

SEPARATION AXIOMS IN  
FUZZY TOPOLOGICAL SPACES

By  
V. SRIVIDHYA

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*G. K. Chandrika*  
Signature of the Guide

*K. N. Meenakshi*  
Signature of the  
Head of the Department

*12/5/93*

*M. V. Malathy*  
Signature of the  
Dean of the Faculty

*12/5/93*

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## **INTRODUCTION**

## INTRODUCTION

Zadeh [6] introduced the concept of fuzzy sets in 1965. Based on this concept, Chang [1] developed the theory of fuzzy topological spaces in 1968. From then on, quite a number of research papers have been published dealing with various aspects of such spaces. Properties like Separation axioms, Compactness, Connectedness in ordinary topological spaces have been generalised to fuzzy topological spaces. In this dissertation we are going to deal with the separation axioms in fuzzy topological spaces. Many authors have obtained generalisations of these axioms in many ways. Here we are going to discuss the generalisations as given in the following articles:

1. Mira Sarkar :

On Fuzzy Topological Spaces [4]

2. Rekha Srivastava , S.N. Lal and Arun K.Srivastava:

Fuzzy  $T_1$ - Topological Spaces [5]

3. M. H. Ghanim, E. E. Kerre and A. S. Mashhour:

Separation Axioms, Subspaces and Sums in Fuzzy  
Topology [2]

In Chapter I we give the preliminaries needed for this dissertation.

Mira Sarkar [4] has defined  $FT_1, FT_2, FT_3$  and  $FT_4$  spaces and has proved that  $FT_4 \Rightarrow FT_3 \Rightarrow FT_2$ . She has defined an  $FT_1$  space to be one in which singletons (both crisp and fuzzy) are closed. It has been proved that in an  $FT_2$  space crisp singletons are closed. All these are discussed in section 1 of chapter II. In section 2, we discuss the notion of fuzzy  $T_1$

spaces due to Rekha Srivastava [5]. " A fuzzy topological space  $(X, \tau)$  is said to be fuzzy  $T_1$  [5] iff for every pair of distinct fuzzy points (not crisp)  $p, q$  in  $X$ , we can find fuzzy open sets  $U$  and  $V$  in  $(X, \tau)$  such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$ ." According to this definition, we get a deviation from ordinary topology that in a fuzzy  $T_1$  space crisp singletons need not be fuzzy closed. But this result is found to be true in topologically generated fuzzy topological spaces. Section 3 of Chapter II deals with the notions of  $FT_0$ ,  $FT_1$ ,  $FT_s$ ,  $FT_2$ ,  $FT_{2\frac{1}{2}}$ ,  $FT_3$  and  $FT_4$  spaces due to Ghanim [2] and the following implications hold.

$$(i) \quad FT_s \implies FT_1 \implies FT_0$$

$$(ii) \quad FT_4 \implies FT_3 \implies FT_{2\frac{1}{2}} \implies FT_2 \implies FT_1 \implies FT_0$$

According to Ghanim's definition it is true that a fuzzy topological space is  $FT_1$  iff fuzzy crisp singletons are fuzzy closed. The definition of  $FT_3$  and  $FT_4$  spaces are equivalent to those due to Mira Sarkar [4]

Chapter III deals with the behaviour of properly compact sets in  $FT_2$  spaces and Fuzzy P-spaces. Chang's definition of compactness does not seem to be quite natural in Hausdorff ( $FT_2$ ) spaces (as introduced by Mira Sarkar), since no subset of an  $FT_2$  space is compact. Hence Mira Sarkar [4] has introduced the notion of proper open cover using which the notion of proper compactness is defined and with this definition generalisations of some of the results in ordinary topological spaces have been obtained.

In chapter IV we discuss two of the most important operations : fuzzy subspaces and fuzzy sums as introduced by Ghanim [2]. Fuzzy subspaces are defined only for ordinary (ie.crisp) subsets and it has been shown that the properties  $FT_0, FT_1, FT_s, FT_2, FT_{2\frac{1}{2}}, FT_3$  are hereditary while fuzzy normality is proved to be hereditary only with respect to closed subspaces. Sum fuzzy topology is defined on the union of pairwise disjoint fuzzy topological spaces and all  $FT_i, (i=0,1,s,2,2\frac{1}{2},3,4)$  properties are proved to be additive.

**CHAPTER I**

CHAPTER I  
PRELIMINARIES

Definition:

Let  $X$  be an arbitrary (nonempty) set. A fuzzy set  $A$  in  $X$  is a function from  $X$  to  $[0,1]$

An ordinary subset of  $X$  can be considered as a fuzzy set by identifying it with its characteristic function.

Hence if  $A$  is an ordinary subset of  $X$ , then the fuzzy set is given by

$$\begin{aligned} A(x) &= 1 \quad \text{if } x \in A \\ &= 0 \quad \text{if } x \notin A \end{aligned}$$

Definition: Let  $A$  and  $B$  be fuzzy sets in  $X$

Then:

$$A = B \iff A(x) = B(x), \quad \text{for all } x \in X$$

$$A \subset B \iff A(x) \leq B(x), \quad \text{for all } x \in X$$

More generally for a family of fuzzy sets,  $\mathcal{A} = (A_i / i \in I)$ , then the union  $C = \bigcup_{i \in I} A_i$ , and the intersection,  $D = \bigcap_{i \in I} A_i$  are

defined by

$$C(x) = \sup_{i \in I} (A_i(x)), \quad x \in X$$

$$D(x) = \inf_{i \in I} (A_i(x)), \quad x \in X$$

Definition: The complement  $\text{co } A$  or  $A'$  of a fuzzy set  $A$  on  $X$  is given by

$$\text{co } A(x) = 1 - A(x), \quad \text{for all } x \in X$$

$$\text{or } A'(x) = 1 - A(x), \quad \text{for all } x \in X$$

**Definition:** Let  $f$  be a function from  $X$  to  $Y$ . Let  $B$  be a fuzzy set in  $Y$ . Then the **inverse of  $B$** , written as  $f^{-1}(B)$ , is a fuzzy set in  $X$  defined by

$$(f^{-1}(B))(x) = B(f(x)), \text{ for all } x \in X$$

Conversely, let  $A$  be a fuzzy set in  $X$ . The **image of  $A$** , written as  $f(A)$ , is a fuzzy set in  $Y$  defined by

$$(f(A))(y) = \sup_{z \in f^{-1}(y)} (A(z)), \text{ if } f^{-1}(y) \text{ is not empty}$$

$$= 0, \text{ otherwise}$$

for all  $y \in Y$ , where  $f^{-1}(y) = \{x / f(x) = y\}$

**Note:** (i)  $A \subset f^{-1}(f(A))$  for every fuzzy set  $A$  in  $X$

(ii)  $B \supset f(f^{-1}(B))$  for any fuzzy set  $B$  in  $Y$

(iii) If  $f$  is onto  $B = f(f^{-1}(B))$

**Definition:** **Support** of a fuzzy set  $A$  in a fts  $(X, \tau)$  is given by

$$\text{Supp } A = \{x \in X / A(x) \neq 0\}$$

**Definition:** A **fuzzy point** or a **fuzzy singleton**  $p$  in  $X$  is a fuzzy set defined by,

$$p(x_p) = r$$

$$= 0 \text{ if } x \neq x_p$$

Where  $r \in (0,1)$ .  $x_p$  is called the **support** of  $p$  and  $r$  its **value**

We say a fuzzy point  $p \in A$  if

$$p(x_p) < A(x_p)$$

**Definition:** If

$$p(x_p) = 1 \text{ and}$$

$$p(x) = 0 \text{ if } x \neq x_p$$

then we say  $p$  is a **crisp point** or **crisp singleton**

Hence we note that a crisp point is nothing else but the characteristic function corresponding to the ordinary singleton set  $(x_p)$ . Hence crisp points are also denoted by  $(x_p)$

We say a **crisp point**  $p \in A$  if  

$$A(x_p) = 1$$

**Theorem 1.1:** Let  $A$  be any non empty fuzzy set in  $X$ . Then

$$A = \bigcup_{x \in \Omega} p_x, \text{ where } \Omega = (x \in X / A(x) \neq 0)$$

and  $p_x$  is a fuzzy point with support  $x$  and value  $A(x)$

**Theorem 1.2:** Every fuzzy set  $A$  can be written as the union of the fuzzy singletons  $p \subset A$

$$\text{ie. } A = \bigcup_{p \subset A} p$$

**Definition (Chang[1]):** Let  $X$  be a set and  $I = [0,1]$ . A **fuzzy topology** on  $X$  is a subset  $\tau \subset I^X$  such that

- (i)  $0, 1 \in \tau$
- (ii) If  $A, B \in \tau$  then,  $A \cap B \in \tau$
- (iii) If  $(A_j)_{j \in J} \subset \tau$  then  $\text{Sup}_{j \in J} A_j \in \tau$

$\tau$  is called a **fuzzy topology** for  $X$  and the pair  $(X, \tau)$  is called a **fuzzy topological space** or fts for short

**Definition(Lowen[3]):**  $\tau \subset I^X$  is a **fuzzy topology** on  $X$  iff

- (i) For every  $\alpha$  constant,  $\alpha \in \tau$
- (ii) If  $A, B \in \tau$  then  $A \cap B \in \tau$
- (iii) If  $(A_j)_{j \in J} \subset \tau$  then  $\text{Sup}_{j \in J} A_j \in \tau$

The pair  $(X, \tau)$  is called a **fuzzy topological space** or **fts** for short.

**Definition:** Members of  $\tau$  are called  **$\tau$  - open fuzzy sets**, Complements of open fuzzy sets are called **closed fuzzy sets**.

**Definition:** The **closure**  $cl A$  and the **interior**  $int A$  of a fuzzy set  $A$  in a fuzzy topological space  $(X, \tau)$  are defined by

$$cl A = \bigcap \{ F / co F \in \tau \text{ and } A \subset F \}$$

$$int A = \bigcup \{ O / O \in \tau \text{ and } O \subset A \}$$

**Definition:** Let  $(X, \tau)$  be an fts.  $\mathcal{B} \subset \tau$  is a **base** for  $\tau$  iff each member of  $\tau$  is a union of members of  $\mathcal{B}$ .  $\mathcal{S} \subset \tau$  is a **subbase** for  $\tau$  iff finite intersections of members of  $\mathcal{S}$  form a base for  $\tau$ ; in this case we say that  $\tau$  is **generated by**  $\mathcal{S}$ .

**Theorem 1.3** For an fts  $(X, \tau)$ ,  $\mathcal{B}$  is a base iff for all  $A \in \tau$  and for all fuzzy point  $p \in A$ , there exists  $B \in \mathcal{B}$  such that  $p \in B \subset A$

**Definition:** Let  $p$  be a fuzzy point in  $(X, \tau)$ . A fuzzy set  $N$  is called a **fuzzy neighbourhood** of  $p$  iff there exists  $A \in \tau$  such that  $p \in A \subset N$

**Theorem 1.4:** A fuzzy set in  $(X, \tau)$  is fuzzy open iff it is a neighbourhood of each of its fuzzy points.

**Definition:** Let  $A$  and  $B$  be fuzzy sets in  $X$  and  $Y$  respectively. The **Cartesian product**  $A \times B$  of  $A$  and  $B$  is a fuzzy set in  $X \times Y$  defined by

$$(A \times B)(x, y) = \inf (A(x), B(y)) \text{ for all } (x, y) \in X \times Y$$

**Definition:** Let  $\{(X_i, \tau_i) / i \in I\}$  be a family of fts and  $X = \prod_{i \in I} X_i$ . The **product fuzzy topology** on  $X$  is the one with basic fuzzy open sets of the form  $\prod_{i \in I} U_i$  where  $U_i \in \tau_i$  and  $U_i = X_i$  except for finitely many  $i$ 's.

Definition: Let  $(X, \tau)$  be a fts. A family  $U$  of fuzzy sets is a cover of a fuzzy set  $B$  iff  $B \subset \bigcup \{A / A \in U\}$ . It is an open cover iff each member of  $U$  is an open fuzzy set. A subcover of  $U$  is a subfamily which is also a cover.

Definition: A function from a fts  $(X, \tau)$  to a fts  $(Y, \sigma)$  is  $F$  - continuous iff the inverse of each  $\sigma$  - open fuzzy set is  $\tau$  - open.

Theorem 1.4  $F$  - continuous image of a compact set is compact.

**CHAPTER II**

## CHAPTER II

### SEPARATION AXIOMS IN FUZZY TOPOLOGICAL SPACES

Separation properties in ordinary topological spaces have been generalised to fuzzy topological spaces in different ways by many authors. In this chapter we are going to study the generalisations due to Mira Sarkar [4], Rekha Srivastava [5] and Ghanim [2]. Though there are some deviations from ordinary topology, many of the results are exact generalisations of the results in ordinary topology. In section 1, we study the results due to Mira Sarkar [4], in section 2 those due to Rekha Srivastava [5] and in the last section we study the results due to Ghanim [2].

#### Section 1.

This section is devoted to the study of  $FT_1, FT_2, FT_3,$  and  $FT_4$  spaces due to Mira Sarkar [4].

She has proved that the following implications hold:

$FT_4 \Rightarrow FT_3 \Rightarrow FT_2$ . She has also proved that in the case of  $FT_2$  spaces, crisp singletons are closed.

#### Remark:

Throughout this section, by points (subsets) of  $X$ , we mean both crisp and fuzzy points (subsets). Points are also referred to as singletons.

Definition: An fts  $(X, \tau)$  is  $FT_1$  iff singletons are closed.

#### Definition:

An fts  $(X, \tau)$  is defined to be Hausdorff or  $FT_2$  iff the following conditions hold:

If  $p, q$  are any two points in  $X$ , then

(i) if  $x_p \neq x_q$ , there exist open sets  $V_p$  and  $V_q$ , such that  $p \in V_p$  and  $q \notin \bar{V}_p$  and  $q \in V_q$ ,  $p \notin \bar{V}_q$ ;

(ii) if  $x_p = x_q$  and  $p(x_p) < q(x_p)$ , then there exists an open set  $V_p$  such that  $p \in V_p$ , but  $q \notin \bar{V}_p$ .

