1

Preliminaries

In this chapter, the basic definitions of intuitionistic fuzzy sets, intuitionistic fuzzy continuous mappings, intuitionistic fuzzy closed mappings, intuitionistic fuzzy homeomorphisms, intuitionistic fuzzy connectedness and some results in intuitionistic fuzzy topological spaces that are used to accomplish the present study are given in detail.

1.1 Intuitionistic fuzzy sets

Definition 1.1.1: [Zadeh, 1965] Let X be a non empty set. A ***fuzzy set*** A in X can be described in the form

$$A = \{\langle x, \mu_A(x)\rangle \,/\, x \in X\}$$

where the function $\mu_A : X \to [0,1]$ is called the membership function and $\mu_A(x)$ denotes the degree to which $x \in A$ and $0 \leq \mu_A(x) \leq 1$ for each $x \in X$.

Definition 1.1.2: [Atanassov, 1986] An ***intuitionistic fuzzy set*** (IFS) A is an object having the form

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to the set A, respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. Denote by IFS(X), the set of all intuitionistic fuzzy sets in X.

An intuitionistic fuzzy set A in X is simply denoted by $A = \langle x, \mu_A, \nu_A\rangle$ instead of denoting $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$.

Definition 1.1.3: [Atanassov, 1986] Let A and B be two IFSs of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle : x \in X\}$. Then,

(a) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$,

(b) $A = B$ if and only if $A \subseteq B$ and $A \supseteq B$,

(c) $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle : x \in X\}$,

(d) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x)\rangle : x \in X\}$,

(e) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)\rangle : x \in X\}$.

The intuitionistic fuzzy sets $0_{\sim} = \langle x, 0, 1\rangle$ and $1_{\sim} = \langle x, 1, 0\rangle$ are respectively the empty set and the whole set of X.

Definition 1.1.4: [Coker, 1995] Let A, B and C be intuitionistic fuzzy sets in X. Then

(i) $(A \subseteq B)$ and $(C \subseteq D) \Rightarrow (A \cup C) \subseteq (B \cup D)$ and $(A \cap C) \subseteq (B \cap D)$

(ii) $A \subseteq B$ and $A \subseteq C \Rightarrow A \subseteq (B \cap C)$

(iii) $A \subseteq C$ and $B \subseteq C \Rightarrow (A \cup B) \subseteq C$

(iv) $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$

(v) $(A \cup B)^c = A^c \cap B^c$

(vi) $(A \cap B)^c = A^c \cup B^c$

(vii) $A \subseteq B \Rightarrow B^c \subseteq A^c$

(viii) $(A^c)^c = A$

(ix) $(0_{\sim})^c = 1_{\sim}$

(x) $(1_{\sim})^c = 0_{\sim}$.

Definition 1.1.5: [Coker, 1997] An ***intuitionistic fuzzy topology*** (IFT) on X is a family τ of IFSs in X satisfying the following axioms:

(i) $0_\sim, 1_\sim \in \tau$,

(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(iii) $\cup G_i \in \tau$ for any family $\{G_i : i \in J\} \subseteq \tau$.

In this case the pair (X, τ) is called intuitionistic fuzzy topological space (IFTS) and any IFS in τ is known as an intuitionistic fuzzy open set (IFOS) in X. The complement A^c of an IFOS A in an IFTS (X, τ) is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 1.1.6: [Coker, 1995] Let (X, τ) be an IFTS and A, B be intuitionistic fuzzy sets in X. Then the following properties hold:

(i) $int(A) \subseteq A$

(ii) $A \subseteq cl(A)$

(iii) $A \subseteq B \Rightarrow int(A) \subseteq int(B)$

(iv) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$

(v) $int(int(A)) = int(A)$

(vi) $cl(cl(A)) = cl(A)$

(vii) $int(A \cap B) = int(A) \cap int(B)$

(viii) $cl(A \cup B) = cl(A) \cup cl(B)$

(ix) $int(1_\sim) = 1_\sim$

(x) $cl(0_\sim) = 0_\sim$

Definition 1.1.7: [Coker, 1997] Let A be an IFS in an IFTS (X, τ). Then the ***interior*** and ***closure*** of A are defined as

$$\mathrm{int}(A) = \cup \{G \,/\, G \text{ is an IFOS in X and } G \subseteq A\},$$

$$\mathrm{cl}(A) = \cap \{K \,/\, K \text{ is an IFCS in X and } A \subseteq K\}.$$

