 , regular, normal, completely regular and completely normal spaces and the relation between them.

DEFINITION: 1.1

Let X be a set and J be a set whose members are subsets of X . J is called a topology on X if it satisfies the following conditions:

(1) $\emptyset \in J$ and $X \in J$.

if $A, B \in J$, then $A \cap B \in J$

(3) The union of any member of J belongs to J . That is J is closed under arbitrary unions.

NOTE :1.2.

Members of J are called J -opensets

EXAMPLE :1.3.

Let $X = \{1, 2\}$ and $J = \{\emptyset, X, \{1\}\}$.

Then J is a topology on X .

DEFINITION :1.4.

If X is a set, and J a topology on X , then the ordered pair (X, J) is called a topological space.

DEFINITION :1.5.

In a topological space X , a set $A \subset X$ is called regularly open if it is the interior of its own closure

or equivalently if it is the interior of some closed sets.

REMARKS :1.6.

Regularly opensets are always open. But the converse need not be true.

EXAMPLE :1.7.

Let $X = \{a, b, c\}$ $J = \{X, \phi, \{b\}\}$

$\{b\}$ is open But is not regularly open.

DEFINITION :1.8.

The complement of a regularly open set is called regularly closed set.

DEFINITION :1.9.

A set is called δ -open if it is expressible as the union of regularly open sets. The complement of δ -open set is called δ -closed.

DEFINITION 1.10.

A point x in a topological space X is said to be δ -adherent point of a set $A \subset X$ if every regularly open set containing x has non-empty intersection with A . The set of all δ adherent pointsof the set A is called $\delta\text{-cl}A$.

NOTE : 1.11.

A is δ -closed if and only if $\delta\text{-cl}A=A$.

DEFINITION :1.12.

Let X be a topological space. Then X is said to be T_0 space if it satisfies the following condition:

$x \neq y \Rightarrow$ There exists a neighbourhood of one of the points not containing the other.

EXAMPLE :1.13.

Any discrete space is T_0 space.

DEFINITION :1.14.

A topological space X is said to be rT_0 if for any two distinct points of X , there exists a regularly open set containing one of the points but not the other or equivalently, There exists a δ -open set containing one of the points but not the other.

obviously every rT_0 space is T_0 space.

Converse need not be true.

EXAMPLE :1.15.

$$\text{Let } X = \{a, b, c\}$$

$$J = \{X, \{a\}, \{b\}\}$$

Then (X, J) is T_0 but not rT_0 space.

DEFINITION :1.16.

A topological space (X, J) is called a T_1 space if for every pair of distinct points x, y there exists a neighbourhood of x which doesnot contain y .

EXAMPLE :1.17.

The real line R with the standard topology is T_1 space.

REMARK :1.18.

Every T_1 space is T_0 space. But the converse need not be true.

EXAMPLE :1.19.

$$\text{Take } X = \{a, b, c\}$$

$$J = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$$

Then (X, J) is T_0 space But it is not T_1 space

DEFINITION :1.20

A space X is said to be rT_1 if whenever x and y are distinct points in X , There exists a regularly openset

containing x but not y or equivalently. There exists a δ -open set containing x but not y .

Always rT_1 spaces are rT_0 spaces.

The converse need not be true.

EXAMPLE :1.21.

Let $X = \{ a, b, c \}$

$J = \{ X, \emptyset, \{a\}, \{b\} \}$

Then (X, J) is rT_0 but not rT_1 space.

DEFINITION :1.22.

A topological space (X, J) is called a T_2 space or hausdorff space if for every pair of distinct points x, y of X , There exists disjoint neighbourhood U and V of x and y respectively.

EXAMPLE :1.23.

Any discrete space is T_2 space.

REMARK :1.24.

Every T_2 space is a T_1 space

But converse need not be true.

EXAMPLE :1.25.

The cofinite space is T_1 space. But it is not T_2 space

THEOREM :1.26

For any space X , the following statements are equivalent to one another.

- (1) X is T_2
- (2) The diagonal Δ in X^2 is closed.
- (3) For every pair of disjoint compact sets A and B there exists disjoint neighbourhoods of A and B .

PROOF:

(1) \Rightarrow (2)

Let X be T_2

To prove diagonal Δ in X^2 is closed.

We prove Δ^c is open.

Let $(x,y) \in \Delta^c$.

$\Rightarrow x \neq y$

Let U, V be disjoint neighbourhoods of x and y respectively.

Then $(x,y) \in U \times V \subseteq \Delta^c$

Assume (2) To Prove (3).

We know if $A \times B$ is a compact subset of $X \times X$ and if N is a neighbourhood of $A \times B$, then there exists open sets U and V such that $A \times B \subseteq U \times V \subseteq N$.

Now Δ^c is an open neighbourhood of AxB , there are open sets U and V such that

$$AxB \subset UxV \subset \Delta^c$$

Then U, V are disjoint neighbourhoods of A, B respectively.

To prove (3) \Rightarrow (1)

As every singleton set is compact we get X is T_2 .

DEFINITION :1.27.

A topological space X is called rT_2 space if for every pair of distinct points x, y of X there exists disjoint regularly open set U and V of x and y respectively.

REMARK :1.28.

Every rT_2 space is rT_1 space

DEFINITION :1.29.

Let X be a set and let $\mathcal{B} \subset \rho(X)$ be non-empty. \mathcal{B} is called a filterbase on X if

$$(1) \quad \emptyset \notin \mathcal{B}$$

(2) if $B_1, B_2 \in \mathcal{B}$, There exists a $B' \in \mathcal{B}$ such that

$$B' \subset B_1 \cap B_2.$$

DEFINITION :1.30.

Let (X, \mathcal{J}) be a topological space and \mathcal{F} be a filter on X . \mathcal{F} is said to \mathcal{J} -Converge to x if $N(x) \in \mathcal{F}$.

DEFINITION :1.31.

An ordered set (D, \leq) is called a directed set if every doubleton $\{a, b\}$ has an upper bound.

DEFINITION :1.32.

If D is directed set and X is any set, a function $\nu: D \rightarrow X$ is called a net in X .

DEFINITION :1.33.

A topological space X is said to be regular if given a point x and a non-empty closed set F not containing x there exists disjoint neighbourhoods of x and F .

EXAMPLE :1.34.

The space \mathbb{R}_1^2 is regular.

DEFINITION :1.35.

A topological space is called a second countable space if there is a countable basis for the topology.

DEFINITION :1.36.

A topological space X is said to be normal if every pair of disjoint non-empty closed subsets A, B of X there exists disjoint neighbourhoods U, V such that $A \subset U$ and $B \subset V$.

EXAMPLE :1.37.

Any indiscrete space is normal.

DEFINITION :1.38.

A topological space X is said to be completely regular if given a point x and a non-empty closed set F not containing x if there exists a continuous function $f: X \rightarrow [0,1]$ such that $f(x) = 1$ $f(F) = 0$, where one point sets are closed in X .

REMARK :1.39.

Every completely regular space is regular.

DEFINITION :1.40

A space which is completely regular and T_0 is called a Tychonoff space.

REMARK :1.41.

Every Tychonoff space is regular.

DEFINITION :1.42.

A space which is normal and T_1 is called T_4 space.

