CHAPTER 2

λ_g^α-Closed Sets and λ_g^α-Open Sets in Topological Spaces

2.1 Introduction

The notion of α-sets in topological spaces was introduced by Njastad [1965] and he studied several fundamental properties. A generalization to the closed set in topological spaces was defined and studied by Levine [1970], which paved the good way to analyse and investigate many stronger and weaker forms of the generalized closed sets. Mashhour et al. [1983] defined the complement of α-sets called α-closed sets by continuing the work of Njastad [1965] and established various properties. Later, Λ-set was introduced by Maki [1986]. Using Λ-set and closed set, Francisco G Arenas et al. [1997] introduced and investigated the notion of λ-closed sets which involves the intersection of Λ-sets and closed sets. Subsequently, Caldas et al. [2007 b] introduced the notion of λ-closure of a set by utilizing the notion of λ-closed sets in topological spaces. Consecutively, generalizations to the Λ-sets such as Λ_g-closed sets and $g\Lambda$-closed sets were introduced by Caldas et al. [2008 a] and examined their corresponding properties.

This chapter deals with yet another new class of sets called λ_g^α-closed sets in topological spaces. At the outset, we have defined λ_g^α-closed sets which is obtained by generalizing λ-closed sets with the help of α-open sets. We have presented the relationships between the newly defined sets and the previously existing sets with corresponding counter examples. From the interrelations we have observed that λ_g^α-closed sets lie exactly between λ-closed sets and $g\Lambda$-closed sets. Also, we have called the complement of λ_g^α-closed sets as λ_g^α-open sets in topological spaces. Further, we have examined the fundamental properties and characterizations of the newly defined sets.

Later the vital properties such as closure and interior of the defined λ_g^α-closed sets and λ_g^α-open sets specified by the notions λ_g^α-closure of a subset and λ_g^α-interior of a subset are also developed. Various properties related to them are also derived.

"The forthcoming Sections 2.2 and 2.3 have been published in the Journal *Malaya Journal of Matematik* entitled On λ_g^α-Closed and λ_g^α-Open Sets in Topological Spaces, Volume 8, No. 4, Year 2020, Pages 2248-2252."

2.2 λ_g^α-Closed Sets

This section is devoted to define the λ_g^α-closed sets in topological spaces. Their dependencies and independencies are derived with counter examples subsequently.

Definition 2.2.1 Let (M, μ) be a topological space. A subset S of (M, μ) is said to be a λ_g^α-closed set if $cl_\lambda(S) \subseteq P$ whenever $S \subseteq P$ and P is α-open in (M, μ).

Example 2.2.2 Let $M = \{i, j, k, l\}$ and $\mu = \{\phi, \{i\}, M\}$. Then the subsets $\phi, \{i\}, \{j, k, l\}, M$ are the λ_g^α-closed sets.

Proposition 2.2.3 Every λ-closed set in (M, μ) is λ_g^α-closed but not conversely.

Proof: Let S be a λ-closed set. Let P be any α-open set containing S in (M, μ). Since S is λ-closed, $cl_\lambda(S) = S \subseteq P$. Therefore S is λ_g^α-closed.

Example 2.2.4 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i, j\}, M\}$. Then the subset $\{i\}$ is λ_g^α-closed but not λ-closed.

Proposition 2.2.5 Every closed set in (M, μ) is λ_g^α-closed but not conversely.

Proof: Obvious from Lemma 1.1.8 (ii) and from Proposition 2.2.3.

Example 2.2.6 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, M\}$. Then the subset $\{i\}$ is λ_g^α-closed but not closed.

Proposition 2.2.7 Every open set in (M, μ) is λ_g^α-closed but not conversely.

Proof: Obvious from Lemma 1.1.8 (ii) and from Proposition 2.2.3.

Example 2.2.8 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, M\}$. Then the subset $\{j, k\}$ is λ_g^α-closed but not open.

Proposition 2.2.9 Every Λ-set in (M, μ) is λ_g^α-closed but not conversely.

Proof: Obvious from Lemma 1.1.8 (i) and from Proposition 2.2.3.

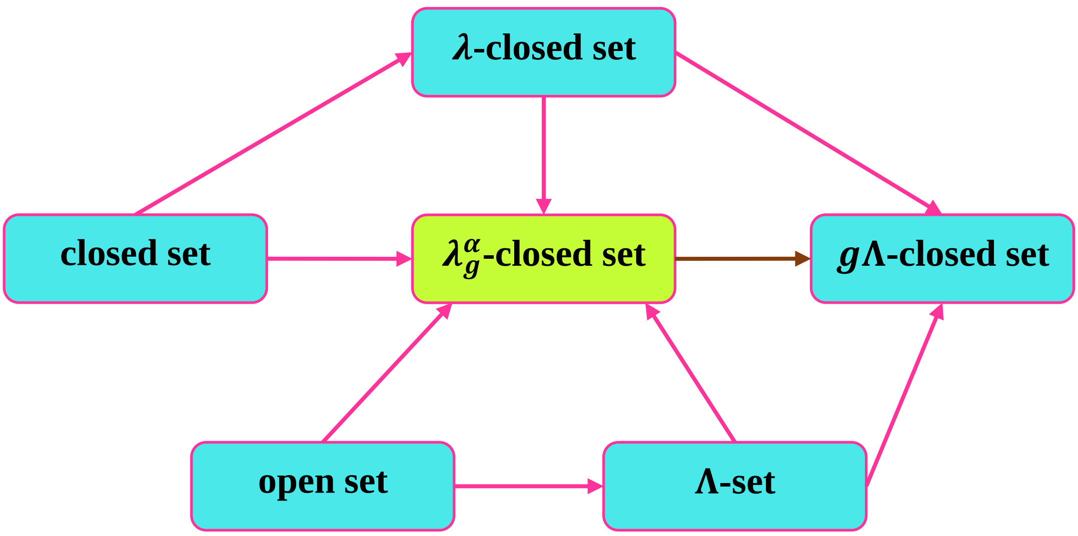
Example 2.2.10 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, \{j\}, \{i, j\}, M\}$. Then the subset $\{k\}$ is λ_g^α-closed but not a Λ-set.