The definitions of regular and normal fuzzy topological spaces are exactly similar to those in ordinary topological spaces.

Definition

An fts  $(X, \tau)$  is **regular** iff for each point  $p \in X$  and for each  $V \in \tau$  with  $p \in V$  there exists  $G \in \tau$  such that  $p \in G \subset \bar{G} \subset V$

Definition

An fts  $(X, \tau)$  is **FT<sub>3</sub>** iff it is **FT<sub>1</sub>** and regular.

Definition

An fts  $(X, \tau)$  is **normal** iff for every closed set  $K$  in  $X$  and  $V \in \tau$  with  $K \subset V$  there exists  $G \in \tau$  such that  $K \subset G \subset \bar{G} \subset V$

Definition

An fts  $(X, \tau)$  is **FT<sub>4</sub>** iff it is **FT<sub>1</sub>** and normal.

Just as in ordinary topological spaces we get that

(i) an **FT<sub>3</sub>** space is **FT<sub>2</sub>**

(ii) an **FT<sub>4</sub>** space is **FT<sub>3</sub>**, while,

(iii) Regarding **FT<sub>2</sub>** spaces we get the following theorem.

Theorem 2.1.1. :

In an **FT<sub>2</sub>** space crisp singletons are closed.

Proof: Let  $p$  be a crisp point in  $X$ .

$$\text{Then } p(x_p) = 1$$

Take any point  $q \in (p)'$

If  $q$  is a fuzzy point, then  $q \in (p)'$

$$\Rightarrow q(x_q) < (p)'(x_q)$$

$$= 1 - p(x_q)$$

Since  $q(x_q)$  is strictly less than  $1 - p(x_q)$ ,  $p(x_q) \neq 1$ .

$$\text{ie. } p(x_q) = 0$$

$$\therefore x_q \neq x_p$$

If  $q$  is a crisp point, then  $q \in (p)'$

$$\Rightarrow (p)'(x_q) = 1$$

$$\Rightarrow p(x_q) = 0$$

$$\Rightarrow x_q \neq x_p$$

Choose a sequence  $(q_n / n \in \mathbb{N})$  of fuzzy points with same support  $x_q$ .

Choose the sequence  $(y_n / n \in \mathbb{N})$  as a decreasing sequence converging to zero.

Define  $q_n(x_q) = y_n$  for all  $n$ .

Consider the fuzzy point  $q_n$ . It has the support  $x_q$  then  $x_{q_n} = x_q \neq x_p$ .

$\therefore x_{q_n} \neq x_p$  for every  $n$ .

Then by the Hausdorff property, there exists a fuzzy open set  $V_{pq_n}$  such that  $p \in V_{pq_n}$  and  $q_n \notin \overline{V_{pq_n}}$

Hence there exists a sequence of fuzzy open sets  $(V_{pq_n} / n \in \mathbb{N})$  such that  $p \in V_{pq_n}$  and  $q_n \notin \overline{V_{pq_n}}$  for all  $n \in \mathbb{N}$ .

$$\therefore V_{pq_n}(x_p) = 1 \quad \text{-----(1)}$$

$$\text{and } q_n(x_q) \geq \overline{V_{pq_n}}(x_q) \quad \text{-----(2)}$$

If  $P = \bigcap_{n \in \mathbb{N}} \overline{V_{pq_n}}$  then  $P$  is a fuzzy closed set.

Claim 1:  $P(x_q) = 0, P(x_p) = 1$

$$\begin{aligned} \text{Consider } P(x_q) &= \bigcap_{n \in \mathbb{N}} \overline{V_{pq_n}}(x_q) \\ &= \inf_{n \in \mathbb{N}} (\overline{V_{pq_n}}(x_q)) \\ &\leq \inf_{n \in \mathbb{N}} (q_n(x_q)) \quad \text{by (2)} \end{aligned}$$

Since  $(q_n(x_q) / n \in \mathbb{N})$  is a decreasing sequence, converging to zero, we have

$$P(x_q) \leq \inf_{n \in \mathbb{N}} (q_n(x_q)) = 0$$

$$\therefore P(x_q) \leq 0$$

But  $P(x_q) \geq 0 \quad \therefore P(x_q) = 0$

$$\begin{aligned} \text{Consider } P(x_p) &= \bigcap_{n \in \mathbb{N}} \overline{V_{pq_n}}(x_p) \\ &= \inf_{n \in \mathbb{N}} (\overline{V_{pq_n}}(x_p)) \end{aligned}$$

By (1),  $V_{pq_n}(x_p) = 1$  for all  $n \in \mathbb{N}$

$$\therefore \overline{V_{pq_n}}(x_p) = 1 \text{ for all } n \in \mathbb{N}$$

Thus  $\bigcap_{n \in \mathbb{N}} \overline{V_{pq_n}}(x_p) = 1$

$$\text{ie. } P(x_p) = 1$$

Hence claim 1.

Claim 2:  $P' \subset (p)'$

ie. To prove :  $P'(x) \leq (p)'(x)$

If  $x = x_p$

$$\begin{aligned} (p)'(x_p) &= 1 - p(x_p) = 0 \\ P'(x_p) &= 1 - P(x_p) \\ &= 1 - 1 \quad (\text{by Claim 1}) \\ &= 0 \end{aligned}$$

If  $x \neq x_p$

$$\begin{aligned}(p)'(x) &= 1 - p(x) \\ &= 1 - 0 = 1\end{aligned}$$

$$\begin{aligned}P'(x) &\leq 1 \\ &= (p)'(x)\end{aligned}$$

$\therefore P'(x) \leq (p)'(x)$  for all  $x \in X$

$\therefore P' \subset (p)'$

Hence claim 2.

Claim 3:  $q \in P'$

If  $q$  is a crisp point, then as

$$\begin{aligned}P'(x_q) &= 1 - P(x_q) \\ &= 1 - 0 \text{ (by claim 1)} \\ &= 1, \text{ We get that } q \in P'\end{aligned}$$

Since the crisp point  $q \in P'$ , any fuzzy point  $q$  with support  $x_q \in P'$ , because, if  $q$  is not crisp

$$\begin{aligned}q(x_q) &< 1 \\ &= 1 - 0 = 1 - P(x_q) \text{ (by claim 1)} \\ &= P'(x_q)\end{aligned}$$

$\therefore q(x_q) < P'(x_q)$

$\therefore$  The fuzzy point  $q$  also  $\in P'$

Hence claim 3.

Hence by claims 2 and 3 we get that, for any point  $q \in (p)'$  there exists a fuzzy open set  $P'$  such that  $q \in P' \subset (p)'$

Hence  $(p)'$  is fuzzy open by theorem 1. 4.

ie.  $(p)$  is fuzzy closed.

$\therefore$  In an  $FT_2$  space crisp singletons are closed.

Theorem 2.1.2: An  $FT_3$  space is an  $FT_2$  space.

Proof : Let  $(X, \tau)$  be the given  $FT_3$  space.

Consider two fuzzy points  $p$  and  $q$ . then there are two cases:

- (i)  $x_p \neq x_q$
- (ii)  $x_p = x_q$  and  $p(x_p) < q(x_p)$

Case (i)  $x_p \neq x_q$

Choose a fuzzy point  $w$  with  $x_w = x_p$  and  $w(x_p) > 1 - p(x_p)$

Since  $(X, \tau)$  is an  $FT_3$  space, it is an  $FT_1$  space. Since Singletons in an  $FT_1$  space are fuzzy closed,  $(w)$  is fuzzy closed in  $(X, \tau)$

$\therefore (w)'$  is fuzzy open.

$$\text{Consider } (w)'(x_p) = 1 - w(x_p) < p(x_p)$$

$$\text{For } x \neq x_p, (w)'(x) = 1 - w(x) = 1 - 0 = 1$$

$$\therefore (w)'(x) < p(x) \text{ if } x = x_p$$

$$(w)'(x) = 1 \text{ otherwise}$$

$$\text{Since } x_q \neq x_p, (w)'(x_q) = 1$$

Since  $q(x_q) < 1$ , we have

$$q(x_q) < (w)'(x_q)$$

$$\implies q \in (w)'$$

$$\text{Since } (w)'(x_p) < p(x_p), p \notin (w)'$$

$$\therefore q \in (w)' \text{ but } p \notin (w)'$$

Since  $(X, \tau)$  is regular, there exists one  $V_q \in \tau$  such that

$$q \in V_q \subset \overline{V_q} \subset (w)'$$

$$\text{Since } p \notin (w)', p \notin \overline{V_q}$$

$\therefore$  We have shown that there exists a fuzzy open set  $V_q$  such

that  $q \in V_q$  and  $p \notin \bar{V}_q$

Similarly by taking a fuzzy point  $w$  with  $x_w = x_q$

and  $w(x_q) > 1 - q(x_q)$  and arguing as above, we can show that there

exists a fuzzy open set  $V_p$  such that  $p \in V_p$  and  $q \notin \bar{V}_p$

Case (ii)  $x_p = x_q$  and  $p(x_p) < q(x_p)$

Choose a fuzzy point  $w$  with  $x_w = x_p$  and  $1 - q(x_p) < w(x_p) < 1 - p(x_p)$

(this is possible since  $p(x_p) < q(x_p)$ )

$$\Rightarrow 1 - q(x_p) < 1 - p(x_p)$$

Since  $1 - q(x_p) < w(x_p)$ ,

$$w'(x_p) = 1 - w(x_p)$$

$$< q(x_p)$$

Hence  $q \notin w'$

Since  $w(x_p) < 1 - p(x_p)$ ,

$$w'(x_p) = 1 - w(x_p)$$

$$> p(x_p)$$

$$\therefore p(x_p) < w'(x_p)$$

Hence  $p \in w'$

$\therefore$  We have  $p \in w'$  but  $q \notin w'$

Since  $(X, \tau)$  is regular there exists a  $V_p \in \tau$  such that  $p \in V_p \subseteq \bar{V}_p \subseteq w'$

Since  $q \notin w'$ ,  $q \notin \bar{V}_p$

$\therefore$  We have shown that there exists a fuzzy open set  $V_p$  such that  $p \in V_p$  but  $q \notin \bar{V}_p$

The other cases can be proved in a similar way.

$\therefore$  The space  $(X, \tau)$  is an  $FT_2$  space.

Theorem 2.1.3. An  $FT_4$  space is an  $FT_3$  space.

Proof : Let  $(X, \tau)$  be an  $FT_4$  space.

Consider a point  $p \in X$  and  $V \in \tau$  such that  $p \in V$

If  $p$  is fuzzy point, then  $p(x_p) < V(x_p)$

Choose a fuzzy point  $w$  with  $x_w = x_p$  and  $p(x_p) < w(x_p) < V(x_p)$

ie.  $p \in w$  and  $w \in V$

Since  $(X, \tau)$  is an  $FT_1$  space,

$w$  is closed in  $(X, \tau)$ .

Since  $(X, \tau)$  is normal, and since  $w \subset V$  there exists  $G \in \tau$  such that  $w \subset G \subset \bar{G} \subset V$ .

Since  $p \in w$ ,  $p \in w \subset G \subset \bar{G} \subset V$

ie.  $p \in G \subset \bar{G} \subset V$

$\therefore$  We have shown that given a point  $p \in X$  and  $V \in \tau$  such that  $p \in V$ , there exists a fuzzy open set  $G$  such that  $p \in G \subset \bar{G} \subset V$ .

If  $p$  is a crisp point, the same argument holds by taking  $w$  as crisp point  $p$  itself.

$\therefore$  The space  $(X, \tau)$  is regular and  $FT_1$

$\therefore$   $(X, \tau)$  is an  $FT_3$  space.

## Section 2:

In this section we study briefly fuzzy  $T_1$  spaces as defined by Rekha Srivastava [5]. We know that in ordinary  $T_1$  topological spaces  $(X, \tau)$ ,  $x \in X$  are closed. Though the parallel result is not true in an arbitrary fuzzy  $T_1$  spaces, it has been shown that it is true in topologically generated fuzzy  $T_1$  spaces.

Note: Throughout this section we follow

1. Lowen's definition of fuzzy topology
2. By two distinct fuzzy points we mean two fuzzy points  $p$  and  $q$  whose supports are distinct.

First we give the definition of  $FT_1$  spaces according to Rekha Srivastava [5].

**Definition** Let  $(X, \tau)$  be a fuzzy topological space. Then  $(X, \tau)$  is called a **fuzzy  $T_1$  topological space** iff for every pair of distinct fuzzy points  $p, q$  in  $X$ , we can find open sets  $U$  and  $V$  in  $(X, \tau)$  such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$ .

The following theorem is an analogous result in ordinary topological spaces.

**Theorem 2.2.1:** Let  $(X, \tau)$  be a fuzzy topological space. Consider the following statements:

(i)  $\Delta_X$ , the diagonal of  $X$ , is fuzzy closed in  $(X \times X, \tau \times \delta)$  where  $\delta$  is discrete fuzzy topology on  $X$  (i.e.  $\delta$  consists of all fuzzy sets in  $X$ )

(ii)  $\{x\}$ , for every  $x \in X$ , is fuzzy closed in  $(X, \tau)$

(iii)  $(X, \tau)$  is fuzzy  $T_1$  topological space

Then (i)  $\Leftrightarrow$  (ii)  $\Rightarrow$  (iii)

**Proof:**

(i)  $\Rightarrow$  (ii)

To prove (ii) it suffices to show that  $X - \{x\}$  is fuzzy open in  $(X, \tau)$  for all  $x \in X$ .