DEFINITION :1.43.

A space X is said to completely normal if every subspace of X is normal.

DEFINITION :1.44.

A space X is perfectly normal if it is normal and each closed subset is a G_δ .

SECTION II

In this section the author analysis the following concepts in bitopological spaces.

pairwise T_0 , pairwise $T_{\frac{1}{2}}$, pairwise rT_0 , pairwise T_1 , pairwise rT_1 , weakly pairwise rT_1 , pairwise T_2 , pairwise rT_2 , pairwise semi- rT_1 , pairwise semi rT_2 , etc. Here we find that many results which are true for topological spaces can be generalised for bitopological spaces also.

DEFINITION :2.1.

Let (X, J_1, J_2) be a bitopological space. Then (X, J_1, J_2) is called pairwise T_0 if for each pair of distinct points of X there is a set which is either J_1 open or J_2 open containing one of the points but not the other.

EXAMPLE : :2.2.

If X is any non singleton set, J_1 is the indiscrete topology on X . and J_2 the discrete topology on X , then (X, J_1, J_2) is pairwise T_0 bitopological space which is not a pair of T_0 topologies.

DEFINITION :2.3.

A biotopological space (X, J_1, J_2) is pairwise $T_{\frac{1}{2}}$ if for each pair x, y of points in X there is either a J_1 openset U such that $x \in U$ and $y \notin U$ or a J_2 open set V that $y \in V$ and $x \notin V$, when $x \neq y$.

Pairwise $T_{\frac{1}{2}} \Rightarrow$ Pairwise T_0

Converse is not true.

EXAMPLE :2.4.

Let X be the set $\{a, b, c\}$ J_1 be the topology $\{\emptyset, \{a\}, X, \{b, c\}\}$ and J_2 the topology $\{\emptyset, \{b\}, X\}$.
Then (X, J_1, J_2) is pairwise T_0 but not pairwise $T_{\frac{1}{2}}$.

For consider the pair of points b, c . The J_1 open set containing b contains c also.

DEFINITION :2.5.

It is obvious that (X, J_1, J_2) is pairwise T_0 if either (X, J_1) or (X, J_2) is T_0 and (X, J_1, J_2) is pairwise $T_{\frac{1}{2}}$ if either (X, J_1) or (X, J_2) is T_1 .

THEOREM :2.6

(X, J_1, J_2) is pairwise T_0 if and only if given two points of X either their J_1 closures are distinct or their J_2 -closures are distinct.

PROOF:

Suppose $x, y \in X$, $x \neq y$ and either $J_1\text{-cl}\{x\} \neq J_1\text{-cl}\{y\}$ or $J_2\text{-cl}\{x\} \neq J_2\text{-cl}\{y\}$.

For this first case there is a point $z \in X$ such that say $z \in J_1\text{-cl}\{y\}$ but $z \notin J_1\text{-cl}\{x\}$. Then $Y \in J_1\text{-cl}\{x\} \Rightarrow z \in J_1\text{-cl}\{x\}$ so that $y \notin J_1\text{-cl}\{x\}$. Then $U = X - J_1\text{-cl}\{x\}$ is a J_1 open set containing y but not x and hence (x, J_1, J_2) is pairwise T_0 .

A similar argument suffices if $Z \in J_1\text{-cl}\{x\}$ and $z \notin J_1\text{-cl}\{y\}$ and also in the case when $J_2\text{-cl}\{x\} \neq J_2\text{-cl}\{y\}$.

Conversly if (X, J_1, J_2) is pairwise T_0 , $x, y \in X$ and $x \neq y$ four cases arise.

- (i) There is J_1 openset U containing x but not y .
- (ii) There is a J_1 openset containing y but not x .
- (iii) There is a J_2 open set containing x but not y .
- (iv) There is a J_2 open set contining y but not x .

Consider case(i). Then $y \in J_1\text{-cl}\{y\} \subset X - U$ while

$x \notin J_1\text{-cl}\{y\}$ as $x \notin X - U$
 So $J_1\text{-cl}\{x\} \neq J_1\text{-cl}\{y\}$.

Similarly we can prove with other cases.

DEFINITION :2.7.

A space (X, J_1, J_2) is said to be pairwise rT_0 if for any two distinct points of X there is a set which is either J_1 -regularly open or J_2 -regularly open set containing one of the points but not the other or equivalently there exists a J_1 - δ open set or J_2 - δ open set containing one of the points but not the other. obviously every pairwise rT_0 space is pairwise T_0 . But converse need not be true.

EXAMPLE :2.8.

Let $X = \{a, b, c\}$, $J_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$

$J_2 = \{X, \emptyset, \{b\}\}$

Then clearly (X, J_1, J_2) is pairwise T_0 but not pairwise rT_0 Because

J_1 regularly opensets are $X, \emptyset, \{a\}, \{a, b\}$

J_2 regularly opensets are X, \emptyset .

for the distinct points $\{b, c\}$ of X there exists J_1 regularly open set $\{b, c\}$ containing b together with c .

therefore, (X, J_1, J_2) is not pairwise rT_0 .

THEOREM :2.9.

A space (X, J_1, J_2) is pairwise rT_0 if and only if given two distinct points of X either J_1 - δ -closures are distinct or their J_2 δ -closures are distinct.

PROOF:

Let (X, J_1, J_2) be a pairwise rT_0 space and let $x, y \in X$ be two distinct points suppose U is a J_1 δ -openset containing x but not y then $Y \in J_1$ - δ cl $\{y\}$ $CX-U$ and so $x \notin J_1$ - δ cl $\{y\}$

Therefore, J_1 - δ cl $\{x\} \neq J_1$ - δ cl $\{y\}$

Conversely, let x, y be two distinct points of X .

then either J_1 - δ cl $\{x\} \neq J_1$ - δ cl $\{y\}$ or

J_2 - δ cl $\{x\} \neq J_2$ - δ cl $\{y\}$

In the former case let p be a point of X such that $P \in J_1$ - δ cl $\{y\}$ and $P \notin J_1$ - δ cl $\{x\}$.

We assert that $y \notin J_1$ - δ cl $\{x\}$

if $y \in J_1$ - δ cl $\{x\}$ then J_1 - δ cl $\{y\} \subset J_1$ - δ cl $\{x\}$

So that $P \in J_1$ - δ cl $\{y\} \subset J_1$ - δ cl $\{x\}$.

This contradicts the fact that $P \notin J_1$ - δ cl $\{x\}$

Hence $y \notin J_1$ - δ cl $\{x\}$.

Thus $U = X - J_1$ - δ cl $\{x\}$ is a J_1 - δ openset containig y but

not x . The case J_2 - δ cl $\{x\} \neq J_2$ δ cl $\{y\}$ can be dealt with similarity.

THEOREM: 2.10.

A space (X, J_1, J_2) is pairwise rT_0 if either (X, J_1) , or (X, J_2) is rT_0 .

PROOF:

Obvious from the definition of pairwise rT_0 spaces, converse is not true as given by the following Example.

EXAMPLE: 2.11.

Let $X = \{a, b, c\}$, $J_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$ and
 $J_2 = \{X, \emptyset, \{c\}, \{a, b\}\}$. Then clearly the
 space (X, J_1, J_2) is pairwise rT_0 but neither (X, J_1) nor
 (X, J_2) is rT_0 .