Proposition 2.2.11 Every λ_g^α-closed set in (M, μ) is $g\Lambda$-closed but not conversely.

Proof: Let S be a λ_g^α-closed set and let P be any open set containing S in (M, μ). As every open set is α-open and S is λ_g^α-closed, we have $cl_\lambda(S) \subseteq P$. Hence S is $g\Lambda$-closed.

Example 2.2.12 Let $M = \{i, j, k, l\}$ and $\mu = \{\phi, \{i\}, \{i, j\}, M\}$. Then the subset $\{k\}$ is $g\Lambda$-closed but not λ_g^α-closed.

Remark 2.2.13 The above implications are depicted in the below diagram.



Proposition 2.2.14 Let S be an α-open subset of (M, μ). Then S is λ-closed if S is λ_g^α-closed.

Proof: Suppose S is λ_g^α-closed. Since $S \subseteq S$ and S is α-open we have $cl_\lambda(S) \subseteq S$. Hence from the fact that $S \subseteq cl_\lambda(S) \subseteq cl(S)$, we have S is λ-closed.

Remark 2.2.15 The following example shows that α-closed sets and λ-closed sets are independent in general.

Example 2.2.16 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, M\}$. Then the subset $\{k\}$ is α-closed but not λ-closed, also the subset $\{i\}$ is λ-closed but not α-closed.

Remark 2.2.17 g-closed (resp. α-closed, αg-closed, $g\alpha$-closed, Λ_g-closed, $\Lambda_{g\alpha}$-closed) sets and λ_g^α-closed sets are independent of each other as observed from the following example.

Example 2.2.18 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, M\}$. Then the subset $\{i\}$ is λ_g^α-closed but not g-closed (resp. α-closed, αg-closed, $g\alpha$-closed, Λ_g-closed, $\Lambda_{g\alpha}$-closed) also the subset $\{j\}$ is g-closed (resp. α-closed, αg-closed, $g\alpha$-closed, Λ_g-closed, $\Lambda_{g\alpha}$-closed) but not λ_g^α-closed.

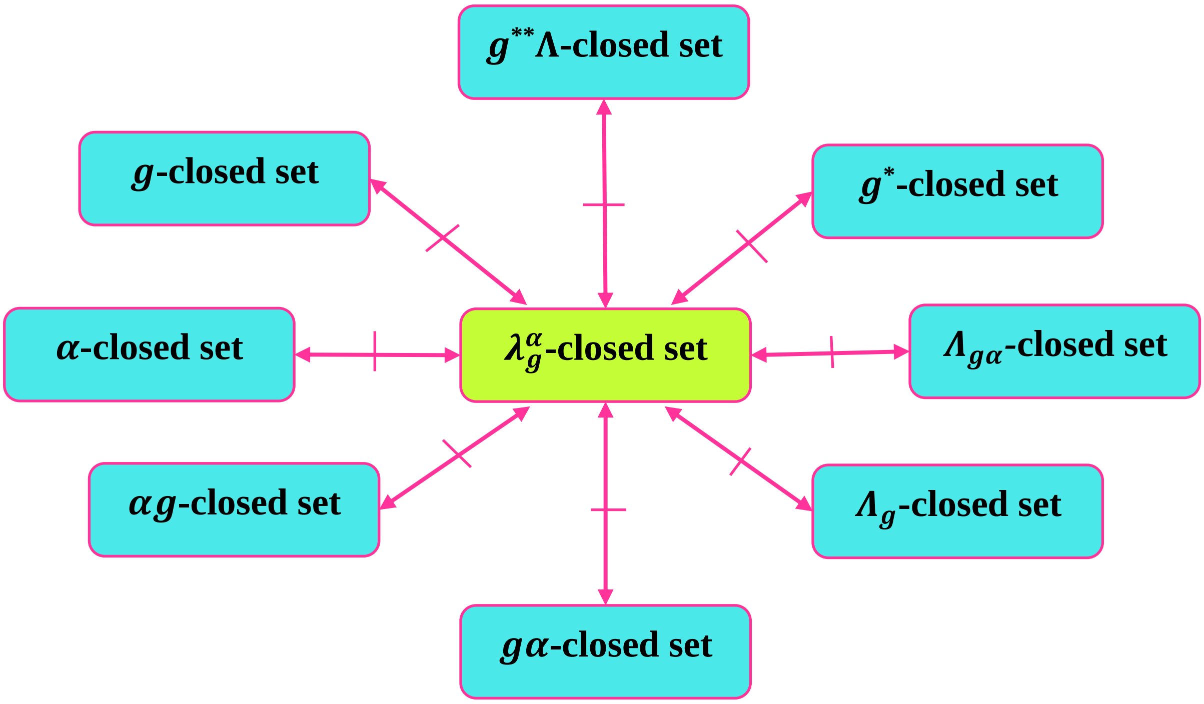
Remark 2.2.19 g^*-closed sets and λ_g^α-closed sets are independent of each other as observed from the following example.

Example 2.2.20 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, \{i, j\}, M\}$. Then the subset $\{i\}$ is λ_g^α-closed but not g^*-closed also the subset $\{i, k\}$ is g^*-closed but not λ_g^α-closed.