Take any fuzzy point  $p \in X - \{x\}$  such that  $p(x_p) = r$

$$\begin{aligned} \therefore p(x_p) &< (X - \{x\})(x_p) \\ &= 1 - (x)(x_p) \end{aligned}$$

$$\text{If } x_p = x, \quad p(x_p) < 1 - 1 = 0$$

ie.  $r < 0$  which is not possible

$$\therefore x_p \neq x$$

So  $(x_p, x) \in (X \times X) - \Delta_X$  where  $\Delta_X = \{ (x, x) / x \in X \}$

Consider a fuzzy point  $q$  of  $X \times X$  Such that  $q(x_p, x) = r$

$$\begin{aligned} ((X \times X) - \Delta_X)(z_1, z_2) &= 0 \quad \text{if } (z_1, z_2) \in \Delta_X \\ &= 1 \quad \text{if } (z_1, z_2) \notin \Delta_X \end{aligned}$$

$$\therefore ((X \times X) - \Delta_X)(x_p, x) = 1 \quad \text{Since } (x_p, x) \in (X \times X) - \Delta_X$$

$$\therefore q(x_p, x) = r < 1$$

$$= ((X \times X) - \Delta_X)(x_p, x)$$

$$\therefore q(x_p, x) < ((X \times X) - \Delta_X)(x_p, x)$$

$$\therefore q \in (X \times X) - \Delta_X$$

Since  $\Delta_X$  is fuzzy closed,  $(X \times X) - \Delta_X$  is fuzzy open.

$\therefore$  By theorem 1.3, there exists a basic open fuzzy set  $U \times V$

of  $(X \times X, \tau \times \delta)$  Such that  $q \in U \times V \subset (X \times X) - \Delta_X$ .

$$\begin{aligned} \Rightarrow q(x_p, x) &< (U \times V)(x_p, x) \\ &= \inf \{ U(x_p), V(x) \} \\ &\leq U(x_p) \end{aligned}$$

$$\Rightarrow r < U(x_p)$$

$$\text{Now } p(x_p) = r < U(x_p)$$

$$\therefore p \in U$$

Claim:  $V(x) \neq 0$  and  $U(x) = 0$

Suppose  $V(x) = 0$

Since  $r < \inf(U(x_p), V(x))$  we get  $r < 0$

Which is not possible.

$$\therefore V(x) \neq 0$$

Since  $U \times V \subset (X \times X) - \Delta_X$ ,

$$\begin{aligned} (U \times V)(x, x) &= \inf(U(x), V(x)) \\ &\leq ((X \times X) - \Delta_X)(x, x) \end{aligned}$$

But  $((X \times X) - \Delta_X)(x, x) = 0$  ( $\because (x, x) \in \Delta_X$ )

$$\therefore \inf (U(x), V(x)) = 0$$

$$\Rightarrow \text{either } U(x) = 0 \text{ or } V(x) = 0$$

But Since  $V(x) \neq 0$ ,  $U(x) = 0$

Claim:  $U \subset X - \{x\}$

Take any  $y \in Y$

If  $y = x$ ,  $(X - \{x\})(y) = 0$

$$= U(x)$$

$$= U(y)$$

If  $y \neq x$ ,  $(X - \{x\})(y) = 1$

$$\geq U(y)$$

$\therefore$  For every  $y$ ,  $U(y) \leq (X - \{x\})(y)$

$$\Rightarrow U \subset X - \{x\}$$

Hence we have shown that given  $p \in X - \{x\}$ , there exists an open set

$U$  such that

$$p \in U \subset X - \{x\}$$

$\therefore$  By theorem 1.4,  $X - \{x\}$  is fuzzy open in  $(X, \tau) \Rightarrow \{x\}$ , for all  $x \in X$ , is fuzzy closed in  $(X, \tau)$ .

(ii)  $\Rightarrow$  (i)

Assume (ii)

To show (i) it suffices to show that  $(X \times X) - \Delta_X$  is fuzzy open in  $(X \times X, \tau \times \delta)$ .

Consider a fuzzy point  $p \in (X \times X) - \Delta_X$  with  $p(x_p, y_p) = r$

If  $x_p = y_p$ ,  $p(x_p, x_p) = r > 0$

$$= ((X \times X) - \Delta_X)(x_p, x_p)$$

$$\text{ie. } p(x_p, x_p) > ((X \times X) - \Delta_X)(x_p, x_p)$$

$\Rightarrow p \notin ((X \times X) - \Delta_X)$ , a contradiction.

$$\therefore x_p \neq y_p$$

Choose two fuzzy points  $q_1$  and  $q_2$  in  $X$  with supports  $x_p$  and  $y_p$  respectively and with values  $q_1(x_p) = q_2(y_p) = r$ .

$$\begin{aligned} (X - \{y_p\})(x_p) &= 1 > r \\ &= q_1(x_p) \end{aligned}$$

$$\begin{aligned} \therefore q_1(x_p) &< (X - \{y_p\})(x_p) \\ \implies q_1 &\in X - \{y_p\} \end{aligned}$$

Since  $\{y_p\}$  is fuzzy closed in  $(X, \tau)$ ,  $X - \{y_p\}$  is fuzzy open.

$\therefore$  By theorem 1.4, there exists fuzzy open set  $U \in \tau$  such that  $q_1 \in U \subset X - \{y_p\}$

Since  $\delta$  is the discrete fuzzy topology  $\{y_p\} \in \delta$

$$\therefore U_x(y_p) \in \tau \times \delta$$

$$\begin{aligned} \text{we have } (U_x(y_p))(x_p, y_p) &= \inf (U(x_p), \{y_p\}(y_p)) \\ &= \inf (U(x_p), 1) \\ &= U(x_p) \quad \text{----> (1)} \end{aligned}$$

To Prove  $\underline{p} \in U \times \{y_p\}$

$$\begin{aligned} \text{Consider } p(x_p, y_p) &= r \\ &= q_1(x_p) \\ &< U(x_p) \quad (\because q_1 \in U) \\ &= (U_x(y_p))(x_p, y_p) \quad \text{by (1)} \end{aligned}$$

$$\begin{aligned} \therefore p(x_p, y_p) &< (U_x(y_p))(x_p, y_p) \\ \implies p &\in U \times \{y_p\} \quad \text{-----> (2)} \end{aligned}$$

To Prove  $\underline{U}_x(y_p) \subset (X \times X) - \Delta_X$

$$\begin{aligned} \text{Since } U \subset X - \{y_p\}, \quad U(y_p) &\leq (X - \{y_p\})(y_p) \\ &= 0 \end{aligned}$$

$$\therefore U(y_p) = 0 \quad \text{-----> (3)}$$

Take  $(x', x') \in \Delta_X$

$$\begin{aligned}
\text{If } x' = y_p \quad (Ux(y_p)) (x', x') &= \inf ( U(x'), (y_p)(x') ) \\
&= \inf ( U(y_p), (y_p)(y_p) ) \\
&\quad \text{(since } x' = y_p) \\
&= 0 \quad \text{by (3)}
\end{aligned}$$

$$\begin{aligned}
\text{If } x' \neq y_p \quad (Ux(y_p)) (x', x') &= \inf ( U(x'), (y_p)(x') ) \\
&= 0 \quad \text{(Since } (y_p)(x') = 0 \text{ as } x' \neq y_p)
\end{aligned}$$

$$\begin{aligned}
\therefore \text{ For all } (x', x') \in \Delta_X, \\
(Ux(y_p)) (x', x') &= 0 \\
&= ((X \times X) - \Delta_X) (x', x')
\end{aligned}$$

$$\begin{aligned}
\text{For } (x_1, x_2) \in (X \times X) - \Delta_X, \\
(Ux(y_p)) (x_1, x_2) &\leq 1 \\
&= ((X \times X) - \Delta_X) (x_1, x_2)
\end{aligned}$$

$$\begin{aligned}
\therefore (Ux(y_p)) (x_1, x_2) &\leq ((X \times X) - \Delta_X) (x_1, x_2) \text{ for all} \\
&\quad (x_1, x_2) \in X \times X.
\end{aligned}$$

$$\therefore Ux(y_p) \subset (X \times X) - \Delta_X \quad \text{-----(4)}$$

Hence by (2) and (4) we get that given a fuzzy point  $p$  in  $(X \times X) - \Delta_X$  there exists a fuzzy open set  $Ux(y_p)$  such that

$$p \in Ux(y_p) \subset (X \times X) - \Delta_X.$$

This is true for all fuzzy points  $p$  in  $(X \times X) - \Delta_X$

$\implies (X \times X) - \Delta_X$  is fuzzy open neighbourhood of  $p$ .

$\therefore$  By theorem 1.4,  $(X \times X) - \Delta_X$  is fuzzy open in  $(X \times X, \tau_X)$

$\implies \Delta_X$  is fuzzy closed in  $(X \times X, \tau_X)$

(ii)  $\implies$  (iii)

Assume (ii)

Let  $p$  and  $q$  be two distinct fuzzy points in  $X$  with  $p(x_p) = r$  and  $q(x_q) = s$

By our assumption  $X-(x_p)$  and  $X-(x_q)$  are fuzzy open in  $(X, \tau)$

Take  $U = X-(x_q)$

Consider  $p(x_p) = r < 1$

$$= (X-(x_q))(x_p)$$

$$\therefore p(x_p) < (X-(x_q))(x_p)$$

$$\Rightarrow p \in X - (x_q)$$

Consider  $q(x_q) = s > 0$

$$= (X-(x_q))(x_q)$$

$$\therefore q \notin X - (x_q)$$

Hence we have  $p \in U$  and  $q \notin U$ .

Similarly by taking  $V = X - (x_p)$  and arguing as above, we can show that  $q \in V$  and  $p \notin V$ . Hence there exists two fuzzy open sets with the required conditions of fuzzy  $T_1$ -ness.

$\therefore (X, \tau)$  is a fuzzy  $T_1$  space.

Now, we consider an example which illustrates that

(iii) need not imply (ii).

Example : Let  $X$  be any set.

For all  $x \in X$  and  $N \in (0, 1)$  define the mapping  $(x, N) : X \rightarrow [0, 1]$  as follows:

$$((x, N))(x) = N \text{ and } ((x, N))(y) = 1 - N, y \neq x$$

Let  $\tau$  denote the fuzzy topology on  $X$  generated by the constant functions or functions of the form  $(x, N)$ ,  $x \in X$ ,  $N \in (0, 1)$

To Prove :  $\tau$  is fuzzy  $T_1$ .

Let  $p$  and  $q$  be two distinct fuzzy points in  $X$  with supports  $x_p$  and  $x_q$  such that  $p(x_p) = r$  and  $q(x_q) = s$ .

Since  $r, s \in (0, 1)$  there exists  $\alpha \in (0, 1)$  such that  $r < \alpha$  and  $1 - s \leq \alpha$

Consider a  $\tau$  - open fuzzy set  $U$  defined by

$$U = (x_p, \alpha), \alpha \in (0, 1)$$

$$\begin{aligned} \text{Consider } U(x_p) &= ((x_p, \alpha))(x_p) \\ &= \alpha \end{aligned}$$

$$\text{Since } p(x_p) = r < \alpha, p(x_p) < U(x_p)$$

$$\implies p \in U$$

$$\begin{aligned} \text{Consider } U(x_q) &= ((x_p, \alpha))(x_q), \alpha \in (0, 1) \\ &= 1 - \alpha \end{aligned}$$

$$\text{Since } 1 - s \leq \alpha, s \geq 1 - \alpha$$

$$\implies q(x_q) \geq U(x_q)$$

$$\implies q \notin U$$

Since  $r, s \in (0, 1)$  there exists  $\alpha' \in (0, 1)$  such that  $s < \alpha'$  and  $1 - r \leq \alpha'$

Consider the  $\tau$ -open fuzzy set  $V$  defined by  $V = (x_q, \alpha')$ ,  $\alpha' \in (0, 1)$

$$\begin{aligned} \text{Consider } V(x_p) &= (x_q, \alpha')(x_p) = 1 - \alpha' \\ &\leq r \\ &= p(x_p) \end{aligned}$$

$$\therefore V(x_p) \leq p(x_p) \implies p \notin V$$

$$\begin{aligned} \text{Consider } V(x_q) &= (x_q, \alpha')(x_q) = \alpha' \\ &> s \\ &= q(x_q) \end{aligned}$$

$$\therefore q(x_q) < V(x_q) \implies q \in V$$

$\therefore$  We have shown that for every pair of distinct fuzzy points  $p, q$  in  $X$ , there exist fuzzy open sets  $U$  and  $V$  in  $(X, \tau)$  such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$   
 $\implies (X, \tau)$  is fuzzy  $T_1$

Claim: Condition (ii) of theorem 2.2.1. is not true

ie. to prove, no point of  $X$  can be fuzzy  $\tau$  - closed.  
 Suppose there exists a point  $x$  in  $X$  which is fuzzy closed in  $(X, \tau)$ .

$\implies X - (x)$  is fuzzy open

choose a fuzzy point  $p$  in  $X$  such that  $p(x_p) = r$  with  $x_p \neq x$

then  $p(x_p) = r$

$$< 1$$

$$= (X - (x))(x_p)$$

$$\therefore p(x_p) < (X - (x))(x_p) \implies p \in X - (x)$$

Hence we have a fuzzy open set  $X - (x)$  with  $p \in X - (x)$

$\therefore$  By theorem 1.3., there exists a basic fuzzy open set  $U$  of  $X$  such that  $p \in U \subset X - (x)$

$$\text{ie. } p(x_p) < U(x_p) \quad \text{-----} \rightarrow (1)$$

$$\text{and } U(x') \leq (X - (x))(x') \text{ for all } x' \in X \quad \text{-----} \rightarrow (2)$$

In (2) if  $x' = x$  then  $(X - (x))(x) = 0$

$$\implies U(x) = 0$$

From the choice of  $\tau$ , this is true only if  $U = \emptyset$ , the empty set.

But this is not possible since  $p \in U$ . Hence our assumption that there exists one fuzzy closed set  $(x)$  is wrong

Hence the claim

Thus (iii) need not imply (ii)

The following definition gives the association between a topology on a set  $X$  and a fuzzy topology on the same set  $X$

Definition: For a topology  $\mathcal{E}$  on  $X$  let  $\omega(\mathcal{E})$  be the set of all lower semicontinuous functions from  $(X, \mathcal{E})$  to  $[0,1]$ ;  $\omega(\mathcal{E})$  turns out to be a fuzzy topology on  $X$  [5]. For a fuzzy topology  $\tau$  on  $X$ , let  $\mathbf{i}(\tau)$  be the topology on  $X$  induced by all functions

$f: X \rightarrow I_r$ , where  $f \in \tau$  and  $I_r$  is  $[0,1]$  with the left half open interval topology.