Since, In (X, J_1) the regularly open set containing b also contains c and in (X, J_2) the regularly open set containing a also contains b .

DEFINITION : 2.12.

The bitopology space (X, J_1, J_2) is pairwise T_1 if for each pair of distinct points x and y of X there is a J_1 open set U and a J_2 open set V so that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

RESULTS :2.13.

A pairwise T_1 bitopological space reduces to a pair of the corresponding topological spaces namely T_1 spaces.

RESULT:2.14.

Pairwise $T_1 \Rightarrow$ Pairwise T_0 .

THEOREM: 2.15.

(X, J_1, J_2) is pairwise T_1 , iff (X, J_1) and (X, J_2) are T_1 .

PROOF:

By the definition of pairwise T_1 spaces, the proof follows.

COROLARY :2.16.

(X, J_1, J_2) is pairwise T_1 iff each singleton set is both J_1 closed and J_2 closed.

If (X, J_1, J_2) is pairwise T_1 then each singleton set is closed as for each point of X , no point other than that point will become limit point.

Conversly if $x \neq y \in X$ are such that $\{x\}$ $\{y\}$ are J_1 closed and J_2 closed then $X - \{x\}$, $X - \{y\}$ are J_1 open and J_2 open sets containing y and x respectively.

Hence the proof.

DEFINITION :2.17.

Let (X, J_1, J_2) be a biotopological space. Then (X, J_1, J_2) is weakly pairwise rT_1 if for every pair of distinct points x, y of X there exists a J_1 regularly open or a J_2 regularly open set containing x but not y . or equivalently there exists a J_1 - ϕ open or J_2 - ϕ open set containing x but not y .

obviously every weakly pairwise rT_1 space is pairwise rT_0 . But the converse need not be true as seen from the following example.

EXAMPLE :2.18.

Let $X = \{a, b, c\}$ $J_1 = \{X, \phi, \{a\}, \{b, c\}\}$ and $J_2 = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. then clearly the space (X, J_1, J_2) is pairwise rT_0 . It is not weakly pairwise rT_1 .

Therefore, J_1 regularly opensets are $X, \phi, \{a\}, \{b, c\}$, J_2 regularly opensets are $X, \phi,$

Now (X, J_1, J_2) is pairwise rT_0 . But consider the pair of points b, c of X we observe that any J_1 -regularly open set containing b also contains c and any J_2 regularly open set containing c also contains b .

Definition: 2.19

A space (X, J_1, J_2) is said to be weakly pairwise T_1 if for every pair of distinct points x and y of X , there exists a J_1 -open set or a J_2 -open set containing x but not y .

Theorem: 2.20

Every weakly pairwise rT_1 bitopological space is weakly pairwise T_1 .

Proof:

As regularly open sets are open the proof follows immediately.

The converse is not true as given by following example.

Example: 2.21

Consider $X = \{a, b\}$

$$J_1 = \{X, \emptyset, \{a\}, \{b\}\}$$

$$J_2 = \{X, \emptyset, \{a\}\}$$

Then clearly (X, J_1, J_2) is weakly pairwise T_1 .

But (X, J_1, J_2) is not weakly pairwise rT_1 . For the J_1 regularly open sets are $X, \emptyset, \{a\}, \{b\}$

J_2 regularly open sets are X, \emptyset

Hence any J_2 regularly open set containing a also contains b and any J_2 regularly open set containing b also contains a .

Theorem 2.22

The following statements are equivalent

- (i) (X, J_1, J_2) is weakly pairwise rT_1 space
(ii) $J_1-\delta cl\{x\} \cap J_2-\delta cl\{x\} = \{x\}$ for all $x \in X$.
(iii) For every $x \in X$, the intersection of all $J_1-\delta$ -neighbourhood and all $J_2-\delta$ -neighbourhoods of x is $\{x\}$.

Proof:

To prove (i) \Rightarrow (ii)

Let $x \in X$ and $y \in J_1-\delta cl\{x\} \cap J_2-\delta cl\{x\}$ where $y \neq x$.

Since X is weakly pairwise rT_1 space, there is a $J_1-\delta$ open set U such that $y \in U$, $x \notin U$ or there is a $J_2-\delta$ open set V such that $y \in V$ and $x \notin V$.

In either case $y \notin J_1-\delta cl\{x\} \cap J_2-\delta cl\{x\}$.

Hence $\{x\} = J_1-\delta cl\{x\} \cap J_2-\delta cl\{x\}$

To prove (ii) \Rightarrow (iii)

If $x, y \in X$ such that $x \neq y$ then $x \notin J_1-\delta cl\{y\} \cap J_2-\delta cl\{y\}$ so there is a $J_1-\delta$ open set or a $J_2-\delta$ open set containing x but not y . Therefore y does not belong to be intersection of all $J_1-\delta$ neighbourhood and all $J_2-\delta$ neighbourhoods of x .

To prove (iii) \Rightarrow (i)

Let x, y be 2 distinct points of X . By hypothesis y does not belong to a $J_1-\delta$ neighbourhood or $J_2-\delta$ neighbourhood of x . Therefore there exists a $J_1-\delta$ open set or a $J_2-\delta$ open set containing x but not y .

Therefore X is weakly pairwise rT_1 space.

Definition: 2.23

A space (X, J_1, J_2) is said to be pairwise rT_1 . If for each pair of distinct points x, y of X there exists a J_1 -regularly open set containing x but not y and a J_2 -regularly open set containing y but not x or equivalently there exists a J_1 - δ open set containing x but not y and a J_2 - δ open set containing y but not x .

Every pairwise rT_1 space is pairwise T_1 but the converse may be false as given by the following example.

Example 2.24

Let R be the set of all real numbers and J the co-countable topology. Then (X, J, J) is pairwise T_1 , but not pairwise rT_1 .

Because the only J -regularly open sets are R and \emptyset

Theorem 2.25

A space (X, J_1, J_2) is pairwise rT_1 if and only if (X, J_1) and (X, J_2) are rT_1 .

Proof:

Let (X, J_1, J_2) be pairwise rT_1 space. Let x, y be distinct points of X . Then there exists a J_1 regularly open set U such that $x \in U, y \notin U$ and similarly J_2 regularly open set V such that $y \in V$ and $x \notin V$.

Thus (X, J_1) is rT_1 and similarly (X, J_2) .

Result: 2.26

It follows from the proceeding theorem that every pairwise rT_1 space is weakly pairwise rT_1 and a pairwise rT_1 space is nothing but a bi- rT_1 space we also note that a weakly pairwise rT_1 space need not be pairwise rT_1 .

Corollary: 2.27

A space (X, J_1, J_2) is pairwise rT_1 if and only if each singleton is both J_1 δ -closed and J_2 δ -closed.

Proof:

By the definition of pairwise rT_1 space the proof follows.

Definition: 2.28

(X, J_1, J_2) is pairwise T_2 if for each pair of distinct points x and y of X there is a J_1 open set U and a J_2 open set V such that $x \in U$, $y \in V$ and U and V are disjoint.

Clearly pairwise $T_2 \Rightarrow$ Pairwise T_1 .

But converse is not true as given by following example.