Remark 2.2.21 $g^{**}\Lambda$-closed sets and λ_g^α-closed sets are independent of each other as observed from the following example.

Example 2.2.22 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j, k\}, \{i, j, k\}, \{j, k, l, m\}, M\}$. Then the subset $\{j\}$ is λ_g^α-closed but not $g^{**}\Lambda$ -closed also the subset $\{j, l\}$ is $g^{**}\Lambda$-closed but not λ_g^α-closed.

Remark 2.2.23 The above independencies are presented in the below diagram.



Remark 2.2.24 The following example shows that the union of any two λ_g^α-closed sets need not be λ_g^α-closed.

Example 2.2.25 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, \{i, j\}, M\}$. Here the subsets $\{i\}$ and $\{k\}$ are λ_g^α-closed sets but their union $\{i, k\}$ is not a λ_g^α-closed set.

Remark 2.2.26 λ_g^α-closed sets will not form a topology, since it does not satisfy the condition that union of λ_g^α-closed sets is a λ_g^α-closed set.

Remark 2.2.27 The following example shows that the difference of two λ_g^α-closed sets need not be λ_g^α-closed.

Example 2.2.28 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j, k\}, \{i, j, k\}, \{j, k, l, m\}, M\}$. Here the subsets $\{i, k, l\}$ and $\{i, k, m\}$ are λ_g^α-closed sets but their difference $\{l\}$ is not a λ_g^α-closed set.

Proposition 2.2.29 If (M, μ) is an α-space, then every α-closed set is λ_g^α-closed.

Proof: Follows from Definition 1.2.1 (i) and Proposition 2.2.5.

Proposition 2.2.30 If (M, μ) is a $T_{1/2}$-space, then every g-closed set is λ_g^α-closed.

Proof: Follows from Definition 1.2.2 (i) and Proposition 2.2.5.

Proposition 2.2.31 If (M, μ) is a $T_{1/2}$-space, then every subset is λ_g^α-closed.

Proof: Follows from Definition 1.2.2 (iii) and Proposition 2.2.3.

Proposition 2.2.32 If (M, μ) is a T_1-space, then every Λ_g-closed set is λ_g^α-closed.

Proof: Follows from Result 1.2.3 (ii) and Proposition 2.2.5.

Proposition 2.2.33 If (M, μ) is an ${}_\alpha T_b$-space, then every αg-closed set is λ_g^α-closed.

Proof: Follows from Definition 1.2.1 (iii) and Proposition 2.2.5.

Proposition 2.2.34 If (M, μ) is a door space, then every subset is λ_g^α-closed.

Proof: Follows from Definition 1.2.1 (v) and Proposition 2.2.5 and 2.2.7.

Proposition 2.2.35 If (M, μ) is a $T_{1/2}^*$-space, then every g^*-closed set is λ_g^α-closed.

Proof: Follows from Definition 1.2.1 (iv) and Proposition 2.2.5.

Definition 2.2.36 [Neiminen, 1977] A partition space is a space where every open set is closed.

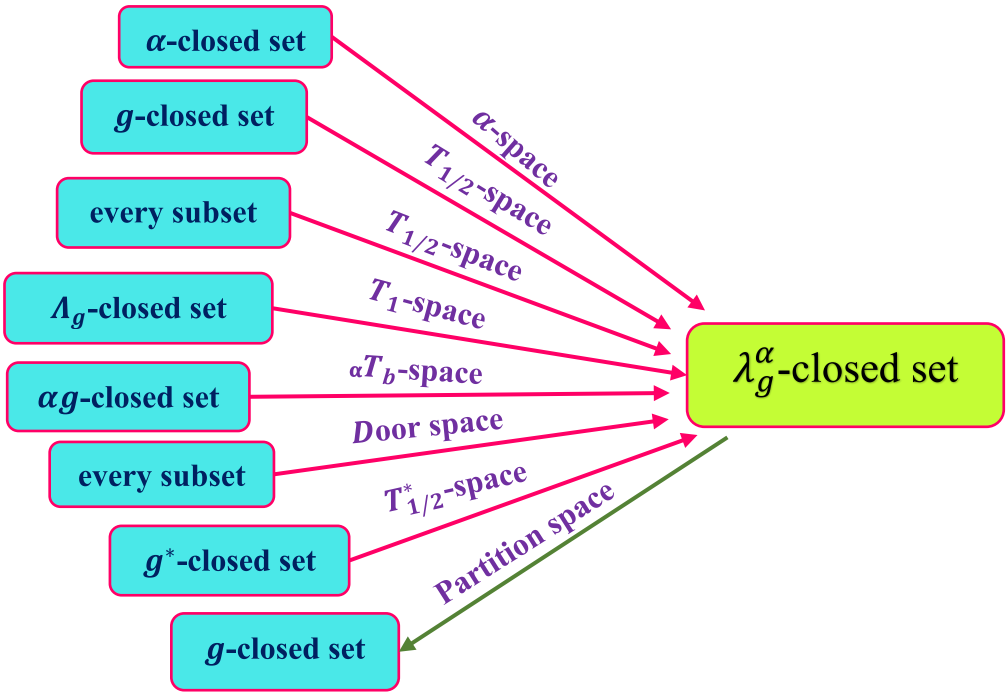
Theorem 2.2.37 Let (M,μ) be an α-space. If (M,μ) is a partition space, then every subset of (M,μ) is a λ_g^α-closed set.

Proof: Let S be any subset of (M,μ) such that $S \subseteq P$ and P is α-open. Since (M,μ) is an α-space, P is open. Since (M,μ) is a partition space, P is closed. As every closed set is λ-closed, P is λ-closed. Hence $cl_\lambda(S) \subseteq cl_\lambda(P) = P$. Hence every subset of (M,μ) is λ_g^α-closed.