**Definition:** A fuzzy topology  $\tau$  on  $X$  is said to be **topologically generated** iff  $\tau = \omega(\mathcal{C})$  for some topology  $\mathcal{C}$  on  $X$ .

When we deal with topologically generated fuzzy topological spaces we see that the defect in theorem 2.2.1. can be rectified with the help of the following theorem.

**Theorem 2.2.2:** (i)  $(X, \mathcal{C})$  is  $T_1 \iff (X, \omega(\mathcal{C}))$  is fuzzy  $T_1$   
(ii)  $(X, \tau)$  is fuzzy  $T_1 \implies i(\tau)$  is  $T_1$

( $\mathcal{C}$  is an ordinary topology on  $X$  and  $\tau$  is fuzzy topology on  $X$ )

**Proof:** (i) First assume that  $(X, \mathcal{C})$  is  $T_1$

To Prove  $(X, \omega(\mathcal{C}))$  is fuzzy  $T_1$

Consider two distinct fuzzy points  $p$  and  $q$  in  $X$ . Then their support points will be different.

ie.  $x_p \neq x_q$

Since  $(X, \mathcal{C})$  is  $T_1$ , there exist  $U, V \in \mathcal{C}$  Such that  $x_p \in U$ ,  $x_q \notin U$  and  $x_q \in V$ ,  $x_p \notin V$

We have  $U(x) = 1$  if  $x \in U$   
 $= 0$  if  $x \notin U$

Consider  $U^{-1}((\epsilon, 1]) = \{x / U(x) \in (\epsilon, 1]\}$   
 $= \{x / x \in U\}$   
 $= U$

$\therefore U^{-1}((\epsilon, 1]) = U$ , an open set in  $(X, \mathcal{C})$

and  $U^{-1}([0, 1]) = X$ , an open set

$\therefore U$  is lower semicontinuous

Similarly we can show that  $V$  is lower semi continuous

$\therefore U, V \in \omega(\mathcal{C})$

$\Rightarrow U$  and  $V$  are fuzzy open sets

Consider,  $p(x_p) < 1$

$$= U(x_p) \text{ (Since } x_p \in U)$$

$$\Rightarrow p \in U$$

$q(x_q) > 0$

$$= U(x_q) \text{ (since } x_q \notin U)$$

$$\Rightarrow q \notin U$$

Similarly, we get  $p \notin V$ .

Hence there exists two fuzzy open sets  $U$  and  $V$  in  $(X, \omega(\mathcal{E}))$

such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$

Hence  $(X, \omega(\mathcal{E}))$  is a fuzzy  $T_1$  space.

**Conversely**, assume that  $(X, \omega(\mathcal{E}))$  is fuzzy  $T_1$

To prove,  $(X, \mathcal{E})$  is  $T_1$ .

Consider two distinct points  $x$  and  $y$  in  $X$  and a pair of fuzzy points  $p$  and  $q$  with supports  $x$  and  $y$  respectively such that

$$p(x) = q(y) = r$$

Since  $(X, \omega(\mathcal{E}))$  is fuzzy  $T_1$ , there exists fuzzy open sets  $U$  and  $V$  in  $(X, \omega(\mathcal{E}))$  such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$ .

$$\Rightarrow p(x) = r < U(x), \quad q(y) = r \geq U(y)$$

$$\text{and } q(y) = r < V(y), \quad p(x) = r \geq V(x)$$

$$U(x) > r$$

$$\Rightarrow x \in U^{-1}((r, 1]) \text{ which is a } \mathcal{E} \text{ - open set}$$

$$U(y) \leq r$$

$$\Rightarrow y \notin U^{-1}((r, 1]) \text{ which is a } \mathcal{E} \text{ - open set}$$

$$V(y) > r$$

$$\Rightarrow y \in V^{-1}((r, 1]) \text{ which is a } \mathcal{E} \text{ - open set}$$

$$V(x) \leq r$$

$$\Rightarrow x \notin V^{-1}((r, 1]) \text{ which is a } \mathcal{E} \text{ - open set}$$

Hence  $V^{-1}((r,1])$  and  $U^{-1}((r,1])$  are open sets in  $(X, \mathcal{E})$  satisfying the required conditions for  $T_1$

Hence  $(X, \mathcal{E})$  is  $T_1$ .

(ii) Assume  $(X, \tau)$  is fuzzy  $T_1$

Consider two distinct points  $x$  and  $y$  in  $X$

Consider two fuzzy points  $p$  and  $q$  in  $X$  with supports  $x$  and  $y$  respectively such that  $p(x) = q(y) = r$

Since  $(X, \tau)$  is fuzzy  $T_1$ , there exists two fuzzy open sets  $U$  and  $V$  in  $(X, \tau)$  such that  $p \in U$ ,  $q \notin U$  and  $q \in V$ ,  $p \notin V$

Then  $r < U(x)$ ,  $r \geq U(y)$

and  $r < V(y)$ ,  $r \geq V(x)$

$\therefore$  As in (i) we have  $x \in U^{-1}((r,1])$ ,  $y \notin U^{-1}((r,1])$  and  $y \in V^{-1}((r,1])$ ,  $x \notin V^{-1}((r,1])$

By the definition  $i(\tau)$ , as  $U, V \in \tau$ ,  $U^{-1}((r,1])$  and  $V^{-1}((r,1])$  are open in  $i(\tau)$  Hence  $(X, i(\tau))$  is  $T_1$

**Theorem 2.2.3** : For topologically generated fuzzy topological spaces, the three statements of the theorem 2.2.1 are equivalent

**Proof:** To prove the theorem, it is enough to show that (iii)  $\implies$  (ii)

Since  $\tau$  is topologically generated,  $\tau = \omega(\mathcal{E})$  for some topology  $\mathcal{E}$  on  $X$ .

By assumption  $(X, \omega(\mathcal{E}))$  is fuzzy  $T_1$

$\therefore$  From theorem 2.2.2(i),  $(X, \mathcal{E})$  is  $T_1$   
 $\implies (x)$ , for all  $x \in X$  is closed in  $(X, \mathcal{E})$

**To prove:**  $(x)$  is fuzzy closed in  $\omega(\mathcal{E})$

ie. to prove  $X - (x)$  is fuzzy open in  $\omega(\mathcal{E})$

ie. to prove  $X - (x) \in \omega(\mathcal{E})$

ie. to prove  $X - (x)$  is lower semicontinuous

for that we have to prove  $(X - \{x\})^{-1}((\epsilon, 1]) \in \mathcal{C}$  for all  $\epsilon > 0$

Take  $y \in (X - \{x\})^{-1}((\epsilon, 1])$

$\Leftrightarrow (X - \{x\})(y) \in ((\epsilon, 1])$  for all  $\epsilon > 0$

$\Leftrightarrow (X - \{x\})(y) = 1$

$\Leftrightarrow y \in X - \{x\}$

Hence  $X - \{x\} = (X - \{x\})^{-1}((\epsilon, 1])$

Since  $X - \{x\}$  is open in  $\mathcal{C}$ ,  $(X - \{x\})^{-1}((\epsilon, 1])$  is open in  $\mathcal{C}$

$\Rightarrow (X - \{x\})^{-1}((\epsilon, 1]) \in \mathcal{C}$

$\therefore X - \{x\} \in \omega(\mathcal{C})$

$\Rightarrow X - \{x\}$  is fuzzy open in  $\omega(\mathcal{C})$

$\Rightarrow \{x\}$  is fuzzy closed in  $\omega(\mathcal{C}) = \tau$

Hence we have (iii)  $\Rightarrow$  (ii)

### Section 3:

In this section we discuss the separation axioms due to Ghanim[2]. According to Ghanim's definition of  $FT_1$  spaces we get that a fts is  $FT_1$  iff every crisp singleton is fuzzy closed. Whereas, we have noted in Section 2, that it is not true according to the definition of  $FT_1$  spaces due to Rekha Srivastava [5]

Terminology: Throughout this section we follow

1. Chang's definition of fuzzy topology.
2. By fuzzy singletons we mean both fuzzy and crisp singletons.

Definition: A fts is said to be

1.  $FT_0$  iff for every pair of fuzzy singletons  $p_1$  and  $p_2$  with different supports, there exists an open fuzzy set  $O$  such that  $p_1 \subset O \subset p_2'$  or  $p_2 \subset O \subset p_1'$
2.  $FT_1$  iff for every pair of fuzzy singletons  $p_1$  and  $p_2$  with

different supports, there exist open fuzzy sets  $O_1$  and  $O_2$  such that  $p_1 \in O_1 \subset P_2'$  and  $P_2 \subset O_2 \subset P_1'$

3.  $FT_5$ (Strong  $FT_1$ ) iff every fuzzy singleton is a closed fuzzy set

4.  $FT_2$ (F-Hausdorff) iff for every pair of fuzzy singletons  $p_1$  and  $p_2$  with different supports, there exists open fuzzy sets  $O_1$  and  $O_2$  such that  $p_1 \in O_1 \subset P_2'$ ,  $P_2 \subset O_2 \subset P_1'$  and  $O_1 \subset O_2'$ . Or, equivalently, iff for every pair of fuzzy singletons  $p_1$  and  $p_2$  with different supports, there exists an open fuzzy set  $O$  such that  $p_1 \in O \subset cl O \subset P_2'$

5.  $FT_{2\frac{1}{2}}$ (F-Urysohn) iff for every pair of fuzzy singletons  $p_1$  and  $p_2$  with different supports, there exists open fuzzy sets  $O_1$  and  $O_2$  such that  $p_1 \in O_1 \subset P_2'$ ,  $P_2 \subset O_2 \subset P_1'$  and  $cl O_1 \subset (cl O_2)'$ .

From the above definitions one can notice that the following chains of implications are true

1.  $(X, \tau)$  is  $FT_5 \implies (X, \tau)$  is  $FT_1 \implies (X, \tau)$  is  $FT_0$
2.  $(X, \tau)$  is  $FT_{2\frac{1}{2}} \implies (X, \tau)$  is  $FT_2 \implies (X, \tau)$  is  $FT_1 \implies (X, \tau)$  is  $FT_0$

In ordinary Topological spaces, the notions  $FT_1$  and  $FT_5$  coincide. But here it is not so which is a deviation from ordinary topology

**Theorem 1.3.1:** A fts is  $FT_1$  iff every crisp singleton is closed

Proof: Assume  $(X, \tau)$  is  $FT_1$

To prove: Crisp singletons are closed

Let  $p_0$  be a crisp point such that  $p_0(x_0)=1$

To prove:  $p_0$  is fuzzy closed

ie. to prove  $p_0'$  is fuzzy open.

By theorem 1.2, since every fuzzy set can be written as the union of the fuzzy singletons, we can write

$$p_0' = \bigcup P$$

$$P \subset p_0'$$

Now  $P \subset p_0'$

$$\implies p(x) \leq p_0'(x) \text{ for all } x \in X$$

$$\implies p(x_0) \leq p_0'(x_0)$$

$$= 1 - p_0(x_0) = 0$$

$$\implies p(x_0) = 0$$

Hence  $x_p \neq x_0$

Hence as  $(X, \tau)$  is  $FT_1$ , there exists fuzzy open sets  $O_0$  and  $O_p$  such that  $p_0 \subset O_0 \subset p'$  and  $p \subset O_p \subset p_0'$

Hence  $p(x) \leq O_p(x) \leq p_0'(x)$

$$\implies \text{Sup } p(x) \leq \text{Sup } O_p(x) \leq p_0'(x)$$

$$P \subset p_0' \quad P \subset p_0'$$

$$\implies p_0'(x) \leq \text{Sup } O_p(x) \leq p_0'(x)$$

$$P \subset p_0'$$

$$\implies p_0' = \bigcup O_p$$

$$P \subset p_0'$$

Since each  $O_p$  is open,  $p_0'$  is open (being union of fuzzy open sets)  $\therefore p_0$  is fuzzy closed.

ie. every crisp singleton is fuzzy closed.

Conversely, Assume that every crisp singleton is closed.

To prove:  $(X, \tau)$  is  $FT_1$

Consider two fuzzy singletons  $p_1$  and  $p_2$  with different supports  $x_1$  and  $x_2$

Also consider two crisp singletons  $q_1$  and  $q_2$  with supports  $x_1$  and  $x_2$  respectively

$$\text{ie. } q_1(x_1) = q_2(x_2) = 1$$

Now by our assumption the fuzzy sets  $q_1'$  and  $q_2'$  are open

$$\begin{aligned} \text{Also } p_1(x_1) &\leq 1 &&= 1 - 0 \\ &&&= 1 - q_2(x_1) \\ &&&= q_2'(x_1) \end{aligned}$$

$$\begin{aligned} \text{and if } x \neq x_1, p_1(x) &= 0 \\ &\leq q_2'(x) \end{aligned}$$

$$\text{Thus } p_1 \subset q_2' \text{ ----- (1)}$$

$$\begin{aligned} \text{If } x \neq x_2, q_2'(x) &= 1 &&= 1 - 0 \\ &&&= 1 - p_2(x) \\ &&&= p_2'(x) \end{aligned}$$

$$\begin{aligned} \text{and if } x = x_2, q_2'(x_2) &= 0 \\ &\leq p_2'(x_2) \end{aligned}$$

$$\text{Thus } q_2' \subset p_2' \text{ ----- (2)}$$

Hence from (1) and (2),  $p_1 \subset q_2' \subset p_2'$

Similarly it can be shown that  $p_2 \subset q_1' \subset p_1'$

Hence there exists open sets  $O_1 = q_2'$  and  $O_2 = q_1'$  such that

$$p_1 \subset O_1 \subset p_2' \text{ and } p_2 \subset O_2 \subset p_1'$$

Hence  $(X, \tau)$  is  $FT_1$

**Theorem 1.3.2:** For every infinite set  $X$  there exists a weakest fuzzy topology such that  $(X, \tau)$  is  $FT_5$

**Proof:** Let  $X$  be an infinite set. define a family  $\tau$  by  
 $\tau = \{ O \mid O \text{ is a fuzzy set in } X \text{ and } \text{Supp } O' \text{ is finite} \}$

It is easily seen that  $\tau$  is a fuzzy topology

**Claim:** Every fuzzy singleton is closed

Consider a fuzzy singleton  $p$

$$\text{Then } \text{Supp } p = \{x_p\},$$

$$\begin{aligned} \therefore \text{Supp } p' &= \{ x \mid p'(x) \neq 0 \} \\ &= \{ x \mid 1 - p(x) \neq 0 \} \\ &= \{ x \mid p(x) \neq 1 \} \end{aligned}$$

$$= \begin{cases} X - \{x_p\} & \text{if } p \text{ is crisp singleton} \\ X & \text{if } p \text{ is a fuzzy singleton} \end{cases}$$

$$= \text{infinite}$$

$$\therefore p \notin \tau$$

$\Rightarrow p$  is  $\tau$ -closed

This is true for every fuzzy singleton on  $X$

$\therefore (X, \tau)$  is  $FT_S$

Claim:  $\tau$  is the weakest fuzzy topology for which  $(X, \tau)$  is  $FT_S$

ie. To prove if  $\sigma$  is another fuzzy topology such that  $(X, \sigma)$  is  $FT_S$  then  $\tau \subseteq \sigma$

Take  $U \in \tau$ . Then  $\text{Supp } U'$  is finite

Let  $\text{Supp } U' = \{x_1, x_2, \dots, x_n\}$

For every  $i \in \{1, 2, \dots, n\}$  define the fuzzy singleton  $p_i$  by

$$p_i(x_i) = U'(x_i)$$

$$p_i(x) = 0 \text{ for } x \neq x_i$$

Since  $(X, \sigma)$  is  $FT_S$  each  $p_i$  is  $\sigma$ -closed.