Example: 2.29

Let X be the set of real numbers. J_1 be \emptyset

together with complements of countable sets and J_2 be the usual topology. Then (X, J_1, J_2) is pairwise T_1 . However any non-empty J_2 open set is uncountable and so intersects any non-empty J_1 open sets. Thus (X, J_1, J_2) is not pairwise T_2 .

If we make J_2 the discrete topology then (X, J_1, J_2) is pairwise T_2 space which is not a pair of T_2 topology.

Proposition: 2.30

The following are equivalent.

- (1) (X, J_1, J_2) is pairwise T_2 .
- (2) Let $x \in X$. For each point y distinct from x there is a J_1 open set U such that $x \in U$ and y does not belong to J_2 -cl U .
- (3) For each $x \in X$, $\{x\} = \bigcap \{ J_2$ -cl $U : x \in U$ and U is J_1 open $\}$
- (4) The diagonal $\Delta = \{ (x, x) : x \in X \}$ is closed in $(X \times X, J_1 \times J_2)$.

Proof:

Assume (1) To prove (2).

Let $x \in X$ and y be a point such that $x \neq y$. Then by the assumption there is a J_1 open set U and a J_2 open set V such that $x \in U$ and $y \in V$, U, V are disjoint.

$$\text{Now } U \cap V = \emptyset$$

$$\Rightarrow U \subset X - V$$

Now $X - V$ is a J_2 closed set such that

$y \notin X - V$ and $x \in X - V$.

Therefore $X - V$ is a J_2 closed set containing x .

Therefore $J_2 - cl U \subset X - V$.

$y \notin X - V \Rightarrow y \notin J_2 - cl U$.

Assume (2). To prove (3)

For any $x \in X$, and for all $y \in X$ distinct from x there is a J_1 open set U such that $x \in U$ and $y \notin J_2 - cl U$ (i.e) $x \in J_2 - cl U$.

For each $x \in X$, $\{x\} = \bigcap \{ J_2 - cl U : x \in U \text{ and } U \text{ is } J_1 \text{ open} \}$

Assume (3) To prove (4).

The diagonal $\Delta = \{ (x,x) : x \in X \}$

To prove Δ is closed.

We know $\{x\} = \bigcap \{ J_2 - cl U : x \in U \text{ and } U \text{ is } J_1 \text{ open.} \}$

Always $\Delta \subset \bar{\Delta}$

To prove $\bar{\Delta} \subset \Delta$

We prove any element not in Δ also does not belong to $\bar{\Delta}$.

Take $(x,y) \in \bar{\Delta}$

$\Rightarrow x \neq y$

$\Rightarrow y \notin \bigcap \{ J_2 - cl U : x \in U, U \text{ is } J_1 \text{ open} \}$.

$\Rightarrow y \notin J_2 - cl U$ for all J_1 open set U .

Now $X - J_2 cl U$ is a J_2 open set containing y .

Now $U \times X - J_2 cl U$ is a nhd of (x,y) .

We prove $U \times X - J_2 cl U \cap \Delta = \emptyset$

Where U and $X - J_2 cl U$ are disjoint.

Let $(x^1, y^1) \in (U \times X - J_2 cl U) \cap \Delta$

$\Rightarrow (x^1, y^1) \in U \times X - J_2 \text{ cl } U$ and $(x^1, y^1) \in \Delta$
 $(x^1, y^1) \in \Delta \Rightarrow x^1 = y^1$

$\Rightarrow U \cap X - J_2 \text{ cl } U \neq \emptyset$ which is a contradiction.

Therefore $U \times X - J_2 \text{ cl } U \cap \Delta = \emptyset$

Therefore $\bar{\Delta} \subset \Delta \Rightarrow \Delta = \bar{\Delta}$.

Therefore Δ is closed.

Assume (4). To prove (1)

To prove (X, J_1, J_2) is pairwise T_2 .

Let $x, y \in X$ be such that $x \neq y$.

Therefore $(x, y) \notin \Delta = \bar{\Delta}$.

$\Rightarrow (x, y) \notin \bar{\Delta}$.

\Rightarrow There exists one open set of (x, y) such that its intersection with Δ is empty.

Take $U \times V$ as the nhd which is $J_1 \times J_2$ open set such that

$U \times V \cap \Delta = \emptyset$

Now U is a J_1 open set such that $x \in U$.

V is a J_2 open set such that $y \in V$

We prove $U \cap V = \emptyset$.

If not let $z \in U \cap V$.

$\Rightarrow z \in V$ and $z \in U$.

Now $(z, z) \in U \times V$ and $(z, z) \in \Delta$ by definition of $\Delta \Rightarrow U \times V \cap \Delta \neq \emptyset$
 a contradiction.

Therefore $U \cap V = \emptyset$

(X, J_1, J_2) is pairwise T_2 .

Pairwise T_2 spaces can be characterized in terms of convergence properties.

Theorem 2.31

(X, J_1, J_2) is pairwise T_2 . iff for each filter base J on X such that J is J_1 convergent to x and J_2 convergent to y then $x = y$.

Proof:

Let (X, J_1, J_2) be pairwise T_2 and $J = \{A_\alpha : \alpha \in I\}$ be a filter base on X which is J_1 convergent to x . Let y be a point of X distinct from x . Then there is a J_1 open set U and a J_2 -open set V disjoint from U such that $x \in U$ and $y \in V$.

There is an $A_\alpha \in J$ such that $A_\alpha \subset U$ and since any two elements A_α, A_β of J have non-empty intersection there is no $A_\beta \in J$ such that $A_\beta \subset V$. Therefore J is not J_2 convergent to y .

Conversely assume that (X, J_1, J_2) is not pairwise T_2 . Then there is a pair of distinct points x, y of X such that each J_1 -open set containing x and each J_2 open set containing y have non-empty intersection.

Then J , the family of all such intersection is a filter base on X which is J_1 convergent to x and J_2 convergent to y .

Corollary: 2.32

(X, J_1, J_2) is pairwise T_2 iff for each net on X which is J_1 convergent to x and J_2 convergent to y , $x = y$

Proof:

Similar to above theorem.

Definition: 2.33

The topological space (X, J_1, J_2) is pairwise semi T_2 if for each pair of distinct points x and y in X there is a J_1 open set U and a J_2 open set V disjoint from U such that either $x \in U, y \in V$ or $y \in U, x \in V$

Remark: 2.34

Clearly pairwise $T_2 \Rightarrow$ Pairwise semi T_2 .

Remark: 2.35

If (X, J_1, J_2) is pairwise semi- T_2 then (X, J_1) and (X, J_2) are T_0 topological spaces.

But converse need not be true always.

Example: 2.36

Let (X, J_1, J_2) be such that X is the set of all real numbers. J_1 be the collection consisting of \emptyset , together with complements of countable sets and J_2 the usual topology. Then in (X, J_1, J_2) , (X, J_1) and (X, J_2) are T_0 bitopological spaces but (X, J_1, J_2) is not pairwise semi- T_2

Definition: 2.37

A space (X, J_1, J_2) is said to be pairwise semi- rT_2 if for every pair of distinct points x, y of X there exists a J_1 regularly open set U and a disjoint J_2 regularly open set V such that $x \in U, y \in V$ or $x \in V, y \in U$

Obviously every pairwise semi- rT_2 space is pairwise semi- T_2 . But not conversely.