It is to be noted that for any IFS A in (X, τ), we have $\mathrm{cl}(A^c) = (\mathrm{int}(A))^c$ and $\mathrm{int}(A^c) = (\mathrm{cl}(A))^c$.

Definition 1.1.8: [Joung Kon Jeon, 2005] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) ***intuitionistic fuzzy semi closed set*** (IFSCS) if $\mathrm{int}(\mathrm{cl}(A)) \subseteq A$,

(ii) ***intuitionistic fuzzy pre closed set*** (IFPCS) if $\mathrm{cl}(\mathrm{int}(A)) \subseteq A$,

(iii) ***intuitionistic fuzzy regular closed set*** (IFRCS) if $\mathrm{cl}(\mathrm{int}(A)) = A$,

(iv) ***intuitionistic fuzzy α closed set*** (IFαCS) if $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(A))) \subseteq A$.

Definition 1.1.9: [Joung Kon Jeon, 2005] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) ***intuitionistic fuzzy semi open set*** (IFSOS) if $A \subseteq \mathrm{cl}(\mathrm{int}(A))$,

(ii) ***intuitionistic fuzzy pre open set*** (IFPOS) if $A \subseteq \mathrm{int}(\mathrm{cl}(A))$,

(iii) ***intuitionistic fuzzy regular open set*** (IFROS) if $\mathrm{int}(\mathrm{cl}(A)) = A$,

(iv) ***intuitionistic fuzzy α open set*** (IFαOS) if $A \subseteq \mathrm{int}(\mathrm{cl}(\mathrm{int}(A)))$.

Definition 1.1.10: [Young Bae Jun and Seak Zun Song, 2005] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) ***intuitionistic fuzzy semipre closed set*** (IFSPCS) if there exists an IFPCS B such that $\mathrm{int}(B) \subseteq A \subseteq B$,

(ii) ***intuitionistic fuzzy semipre open set*** (IFSPOS) if there exists an IFPOS such that $B \subseteq A \subseteq \mathrm{cl}(B)$.

Definition 1.1.11: [Hanafy, 2009] An IFS $A = \langle x, \mu_A, \nu_A \rangle$ in an IFTS (X, τ) is said to be an

(i) ***intuitionistic fuzzy γ closed set*** (IFγCS) if $int(cl(A)) \cap cl(int(A)) \subseteq A$

(ii) ***intuitionistic fuzzy γ open set*** (IFγOS) if $A \subseteq int(cl(A)) \cup cl(int(A))$

Definition 1.1.12: [Hanafy, 2009] Let A be an IFS in an IFTS (X, τ). Then the ***γ-interior*** and ***γ-closure*** of A are defined as

$$\gamma int(A) = \cup \{G \,/\, G \text{ is an IF}\gamma\text{OS in } X \text{ and } G \subseteq A\},$$

$$\gamma cl(A) = \cap \{K \,/\, K \text{ is an IF}\gamma\text{CS in } X \text{ and } A \subseteq K\}.$$

It is to be noted that for any IFS A in (X, τ), we have $\gamma cl(A^c) = (\gamma int(A))^c$ and $\gamma int(A^c) = (\gamma cl(A))^c$.

Result 1.1.13: [Kanimozhi and Jayanthi, 2016] Let A be an IFS in (X, τ). Then

(i) $\gamma cl(A) \supseteq A \cup (int(cl(A)) \cap cl(int(A)))$

(ii) $\gamma int(A) \subseteq A \cap (cl(int(A)) \cup int(cl(A)))$

Definition 1.1.14: [Thakur and Rekha Chaturvedi, 2006] Two IFSs A and B are said to be ***q-coincident*** ($A_q B$) if and only if there exists an element $x \in X$ such that $\mu_A(x) > \nu_B(x)$ or $\nu_A(x) < \mu_B(x)$.