Claim:  $U' = \bigcup_{i=1}^n p_i$

Take any  $x$ .

Suppose  $x \neq x_i$  for every  $i = 1, 2, \dots, n$

$$\text{Consider } \left( \bigcup_{i=1}^n p_i \right) (x) = \text{Sup}_{i=1, 2, \dots, n} (p_i(x)) = 0$$

$$= U'(x)$$

Suppose  $x = x_j$  for some  $j$

$$\text{Consider } \left( \bigcup_{i=1}^n p_i \right) (x) = \text{Sup}_{i=1, 2, \dots, n} (p_i(x))$$

$$= p_j(x_j)$$

$$= U'(x_j)$$

$$= U'(x)$$

$$\therefore \text{We have } U' = \bigcup_{i=1}^n p_i$$

$$i=1$$

Since  $\bigcup_{i=1}^n p_i$  is a finite union of  $\sigma$  - closed fuzzy sets,  $U'$  is  $\sigma$  - closed.

$\implies U \in \sigma$

$\therefore$  We have  $\tau \subset \sigma$

$\therefore$   $\tau$  is the weakest fuzzy topology for which  $(X, \tau)$  is  $FT_5$

Hence the theorem

**Definition:** A fuzzy topological space is said to be FR (fuzzy regular) iff for every fuzzy singleton in  $X$  and every closed fuzzy set  $F$  in  $X$  such that  $p \subset F'$  there exist open fuzzy sets  $U$  and  $V$  such that  $p \subset U$ ,  $F \subset V$  and  $U \subset V'$ . Or, equivalently, iff for every fuzzy singleton  $p$  in  $X$  and for every open fuzzy set  $U$  containing  $p$ , there exists an open fuzzy set  $W$  such that

$$p \subset W \subset \text{cl } W \subset U$$

An FR fts which is also  $FT_5$  is said to be  $FT_3$ ,

This definition of FR and  $FT_3$  spaces coincide with those due to Mira Sarkar[4] given in Section I

**Theorem 1.3.3:** For every closed fuzzy set  $F$  in a FR fts and any fuzzy singleton  $p \subset F'$  there exists open fuzzy sets  $U$  and  $V$  such that  $p \subset U$ ,  $F \subset V$  and  $\text{cl } U \subset (\text{cl } V)'$

**Proof:** Let  $(X, \tau)$  be the given FR space. Since  $(X, \tau)$  is FR, there exists an open fuzzy set  $U$  such that  $p \subset U \subset \text{cl } U \subset F'$ . By taking  $V = (\text{cl } F)'$ , we see that the required conditions are satisfied.

**Corollary:** Every  $FT_3$  fts is  $FT_{2\frac{1}{2}}$

**Proof:** Let  $(X, \tau)$  be an  $FT_3$  space

$\implies (X, \tau)$  is FR and  $FT_5$

Let  $p_1$  and  $p_2$  be any pair of fuzzy singletons with different supports  $x_1$  and  $x_2$  respectively. Since  $(X, \tau)$  is  $FT_5$ , the singleton  $p_2$  is a closed fuzzy set

Claim:  $p_1 \subset p_2'$

Suppose  $x = x_1$

Consider  $p_1(x_1) \leq 1$

$$= p_2'(x_1)$$

$$\therefore p_1(x_1) \leq p_2'(x_1)$$

Suppose  $x \neq x_1$

Consider  $p_1(x) = 0$

$$\leq p_2'(x)$$

$$\therefore p_1 \subset p_2'$$

Hence by the above theorem there exists open fuzzy sets  $U$  and  $V$  such that  $p_1 \subset U$ ,  $p_2 \subset V$  and  $cl U \subset (cl V)'$  -----(1)

Since  $U \subset cl U$  and  $(cl V)' \subset V'$ , we get

from (1)  $U \subset cl U \subset (cl V)'$

$$\subset V'$$

$$\text{ie. } U \subset V'$$

$$\implies V \subset U'$$

Thus we have  $p_1 \subset U \subset V'$  and  $p_2 \subset V \subset U'$  with  $cl U \subset (cl V)'$

Hence  $(X, \tau)$  is  $FT_{2\frac{1}{2}}$

In ordinary topology we know that every regular  $T_0$  topological space is also  $T_3$ . But this is not true in fuzzy topology. Hence we get a deviation from ordinary topology. But we have only the following results.

theorem 1.3.4: Let  $X$  be an FR fts which is also  $FT_0$ . Then  $X$  is  $FT_{2\frac{1}{2}}$

Proof: Let  $p$  and  $q$  be any pair of fuzzy singletons with different supports, since  $X$  is  $FT_0$ , there exists an open fuzzy

set  $U$  such that  $p \in U \subseteq q'$

Take  $F = U'$ . Then  $F$  is fuzzy closed and  $U = F'$

$\therefore p \in F'$

By theorem 1.3.3, there exist two open fuzzy sets  $G$  and  $H$  such that  $p \in G$ ,  $F \subseteq H$  and  $\text{cl } G \subseteq (\text{cl } H)'$

Since  $U \subseteq q'$  we have  $q \in U'$

ie.  $q \in F$

Hence  $q \in F \subseteq H \implies q \in H$

Hence there exists open fuzzy sets  $G$  and  $H$  such that  $p \in G$ ,  $q \in H$

and  $\text{cl } G \subseteq (\text{cl } H)'$

as in the proof of the previous theorem it can be shown that

$(X, \tau)$  is  $FT_{2\frac{1}{2}}$

**CHAPTER III**

## CHAPTER III PROPERLY COMPACT SETS

Mira Sarkar [4] has defined the notion of properly compactness which is more meaningful (in Hausdorff spaces) than Chang's definition of compactness. According to Chang's definition of compactness no subset of a Hausdorff space is compact. But according to Mira Sarkar's definition of proper compactness

1. Every singleton is properly compact and
2. A properly compact set in a Hausdorff fts is closed

In this chapter we have also discussed the behaviour of properly compact sets in Fuzzy P - spaces

Terminology: Throughout this section,

(i) by points (subsets) of  $X$ , we mean both crisp and fuzzy points (subsets). Points are also referred to as singletons.

(ii) We follow the definitions of separation properties as given in Section 1 of chapter II

### Section 1

#### Properly compact sets and Hausdorff spaces

In this section we discuss the behaviour of properly compact sets in  $FT_2$  spaces. Before giving the definition of properly compact sets we prove that according to Chang's definition of compactness no subset in a Hausdorff fts is compact

Definition(Chang[1]): Let  $(X, \tau)$  be a fts. A family  $U$  of fuzzy sets is a cover of a fuzzy set  $B$  iff  $B \subset \cup \{A / A \in U\}$ . It is an open cover iff each member of  $U$  is an open fuzzy set. A subcover of  $U$  is a subfamily which is also a cover.

Definition (Chang[1]) A fts is compact iff each open cover of the space has a finite subcover

Theorem 3.1.1: No subset of Hausdorff fts can be compact (countably compact)

Proof: Let  $A$  be a subset of the Hausdorff fts  $(X, \tau)$  such that

$A(x_A) > 0$  for some  $x_A \in X$

Choose the sequence  $(y_n / n \in \mathbb{N})$  as a strictly increasing sequence converging to  $A(x_A)$ . Define a sequence  $(p_n)$  of fuzzy points all having the support  $x_A$  as follows

$$p_n(x_A) = y_n \text{ for all } n.$$

Consider two fuzzy points  $p_n$  and  $p_{n+1}$ .

They have the same support  $x_A$  and  $p_n(x_A) < p_{n+1}(x_A)$ .

Then from the Hausdorff property there exists an open set  $V_{x_{A_n}}$  such that  $p_n \in V_{x_{A_n}}$  and  $p_{n+1} \notin \overline{V_{x_{A_n}}}$ . Hence there exists a sequence of open sets  $(V_{x_{A_n}} / n \in \mathbb{N})$  such that

$$p_n \in V_{x_{A_n}} \text{ and } p_{n+1} \notin \overline{V_{x_{A_n}}}$$

$$\text{i.e. } p_n(x_A) < V_{x_{A_n}}(x_A) \text{ and } p_{n+1}(x_A) \geq \overline{V_{x_{A_n}}}(x_A)$$

Since the space is an  $FT_2$  space, crisp singletons are closed.

Hence  $(x_A)$  is closed.

$\therefore$  The complement of the crisp point at  $x_A$  is open.

Claim 1: The sequence  $(V_{x_{A_n}} / n \in \mathbb{N})$  together with the complement of the crisp point at  $x_A$  forms an open cover of  $A$ .

For that we have to prove  $A \subset (U V_{x_{A_n}}) \cup (x - (x_A))$

i.e. to prove  $A(x) \leq \sup \{ \sup V_{x_{A_n}}(x), (x - (x_A))(x) \}$  for all  $x \in X$ .

When  $x = x_A$

We have  $(p_n(x_A) / n \in \mathbb{N})$  is an increasing sequence which converges to  $A(x_A)$ .

$$\begin{aligned} \therefore A(x_A) &= \sup p_n(x_A) \\ &\leq \sup V_{x_{A_n}}(x_A) \\ &= \sup \{ \sup V_{x_{A_n}}(x_A), 0 \} \end{aligned}$$

Since  $(x_A)'(x_A) = 1 - (x_A)(x_A)$

$$= 1 - 1 = 0$$

We have  $A(x_A) \leq \sup \{ \sup V_{x_{A_n}}(x_A), (x_A)'(x_A) \}$  --- (1)

When  $x \neq x_A$ .

$$\begin{aligned}
& \text{Consider } \sup \{ \sup V_{x_{A_n}}(x), (x_A)'(x) \} \\
&= \sup \{ \sup V_{x_{A_n}}(x), (1 - (x_A))'(x) \} \\
&= \sup \{ \sup V_{x_{A_n}}(x), (1 - 0) \} \\
&= 1 \quad (\because \sup V_{x_{A_n}} \leq 1)
\end{aligned}$$

$\therefore$  We have  $A(x) \leq 1$

$$= \sup \{ \sup V_{x_{A_n}}(x), (x_A)'(x) \}$$

$$\therefore A(x) \leq \sup \{ \sup V_{x_{A_n}}(x), (x_A)'(x) \} \quad \text{---(2)}$$

From (1) and (2) we get

$$A \subseteq (U V_{x_{A_n}}) \cup (X - (x_A)')$$

Hence claim 1

**Claim 2:** No finite subfamily can cover A.

Take any finite subfamily  $\{ V_{x_{B_1}}, V_{x_{B_2}}, \dots, V_{x_{B_k}} \}$

Then  $B_1 = A_{i_1}; B_2 = A_{i_2}; \dots, B_k = A_{i_k}$

We have  $p_n \in V_{x_{A_n}}$  and  $p_{n+1} \notin \overline{V_{x_{A_n}}}$

ie.  $y_n < V_{x_{A_n}}(x_A)$  and  $y_{n+1} \geq \overline{V_{x_{A_n}}}(x)$

$$\begin{aligned}
\text{Now } \sup_{j=1,2,\dots,n} V_{x_{A_{ij}}}(x_A) &\leq \sup_{j=1,2,\dots,n} y_{j+1} \\
&< A(x_A) \quad [\text{since } \{y_n / n \in \mathbb{N}\} \text{ is an increasing} \\
&\quad \text{sequence which converges to } A(x_A)]
\end{aligned}$$

$$\text{ie. } A(x_A) > \sup_{j=1,2,\dots,n} V_{x_{A_{ij}}}(x_A)$$

$$\therefore A \not\subseteq \sup_{j=1,2,\dots,n} V_{x_{A_{ij}}}$$

$\therefore$  The finite sequence  $\{ V_{x_{B_1}}, V_{x_{B_2}}, \dots, V_{x_{B_k}} \}$  does not cover A.

$\therefore$  The subset A is not compact

**Corollary:** Singletons in an  $FT_2$  space are not compact (countably compact)

**Proof:** Since no subset of a Hausdorff fts can be compact, singletons being particular subsets of Hausdorff fts can not be compact.

Now we give the definition of proper open cover using which

properly compact sets can be defined.