Example: 2.38

Let $X = \{a, b\}$, J_1 is the discrete topology on X and $J_2 = \{X, \emptyset, \{a\}\}$. Then (X, J_1, J_2) is pairwise semi- rT_2 but not pairwise semi- rT_2 .

Since Any J_2 regularly open set containing a also contains b .

Theorem: 2.39

If (X, J_1, J_2) is pairwise semi- rT_2 then (X, J_1) and (X, J_2) are both rT_0 spaces.

Proof:

Obvious from the definition of pairwise semi- rT_2 spaces.

Corollary: 2.40

Every pairwise semi- rT_2 space is weakly pairwise rT_1 .

proof follows from the definition of semi rT_2 spaces and weakly pairwise rT_1 spaces.

Definition: 2.41

The bitopological space (X, J_1, J_2) is said to be pairwise rT_2 if for every pair of distinct points x, y of X there exists J_1 -regularly open set U and a J_2 -regularly open set V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

Theorem: 2.42

Every pairwise rT_2 space is bi- rT_1 .

Proof:

From the definition of pairwise rT_2 space, the proof follows:

Corollary: 2.43

Every pairwise rT_2 space is pairwise rT_1 . The Converse may not be true.

For consider the example given below.

Example: 2.44

Let $X = \{a, b, c\}$.

$$J_1 = \{X, \emptyset, \{a\}, \{c\}, \{a, c\}\}$$

$$J_2 = \{X, \emptyset, \{b, c\}, \{a\}, \{a, c\}, \{c\}, \{a, b\}, \{b\}\}.$$

J_1 -regularly open sets are $X, \emptyset, \{a, b\}, \{c, b\}$

J_2 -regularly open sets are $X, \emptyset, \{a, c\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}$

Now Consider the points $a \neq b$ of X .

Then there exists a J_2 regularly open set $\{a, c\}$ contains a but not b and J_1 -regularly open set $\{b, c\}$

containing b not a and similarly with other distinct pair of points of X .

Hence (X, J_1, J_2) is pairwise rT_1 .

But $\{a, c\} \cap \{b, c\} \neq \emptyset$.

Therefore (X, J_1, J_2) is not pairwise rT_2 .

Definition: 2.45

A subset A of a space (X, J_1, J_2) is said to be N -closed relative to J if every cover of A by regular open sets of X has a finite subcover.

Theorem: 2.46

Let A be a subset of a pairwise rT_2 space (X, J_1, J_2) which is N -closed relative to J_2 , then A is J_2 - δ closed.

Proof:

If $A = X$, then A is obviously $J_2\delta$ -closed.

If $A \neq X$, then there is a point $x \in X-A$.

Since X is pairwise rT_2 for each $y \in A$, there exists open set V_y such that $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$.

Then $\{V_y : y \in A\}$ is a J_1 regularly open cover of the set A which is N -closed relative to J_1 .

so it has a finite subcover say $V_{y_1}, V_{y_2}, \dots, V_{y_n}$.

$$\text{Let } U = \bigcap_{i=1}^n U_{y_i}, \quad V = \bigcap_{i=1}^n V_{y_i}$$

Then U is a J_2 open, V is J_1 -open and $x \in U$, $A \subset V$ and $U \cap V = \emptyset$.

Thus $x \in U \subset X-A$ and so $X-A$ is J_2 - δ open

$\Rightarrow A$ is $J_2 \delta$ -closed.

Hence the proof.

The implications between pairwise separation axioms are given below.

$$\begin{array}{ccc}
 \text{Pairwise semi-} T_2 & \Rightarrow & \text{weakly pairwise } T_1 \\
 \Uparrow & & \Uparrow \\
 \text{Pairwise semi-} rT_2 & \Rightarrow & \text{weakly pairwise - } rT_1 \\
 \Uparrow & & \Downarrow \\
 \text{Pairwise } rT_2 & \Rightarrow & \text{Pairwise } rT_1 \Rightarrow \text{Pairwise } rT_0 \\
 \Downarrow & & \Downarrow \\
 \text{Pairwise } T_2 & \Rightarrow & \text{Pairwise } T_1 \Rightarrow \text{Pairwise } T_0
 \end{array}$$

Section 3

In this section pairwise regular, Pairwise T_3 , Pairwise normal, pairwise T_4 , Pairwise completely normal, pairwise T_5 , Pairwise perfectly normal and pairwise Tychonoff spaces are studied.

defintion: 3.1

Let (X, J_1, J_2) be a bitopological space. Then J_1 is regular with respect to J_2 if for each point x in X and each J_1 -closed set P such that $x \notin P$ there is a J_1 -open set U and J_2 open set V disjoint from U such that $x \in U$ and $P \subset V$. (X, J_1, J_2) is pairwise regular if J_1 is regular with respect to J_2 and J_2 is regular with respect to J_1 .

Proposition: 3.2

If (X, J_1, J_2) is a bitopological space the following are equivalent.

1. J_1 is regular with respect to J_2 .
2. For each point $x \in X$ and J_1 open set G containing x there is a J_1 open set H such that $x \in H \subset J_2\text{-cl}H \subset G$.
3. For each point $x \in X$ and J_1 closed set K such that $x \notin K$ there is a J_1 -open set M such that $x \in M$ and $(J_2\text{-cl}M) \cap K = \emptyset$.

Proof:

Assume (1). To prove (2)

Take $x \in X$ and J_1 open set G such that $x \in G$

Now $X - G$ is a J_1 closed set and $x \notin X - G$

By assempcion there exists a J_1 open set H and J_2 -open set V such that $x \in H$ and $X - G \subset V$.

where $H \cap V = \varnothing$.

$H \cap V = \varnothing$.

$\Rightarrow H \subset X - V$, $X - V$ is J_2 closed set containing H .

Therefore $J_2\text{-cl } H \subset X - V \subset G$ Since $X - G \subset V$.

$x \in H \subset J_2\text{-cl } H \subset G$

Assume (2). To prove (3)

Take $x \in X$ and J_1 closed set K such that $x \notin K$.

Now $X - K$ is J_1 open set containing x .

Therefore By the assumption there exists a J_1 open set M such that $x \in M$ and $M \subset J_2\text{-cl } M \subset X - K$.

$\Rightarrow x \in M$ and $(J_2\text{-cl } M) \cap K = \varnothing$.

Assume (3). To prove (3)

Let $x \in K$ and P be a J_1 closed set such that $x \notin P$.

By assumption there exists a J_1 open set M such that $x \in M$ and $(J_2\text{-cl } M) \cap P = \varnothing$.

$\Rightarrow J_2\text{-cl } M \subset X - P$

$\Rightarrow P \subset X - J_2\text{-cl } M$

Now $J_2\text{-cl } M$ is a J_2 closed set and hence $X - J_2\text{-cl } M$ is J_2 open set containing P in which M and $X - J_2\text{-cl } M$ are disjoint.

Hence the proof

Defintion 3.3

A bitopological space (X, J_1, J_2) is pairwise T_3 if it is pairwise regular and pairwise T_1 .

Remark: 3.4

Pairwise $T_3 \Rightarrow$ Pairwise T_2

But the converse need not be true is shown by the following example.