Theorem 2.2.38 Let (M,μ) be a partition space. Then every λ_g^α-closed set in (M,μ) is g-closed.

Proof: Let S be a λ_g^α-closed set and $S \subseteq P$ and P is open. Since every open set is α-open and S is λ_g^α-closed, we have $cl_\lambda(S) \subseteq P$. Since (M,μ) is a partition space, the class of λ-closed sets coincide with the class of closed sets. Therefore, we have $cl(S) = cl_\lambda(S) \subseteq P$. Hence S is g-closed.

Remark 2.2.39 The above results are exhibited in the below diagram.



2.3 Properties of λ_g^α-Closed Sets

Fundamental properties and standard note-worthy theorems are derived for λ_g^α-closed sets in this section.

Theorem 2.3.1 If a subset S is λ_g^α-closed, then $cl_\lambda(S) \setminus S$ does not contain any non-empty closed set in (M, μ).

Proof: Let S be a λ_g^α-closed set in (M, μ). Suppose T is a non-empty closed set contained in $cl_\lambda(S) \setminus S$, which implies $S \subseteq T^c$, where T^c is open. Since S is λ_g^α-closed and as every open set is α-open, we have $cl_\lambda(S) \subseteq T^c$. Hence $T \subseteq M \setminus cl_\lambda(S)$. Also, we have $T \subseteq cl_\lambda(S)$. Therefore $T \subseteq [M \setminus cl_\lambda(S)] \cap cl_\lambda(S) = \phi$. Hence $cl_\lambda(S) \setminus S$ does not contain any non-empty closed set.

Remark 2.3.2 Converse part of Theorem 2.3.1 need not be true as seen from the following example.

Example 2.3.3 Let $M = \{i, j, k\}$ and $\mu = \{\phi, \{i\}, M\}$. If $S = \{j\}$, then $cl_\lambda(S) = \{j, k\}$ and $cl_\lambda(S) \setminus S = \{k\}$, which does not contain any non-empty closed set, but S is not λ_g^α-closed.

Theorem 2.3.4 If a subset S is λ_g^α-closed, then $cl_\lambda(S) \setminus S$ does not contain any non-empty α-closed set in (M, μ).

Proof: Let S be a λ_g^α-closed set in (M, μ). Suppose T is an α-closed set contained in $cl_\lambda(S) \setminus S$, which implies $S \subseteq T^c$, where T^c is α-open. Since S is λ_g^α-closed, $cl_\lambda(S) \subseteq T^c$. Hence $T \subseteq M \setminus cl_\lambda(S)$. Also, we have $T \subseteq cl_\lambda(S)$. Therefore $T \subseteq [M \setminus cl_\lambda(S)] \cap cl_\lambda(S) = \phi$. Hence $cl_\lambda(S) \setminus S$ does not contain any non-empty α-closed set.

Remark 2.3.5 Converse part of the Theorem 2.3.4 need not be true as seen from the following example.

Example 2.3.6 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j, k\}, \{i, j, k\}, \{j, k, l, m\}, M\}$. If $S = \{m\}$, then $cl_\lambda(S) = \{l, m\}$ and $cl_\lambda(S) \setminus S = \{l\}$, which does not contain any non-empty α-closed set, but S is not λ_g^α-closed.

Theorem 2.3.7 In a topological space (M, μ), for each $m \in M$, $\{m\}$ is α-closed or λ_g^α-open.

Proof: Suppose $\{m\}$ is not α-closed then $M \setminus \{m\}$ is not α-open, then the only α-open set containing $M \setminus \{m\}$ is M i.e., $[M \setminus \{m\}] \subseteq M$. Obviously $cl_\lambda(M \setminus \{m\}) \subseteq M$. Hence $M \setminus \{m\}$ is λ_g^α-closed and $\{m\}$ is λ_g^α-open.

Theorem 2.3.8 Let S be a λ_g^α-closed set in a topological space (M, μ). Then S is λ-closed if and only if $cl_\lambda(S) \setminus S$ is closed.

Proof: (Necessity) Suppose S is λ_g^α-closed and λ-closed in a topological space (M, μ). S is λ-closed implies $cl_\lambda(S) = S$. Hence $cl_\lambda(S) \setminus S = \phi$, which is a closed set.

Sufficiency: Suppose S is λ_g^α-closed and $cl_\lambda(S) \setminus S$ is closed. Then by Theorem 2.3.1, $cl_\lambda(S) \setminus S$ contains no non-empty closed subset. Hence $cl_\lambda(S) \setminus S = \phi$, which implies $cl_\lambda(S) = S$. Therefore S is λ-closed.

Theorem 2.3.9 If every λ_g^α-closed set is λ-closed then for each $m \in (M, \mu)$ either $\{m\}$ is α-closed or λ-open.

Proof: Suppose $\{m\}$ is not α-closed, then $M \setminus \{m\}$ is not α-open. Hence, we have M is the only α-open set containing $M \setminus \{m\}$. Obviously $cl_\lambda(M \setminus \{m\}) \subseteq M$. Therefore $M \setminus \{m\}$ is λ_g^α-closed. By hypothesis $M \setminus \{m\}$ is λ-closed. Hence $\{m\}$ is λ-open.

Theorem 2.3.10 Let S be α-open and λ_g^α-closed in a topological space (M, μ). If T is λ-closed then $S \cap T$ is λ_g^α-closed.

Proof: By Proposition 2.2.14, we have if a set S is both α-open and λ_g^α-closed then S is λ-closed. Since T is λ-closed, $S \cap T$ is λ-closed as the intersection of λ-closed sets is a λ-closed set. Hence by Proposition 2.2.3, $S \cap T$ is λ_g^α-closed.