Definition 1.1.15: [Thakur and Rekha Chaturvedi, 2006] Two IFSs A and B are said to be ***not q-coincident*** ($A_{\bar{q}}$ B in short) if and only if $A \subseteq B^c$.

Definition 1.1.16: [Coker and Demirci, 1995] An ***intuitionistic fuzzy point*** (IFP), written as $p_{(\alpha,\beta)}$, is defined to be an intuitionistic fuzzy set of X given by

$$p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha, \beta) & \text{if } x = p, \\ (0, 1) & \text{otherwise.} \end{cases}$$

An intuitionistic fuzzy point $p_{(\alpha,\beta)}$ is said to belong to a set A if $\alpha \leq \mu_A$ and $\beta \geq \nu_A$.

Definition 1.1.17: [Santhi and Jayanthi, 2012] An IFS A in (X, τ) is an ***IFQ-set*** if int(cl(A)) = cl(int(A)).

Definition 1.1.18: [Thakur and Dhavaseelan, 2015] Let A be an IFS in X. If A is an ***intuitionistic fuzzy nowhere dense set*** in X, then int(A) = $0_\sim$.

Definition 1.1.19: [Thakur and Dhavaseelan, 2015] An IFS A in (X, τ) is an ***intuitionistic fuzzy nowhere dense set*** if there exists no IFOS U such that U ⊆ cl(A), that is int(cl(A)) = $0_\sim$.

Definition 1.1.20: [Seok Jong Lee and Eun Pyo Lee, 2000] Let $p_{(\alpha,\beta)}$ be an IFP in (X, τ). An IFS A of X is called an ***intuitionistic fuzzy neighbourhood*** (IFN) of $p_{(\alpha,\beta)}$ if there exists an IFOS B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Definition 1.1.21: [Hanafy, 2009] Let $p_{(\alpha,\beta)}$ be an IFP in (X, τ). An IFS A of X is called an ***intuitionistic fuzzy γ neighbourhood*** (IFγN) of $p_{(\alpha,\beta)}$ if there exists an IFγOS B in X such that $p_{(\alpha,\beta)} \in B \subseteq A$.

Corollary 1.1.22: [Coker, 1995] Let A, A_i (i ∈ J) be intuitionistic fuzzy sets in X and B, B_j (j ∈ K) be intuitionistic fuzzy sets in Y and f: X → Y be a mapping. Then

a) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

b) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

c) $A \subseteq f^{-1}(f(A))$ [If f is injective, then $A = f^{-1}(f(A))$]

d) $f(f^{-1}(B)) \subseteq B$ [If f is surjective, then $B = f(f^{-1}(B))$]

e) $f^{-1}(\cup B_j) = \cup f^{-1}(B_j)$

f) $f^{-1}(\cap B_j) = \cap f^{-1}(B_j)$

g) $f^{-1}(0_\sim) = 0_\sim$

h) $f^{-1}(1_\sim) = 1_\sim$

i) $f^{-1}(B^c) = (f^{-1}(B))^c$

Definition 1.1.23: [Coker, 1995] Let (X, τ) be an IFTS and $A=\langle x, \mu_A, \nu_A\rangle$ be an IFS in X. Then ***intuitionistic fuzzy kernel*** of A is the intersection of all IFOSs containing A.

Definition 1.1.24: [Thakur and Rekha Chaturvedi, 2008] An IFTS (X, τ) is said to be an ***intuitionistic fuzzy $T_{1/2}$ space*** if every IFGCS is an IFCS in (X, τ).