Definition: A collection  $U = \{ V_\alpha, \alpha \in \Lambda, V_\alpha \in \tau \}$  is said to be a proper open cover of the set  $A$  in the fts  $(X, \tau)$  iff for each  $x \in X$ , there exists

$V_{\alpha_x} \in U$ , such that  $V_{\alpha_x}(x) \geq A(x)$ .  $U$  is a countable (finite) proper open cover of  $A$  if  $U$  is countable (finite). A subcollection  $U_1$  of  $U$  is a proper open subcover of  $U$  if it is a proper open cover of  $A$  in its own right.

Clearly, a proper cover of  $A$  is always a cover of  $A$ , but not conversely.

Definition: A set  $A$  is properly (countably) compact in the fts  $(X, \tau)$  iff every (countable) proper open cover of  $A$  has a proper open finite subcover.

Theorem 3.1.2: Every singleton (hence a subset with finite support) in an fts is properly compact.

Proof: Let  $p$  be a point of  $X$  with support  $x_p$

Let  $(V_\alpha, \alpha \in \Lambda, V_\alpha \in \tau)$  be a proper open cover of the crisp point  $p$

Now  $x_p \in X$

$\therefore$  There exists a  $V_{\alpha_{x_p}} \in (V_\alpha, \alpha \in \Lambda, V_\alpha \in \tau)$

Such that  $p(x_p) \leq V_{\alpha_{x_p}}(x_p)$

If  $x \neq x_p$ , then  $p(x) = 0$

$$\leq V_{\alpha_{x_p}}(x)$$

$\therefore \{V_{\alpha_{x_p}}\}$  consisting of only  $V_{\alpha_{x_p}}$  forms a proper open subcover of  $p$

$\therefore p$  is properly compact

Hence the theorem.

Theorem 3.1.3: Let  $(X, \tau)$  be an fts and  $A$  a properly compact set in  $X$ . If  $f: (X, \tau) \rightarrow (Y, \sigma)$  is an  $F$ -continuous (definition in Chapter-I) surjection, then  $f(A)$  is a properly compact set in  $Y$ .

Proof: Let  $U = \{ V_\alpha, \alpha \in \Lambda, V_\alpha \in \tau \}$  be a proper open cover of the set  $f(A)$  in  $(Y, \sigma)$ . Consider the collection

$$W = \{ f^{-1}(V_{\alpha}) / \alpha \in \Delta \}$$

Since  $f$  is  $F$ -continuous,  $f^{-1}(V_{\alpha}) \in \tau$ .

To prove the collection,  $W$  is a proper open cover of the set  $A$  in  $(X, \tau)$

Take any  $x \in X$ . Then  $f(x) \in Y$

$\therefore$  There exists a  $V_{\alpha_{f(x)}}$  such that

$$\begin{aligned} (f(A))(f(x)) &\leq V_{\alpha_{f(x)}}(f(x)) \\ &= (f^{-1}(V_{\alpha_{f(x)}}))(x) \end{aligned}$$

[since by definition,  $(f^{-1}(B))(x) = B(f(x))$  for all  $x \in X$ ]

$$\text{ie. } \sup_{z \in f^{-1}(f(x))} A(z) \leq (f^{-1}(V_{\alpha_{f(x)}}))(x)$$

$$\text{Since } A(x) \leq \sup_{z \in f^{-1}(f(x))} A(z),$$

$$A(x) \leq (f^{-1}(V_{\alpha_{f(x)}}))(x)$$

This is true for every  $x \in X$

$\therefore W$  is a proper open cover of  $A$  since  $A$  is properly compact in  $(X, \tau)$ , there exists a proper open finite subcover

$$W' = \{ f^{-1}(V_{\alpha_i}) / i=1, 2, \dots, k \}$$

To prove  $\{ V_{\alpha_i} / i=1, 2, \dots, k \}$  is a proper open cover of  $f(A)$

Take  $y \in Y$

Since  $f$  is onto there exists  $x \in X$  such that  $y = f(x)$

Since  $W'$  is a proper open cover of  $A$ , for every  $z \in X$  there exists an  $i$  such that  $A(z) \leq f^{-1}(V_{\alpha_i})_z(z)$

$$\text{Now Consider } f(A)(y) = f(A)(f(x)) = \sup_{z \in f^{-1}(f(x))} A(z)$$

$$\begin{aligned} &\leq \sup_{z \in f^{-1}(f(x))} (f^{-1}(V_{\alpha_i})_z(z)) \\ &= \sup_{z \in f^{-1}(f(x))} ((V_{\alpha_i})_z(f(z))) \\ &= \sup_{z \in f^{-1}(f(x))} ((V_{\alpha_i})_z(f(x))) \\ &= V_{\alpha_j}(f(x)) \text{ (say)} \\ &= V_{\alpha_j}(y) \end{aligned}$$

$$\therefore f(A)(y) \leq \bigvee \alpha_j (y)$$

$\implies ( \bigvee \alpha_i / i=1,2,\dots,k )$  is a proper open cover of  $f(A)$

Hence  $f(A)$  is a properly compact set in  $Y$ .

Theorem 3.1.4 A properly compact set in an  $FT_2$  space is closed

Proof: Let  $(X, \tau)$  be the given  $FT_2$  space. Let  $A \neq \emptyset$  be a properly compact subset in  $(X, \tau)$ , then there exists a point  $x_p \in X$  such that  $A(x_p) \neq 1$

Choose  $p$  in  $X$  such that  $A(x_p) < p(x_p)$  ---(1)

By theorem 1.1, We can take  $A$  as

$$A = \bigcup_{x \in X} \left\{ q_x / q_x \text{ is a fuzzy point with support } x, \text{ value } A(x) \right\}$$

$$A(x) \neq 0$$

Take  $q_x$  in  $A$ .

For any  $y \in X$ ,  $q_x(y) \leq A(y)$

In particular,  $q_x(x_p) \leq A(x_p)$  But  $A(x_p) < p(x_p)$

$$\therefore q_x(x_p) < p(x_p)$$

By Hausdorff property there exists an open set  $V_p$  such that

$$q_x \in V_p \text{ but } p \notin \overline{V_p}$$

$$\implies q_x(x_p) < V_p(x_p) \text{ and } p(x_p) \geq \overline{V_p}(x_p)$$

Proceeding like this, to each point  $p$  in  $X$  satisfying the condition  $p(x_p) > A(x_p)$  we get a sequence of open sets  $(V_{pq}, x_q \in X)$

Such that  $A(x_q) < V_{pq}(x_q)$  for all  $x_q \in X$

and  $p(x_p) \geq \overline{V_{pq}}(x_p)$  for all  $q$  such that  $x_q \in X$  ---(2)

If  $A(x_q) = 1$ , then we must have  $A(x_q) = V_{pq}(x_q)$  for all  $x_q \in X$

(since  $V_{pq}(x_q) \neq 1$  for all  $x_q \in X$ )

Hence  $A(x_q) < V_{pq}(x_q)$  for all  $x_q \in X$

$$\leq \bigcup_{x_q \in X} V_{pq}(x_q)$$

$$x_q \in X$$

$$\implies A \subseteq \bigcup_{x_q \in X} V_{pq}$$

$$x_q \in X$$

$\Rightarrow$  The collection  $\{V_{pq}, x_q \in X\}$  forms a proper open cover of  $A$ , since  $A$  is properly compact this cover has a proper open finite sub-cover  $\{V_{pq_1}, V_{pq_2}, \dots, V_{pq_n}\}$

$$\text{ie. } A \subset \bigcup_{\substack{x_{q_k} \in X \\ k=1, 2, \dots, n}} V_{pq_k}$$

$$\Rightarrow A \subset \bigcup_{k=1, 2, \dots, n} \overline{V_{pq_k}}$$

Take  $\bigcup_{k=1, 2, \dots, n} \overline{V_{pq_k}} = F_p$  So that  $F_p$  is closed

Hence  $A \subset F_p$

$$\Rightarrow A(x_q) \leq F_p(x_q) \text{ for all } x_q \in X \text{ ---(3)}$$

From (2),  $p(x_p) \geq \overline{V_{pq}}(x_p)$  for all  $q, x_q \in X$  and also we have

$$\bigcup_{k=1, 2, \dots, n} \overline{V_{pq_k}} = F_p$$

$$\therefore p(x_p) \geq F_p(x_p) \text{ ---(4)}$$

Considering all the points  $p$  satisfying (1) we get the collection  $\{F_p / F_p' \in \tau\}$  such that (3) and (4) are true

Then by using (3) and (4) it can be shown that

$$A(x_q) = \bigcap_p F_p(x_q) \text{ for all } x_q \in X$$

Hence  $A = \bigcap_p F_p \Rightarrow A$  is closed.  $A$  is an arbitrary set in  $(X, \tau)$

$\therefore$  Every properly compact set in  $(X, \tau)$  is closed.

Hence the theorem.

## Section 2

### Properly compact sets and Fuzzy P-space.

In this section we study the behaviour of properly compact sets in Fuzzy P-spaces.

**Definition** :  $(X, \tau)$  is a Fuzzy P-space if the countable union of closed sets is closed.

**Theorem 3.2.1.** : Properly compact sets in an  $FT_1$ , P-space have finite supports.

**Proof** : Let  $(X, \tau)$  be an  $FT_1$ , P-space.

Let  $A$  be a fuzzy set on  $X$  having atleast a countable support  $(x_1, x_2, \dots, x_n, \dots)$

Define a set  $F_r$  by

$$F_r(x) = \begin{cases} 1 & \text{if } x = x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n, \dots \\ 0 & \text{otherwise.} \end{cases}$$

$$\text{Let } H_r = \bigcup_{i \neq r}^{\infty} (x_i)$$

Considering  $H_r$  as a fuzzy set, we have

$$H_r(x) = \begin{cases} 1 & \text{if } x \in H_r \\ 0 & \text{otherwise} \end{cases}$$

Hence we get that  $F_r = H_r$

$$\text{ie. } F_r = \bigcup_{i \neq r}^{\infty} (x_i)$$

In an  $FT_1$  space crisp singletons are closed.

$\therefore \bigcup_{i \neq r}^{\infty} (x_i)$  is a countable union of closed sets in  $(X, \tau)$ .

Since  $(X, \tau)$  is a P-space, we get  $\bigcup_{i \neq r}^{\infty} (x_i)$  is closed

ie  $F_r$  is closed.

Take  $G_r = F_r'$

**Claim** :  $(G_n, n \in \mathbb{N})$  is a proper open cover of  $A$ .

Since  $G_r(x) = 1 - F_r(x)$ ,

$$\left. \begin{aligned} G_r(x) &= 1 \text{ if } x \notin H_r \\ &= 0 \text{ if } x \in H_r \end{aligned} \right\} \text{-----} (1)$$

Since  $x_r \notin H_r$ ,  $G_r(x_r) = 1$ .

Suppose  $x \in \text{supp } A$ .

$\Rightarrow x = x_r$  for some  $r$ .

Take corresponding  $G_r$ .

Then  $G_r(x) = G_r(x_r) = 1$

$$\geq A(x)$$

$$\therefore A(x) \leq G_r(x)$$

Suppose  $x \notin \text{supp } A$ .

$\Rightarrow A(x) = 0$

$$\leq G_n(x) \text{ for all } n \in \mathbb{N}.$$

$\therefore$  For each  $x \in X$ , there exists a  $G_r$  such that  $A(x) \leq G_r(x)$ .

$\therefore (G_n, n \in \mathbb{N})$  is a proper open cover of  $A$ .

Claim : No finite subcollection of  $(G_n, n \in \mathbb{N})$  is a proper sub-cover of  $A$ .

Take any finite subcollection of  $(G_n, n \in \mathbb{N})$

Let it be  $(G_{i_1}, G_{i_2}, \dots, G_{i_k})$

Choose  $n = \max\{i_1, i_2, \dots, i_k\}$

Then  $n+1 \neq i_1, i_2, \dots, i_k$ .

Let  $x = x_{n+1}$ .

$\therefore x_{n+1} \notin (x_{i_1}, x_{i_2}, \dots, x_{i_k})$

Since  $x_{n+1} \neq x_{i_1}$ ,  $x_{n+1} \notin H_{i_1}$

Similarly,  $x_{n+1} \notin H_{i_2}, x_{n+1} \notin H_{i_3}, \dots, x_{n+1} \notin H_{i_k}$

$\therefore$  From (1),  $G_{i_1}(x_{n+1}) = 0$

$$G_{i_2}(x_{n+1}) = 0$$

:

:

$$G_{i_k}(x_{n+1}) = 0$$

Now  $A(x_{n+1}) > 0$

$$= G_{i_j}(x_{n+1}), j=1, 2, \dots, k.$$

$\therefore$  The subcollection  $(G_{i_1}, G_{i_2}, \dots, G_{i_k})$  is not a proper open finite

subcover of  $A$ . Hence no finite subcollection can properly cover  $A$ .  
 $\therefore A$  is not properly compact.

Hence we conclude that every properly compact set in an  $FT_1$ ,  $P$ -space has finite support.

Corollary  $\dot{}$  Properly compact sets in a Hausdorff fuzzy  $P$ -space have finite supports.

Proof  $\dot{}$  By theorem 2.1.1, in an  $FT_2$  space crisp singletons are closed. Following the ~~same~~ proof as in the above theorem, we get the result.

**CHAPTER IV**

## CHAPTER IV

### FUZZY SUBSPACES AND FUZZY SUMS

This chapter is devoted to the study of two of the most important operations: the subspace of a fuzzy topological space and the sum of fuzzy topological spaces. The results are due to Ghanim [2]. He has proved that the properties  $FT_0, FT_1, FT_5, FT_2, FT_{2\frac{1}{2}}, FT_3$  are hereditary and regarding fuzzy normality, he has proved that it is hereditary with respect to closed subspaces. Regarding fuzzy sums, he has proved that all these separation properties are additive.

Terminology : Throughout this section we follow

1. Chang's definition of fuzzy topology.
2. By fuzzy singletons we mean both fuzzy and crisp singletons.

#### Section 1.

##### Fuzzy subspaces

In this section we study briefly the notion of Fuzzy subspaces. Fuzzy subspaces are defined only for ordinary subsets of  $X$ .