Example 3.5

Let X be the set of all real numbers. J_1 be the left hand topology and J_2 be the right hand topology. It is easy to show that (X, J_1, J_2) is pairwise regular and pairwise T_0 .

However it is not pairwise T_2 for if $x, y \in X$ and $x > y$ then every J_1 open set containing x contains y and every J_2 open set containing y also contains x .

Theorem: 3.6

A bitopological space (X, J_1, J_2) is pairwise T_3 iff it is pairwise regular and pairwise $T_{\frac{1}{2}}$

Proof

We only need to establish the sufficiency.

Let x and y be distinct points X .

Since (X, J_1, J_2) is pairwise $T_{\frac{1}{2}}$ two cases arise.

(i) There is a J_1 open set U containing x but not y .

Since $x \notin X - U$, pairwise regularity gives a J_1 open set M containing x and a J_2 open set N disjoint from M such that $X - U \subset N$.

Thus, $x \in M$, $y \notin M$ and $x \notin N$.

(ii) There is a J_2 open set containing y but not x .

A similar proof suffices.

Definition 3.7

Let (X, J_1, J_2) be a bitopological space. Then (X, J_1, J_2) is pairwise normal if for each J_1 closed set A and a J_2 closed set B disjoint from A there is a J_1 open set V containing B and a J_2 open set U disjoint from V containing A .

Proposition 3.8

The following are equivalent

- (1) (X, J_1, J_2) is pairwise normal.
- (2) For each J_1 closed set A and a J_2 open set H containing A there is a J_2 open set U such that $A \subset U \subset J_1\text{-cl} U \subset H$.
- (3) For each J_1 closed set A and J_2 closed set B disjoint from A there is a J_2 open set U containing A and such that $(J_1\text{-cl } U) \cap B = \emptyset$.
- (4) For each J_1 closed set A and J_2 closed set B disjoint from A there is a J_1 open set V containing B and a J_2 open set U containing A such that $(J_1\text{-cl } U) \cap (J_2\text{-cl } V) = \emptyset$.

Proof:

Assume (1). To prove (2)

Let A be a J_1 closed set and H J_2 open set containing A . Then $X - H$ is J_2 closed set not containing A .

Assumption \Rightarrow There exists a J_1 open set V containing $X - H$ and J_2 open set U containing A .

(i.e) $A \subset U$ and $X - H \subset V \Rightarrow X - V \subset H$ and

$U \cap V = \emptyset \Rightarrow U \subset X - V$

Now $X - V$ is J_1 closed set containing U .

Therefore J_1 -cl $U \subset X - V \subset H$

Therefore $A \subset U \subset J_1$ -cl $U \subset X - V \subset H$

(i.e) $A \subset U \subset J_1$ -cl $U \subset H$

Assume (2) To prove (3)

Take A be J_1 closed and B , J_2 closed set disjoint from A .

Now $X - B$ is J_2 open set containing A .

Therefore By the assumption there is a J_2 open set U such that $A \subset U \subset J_2$ -cl $U \subset X - B$.

(i.e) J_1 -cl $U \subset X - B$.

$\Rightarrow (J_1$ -cl $U) \cap B = \emptyset$.

Assume (3). To prove (4).

Take J_1 closed set A and J_2 closed set B and disjoint from A .

By assumption there is a J_2 open set U containing A and such that $(J_1$ -cl $U) \cap B = \emptyset$.

$\Rightarrow B \subset X - J_1$ -cl U

Take $X - J_1$ -cl $U = V$. Then V is J_1 open set containing B .

We know $V \subset J_2$ -cl V .

(i.e) $X - J_1$ -cl $U \subset J_2$ -cl V

$\Rightarrow (J_1$ -cl $U) \cap (J_2$ -cl $V) = \emptyset$.

Assume (4). To prove (1)

Let A be a J_1 closed set and B be a J_2 closed set disjoint from A .

By assumption there exists a J_1 open set V containing B and J_2 open set U containing A such that

$$(J_1\text{-cl } U) \cap (J_2\text{-cl } V) = \emptyset.$$

Therefore U, V are disjoint

Hence (X, J_1, J_2) is pairwise normal.

Definition 3.9

(X, J_1, J_2) is pairwise T_4 if it is pairwise normal and pairwise T_1 .

Remark 3.10

Always Pairwise $T_4 \Rightarrow$ Pairwise T_3

But the converse (i.e) pairwise normality does not imply pairwise regularity. For there are topological spaces which are not regular.

Theorem 3.11

(X, J_1, J_2) is pairwise normal iff for each J_2 closed set F and J_1 closed set H disjoint from F there is a real valued function g on X such that $g(F) = 0$, $g(H) = 1$, $g(X) \subset [0, 1]$ and g is J_1 upper semi continuous denoted as u.s.c and J_2 lower semi continuous (l.s.c).

Proof:

Necessary Part.

Let (X, J_1, J_2) be a pairwise normal bitopological

space. Let H and F be subsets of X such that H is J_1 -closed, F is J_2 -closed and $H \cap F = \emptyset$.

Let $F_0 = F$ and let $K_1 = X/H$. Then F_0 is J_2 -closed. K_1 is J_1 open and $F_0 \subseteq K_1$, since (X, J_1, J_2) is pairwise normal, there exists a J_1 open set $K_{1/2}$ and a J_2 -closed set $F_{1/2}$ such that $F_0 \subseteq K_{1/2} \subseteq F_{1/2} \subseteq K_1$. Applying our hypothesis on (X, J_1, J_2) to each pair of sets $F_0, K_{1/2}$ and $F_{1/2}, K_1$. We obtain J_1 open sets $K_{1/4}, K_{3/4}$ and J_2 -closed sets $F_{1/4}, F_{3/4}$ such that

$$F_0 \subseteq K_{1/4} \subseteq F_{1/4} \subseteq K_{1/2} \subseteq F_{1/2} \subseteq K_{3/4} \subseteq F_{3/4} \subseteq K_1.$$

continuing this process, we obtain two families $\{F_s\}$

and $\{K_s\}$, where $s = p/2^q$ where $p = 1, 2, \dots, 2^q - 1$:

$q = 1, 2, \dots$ if s is any other dyadic rational, let

$$K_s = \emptyset (s \leq 0), K_s = X (s > 1) \text{ and } F_s = \emptyset (s < 0),$$

$$F_s = X (s \geq 1). \text{ Then } K_r \subseteq K_s \subseteq F_s \subseteq F_t (r \leq s \leq t) \text{ and}$$

$$F_s \subseteq K_t (s < t)$$

Let g be the function defined on X to $[0, 1]$

$$\text{by } g(x) = \inf \{ t/x \in K_t \}, x \in X$$

$$\text{Then } g(x) = \inf \{ t/x \in F_t \}, x \in X$$

Then clearly $0 \leq g(x) \leq 1 (x \in X)$, $g(x) = 0 (x \in F)$ and

$$g(x) = 1 (x \in [x/k_1] = H)$$

Using the sets K_s and F_s we can prove that g is J_1 - upper semi continuous and g is J_2 - lower semi continuous as it is proved in the classical Urysohn lemma.

