Chapter II

Soft Topological Spaces

CHAPTER – II

SOFT TOPOLOGICAL SPACES

Definition : 2.1

A soft topology τ is a family of soft sets over U satisfying the following properties :

T_1) $\Phi, \tilde{E} \in \tau$.

T_2) If $F_A, G_B \in \tau$, then $F_A \tilde{\cap} G_B \in \tau$.

T_3) If $(F_A)_\lambda \in \tau, \forall \lambda \in \Lambda$, then $\tilde{\bigcup}_{\lambda \in \Lambda} (F_A)_\lambda \in \tau$.

(U, τ) is called a **soft topological space**. Every member of τ is called soft open.

A soft set G_B is called soft closed in (X, τ) if $G_B^c \in \tau$.

Indiscrete soft topology, denoted by τ^0 contains only Φ and $\tilde{E}$ while the discrete soft topology, denoted by τ^1 contains all soft sets over X.

Note : 2.2

If $\tau \subseteq SS(U)_A$, where A is a fixed set of parameters, then the soft topological space is denoted by (U, τ, A).

Example : 2.3

Let R be the set of real numbers, $E = R^+$ be the positive real numbers and $(F_E)_\lambda = \{x, (x - \lambda, x + \lambda)) : x \in E\}$. If $\tau = \{(F_E)_\lambda : \lambda \in E\} \cup \{\Phi, \tilde{E}\}$ then the pair (R, τ) is a soft topological space.

Example : 2.4

Let R be the real numbers and let E be a countable set. Consider the family $\tau_* = \{F_A : \underset{e \in E}{\cup} X \setminus F_A(e) \text{ is countable}\} \cup \{\Phi\}$, then the pair (R, τ_*) is a soft topological space.

Definition : 2.5

Let (X, τ) be a soft topological space. A sub collection $\mathscr{B}$ of τ is called a **base** for τ if every member of τ can be expressed as a union of members of $\mathscr{B}$.

Definition : 2.6

Let (X, τ_1) and (Y, τ_2) be two soft topological spaces. A soft mapping $(\varphi, \psi) : (X, \tau_1) \to (Y, \tau_2)$ is called **soft continuous** if $(\varphi, \psi)^{-1} (G_B) \in \tau_1$, for all $G_B \in \tau_2$.

If $(\varphi, \psi) : (X, \tau_1) \to (Y, \tau_2)$ and $(\varphi_1, \psi_1) : (Y, \tau_2) \to (Z, \tau_3)$ are continuous then clearly $(\varphi_1, \psi_1) \circ (\varphi, \psi)$ is soft continuous.

A soft mapping $(\varphi, \psi) : (X, \tau_1) \to (Y, \tau_2)$ is called **soft open** if

$(\varphi, \psi) (F_A) \in \tau_2$, for all $F_A \in \tau_1$.

Definition : 2.7

Let (X, τ) be a soft topological space. A sub collection S of τ is said to be **sub base** for τ if the family of all finite intersections of members of S forms a base for τ.

Theorem : 2.8

Let S be a family of soft set over U such that $\Phi, \tilde{E} \in S$. Then S is a base for the topology τ, whose members are of the form $\underset{i \in J}{\tilde{\cup}} (\underset{\lambda \in \Lambda_i}{\tilde{\cap}} (F_A)_{i, \lambda})$ where J is arbitrary index set and for each $i \in J$, Λ_i is a finite index set. $(F_A)_{i, \lambda} \in \mathcal{S}$ for $i \in J$ and $\lambda \in \Lambda_i$.

Definition : 2.9

Let $\{(\varphi, \psi)_i : \mathcal{S}(X, E) \rightarrow (Y_i, \tau_i)\}_{i \in J}$ be a family of soft mappings and $\{(Y_i, \tau_i)\}_{i \in J}$ is a family of soft topological spaces. Then the topology τ generated from the sub base $\mathcal{S} = \{(\varphi, \psi)_i^{-1} (F_A) : F_A \in \tau_i, i \in J\}$ is called the **soft topology (or initial soft topology) induced by the family of soft mappings** $\{(\varphi, \psi)_i\}_{i \in J}$.