Agreement : Let  $M$  be an ordinary subset of  $X$ . If  $A$  is a fuzzy set in  $M$ , then it is convenient to consider  $A$  as a fuzzy set in the superset  $X$  by setting  $A(x) = 0$  for every  $x \in X \setminus M$ .

Definition : Associated with a given fts  $(X, \tau)$  and an arbitrary ordinary subset  $M$  of  $X$  we define the induced fuzzy topology for  $M$  or the relative fuzzy topology for  $M$  by

$\tau_M = (M \cap O / O \in \tau)$ . The corresponding pair  $(M, \tau_M)$  is called a Subspace of  $(X, \tau)$ . A subspace  $(M, \tau_M)$  of a fts  $(X, \tau)$  is called an Open (closed) subspace iff the basic set  $M$  is  $\tau$ -open ( $\tau$ -closed).

One easily verifies the transitivity of the relation "being a fuzzy subspace for a fts"

Definition : The relative complement  $co_M A$  of a fuzzy set  $A$  in  $X$  with respect to an ordinary subset  $M$  of  $X$  is defined by  $co_M A = M \cap co A$ . It can be easily shown that

$co_M (co_M A) = M \cap A$  for every fuzzy set  $A$  in  $X$ .

Theorem 4.1.1. Let  $(M, \tau_M)$  be a subspace of a fts  $(X, \tau)$  and  $A$  a fuzzy set on  $M$ . Then we have  $co_M (M \cap A) = M \cap co A$

Proof : Let  $x \in M$

$$\begin{aligned} (M \cap co A)(x) &= \inf(M(x), 1 - A(x)) \\ &= 1 - A(x) \\ &= 1 - \inf(M(x), A(x)) \\ &= 1 - (M \cap A)(x) \\ &= co (M \cap A)(x) \\ &= M \cap co (M \cap A)(x) \\ &= co_M (M \cap A)(x) \end{aligned}$$

If  $x \notin M$ , both the values are zero.

Hence  $co_M (M \cap A) = M \cap co A$ .

Theorem 4.1.2 : Let  $(M, \tau_M)$  be a subspace of a fts  $(X, \tau)$  and  $A$  a fuzzy set on  $M$ . then we have:  $A$  is relatively closed iff  $A$  is the trace of a closed set on  $X$ . Furthermore the  $\tau_M$  closure of  $A$ ,  $cl_M A$  and  $\tau$ -closure of  $A$ ,  $cl A$ , are related by  $cl_M A = M \cap cl A$ .

Proof : Assume  $A$  is the trace of a closed set on  $X$ .

ie.  $A = M \cap F$ ,  $F$  closed in  $\tau$

Then  $co_M A = M \cap co F$  Hence by definition  $co_M A$  is  $\tau_M$ -open.

ie.  $A$  is  $\tau_M$ -closed

ie.  $A$  is relatively closed.

Conversely Assume that  $A$  is  $\tau_M$ -closed.

$\implies co_M A$  is  $\tau_M$ -open

$\implies co_M A = M \cap O$ ,  $O \in \tau$ .

$\implies A = co_M co_M A$

$= M \cap co O$

=  $M \cap F$  where  $F = \text{co } 0, 0 \in \tau$

$\implies F$  is  $\tau$ -closed

$\implies A$  is the trace of a closed set on  $X$ .

From the definition of closure of  $A$ , it follows that  $\text{cl}_M A = M \cap \text{cl } A$

**Definition** : A fuzzy topological property  $P$  is called **hereditary** (hereditary with respect to closed subspaces, hereditary with respect to open subspaces), iff each subspace (closed subspace, open subspace) of a fts with property  $P$  also has property  $P$ .

**Definition** : A fts  $(X, \tau)$  is called **FN (Fuzzy Normal)** iff for every pair of fuzzy closed sets  $F_1$  and  $F_2$  such that  $F_1 \subseteq \text{co } F_2$ , there exist two open sets  $O_1$  and  $O_2$  such that  $F_1 \subseteq O_1$ ,  $F_2 \subseteq O_2$  and  $O_1 \subseteq \text{co } O_2$ . A fuzzy normal fts which is also  $FT_5$  is said to be  $FT_4$ . clearly every  $FT_4$  fts is also  $FT_3$ .

**Theorem 4.1.3** : For every  $i \in \{0, 1, s, 2, 2\frac{1}{2}, 3\}$  the corresponding  $FT_i$  property is hereditary.

**Proof** : We will prove the result for  $i = 2$

ie. For  $FT_2$  spaces.

Let  $(M, \tau_M)$  be a subspace of the  $FT_2$  space  $(X, \tau)$ . Consider two fuzzy singletons  $p$  and  $q$  with different supports  $x_p$  and  $x_q$  in  $M$ . since  $M$  is an ordinary subset of  $X$ ,  $x_p$  and  $x_q$  belong to  $X$  also.

$\implies$  the fuzzy singletons  $p$  and  $q$  belong to  $X$ .

Since  $(X, \tau)$  is  $FT_2$ , corresponding to these fuzzy singletons there exists open fuzzy sets  $O_p$  and  $O_q$  such that  $p \subseteq O_p \subseteq \text{co } q$ ,

$q \subseteq O_q \subseteq \text{co } p$  and  $O_p \subseteq \text{co } O_q$

Since  $p \subseteq O_p \subseteq \text{co } q$ ,  $M \cap p \subseteq M \cap O_p \subseteq M \cap \text{co } q$

Since  $x_p \in M$ ,  $p = M \cap p$

$\subseteq M \cap O_p$

$\subseteq \text{co}_M q$  ( Since  $M \cap \text{co } q = \text{co}_M q$  )

$\therefore p \subseteq M \cap O_p \subseteq \text{co}_M q$

Let  $O = M \cap O_p \quad \therefore O \in \tau$

Since  $q \subseteq O_q \subseteq \text{co } p$ ,  $M \cap q \subseteq M \cap O_q \subseteq M \cap \text{co } p$ .

Since  $x_q \in M$ ,  $q = M \cap q$

$$\subset M \cap O_q$$

$$\subset \text{co}_M p \quad (\because M \cap \text{co } p = \text{co}_M p)$$

$\therefore$  We have  $q \subset M \cap O_q \subset \text{co}_M p$

Let  $O_Q = M \cap O_q \quad \therefore O_Q \in \tau_M$

Since  $O_p \subset \text{co } O_q$ ,  $O_p \cap M \subset (\text{co } O_q) \cap M$ .

Since, by theorem 4.1.1  $M \cap \text{co } O_q = \text{co}_M(M \cap O_q)$ , we get

$$O_R \subset \text{co}_M O_q$$

Hence there exists  $\tau_M$ -open fuzzy sets  $O_R$  and  $O_Q$  satisfying the required conditions. Hence  $(M, \tau_M)$  is  $FT_2$ . Thus we get that  $FT_2$  is a hereditary property.

The proofs for  $i = 0, 1, 2, 3$  are similar to the above proof.

Regarding fuzzy normality we have the following theorem.

**Theorem 4.1.4** : Fuzzy normality is hereditary with respect to closed subspaces.

**Proof** : Let  $M$  be a closed subspace of a FN fts  $(X, \tau)$ .

Let  $F_1$  and  $F_2$  be  $\tau_M$ -closed fuzzy sets such that  $F_1 \subset \text{co}_M F_2$ .

Since  $M$  is a closed subspace,  $F_1$  and  $F_2$  become two  $\tau$ -closed sets such that  $F_1 \subset \text{co } F_2$ .

Since  $(X, \tau)$  is FN, for these closed sets there exist two  $\tau$ -open fuzzy sets  $U$  and  $V$  such that  $F_1 \subset U$ ,  $F_2 \subset V$  and  $U \subset \text{co } V$ .

Now  $M \cap U$  and  $M \cap V$  are  $\tau_M$ -open fuzzy sets.

**Claim** :  $F_1 \subset M \cap U$  and  $F_2 \subset M \cap V$ .

$$\begin{aligned} (M \cap U)(x) &= \inf (M(x), U(x)) \\ &= U(x) \quad \text{if } x \in M \\ &= 0 \quad \text{if } x \notin M. \end{aligned}$$

We have  $F_1(x) \leq U(x)$

$$= (M \cap U)(x) \quad \text{if } x \in M.$$

If  $x \notin M$ ,  $F_1(x) = 0$

$$= (M \cap U)(x).$$

$$\therefore F_1(x) \leq (MNU)(x)$$

$$\Rightarrow F_1 \subset MNU.$$

Similarly,  $F_2 \subset MNV$

Since  $U \subset co V$ ,  $MNU \subset MNco V$

By theorem 4.1.1,  $MNco V = co_M(MNV)$ , we get  $MNU \subset co_M(MNV)$

Hence there exist  $\tau_M$ -open fuzzy sets  $MNU$  and  $MNV$  satisfying the required conditions.

Hence  $(M, \tau_M)$  is FN.

Hence fuzzy normality is hereditary with respect to closed subspaces.

## Section 2

### Fuzzy Sums

In this Section we study the notion of Sum fuzzy topology on  $X[2]$  and the main result proved is that all the separation properties are additive. First we give the definition of sum fuzzy topology.

Let  $((X_i, \tau_i))_{i \in I}$  be a family <sup>of</sup> pairwise disjoint fts's. Consider the set  $X = \bigcup_{i \in I} X_i$  and the class

$$\tau = \bigoplus_{i \in I} \tau_i = \{ \theta / \theta \text{ is a fuzzy set in } X \text{ and for every } i \in I, \theta|_{X_i} \in \tau_i \}$$

It can easily be seen that  $\tau$  defines a fuzzy topology  $X$ .

So the following definition makes sense.

Definition: The fuzzy topology  $\bigoplus_{i \in I} \tau_i$  described above is called

the **sum fuzzy topology** on  $X$  and the corresponding pair

$(X, \bigoplus_{i \in I} \tau_i)$  is called the **sum fts** of the family  $((X_i, \tau_i))_{i \in I}$

We use the notation  $\bigoplus_{i \in I} X_i$  to denote the sum fts.

Theorem 4.2.1: Let  $\bigoplus_{i \in I} X_i$  be the sum fts of the family

$((X_i, \tau_i))_{i \in I}$ . Then a fuzzy set  $A$  on  $X = \bigcup_{i \in I} X_i$  is  $\bigoplus_{i \in I} \tau_i$ -closed iff

the intersection  $X_i \cap A$  is  $\tau_i$ -closed for every  $i \in I$ .

Proof: By theorem 4.1.1, we get

$$X_i \cap \text{co } A = X_i \setminus (X_i \cap A)$$

Suppose  $A$  is  $\tau$ -closed

Then  $\text{co } A$  is  $\tau$ -open

$\implies X_i \cap \text{co } A$  is  $\tau_i$ -open for every  $i$

$$\text{But } X_i \cap \text{co } A = \text{co } X_i A.$$

$\therefore \text{co } X_i A$  is  $\tau_i$ -open.

$\implies A$  is  $\tau_i$ -closed

Conversely, assume  $A$  is  $\tau_i$ -closed for every  $i$ .

Then  $\text{co } X_i A$  is  $\tau_i$ -open for every  $i$ .

$$\text{But } \text{co } X_i A = X_i \cap \text{co } A \text{ for every } i.$$

$\therefore X_i \cap \text{co } A$  is  $\tau_i$ -open for every  $i$ .

By definition,  $\text{co } A \in \tau$ .

$\implies A$  is  $\tau$ -closed

Hence the theorem.

Corollary: Every  $X_i$  is a clopen subspace of the sum fts  $\bigoplus_{i \in I} X_i$ .

Proof: To prove  $X_i$  is a clopen subspace we have to prove  $X_i$  is  $\tau$ -closed and  $\tau$ -open.

To prove:  $X_i$  is  $\tau$ -closed.

Consider  $X_i \cap X_j$

If  $j=i$ , then  $X_i \cap X_i = X_i$  and we know that  $X_i$  is  $\tau_j$ -closed.

If  $j \neq i$  then  $X_i \cap X_j = \emptyset$  and  $\emptyset$  is  $\tau_j$ -closed

$\therefore X_i \cap X_j$  is  $\tau_j$ -closed for all  $j$ .

Hence by theorem 4.2.1,  $X_i$  is  $\tau$ -closed

Consider,  $\text{co } X_i \cap X_j$ .

If  $j = i$ ,  $\text{co } X_i \cap X_j = \emptyset$  and  $\emptyset$  is  $\tau_j$ -closed

If  $j \neq i$ ,  $\text{co } X_i \cap X_j = X_j$ , which is  $\tau_j$ -closed

$\therefore \text{co } X_i \cap X_j$  is  $\tau_j$ -closed for every  $j$ .

Hence by theorem 4.2.1,  $\text{co } X_i$  is  $\tau$ -closed.

$\implies X_i$  is  $\tau$ -open.

$\therefore X_i$  is a clopen subspace of  $\bigoplus_{i \in I} X_i$

**Theorem 4.2.2:** Every  $\tau_j$ -open set is  $\tau$ -open

**Proof:** Let  $O_j$  be a  $\tau_j$ -open set.

ie.  $O_j : X_j \rightarrow [0,1]$ . We can extend  $O_j$  to  $X$

by defining  $O_j(x) = 0$  if  $x \notin X_j$

Consider  $O_j \cap X_i$

If  $i = j$   $O_j \cap X_j = O_j$  which is  $\tau_i$ -open

if  $i \neq j$   $O_j \cap X_i(x) = \inf(O_j(x), X_i(x))$

If  $x \in X_i$ , as  $x \notin X_j$ ,  $O_j(x) = 0$

If  $x \notin X_i$ ,  $X_i(x) = 0$

Hence in either case,  $O_j \cap X_i = \emptyset$ , which is  $\tau_i$ -open

Thus  $O_j \cap X_i$  is  $\tau_i$ -open for every  $i$ .

Hence by definition,  $O_j \in \tau$

ie.  $O_j$  is  $\tau$ -open

**Theorem 4.2.3:** Let  $((X_i, \tau_i))_{i \in I}$  be a family of pairwise disjoint fts's. For every  $i \in I$ . Let  $M_i$  be a subspace of  $X_i$ . Then the two fuzzy topologies defined for the set  $M = \bigcup_{i \in I} M_i$ , viz., the sum fuzzy

topology of the subspace  $((M_i, \tau_i/M_i))_{i \in I}$  and the relative fuzzy topology of the subspace  $M$  of  $\bigoplus_{i \in I} X_i$ , coincide.