Definition 1.1.25: [Coker, 1997] Let X and Y be two non empty sets and f: $X \rightarrow Y$ be a mapping. If $A = \{\langle x, (\mu_A(x), \nu_A(x) / x \in X\rangle\}$ is an IFS in X, then the ***image*** of A under f, denoted by f(A), is the IFS in Y defined by

$$f(A) = \{\langle y, f(\mu_A)(y), f_-(\nu_A)(y) / y \in Y\rangle\}, \text{ where } f_-(\nu_A) = 1 - f(1- \nu_A).$$

Definition 1.1.26: [Coker, 1995] Let X and Y be two non empty sets and f: $X \rightarrow Y$ be a mapping. If $B = \{\langle y, (\mu_B(y), \nu_B(y) / y \in Y\rangle\}$ is an IFS in Y, then the ***preimage*** of B under f is denoted and defined by

$$f^{-1}(B) = \{\langle x, f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x) / x \in X\rangle\}$$

where $f^{-1}(\mu_B)(x) = \mu_B(f(x))$ for every $x \in X$.

Definition 1.1.27: [Coker, 1995] An IFS A is said to be an ***intuitionistic fuzzy dense*** (IFD) in another IFS B in an IFTS (X, τ), if cl(A) = B.

Definition 1.1.28: [Hanafy, 2009] Let X and Y be two IFTSs. Let $A = \{\langle x, \mu_A(x), \nu_A(x) / x \in X\rangle\}$ and $B= \{\langle y, \mu_B(y), \nu_B(y) / y \in Y\rangle\}$ be IFSs of X and Y respectively. Then A x B is an IFS of X x Y defined by

$$(A \times B)(x,y) = \langle (x, y), \min(\mu_A(x), \mu_B(y)), \max(\nu_A(x), \nu_B(y))\rangle$$

Definition 1.1.29: [Hanafy, 2009] Let $f_1: X_1 \rightarrow Y_1$ and $f_2: X_2 \rightarrow Y_2$. Then the product $f_1 \times f_2: X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is defined by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ for every $(x_1, x_2) \in X_1 \times X_2$.

Definition 1.1.30: [Thakur and Rekha Chaturvedi, 2008] An IFS A in an IFTS (X, τ) is said to be an ***intuitionistic fuzzy generalized closed set*** (IFGCS for short) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an IFOS in (X, τ).

Definition 1.1.31: [Kanimozhi and Jayanthi, 2016] An IFS A in an IFTS (X, τ) is said to be an ***intuitionistic fuzzy generalized γ closed set*** (IFGγCS for short) if γcl(A) ⊆ U whenever A ⊆ U and U is an IFOS in (X, τ).

1.2 Intuitionistic fuzzy continuous mappings

Definition 1.2.1: [Gurcay, Coker and Hayder, 1997] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy continuous*** (IF continuous) ***mapping*** if f^{-1} (V) is an IFCS in(X, τ) for every IFCS V of (Y, σ).

Definition 1.2.2: [Joung Kon Jeon et.al., 2005] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) ***intuitionistic fuzzy semi continuous*** (IFS continuous) ***mapping*** if f^{-1} (V) is an IFSCS in (X, τ) for every IFCS V of (Y, σ),

(ii) ***intuitionistic fuzzy α continuous*** (IFα continuous) ***mapping*** if f^{-1} (V) is an IFαCS in (X, τ) for every IFCS V of (Y, σ),

(iii) ***intuitionistic fuzzy pre continuous*** (IFP continuous) ***mapping*** if f^{-1} (V) is an IFPCS in (X, τ) for every IFCS V of (Y, σ).

Definition 1.2.3: [Hanafy, 2009] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy γ continuous*** (IFγ continuous) ***mapping*** if f^{-1} (V) is an IFγCS in (X, τ) for every IFCS V of (Y, σ).

Definition 1.2.4: [Young Bae Jun and Seok-Zun Song, 2005] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy semipre continuous*** (IFSP continuous) ***mapping*** if f^{-1} (V) is an IFSPCS in (X, τ) for every IFCS V of (Y, σ).