Theorem 2.3.11 If S is λ_g^α-closed then $cl_\alpha(\{m\}) \cap S \neq \phi$ for every $m \in cl_\lambda(S)$.

Proof: Let S be a λ_g^α-closed set. Suppose $cl_\alpha(\{m\}) \cap S = \phi$ for some $m \in cl_\lambda(S)$. Then $M \setminus cl_\alpha(\{m\})$ is an α-open set containing S. Further $m \in cl_\lambda(S)$ and $m \notin cl_\alpha(\{m\})$ implies $cl_\lambda(S) \nsubseteq M \setminus cl_\alpha(\{m\})$ is a contradiction to S is a λ_g^α-closed set. Therefore $cl_\alpha(\{m\}) \cap S \neq \phi$ for every $m \in cl_\lambda(S)$.

Theorem 2.3.12 In a topological space (M, μ) the following are equivalent:

1. Every α-open set is λ-closed.
2. Every subset is λ_g^α-closed.

Proof: 1⟹2: Let S be any subset of (M, μ) such that $S \subseteq T$, where T is α-open. Therefore, we have $cl_\lambda(S) \subseteq cl_\lambda(T)$. By hypothesis T is λ-closed, then $cl_\lambda(T) = T \Longrightarrow cl_\lambda(S) \subseteq T$. Hence S is λ_g^α-closed.

2⟹1: Let S be an α-open set. By hypothesis S is λ_g^α-closed. Then we have $cl_\lambda(S) \subseteq S$. Therefore S is λ-closed. Hence every α-open set is λ-closed.

Proposition 2.3.13 In T_1-space, every λ_g^α-closed set is a λ-closed set.

Proof: Let T be a λ_g^α-closed set in T_1-space. Suppose T is not a λ-closed set, then there exists a $m \in M$ such that $\{m\} \in cl_\lambda(T) \setminus T$. By Definition 1.2.2 (ii), $\{m\}$ is closed in (M, μ). But by Theorem 2.3.1, we have $cl_\lambda(T) \setminus T$ contains no non-empty closed subset, which is a contradiction. Hence T is a λ-closed set.

Definition 2.3.14 Let S be a subset of a topological space (M, μ). Then the α-kernel of the set S denoted by α-ker(S) is the intersection of all α-open supersets of S.

Theorem 2.3.15 A subset S of a topological space (M, μ) is λ_g^α-closed if and only if $cl_\lambda(S) \subseteq \alpha\text{-ker}(S)$.

Proof: (Necessity) Suppose S is λ_g^α-closed in (M, μ). Let $m \in cl_\lambda(S)$ but $m \notin \alpha\text{-ker}(S)$. Then $\exists$ an α-open set $Q \supseteq S$, such that $m \notin Q$. Since S is λ_g^α-closed, $cl_\lambda(S) \subseteq Q$ and Q is an α-open set containing S. So, we have $m \in cl_\lambda(S)$ and $m \notin Q$ which is a contradiction. Therefore $cl_\lambda(S) \subseteq \alpha\text{-ker}(S)$.

Sufficiency: Let $cl_\lambda(S) \subseteq \alpha\text{-ker}(S)$ and let P be an α-open set containing S. Then $\alpha\text{-ker}(S) \subseteq P$, implies $cl_\lambda(S) \subseteq P$. Thus S is λ_g^α-closed.

Theorem 2.3.16 Let $S \subseteq T \subseteq cl_\lambda(S)$. If S is λ_g^α-closed in (M, μ) then T is also λ_g^α-closed in (M, μ).

Proof: Let S be a λ_g^α-closed set in (M, μ) and $T \subseteq P$, where P is α-open in (M, μ). Then $S \subseteq P$, where P is α-open in (M, μ). Since S is λ_g^α-closed we have $cl_\lambda(S) \subseteq P$. By hypothesis $T \subseteq cl_\lambda(S)$ implies $cl_\lambda(T) \subseteq cl_\lambda(cl_\lambda(S)) = cl_\lambda(S)$, that is $cl_\lambda(T) \subseteq cl_\lambda(S) \subseteq P$. Hence T is λ_g^α-closed.

2.4 λ_g^α-Open Sets

The complement of λ_g^α-closed sets called λ_g^α-open sets are defined in this section. Interrelationships similar to λ_g^α-closed sets are derived and their prominent properties are established.

Definition 2.4.1 Let (M, μ) be a topological space. A subset S of (M, μ) is said to be a λ_g^α-open set if $int_\lambda(S) \supseteq P$ whenever $S \supseteq P$, where P is α-closed in (M, μ) and $int_\lambda(S)$ is the union of all λ-open sets contained in S. Equivalently, a subset S of a topological space (M, μ) is said to be λ_g^α-open if its complement S^c is λ_g^α-closed.

Theorem 2.4.2 A subset S of a topological space (M, μ) is λ_g^α-open if and only if $Q \subseteq int_\lambda(S)$ whenever Q is α-closed in (M, μ) and $Q \subseteq S$.

Proof: (Necessity) Let Q be an α-closed set contained in S and let S be λ_g^α-open. Then S^c is λ_g^α-closed, $S^c \subseteq Q^c$, where Q^c is α-open. Since S^c is λ_g^α-closed, $cl_\lambda(S^c) \subseteq Q^c$, implies $Q \subseteq \big(M \setminus cl_\lambda(S^c)\big) = int_\lambda(M \setminus S^c) = int_\lambda(S)$.