**Proof:** Given  $M = \bigcup_{i \in I} M_i$

The sum fuzzy topology  $\tau'$  of the subspace is given by

$\tau' = \{ O / O \text{ is a fuzzy set in } M \text{ and for every } i \in I,$

$O \cap M_i \in \tau_i / M_i \}$

The relative fuzzy topology of the subspace  $M$  is given by

$\tau_M = \{ M \cap O / O \in \tau \}$

Take  $A \in \tau_M$

**To prove:**  $A \in \tau'$

$A \in \tau_M \implies A = M \cap O, O \in \tau$

$O \in \tau \implies O \cap X_i \in \tau_i$  for all  $i$

$$\implies (O \cap X_i) \cap M_i \in \tau_i / M_i$$

Claim:  $A \cap M_i = O \cap X_i \cap M_i$

If  $x \in M_i$

$$\begin{aligned} M \cap O \cap M_i(x) &= \inf (M(x), O(x), M_i(x)) \\ &= O(x) \end{aligned}$$

$$\begin{aligned} O \cap X_i \cap M_i(x) &= \inf (O(x), X_i(x), M_i(x)) \\ &= O(x) \end{aligned}$$

If  $x \notin M_i$  then  $M \cap O \cap M_i(x) = 0 = O \cap X_i \cap M_i(x)$

Hence we have  $M \cap O \cap M_i(x) = O \cap X_i \cap M_i(x)$

$$\begin{aligned} \implies M \cap O \cap M_i &= (O \cap X_i) \cap M_i \\ &\in \tau_i / M_i \end{aligned}$$

ie.  $A \cap M_i \in \tau_i / M_i$  for every  $i$ .

$$\implies A \in \tau'$$

Hence  $\tau_M \subset \tau'$

Conversely, Take  $A \in \tau'$

To Prove:  $A \in \tau_M$ .

$A \in \tau' \implies A \cap M_i \in \tau_i / M_i$  for every  $i$ .

$$\implies A \cap M_i = O_i \cap M_i, O_i \in \tau_i$$

$$\implies A(x) = O_i(x) \text{ for all } x \in M_i \quad \text{-----(1)}$$

This is true for every  $i$ .

Since  $O_i \in \tau_i$ , by theorem 4.2.2,  $O_i \in \tau$

Thus  $O = \bigcup_i O_i \in \tau$

Claim:  $A = M \cap O$

Take  $x \in M$

$\implies$  There exists a  $k \in I$  such that  $x \in M_k$

$\implies x \notin X_i$  if  $i \neq k$ .

$$\therefore O_i(x) = 0 \text{ if } i \neq k.$$

$$\begin{aligned} \text{Hence } \bigcup_{i \in I} O_i(x) &= \sup_{i \in I} (O_i(x)) \\ &= O_k(x) \end{aligned}$$

$$= A(x) \text{ by } (1)$$

$$\Rightarrow M \cap \left( \bigcup_{i \in I} O_i \right) = A$$

Thus  $A = M \cap O$ , with  $O \in \tau$

$\therefore A \in \tau_M$

$$\Rightarrow \tau' \subset \tau_M$$

Hence  $\tau' = \tau_M$

Hence the proof

**Theorem 4.2.4 :** Let  $X$  be a fts such that it can be represented as the union of a family  $(\langle X_i, \tau_i \rangle)_{i \in I}$  of pairwise disjoint open subspaces.

Then  $X = \bigoplus X_i$ .

**Proof :** Let  $\tau$  be the given fuzzy topology on  $X$  and let  $\tau'$  be the sum topology.

Take  $O \in \tau$ .

$$\Rightarrow O \cap X_i \in \tau_i.$$

This is true for every  $i$ .

$$\Rightarrow O \in \tau' \text{ Hence } \tau \subset \tau' \quad \text{----(1)}$$

Conversely,

Take  $O \in \tau'$

$$\Rightarrow O \cap X_i \in \tau_i \text{ for every } i.$$

Since  $X_i$  is open in  $X$ ,  $O \cap X_i$  is open in  $X$  ----(2)

**Claim :**  $O = \bigcup_{i \in I} (O \cap X_i)$

Take  $x \in X \Rightarrow x \in X_j$  for some  $j$ .

$$\bigcup_{i \in I} (O \cap X_i)(x) = \sup_{i \in I} \{(O \cap X_i)(x)\} \dots$$

Corresponding to the suffix  $i \neq j$ ,

$$(O \cap X_i)(x) = 0$$

$$\text{If } i=j \text{ } (O \cap X_i)(x) = O(x)$$

$$\therefore \bigcup_{i \in I} (O \cap X_i)(x) = O(x)$$

This is true for all  $x \in X$ .

$$\therefore O = \bigcup_{i \in I} (O \cap X_i)$$

Hence by using (2), we get  $O \in \tau$ , being union of open sets.

$$\text{Thus } \tau' \subseteq \tau \text{ ----(3)}$$

Hence by (1) and (3), we get  $\tau = \tau'$

**Definition 1** : A fuzzy topological property P is additive iff the sum  $\bigoplus_{i \in I} X_i$  of every family  $(X_i)_{i \in I}$  of pairwise disjoint fts's enjoying property P also has property p.

**Theorem 4.2.6** : Let  $(X_i)_{i \in I}$  be a family of pairwise disjoint nonempty sets and let X be its union. Let, further on,  $(U_i)_{i \in I}$ ,  $(V_i)_{i \in I}$  be two families of fuzzy sets in  $(X_i)_{i \in I}$  such that for every  $i \in I$ ,  $U_i \subseteq \text{co}_{X_i} V_i$ . Then  $U \subseteq \text{co} V$ , where  $U = \bigcup_{i \in I} U_i$  and  $V = \bigcup_{i \in I} V_i$ .

**Proof** : Assume  $U \not\subseteq \text{co} V$ . Then there exists an element  $x_0 \in X$  such that  $U(x_0) > \text{co} V(x_0)$  and hence  $x_0 \in X_{i_0}$  for some unique  $i_0 \in I$ . This implies that for  $i \neq i_0$  one has  $U_i(x_0) = V_i(x_0) = 0$ . Now it can be shown that  $U(x_0) = U_{i_0}(x_0)$  and  $\text{co} V(x_0) = \text{co}_{X_{i_0}} V_{i_0}(x_0)$ . And so the hypothesis leads to  $U_{i_0}(x_0) > \text{co}_{X_{i_0}} V_{i_0}(x_0)$ , which contradicts  $U_{i_0} \subseteq \text{co}_{X_{i_0}} V_{i_0}$ .

**Theorem 4.2.7** : For every  $i \in (0, 1, s, 2, 2\frac{1}{2}, 3, 4)$  the corresponding  $FT_i$  property is additive.

**Proof** : The cases  $i = 0, i = 1, i = 2, i = 2\frac{1}{2}$  have similar proofs. We will prove the result for  $i = 2$ . i.e. for  $FT_2$  spaces.

Let  $\bigoplus_{i \in I} X_i$  be the sum fts of the family  $((X_i, \tau_i))_{i \in I}$  of pairwise disjoint  $FT_i$  fts's and let  $\tau = \bigoplus_{i \in I} \tau_i$

Suppose  $X_i$  is  $FT_2$  for all  $i \in I$ .

**To prove** :  $\bigoplus_{i \in I} X_i$  is  $FT_2$

Take two fuzzy singletons p and q in  $X = \bigcup_{i \in I} X_i$  with different

Supports  $x_p$  and  $x_q$  respectively.

**Case (i)** : Suppose  $x_p \in X_i$  and  $x_q \in X_j$  with  $i \neq j$

Claim :  $p \subset X_i \subset co\ q$

$$p(x_p) \leq 1$$

$$\therefore p(x_p) \leq X_i(x_p) \quad \because x_p \in X_i \quad \text{-----(1)}$$

$$\text{If } x \neq x_p, p(x) = 0 \leq X_i(x) \quad \text{-----(2)}$$

From (1) and (2),  $p \subset X_i$

To Prove :  $X_i \subset co\ q$

$$X_i(x_p) = 1$$

$$= co\ q(x_p) \quad (\because co\ q(x_p) = 1 - q(x_p) = 1 - 0 = 1)$$

If  $x \neq x_p$ , two cases arise.

Suppose  $x \in X_i$

$$X_i(x) = 1$$

$$= 1 - 0$$

$$= 1 - q(x) \text{ as } x \notin X_j \implies x \notin X_q.$$

$$= co\ q(x)$$

Suppose  $x \notin X_i$

$$X_i(x) = 0$$

$$\leq co\ q(x)$$

Hence  $X_i \subset co\ q$

Thus we get  $p \subset X_i \subset co\ q$

similarly it can be shown that

$$q \subset X_j \subset co\ p$$

To prove :  $X_i \subset co\ X_j$

Take  $x \in X$

$$\text{If } x \in X_i, X_i(x) = 1$$

$$= co\ X_j(x)$$

$$\text{If } x \notin X_i, X_i(x) = 0$$

$$\leq co\ X_j(x)$$

$$\therefore X_i \subset co\ X_j.$$

By corollary of theorem 4.2.1.  $X_i, X_j$  are both  $\tau = \bigoplus_{i \in I} \tau_i$ -open fuzzy sets in  $X$ .

Hence there exist open fuzzy sets  $X_i, X_j$  such that  $p \in X_i \subset \text{co } q$ ,  
 $q \in X_j \subset \text{co } p$  and  $X_i \subset \text{co } X_j$

Case (ii) : Suppose  $x_p$  and  $x_q$  belong to the same  $FT_2$   
fts  $(X_{i_0}, \tau_{i_0})$ .

Then by the definition of  $FT_2$ , there exist two  $\tau_{i_0}$ -open fuzzy sets  
 $U_0$  and  $V_0$  such that  $p \in U_0 \subset \text{co}_{X_{i_0}} q \subset \text{co } q$ ,  
 $q \in V_0 \subset \text{co}_{X_{i_0}} p \subset \text{co } p$ , and  
 $U_0 \subset \text{co}_{X_{i_0}} V_0 \subset \text{co } V_0$ .

Since by theorem 4.2.2 every  $\tau_{i_0}$ -open set is  $\tau$ -open we get  $U_0$  and  
 $V_0 \in \tau$ .

Hence given two fuzzy singletons  $p$  and  $q$ , we can find two  $\tau$ -open  
sets  $U_0$  and  $V_0$  such that  $p \in U_0 \subset \text{co } q$ ,  $q \in V_0 \subset \text{co } p$  and  
 $U_0 \subset \text{co } V_0$ .

Hence  $\bigoplus_{i \in I} X_i$  is  $FT_2$ .

(b) the case for  $i = s$

Let  $p$  be a fuzzy singleton in  $X$  with support  $x_p$ . If  $x_p \in X_i$  then  $p$   
is a fuzzy singleton in  $X_i$

As each  $X_i$  is  $FT_s$

$p$  is  $\tau_i$ -closed.

ie.  $p \cap X_i$  is  $\tau_i$ -closed.

If  $i \neq j$ ,  $p \cap X_j = X_j$   
 $= \tau_j$  - closed.

Thus  $p \cap X_i$  is  $\tau_i$ -closed for every  $i$ .

Hence by theorem 4.2.1,  $p$  is  $\tau$ -closed.

Thus  $\bigoplus_{i \in I} X_i$  is  $FT_s$

(c) Let  $X_i$  be  $FT_3$  for every  $i \in I$ .

Consider a fuzzy singleton  $p$  and a  $\tau$ -closed fuzzy set  $F$  such that  
 $p \subset \text{co } f$ .

Assume  $\text{supp } p \in X_{i_0}$ .

Now  $X_{i_0} \cap \text{co } F$  will be an open set containing  $p$ . since  $X_{i_0}$  is  $FR$ , there exists  $U \in \tau_{i_0}$  such that  $p \in U \subset \text{cl}_{X_{i_0}} U \subset X_{i_0} \cap \text{co } F$ .

since  $X_i$  is clopen in  $\bigoplus_{i \in I} X_i$ ,  $U \in \tau$  and  $\text{cl}_{X_{i_0}} U = \text{cl } U$   
 $\therefore p \in U \subset \text{cl } U \subset \text{co } F$ .

thus  $\bigoplus_{i \in I} X_i$  is  $FT_3$ .

By (b), it is  $FT_5$  also.

Hence it is  $FT_3$ .

(d) Let  $X_i$  be  $FT_4$  for every  $i \in I$ .

Consider two  $\bigoplus_{i \in I} \tau_i$ -closed fuzzy sets  $F_1, F_2$  in  $X$  such that  $F_1 \subset \text{co } F_2$ . Then for every  $i \in I$ ,  $X_i \cap F_1, X_i \cap F_2$  are  $\tau_i$ -closed and  $X_i \cap F_1 \subset X_i \cap \text{co } F_2 = \text{co}_{X_i}(X_i \cap F_2)$ .

Since  $X_i$  is  $FN$  for every  $i \in I$  there exist, for every  $i \in I$ , two  $\tau_i$ -open fuzzy sets  $U_i$  and  $V_i$  such that  $X_i \cap F_1 \subset U_i$ ,

$X_i \cap F_2 \subset V_i$  and  $U_i \subset \text{co}_{X_i} V_i$ .

With the above theorem 4.2.6. it can be shown that  $F_1 \subset U, F_2 \subset V, U \subset \text{co } V$ . Take  $U = \bigcup_{i \in I} U_i, V = \bigcup_{i \in I} V_i$ .

But  $U$  and  $V$  are unions of  $\bigoplus_{i \in I} \tau_i$ -open fuzzy sets and so consequently  $U \in \bigoplus_{i \in I} \tau_i, V \in \bigoplus_{i \in I} \tau_i$ .

From (b) it follows that  $\bigoplus_{i \in I} X_i$  is also  $FT_5$  and hence  $FT_4$ .

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