To prove the sufficiency, let F, H and g be as described

Then $U = \{x \in X / g(x) \leq \frac{1}{2}\}$ and

$V = \{x \in X / g(x) > \frac{1}{2}\}$ are approximately open, disjoint and separate F and H .

Definition 3.12

(X, J_1, J_2) is pairwise completely normal if for each pair A, B of subsets of X satisfying $(A \cap J_2 - \text{cl } B) \cup (J_1 - \text{cl } A \cap B) = \emptyset$, there is a J_2 open set U containing A and J_1 open set V containing B and disjoint from U .

definition 3.1

(X, J_1, J_2) is pairwise T_5 if it is pairwise completely normal and pairwise T_1 .

Example 3.14

Any quasi-metric bitopological space is pairwise T_5 .

Remark 3.15

It is clear that pairwise T_5 implies pairwise T_4 .

Defintion 3.16

(X, J_1, J_2) is pairwise perfectly normal if it is pairwise normal, each J_1 closed set is J_2 G_δ and each J_2 closed set is a J_1 G_δ .

Theorem 3.17

If (X, J_1, J_2) is pairwise perfectly normal then it is pairwise completely normal.

Proof:

Assume (X, J_1, J_2) be pairwise perfectly normal.

To prove (X, J_1, J_2) is pairwise completely normal.

Take A, B be subsets of X satisfying

$$(A \cap J_2\text{-cl } B) \cup (J_1\text{-cl } A \cap B) = \varphi.$$

Then to prove the existance of J_2 open set U containing A and J_1 open set V containing B that is disjoint from U .

Now $J_2\text{-cl } B$ is a J_2 closed set and $J_1\text{-cl } A$ is a J_1 closed set.

By assempcion $J_2\text{-cl } B$ is J_1 G_δ and $J_1\text{-cl } A$ is J_2 G_δ .

(i.e) $J_2\text{-cl } B = \bigcap J_1\text{-}\delta$ open sets V_i , where i varies over a countable set.

$$J_1\text{-cl } A = \bigcap_{i \in \mathbb{Z}_+} J_2 \text{ open sets } U_i$$

$$\text{Take } \bigcap_{i \in \mathbb{Z}_+} U_i = U, \quad \bigcap_{i \in \mathbb{Z}_+} V_i = V$$

$$\text{Then } A \subset J_1\text{-cl } A = U$$

$$B \subset J_2\text{-cl } B = V$$

U is J_2 open set containing A and V is a J_1 open set containing B .

Since Countable intersection of open sets is open

$$\text{Also } (A \cap J_2\text{-cl } B) \cup (J_1\text{-cl } A \cap B) = \varnothing.$$

$$\Rightarrow U \cap V = \varnothing.$$

Hence (X, J_1, J_2) is pairwise completely normal.

Theore 3.18

Assume (X, J_1, J_2) is pairwise completely normal iff every subspace is pairwise normal.

Proof:

Assume (X, J_1, J_2) be pairwise completely normal

Let Y be a subspace of X . To prove Y is pairwise normal.

Let A be J_1 closed set and B be J_2 closed set in Y where A, B are disjoint.

$$\text{Then } A = A^1 \cap Y, \quad B = B^1 \cap Y$$

where A^1 is J_1 closed set in X and B^1 is J_2 closed set in X and A^1, B^1 are disjoint.

$$\text{Therefore } (A^1 \cap J_2\text{-cl } B^1) \cup (B^1 \cap J_1\text{-cl } A^1) = \varnothing.$$

Since (X, J_1, J_2) is pairwise completely normal, there exists a J_2 open set U containing A^1 and J_1 open set V containing B^1 and disjoint from U .

$$\text{Therefore } A^1 \subset U, \quad B^1 \subset V$$

$Y \cap A^1 \subset U \cap Y$ and $B^1 \cap Y \subset V \cap Y$.

Now $U \cap Y, V \cap Y$ are J_2 open set and J_1 open set in Y respectively.

Therefore $A \subset U \cap Y, B \subset V \cap Y$

Hence Y is pairwise normal.

Conversely assume Y be pairwise normal.

To prove (X, J_1, J_2) is pairwise completely normal.

Let A, B be subsets of X satisfying the condition

$$(A \cap J_2\text{-cl } B) \cup (J_1\text{-cl } A \cap B) = \emptyset.$$

To prove the existence of J_2 open set U containing A and

J_1 open set V containing B such that $U \cap V = \emptyset$.

Now $J_1\text{-cl } A$ and $J_2\text{-cl } B$ are closed in X .

Now $J_1\text{-cl } A \cap Y$ and $J_2\text{-cl } B \cap Y$ are J_1 closed and J_2 closed in Y respectively

Since Y is pairwise normal, as $J_1\text{-cl } A \cap Y$ and $J_2\text{-cl } B \cap Y$

are disjoint implies there exists a J_1 -open set $V \cap Y$

containing $J_2\text{-cl } B \cap Y$ and J_2 open set $U \cap Y$ containing

$J_1\text{-cl } A \cap Y$ where V, U are J_1 open and J_2 open respectively

in X .

(i.e) $J_2\text{-cl } B \cap Y \subset V \cap Y$ and $J_1\text{-cl } A \cap Y \subset U \cap Y$

(i.e) $J_2\text{-cl } B \subset V, J_1\text{-cl } A \subset U$

(i.e) $A \subset J_1\text{-cl } A \subset U$

and $B \subset J_2\text{-cl } B \subset V$ and $U \cap V = \emptyset$.

Therefore there exists J_2 open set U containing A and J_1 open set V containing B such that $U \cap V = \varphi$.

$\Rightarrow (X, J_1, J_2)$ is pairwise completely normal

Hence the proof

Theorem 3.19

(X, J_1, J_2) is pairwise perfectly normal iff for each non-empty J_1 closed set A and each point $x_0 \notin A$ there is a real valued function f on X such that $f^{-1}(0) = A$, $f(x_0) = 1$, f is J_2 u.s.c and J_1 l.s.c and $f(X) \subset [0,1]$ and for each non-empty J_2 closed set B and each point $y_0 \notin B$ there is a real valued function g on X such that $g^{-1}(0) = B$, $g(y_0) = 1$, g is J_1 u.s.c and J_2 l.s.c and $g(X) \subset [0,1]$.

Theore. 3.20

Let (X, J_1, J_2) be a bitopological space. If J_1 is regular with respect to J_2 and (X, J_1) is second countable then every J_1 closed set is a J_2 G_δ

Proof:

Let A be a J_1 closed set.

If $A = X$, there is nothing to prove

Otherwise for each $x \notin A$, there is a J_1 open set U_x such

that $x \in U_x \subset J_2\text{-cl } U_x \subset X - A$ by proposition 3.2

Let $\{V_n : n \in \mathbb{N}, \text{ the integers}\}$ be a countable base for (X, J_1)

Then $U_x = \bigcup_m \{V_m : m \in M \subset \mathbb{N}\}$

Hence $x \in V_{n(x)}$ for some integer $n(x)$

$V_{n(x)} \subset J_2\text{-cl } U_x \Rightarrow J_2\text{-cl } V_{n(x)} \cap A = \emptyset$ and hence that

$$A = \bigcap \{X - J_2\text{-cl } V_{n(x)} : x \notin A\}$$

Thus A is J_2 G_δ since the number of distinct integers $n(x)$ is countable.