Theorem : 2.10

The initial soft topology τ on X induced by the family $\{(\varphi, \psi)_i\}$ from X to Y_i respectively, is the coarsest soft topology making $(\varphi, \psi)_i : (X, \tau) \rightarrow (Y_i, \tau_i)$ continuous, for all $i \in J$.

Definition : 2.11

Let $\{(X, \tau_i)\}_{i \in J}$ be a family of soft topological spaces. Then the initial soft topology on X $(= \prod_{i \in J} X_i)$ generated by the family $\{(p, q)_i\}_{i \in J}$ is called **product soft topology** on X.

The product soft topology is denoted by $\prod_{i \in J} \tau_i$.

Definition : 2.12

Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U.

a) The **soft closure** of (G, A) is the soft set. $\overline{(G, A)} = \tilde{\cap} \{(S, A) : (S, A)$ is soft closed and $(G, A) \tilde{\subseteq} (S, A)\}$.

b) The **soft interior** of (G, A) is the soft set $(G, A)^0 = \tilde{\cup}\ \{(S, A) : (S, A)$ is soft open and $(S, A) \tilde{\subseteq} (G, A)\}$

By the property T_3 for soft open sets, $(G, A)^0$ is soft open. It is the largest soft open set contained in (G, A).

Definition : 2.13

Let (X, τ) be a soft topological space, F_A be a soft set over X and $x \in X$. Then F_A is called a **soft neighborhood** of x if there exists a soft open set G_B such that $x\ \tilde{\in}\ G_B \tilde{\subseteq} F_A$. The neighborhood system of a point x, denoted by $\mathcal{N}_\tau(x)$, is the family of all its neighborhoods.

Theorem : 2.14

Let (X, τ) be a soft topological space. The neighborhood system $\mathcal{N}_\tau(x)$ in (X, τ) has the following properties.

1. Each $x \in X$ has a soft neighborhood.
2. If $F_A, G_B \in \mathcal{N}_\tau(x)$, then $F_A \tilde{\cap} G_B \in \mathcal{N}_\tau(x)$.
3. If $F_A \in \mathcal{N}_\tau(x)$ and $F_A \tilde{\subseteq} G_B$, then $G_B \in \mathcal{N}_\tau(x)$.

Definition : 2.15

Let (X, τ) be a soft topological space and M be a non-empty subset of X. The set $\tau_M = \{\tilde{E}_M \tilde{\cap} F_A : F_A \in \tau\}$ is called the **soft relative topology** on M and (M, τ_M) is called **soft subspace** of (X, τ).

Theorem : 2.16

Let (M, τ_M) be a soft subspace of (X, τ) and $F_A \in \mathcal{S}(X, E)$. Then F_A is soft open in M iff $F_A = \tilde{E}_M \tilde{\cap} G_B$, for some $G_B \in \tau$.

Definition : 2.17

Let (X, τ) be a soft topological space, $(X_n) \subset X$ be a sequence and $x_0 \in X$. (x_n) **converges** to x_0 in (X, τ) if for all $F_A \in \mathcal{N}(x_0)$ there exists $n_0 \in N$ such that $x_n \tilde{\in} F_A$ for all $n \geq n_0$.

Example : 2.18

Let (R, τ^0) be an indiscrete soft topological space and (x_n) converges to x_0 in (R, τ^0). In this topological space $\mathcal{N}(x_0) = \{\tilde{E}\}$, so $x_n \tilde{\in} \tilde{E}$ for all $n \in N$. Hence every sequence converges to every point in (R, τ^0).

Example : 2.19

Let (R, τ_*) be a soft topological space which is defined in Example 2.4 and let (x_n) converges to x_0 in (R, τ_*). Then for all $G_B \in \mathcal{N}(x_0)$ there exists $n_0 \in N$ such that $x_n \tilde{\in} G_B$ for all $n \geq n_0$. Let define the soft set $F_A : E \rightarrow P(x)$ by

$F_A(e) = \{x_n : x_n \neq x_0 \text{ and } n \in N\}$, $\forall\, e \in E$. From here, $x_0, \notin F_A(e)$ and $x_0 \in X \setminus F_A(e)$. So $x_0 \tilde{\in} F_A^c \in \tau^*$. Let take $G_B = F_A^c \in \mathcal{N}(x_0)$ then $x_n \tilde{\in} F_A^c$, for all $n \geq n_0$ and we obtain $x_n \tilde{\notin} F_A$, for all $n \geq n_0$.