Sufficiency: Let $Q \subseteq int_\lambda(S)$, where Q is an α-closed set contained in S. We have $S^c \subseteq Q^c$ and $\big(M \setminus int_\lambda(S)\big) \subseteq (S \setminus Q)$ implies $cl_\lambda(S^c) \subseteq Q^c$. Thus, by definition S^c is λ_g^α-closed and hence S is λ_g^α-open.

Proposition 2.4.3

(i) Every λ-open (resp. closed, open, Λ-) set in (M, μ) is λ_g^α-open.

(ii) Every λ_g^α-open set in (M, μ) is $g\Lambda$-open.

Theorem 2.4.4 Let S be an α-closed subset of a topological space (M, μ). If S is λ_g^α-open, then S is λ-open.

Proof: Let S be λ_g^α-open and α-closed. Since $S \subseteq S$ and S is λ_g^α-open we have $S \subseteq int_\lambda(S)$. Then we get $M \setminus int_\lambda(S) \subseteq M \setminus S$. By the fact that $M \setminus int_\lambda(S) = cl_\lambda(M \setminus S)$, we have $cl_\lambda(M \setminus S) \subseteq M \setminus S$. Therefore $M \setminus S$ is λ-closed and hence S is λ-open.

Theorem 2.4.5 If $int_\lambda(S) \subseteq T \subseteq S$ and S is λ_g^α-open, then T is λ_g^α-open.

Proof: $int_\lambda(S) \subseteq T \subseteq S$ implies $S^c \subseteq T^c \subseteq cl_\lambda(S^c)$. Since S is λ_g^α-open, S^c is λ_g^α-closed. By Theorem 2.3.16, T^c is λ_g^α-closed. Therefore T is λ_g^α-open.

Theorem 2.4.6 If a subset S of a topological space (M, μ) is λ_g^α-open in (M, μ) then $Q = M$, whenever Q is α-open and $int_\lambda(S) \cup S^c \subseteq Q$.

Proof: Let S be λ_g^α-open, Q be α-open and $int_\lambda(S) \cup S^c \subseteq Q$. This gives $Q^c \subseteq [M \setminus int_\lambda(S)] \cap S = cl_\lambda(S^c) \cap S = cl_\lambda(S^c) \setminus S^c$. i.e., $Q^c \subseteq cl_\lambda(S^c) \setminus S^c$. Since Q^c is α-closed, S^c is λ_g^α-closed and by Theorem 2.3.4, we have $Q = \phi$. Hence $Q = M$.

Theorem 2.4.7 If a subset S of a topological space (M,μ) is λ_g^α-closed, then $cl_\lambda(S) \setminus S$ is λ_g^α-open.

Proof: Let $S \subseteq M$ be λ_g^α-closed. Let Q be α-closed such that $Q \subseteq (cl_\lambda(S) \setminus S)$. Then by Theorem 2.3.4, $Q = \phi$. So $\phi = Q \subseteq int_\lambda(cl_\lambda(S) \setminus S)$. This shows that $cl_\lambda(S) \setminus S$ is λ_g^α-open.

2.5 Properties of λ_g^α-Closure and λ_g^α-Interior Operators

The mandatory concepts to study are closure and interior of a subset with respect to the defined closed and open sets. Likewise, λ_g^α-closure and λ_g^α-interior are defined in view of λ_g^α-closed and λ_g^α-open sets respectively and their most effective properties and theorems are derived in this section.

Definition 2.5.1 For a subset S of a topological space (M,μ), the **λ_g^α-closure** of S (briefly $\lambda_g^\alpha cl(S)$) is defined to be the intersection of all λ_g^α-closed sets containing S.

i.e., $\lambda_g^\alpha cl(S) = \cap \{Q \subseteq M \mid S \subseteq Q$ and Q is λ_g^α-closed in $(M,\mu)\}$

Proposition 2.5.2 Let S and T be any two subsets of a topological space (M,μ). Then the following properties hold:

(i) $\lambda_g^\alpha cl(\phi) = \phi$ and $\lambda_g^\alpha cl(M) = M$

(ii) If $S \subseteq T$, then $\lambda_g^\alpha cl(S) \subseteq \lambda_g^\alpha cl(T)$

(iii) $S \subseteq \lambda_g^\alpha cl(S)$

(iv) $\lambda_g^\alpha cl\left(\lambda_g^\alpha cl(S)\right) = \lambda_g^\alpha cl(S)$

(v) For $S \subseteq M$, $\lambda_g^\alpha cl(S) \subseteq cl_\lambda(S)$

Proof: (i), (ii), (iii) and (iv) follow from Definition 2.5.1 and (v) follows from Proposition 2.2.3.

Remark 2.5.3 For a subset $S \subseteq M$, $\lambda_g^\alpha cl(S)$ need not be the smallest λ_g^α-closed set containing S as observed from the following example.

Example 2.5.4 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j, k\}, \{i, j, k\}, \{j, k, l, m\}, M\}$. Then λ_g^α-closed sets are $\phi, \{i\}, \{j\}, \{k\}, \{i, j\}, \{i, k\}, \{i, l\}, \{i, m\}, \{j, k\}, \{l, m\}, \{i, j, k\}, \{i, j, l\}, \{i, j, m\}, \{i, k, l\}, \{i, k, m\}, \{i, l, m\}, \{j, l, m\}, \{k, l, m\}, \{i, j, k, l\}, \{i, j, l, m\}, \{i, j, k, m\}, \{i, k, l, m\}, \{j, k, l, m\}, M$. Let $S = \{l\}$. Then $\lambda_g^\alpha cl(S) = \{i, l\} \cap \{l, m\} \cap \{i, j, l\} \cap \{i, k, l\} \cap \{i, l, m\} \cap \{j, l, m\} \cap \{k, l, m\} \cap \{i, j, k, l\} \cap \{i, j, l, m\} \cap \{i, k, l, m\} \cap \{j, k, l, m\} \cap M = \{l\}$ which is not the smallest λ_g^α-closed set containing S.