Definition 1.2.5: [Krsteska and Ekici, 2007] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) ***intuitionistic fuzzy contra continuous*** (IF contra continuous) ***mapping*** if f^{-1}(V) is an IFCS in X for every IFOS V in Y,

(ii) ***intuitionistic fuzzy contra α continuous*** (IF contra α continuous) ***mapping*** if $f^{-1}(V)$ is an IFαCS in X for every IFOS V in Y,

(iii) ***intuitionistic fuzzy contra pre continuous*** (IF contra P continuous) ***mapping*** if $f^{-1}(V)$ is an IFPCS in X for every IFOS V in Y,

(iv) ***intuitionistic fuzzy contra semi continuous*** (IF contra S continuous) ***mapping*** if $f^{-1}(V)$ is an IFSCS in X for every IFOS V in Y.

Definition 1.2.6: [Hanafy, 2009] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy contra γ continuous*** (IF contra γ continuous) ***mapping*** if $f^{-1}(B) \in$ IFγO(X) for each IFCS B in Y.

Definition 1.2.7: [Hanafy and El Arish, 2003] Let f: (X, τ) → (Y, σ) be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). The mapping f is called an ***intuitionistic fuzzy completely continuous mapping*** if $f^{-1}(B)$ is an IFROS in X for each IFOS B in Y.

1.3 Intuitionistic fuzzy closed mappings

Definition 1.3.1: [Gurcay, Coker and Hayder, 1997] A mapping f: X→ Y is called an ***intuitionistic fuzzy closed mapping*** (IFCM) if f(A) is an IFCS in Y for each IFCS A in X.

Definition 1.3.2: [Joung Kon Jeon, Young Bae Jun and Jin Han Park, 2005] Let f be a mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an

(i) ***intuitionistic fuzzy pre closed mapping*** (IFPCM) if f(A) is an IFPCS in Y for each IFCS A in X,

(ii) ***intuitionistic fuzzy α closed mapping*** (IFαCM) if f(A) is an IFαCS in Y for each IFCS A in X,

(iii) ***intuitionistic fuzzy semi closed mapping*** (IFSCM) if f(A) is an IFSCS in Y for each IFCS A in X.

Definition 1.3.3: [Hanafy, 2009] A map f: X→ Y is called an ***intuitionistic fuzzy γ closed mapping*** (IFγCM) if f(A) is an IFγCS in Y for each IFCS A in X.

1.4 Intuitionistic fuzzy homeomorphisms

Definition 1.4.1: [Santhi and Sakthivel, 2011b] Let f be a bijection mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy homeomorphism*** (IF homeomorphism in short) if both f and f^{-1} are IF continuous mappings.

Definition 1.4.2: [Santhi and Sakthivel, 2011b] Let f be a bijection mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy semi homeomorphism*** (IFS homeomorphism in short) if both f and f^{-1} are IFS continuous mappings.

Definition 1.4.3: [Santhi and Sakthivel, 2011b] Let f be a bijection mapping from an IFTS (X, τ) into an IFTS (Y, σ). Then f is said to be an ***intuitionistic fuzzy α homeomorphism*** (IFα homeomorphism in short) if both f and f^{-1} are IFα continuous mappings.

1.5 Intuitionistic fuzzy connectedness

Definition 1.5.1: [Coker, 1997] An IFTS (X, τ) is said to be an ***intuitionistic fuzzy C_5-connected space*** if the only IFSs which are both IFOS and IFCS are $0_\sim$ and $1_\sim$.

Definition 1.5.2: [Thakur and Rekha Chaturvedi, 2008] An IFTS (X, τ) is said to be an ***intuitionistic fuzzy GO-connected space*** if the only IFSs which are both IFGOS and IFGCS are $0_\sim$ and $1_\sim$.

Definition 1.5.3: [Santhi and Jayanthi, 2012] An IFTS (X, τ) is an ***intuitionistic fuzzy C_5-connected between two IFSs A and B*** if there is no IFOS E in (X, τ) such that $A \subseteq E$ and $E\ {}_q^{c}\ B$.