Hence $x_n = x_0$, for all $n \geq n_0$.

Definition : 2.20

A soft set (G, A) in a soft topological space (U, τ, A) is called a **soft neighborhood of the soft point** $e_F \tilde{\in} U_A$, if there exists a soft open set (H, A) such that $e_F \tilde{\in} (H, A) \tilde{\subseteq} (G, A)$.

The **neighborhood system** of a soft point e_F, denoted by $N_\tau(e_F)$, is the family of all its neighborhoods.

Definition : 2.21

A soft set (G, A) in a soft topological space (U, τ, A) is called a **soft neighborhood of the soft set** (F, A) if there exists a soft open set (H, A) such that (F, A) $\tilde{\subseteq}$ (H, A) $\tilde{\subseteq}$ (G, A).

Theorem : 2.22

The neighborhood system $N_\tau(e_F)$ at e_F in a soft topological space (U, τ, A) has the following properties :

a) If (G, A) $\in N_\tau(e_F)$, then $e_F \tilde{\in}$ (G, A),

b) If (G, A) $\in N_\tau(e_F)$ and (G, A) $\tilde{\subseteq}$ (H, A), then (H, A) $\in N_\tau(e_F)$,

c) If (G, A), (H, A) $\in N_\tau(e_F)$, then (G, A) $\tilde{\cap}$ (H, A) $\in N_\tau(e_F)$,

d) If (G, A) $\in N_\tau(e_F)$, then there is a (M, A) $\in N_\tau(e_F)$ such that (G, A) $\in N_\tau(e_H^c)$ for each $e_H^c \tilde{\in}$ (M, A).

Proof

a) If (G, A) $\in N_\tau(e_F)$, then there is a (H, A) $\in \tau$ such that $e_F \tilde{\in}$ (H, A) $\tilde{\subseteq}$ (G, A).

Therefore, we have $e_F \tilde{\in}$ (G, A).

b) Let (G, A) $\in N_\tau(e_F)$ and (G, A) $\tilde{\subseteq}$ (H, A). Since (G, A) $\in N_\tau(e_F)$, then there is a (M, A) $\in \tau$ such that $e_F \tilde{\in}$ (M, A) $\tilde{\subseteq}$ (G, A).

Therefore, we have $e_F \tilde{\in}$ (M, A) $\tilde{\subseteq}$ (G, A) $\tilde{\subseteq}$ (H, A) and so (H, A) $\in N_\tau(e_F)$.

c) If (G, A), (H, A) $\in N_\tau(e_F)$, then there exists (M, A), (S, A) $\in \tau$ such that $e_F \tilde{\in}$ (M, A) $\tilde{\subseteq}$ (G, A) and $e_F \tilde{\in}$ (S, A) $\tilde{\subseteq}$ (H, A).

Hence $e_F \tilde{\in}$ (M, A) $\tilde{\cap}$ (S, A) $\tilde{\subseteq}$ (G, A) $\tilde{\cap}$ (H, A).

Since (M, A) $\tilde{\cap}$ (S, A) $\in \tau$, we have (G, A) $\tilde{\cap}$ (H, A) $\in N_\tau(e_F)$.

d) If $(G, A) \in N_\tau(e_F)$, then there is a $(S, A) \in \tau$ such that $e_F \tilde{\in} (S, A) \tilde{\subseteq} (G, A)$.

Put $(M, A) = (S, A)$.

Then for every $e_H^c \tilde{\in} (M, A)$, $e_H^c \tilde{\in} (M, A) \tilde{\subseteq} (S, A) \tilde{\subseteq} (G, A)$.

This implies $(G, A) \in N_\tau(e_H^c)$.

Corollary : 2.23

Let (U, τ, A) be a soft topological space and let (F, A) and (G, A) be soft sets over U. Then

a) (F, A) is soft closed iff $(F, A) = (\overline{F, A})$

b) (G, A) is soft open iff $(G, A) = (G, A)^0$.

Theorem : 2.24

A soft set (G, A) is soft open iff for each soft set (F, A) contained in (G, A), (G, A) is a soft neighborhood of (F, A).