Proposition 2.5.5 If a subset S of (M, μ) is λ_g^α-closed then $\lambda_g^\alpha cl(S) = S$, but not conversely.

Proof: Let S be λ_g^α-closed in (M, μ). By definition, $\lambda_g^\alpha cl(S) = \cap \{Q \subseteq M \mid S \subseteq Q$ and Q is λ_g^α-closed in $(M, \mu)\}$. Since S is a λ_g^α-closed set, Q in the above intersection is S and hence $\lambda_g^\alpha cl(S) = S$.

Example 2.5.6 Consider (M, μ) as in Example 2.5.4. Let $S = \{m\}$ then $\lambda_g^\alpha cl(S) = S$ but $S = \{m\}$ is not a λ_g^α-closed set.

Remark 2.5.7 $\lambda_g^\alpha cl(S)$ need not be a λ_g^α-closed set as observed from the Examples 2.5.4 and 2.5.6.

Proposition 2.5.8 For the subsets S and T of a topological space (M, μ), $\lambda_g^\alpha cl(S \cap T) \subseteq \lambda_g^\alpha cl(S) \cap \lambda_g^\alpha cl(T)$.

Proof: Since $S \cap T \subseteq S$ and $S \cap T \subseteq T$, by Proposition 2.5.2 (ii) we have $\lambda_g^\alpha cl(S \cap T) \subseteq \lambda_g^\alpha cl(S)$ and $\lambda_g^\alpha cl(S \cap T) \subseteq \lambda_g^\alpha cl(T)$. Hence $\lambda_g^\alpha cl(S \cap T) \subseteq \lambda_g^\alpha cl(S) \cap \lambda_g^\alpha cl(T)$.

Remark 2.5.9 The reverse inclusion of Proposition 2.5.8 may not be true as observed from the following example.

Example 2.5.10 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j\}, \{i, j\}, \{j, k\}, \{i, j, k\}, \{j, k, l\}, \{i, j, k, l\}, \{j, k, l, m\}, M\}$. Then λ_g^α-closed sets are $\phi, \{i\}, \{j\}, \{k\}, \{l\}, \{m\}, \{i, j\}, \{i, k\}, \{i, l\}, \{i, m\}, \{j, k\}, \{k, l\}, \{l, m\}, \{i, j, k\}, \{i, k, l\}, \{i, l, m\}, \{j, k, l\}, \{k, l, m\}, \{i, j, k, l\}, \{i, j, k, l\}, \{j, k, l, m\}, M$. Let $S = \{j, l\}$ and $T = \{j, m\}$. Then $S \cap T = \{j\}$, $\lambda_g^\alpha cl(S \cap T) = \{j\}$, $\lambda_g^\alpha cl(S) = \{j, k, l\}$ and $\lambda_g^\alpha cl(T) = \{j, k, l, m\}$. Therefore $\lambda_g^\alpha cl(S) \cap \lambda_g^\alpha cl(T) = \{j, k, l\}$ but $\lambda_g^\alpha cl(S \cap T) = \{j\}$. Hence $\lambda_g^\alpha cl(S) \cap \lambda_g^\alpha cl(T) \nsubseteq \lambda_g^\alpha cl(S \cap T)$.

Proposition 2.5.11 For the subsets S and T of a topological space (M, μ), $\lambda_g^\alpha cl(S) \cup \lambda_g^\alpha cl(T) \subseteq \lambda_g^\alpha cl(S \cup T)$.

Proof: Since $S \subseteq S \cup T$ and $T \subseteq S \cup T$, by Proposition 2.5.2 (ii), $\lambda_g^\alpha cl(S) \subseteq \lambda_g^\alpha cl(S \cup T)$ and $\lambda_g^\alpha cl(T) \subseteq \lambda_g^\alpha cl(S \cup T)$. Hence $\lambda_g^\alpha cl(S) \cup \lambda_g^\alpha cl(T) \subseteq \lambda_g^\alpha cl(S \cup T)$.

Remark 2.5.12 The reverse inclusion of Proposition 2.5.11 may not be true as observed from the following example.

Example 2.5.13 Consider (M, μ) as in Example 2.5.10. Let $S = \{j\}$ and $T = \{l\}$. Then $S \cup T = \{j, l\}$, $\lambda_g^\alpha cl(S) = \{j\}$, $\lambda_g^\alpha cl(T) = \{l\}$ and $\lambda_g^\alpha cl(S \cup T) = \{j, k, l\}$. Therefore $\lambda_g^\alpha cl(S) \cup \lambda_g^\alpha cl(T) = \{j, l\}$ but $\lambda_g^\alpha cl(S \cup T) = \{j, k, l\}$. Hence $\lambda_g^\alpha cl(S \cup T) \nsubseteq \lambda_g^\alpha cl(S) \cup \lambda_g^\alpha cl(T)$.

Remark 2.5.14 λ_g^α-closure operator is not a Kuratowski closure operator as it does not satisfy $\lambda_g^\alpha cl(S \cup T) = \lambda_g^\alpha cl(S) \cup \lambda_g^\alpha cl(T)$.

Definition 2.5.15 For a subset S of a topological space (M, μ), the $\boldsymbol{\lambda_g^\alpha}$**-interior** of S (briefly $\lambda_g^\alpha int(S)$) is defined to be the union of all λ_g^α-open sets contained in S.

i.e. $\lambda_g^\alpha int(S) = \cup \{Q \subseteq M \mid Q \subseteq S$ and Q is λ_g^α-open in $(M, \mu)\}$

Proposition 2.5.16 Let S and T be any two subsets of a topological space (M, μ). Then the following properties hold:

(i) $\lambda_g^\alpha int(\phi) = \phi$ and $\lambda_g^\alpha int(M) = M$.