Theorem 3.21

If (X, J_1, J_2) is pairwise regular and second countable (Both (X, J_1) and (X, J_2) are second countable) then it is pairwise perfectly normal.

Proof:

We prove (X, J_1, J_2) is pairwise normal.

Let $\{P_n\}$ and $\{Q_n\}$ ($n = 1, 2, \dots$) be countable bases for J_1 and J_2 respectively

Let A be a J_1 closed set and B be J_2 closed set with $A \cap B = \emptyset$.

Since (X, J_1, J_2) is pairwise regular, for each x in A there is a J_1 open set Q , $Q \in \{Q_n\}$ such that

$$x \in Q \subset J_1\text{-cl } Q \subset [X/B]$$

Similarly, for each y in B , there is a J_1 open set P .

$$P \in \{P_n\}, \text{ such that } y \in P \subset J_2\text{-cl } (P) \subset [X/A]$$

Proceeding as in the analogous classical lemma [8], we obtain a countable J_2 open cover of A and a countable J_1 open cover of B which can be used to construct required disjoint sets U and V such that U contains A and is J_2 - open whilst V contains B and is J_1 - open.

Hence (X, J_1, J_2) is pairwise normal

Above theorem implies that each J_1 closed set is J_2 G_δ

and each J_2 closed set is J_1 G_δ

Hence (X, J_1, J_2) is pairwise perfectly normal.

Definition 3.22

Let (X, J_1, J_2) be a bitopological space. Then J_1 is completely regular with respect to J_2 if for each J_1 -closed set C and each point $x \notin C$ there is a real valued function f on X into $[0,1]$ such that $f(x) = 0$, $f(c) = 1$ and f is J_1 u.s.c and J_2 l.s.c. (X, J_1, J_2) is pairwise completely regular if J_1 is completely regular with respect to J_2 and J_2 is completely regular with respect to J_1 .

Definition 3.23

(X, J_1, J_2) is pairwise Tychonoff (or pairwise $T_{3-\frac{1}{2}}$) if it is pairwise completely regular and pairwise T_1 .

Result 3.24

Pairwise completely regular implies pairwise regularity

Result 3.25

Pairwise T_4 spaces are pairwise tychonoff spaces.

Proposition 3.26

If (X, J_1, J_2) is pairwise normal, then J_1 is completely regular with respect to J_2 iff J_1 is regular with respect to J_2 .

Proof:

We only need to prove the sufficiency.

Let C be any J_1 closed subset of X and $x \notin C$.

Then proposition 3.2 provides a J_1 open set H such that $x \in H \subset J_2\text{-cl } H \subset X - C$.

By theorem 3.11 there is a function g on X into $[0,1]$ such that $g(J_2\text{-cl } H) = 0, g(C) = 1$ and g is J_1 u.s.c and J_2 l.s.c

In particular $g(x) = 0$, so that J_1 is completely regular with respect to J_2 .

Corollary 3.27

If (X, J_1, J_2) is pairwise normal then it is pairwise completely regular iff it is pairwise regular.

Note 3.28

Pairwise Tychonoff property characterizes subspaces of pairwise T_4 spaces.

Note 3.29

Pairwise T_4 is not bitopologically hereditary since normality is not topologically hereditary.

Proposition 3.30

Suppose (X, J_1, J_2) is a bitopological space and $A \subset X$. If J_1 is completely regular with respect to J_2 then J_1/A is completely regular with respect to J_2/A .

Proof:

If C is a J_1/A closed subset of A and $a \in A - C$, then $C = A \cap P$ where P is J_1 closed in X and $a \notin P$. So there is a function $f: X \rightarrow [0,1]$ such that $f(P) = 1$, $f(a) = 0$ and f is J_1 u.s.c and J_2 l.s.c. If g is f/A the restriction of f to A then $g(C) = 1$, $g(a) = 0$ and g is J_1/A u.s.c and J_2/A l.s.c as required.

Corollary 3.31

1. Any subspace of pairwise Tychonoff space is pairwise Tychonoff.
2. Any subspace of a pairwise T_4 space is pairwise Tychonoff

Proposition 3.32

Let (X, J_1, J_2) be such that (X, J_1) is T_1 and J_1 is completely regular with respect to J_2 . Then if x and y are distinct points of X , there is a real valued

function f on X such that $f(x) \neq f(y)$ and f is J_1 u.s.c and J_2 l.s.c

Proof:

Now $\{y\}$ is J_1 closed and $x \notin \{y\}$

Therefore by invoking 3.32 we get the proof of this proposition

Corollary 3.33

If x and y are distinct points of a pairwise tychonoff space (X, J_1, J_2) then there are real valued functions f and g on X such that $f(x) \neq f(y)$, $g(x) \neq g(y)$ f is J_1 u.s.c and J_2 l.s.c and g is J_1 l.s.c and J_2 u.s.c.

Definition 3.34

Let f be a real valued function on (X, J_1, J_2) which is J_1 u.s.c and J_2 l.s.c Then $\{x \in X : f(x) < 0\}$ is a J_1 cozero set and $\{x \in X : f(x) > 0\}$ is a J_2 cozero set. Similarly for the case when f is J_1 l.s.c and J_2 u.s.c.

Theorem 3.35

If (X, J_1, J_2) is pairwise T_1 then it is pairwise Tychonoff iff the J_1 cozero sets form a base for the J_1 topology and the J_2 cozero sets form a base for J_2 .

Proof:

Let (X, J_1, J_2) be pairwise Tychonoff and $x \in U$ which is J_1 open. There is a function $f : X \rightarrow [0, 1]$ such that $f(x) = 0, f(X - U) = 1$ and f is J_1 u.s.c and J_2 l.s.c. Let $g = 1 - f$. Then $g(x) = 1, g(X - U) = 0$ and g is J_1 l.s.c and J_2 u.s.c. Then J_1 cozero set of g contains x and is contained in U .

Since each J_1 cozero set is J_1 open and their union is X , the family of all such J_1 cozero sets form a base for the topology J_1 .

Similarly for J_2 .

Conversely, Suppose the J_1 cozero sets form a base for J_1 and let P be J_1 closed and $x_0 \in P$.

There is a J_1 cozero set C of some real valued function f on X such that $x_0 \in C \subset X - P$. Two cases arise.

(i) f is J_1 u.s.c and J_2 l.s.c so that $C = \{x \in X : f(x) < 0\}$

Let $g = -\min \{f, 0\}$

(ii) f is J_1 l.s.c and J_2 u.s.c so that

$C = \{x \in X : f(x) > 0\}$ let $g = \max \{f, 0\}$

In each case, g is J_1 l.s.c and J_2 u.s.c $g(X - C) = 0$ and $g(x) > 0$ for all $x \in C$.

Define the function h on X by $h(x) = 1 - \frac{g(x)}{g(x_0)}$ for each

$x \in X$.

Then $h(x_0) = 0$ $h(P) = 1$ and h is J_1 u.s.c and J_2 l.s.c.

Hence J_1 is completely regular with respect to J_2 .

Similarly J_2 is completely regular with respect to J_1

and hence (X, J_1, J_2) is pairwise Tychonoff.

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