(ii) If $S \subseteq T$, then $\lambda_g^\alpha int(S) \subseteq \lambda_g^\alpha int(T)$.

(iii) $\lambda_g^\alpha int(S) \subseteq S$.

(iv) $\lambda_g^\alpha int\left(\lambda_g^\alpha int(S)\right) = \lambda_g^\alpha int(S)$.

(v) For $S \subseteq M$, $int_\lambda(S) \subseteq \lambda_g^\alpha int(S)$.

Proof: Obvious.

Remark 2.5.17 For a subset $S \subseteq M$, $\lambda_g^\alpha int(S)$ need not be the largest λ_g^α-open set contained in S as observed from the following example.

Example 2.5.18 Let $M = \{i, j, k, l, m\}$ and $\mu = \{\phi, \{i\}, \{j, k\}, \{i, j, k\}, \{j, k, l, m\}, M\}$. Then λ_g^α-open sets are $\phi, \{i\}, \{j\}, \{k\}, \{l\}, \{m\}, \{i, j\}, \{i, k\}, \{j, k\}, \{j, l\}, \{j, m\}, \{k, l\}, \{k, m\}, \{l, m\}$, $\{i, j, k\}, \{i, l, m\}, \{j, k, l\}, \{j, k, m\}, \{j, l, m\}, \{k, l, m\}, \{i, j, l, m\}, \{i, k, l, m\}, \{j, k, l, m\}, M$. Let $M = \{i, j, l\}$. Then $\lambda_g^\alpha int(M) = \{i\} \cup \{j\} \cup \{l\} \cup \{i, j\} \cup \{j, l\} = \{i, j, l\}$, which is not the largest λ_g^α-open set contained in S.

Proposition 2.5.19 If a subset S of (M, μ) is λ_g^α-open then $\lambda_g^\alpha int(S) = S$, but not conversely.

Proof: Obvious.

Example 2.5.20 Consider (M, μ) as in Example 2.5.18. Let $S = \{i, m\}$. Then $\lambda_g^\alpha int(S) = S$ but $S = \{i, m\}$ is not a λ_g^α-open set.

Remark 2.5.21 $\lambda_g^\alpha int(S)$ need not be a λ_g^α-open set as observed from the Examples 2.5.18 and 2.5.20.

Proposition 2.5.22 For the subsets S and T of a topological space (M, μ), $\lambda_g^\alpha int(S \cup T) \supseteq \lambda_g^\alpha int(S) \cup \lambda_g^\alpha int(T)$.

Proof: Obvious.

Remark 2.5.23 The reverse inclusion of Proposition 2.5.22 may not be true as observed from the following example.

Example 2.5.24 Consider (M, μ) as in Example 2.5.10. Here λ_g^α-open sets are $\phi, \{i\}, \{j\}, \{m\}, \{i, j\}, \{i, m\}, \{j, k\}, \{j, m\}, \{l, m\}, \{i, j, k\}, \{i, j, m\}, \{i, l, m\}, \{j, k, l\}, \{j, k, m\}$, $\{j, l, m\}, \{k, l, m\}, \{i, j, k, l\}, \{i, j, k, m\}, \{i, j, l, m\}, \{i, k, l, m\}, \{j, k, l, m\}, M$. Let $S = \{k\}$ and $T = \{l, m\}$. Then $S \cup T = \{k, l, m\}$, $\lambda_g^\alpha int(S \cup T) = \{k, l, m\}$, $\lambda_g^\alpha int(S) = \phi$ and $\lambda_g^\alpha int(T) = \{l, m\}$. Therefore $\lambda_g^\alpha int(S) \cup \lambda_g^\alpha int(T) = \{l, m\}$ but $\lambda_g^\alpha int(S \cup T) = \{k, l, m\}$. Hence $\lambda_g^\alpha int(S \cup T) \nsubseteq \lambda_g^\alpha int(S) \cup \lambda_g^\alpha int(T)$.

Proposition 2.5.25 For the subsets S and T of a topological space (M, μ), $\lambda_g^\alpha int(S \cap T) \subseteq \lambda_g^\alpha int(S) \cap \lambda_g^\alpha int(T)$.

Proof: Obvious.

Remark 2.5.26 The reverse inclusion of Proposition 2.5.25 may not be true as observed from the following example.

Example 2.5.27 Consider (M, μ) as in Example 2.5.24. Let $S = \{j, k, l\}$ and $T = \{k, l, m\}$. Then $S \cap T = \{k, l\}$, $\lambda_g^\alpha int(S \cap T) = \{\phi\}$, $\lambda_g^\alpha int(S) = \{j, k, l\}$ and $\lambda_g^\alpha int(T) = \{k, l, m\}$. Therefore $\lambda_g^\alpha int(S) \cap \lambda_g^\alpha int(T) = \{k, l\}$ but $\lambda_g^\alpha int(S \cap T) = \{\phi\}$. Hence $\lambda_g^\alpha int(S) \cap \lambda_g^\alpha int(T) \nsubseteq \lambda_g^\alpha int(S \cap T)$.

Lemma 2.5.28 For a subset S of (M, μ), the following properties hold:

(i) $\lambda_g^\alpha cl(S^c) = (\lambda_g^\alpha int(S))^c$.

(ii) $(\lambda_g^\alpha cl(S^c))^c = \lambda_g^\alpha int(S)$.

(iii) $\lambda_g^\alpha cl(S) = (\lambda_g^\alpha int(S^c))^c$.

Proof: Obvious.