Proof

Each soft set (F, A) contained in (G, A), (G, A) is a soft neighbourhood of (F, A) since $(G, A) \tilde{\subseteq} (G, A)$, there exists a soft open set (H, A) such that $(G, A) \tilde{\subseteq} (H, A) \tilde{\subseteq} (G, A)$. Hence $(H, A) = (G, A)$ and (G, A) is soft open.

Theorem : 2.25

Let (U, τ, A) be a soft topological space and let (F, A) and (G, A) be a soft sets over U. Then

a) $((\overline{G, A}))^c = ((G, A)^c)^0$

b) $((G, A)^0)^c = ((\overline{G, A})^c)$

Proof

By theorem 1.26,

$$\begin{aligned} ((\overline{G, A}))^c &= (\tilde{\cap}\ \{(S, A) : (S, A) \text{ is soft closed and } (G, A) \tilde{\subseteq} (S, A)\})^c \\ &= \tilde{\cup}\{(S, A)^c : (S, A) \text{ is soft closed and } (G, A) \tilde{\subseteq} (S, A)\} \\ &= \tilde{\cup}\{(S, A)^c : (S, A)^c \text{ is soft open and } (S, A)^c \tilde{\subseteq} (G, A)^c\} \\ &= ((G, A)^c)^0 \end{aligned}$$

Similar proof holds for (b).

Theorem : 2.26

Let $e_F \tilde{\in} U_A$ for all $e \in A$ and (G, A) be a soft open set in a topological space (U, τ, A). Then the following statements hold :

a) Every soft point $e_F \tilde{\in}$ (G, A) is a soft interior point.

b) For each $e \in A$, let us consider a mapping $(e)_G : A \to P(U)$ defined as follows :

$$(e)_G\,(e_1) = \begin{cases} G(e) \text{ if } e_1 = e \\ \Phi, \text{ if } e_1 \neq e \end{cases}$$

Then obviously $(e)_G$ is a soft interior point of (G, A) and $(e)_G = \tilde{\cup}\, e_F$ for every soft interior point e_F of (G, A).

c) $\underset{e \in A}{\tilde{\cup}} (e)_G = (G, A)$.

Proof

a) Obvious

b) Since (G, A) is a soft open set, the soft interior point $(e)_G$ is the largest soft interior point of (G, A) determined by $e \in A$ and so $(e)_G = \tilde{\cup}\, e_F$ for every soft interior point e_F of (G, A).

c) Obvious.

Theorem : 2.27

Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U. Then $(G, A)^0 = \tilde{\bigcup}_{e \in A} \{e_F : e_F$ is any soft interior point of (G, A) for $e \in A\}$.

Proof

Let $(G, A)^0 = (H, A)$, where $H(e) = \cup S(e)$ for each soft open set (S, A) such that $(S, A) \tilde{\subseteq} (G, A)$. Since $(G, A)^0$ is a soft open set, by the Theorem 2.26 (c), $(G, A)^0 = \tilde{\bigcup}_{e \in A} (e)_H$ and for each $(e)_H$, $(e)_H$ is a soft interior point of (G, A) because of $(e)_H \tilde{\in} (G, A)^0 \tilde{\subseteq} (G, A)$. Therefore, $(G, A)^0 \tilde{\subseteq} \tilde{\bigcup}_{e \in A} \{e_F : e_F$ is any soft interior point of (G, A) for $e \in A\}$.

For the other hand, let e_{F_i} be any soft interior point of (G, A) for each $e \in A$. Then there exists a soft open set $(K^e_{Fi}, A) \in SS(U)_A$ for each $e \in A$ such that $e_{F_i} \tilde{\in} (K^e_{Fi}, A) \tilde{\subseteq} (G, A)$. So for each $e \in A$, we have $\tilde{\bigcup}_i e_{F_i} \tilde{\subseteq} \tilde{\bigcup}_i (K^e_{Fi}, A) \tilde{\subseteq} (G, A)$ and it implies

$\tilde{\bigcup}_{e \in A} \{e_F : e_F$ is any soft interior point of (G, A) for $e \in A\}$

$= \tilde{\bigcup}_e \tilde{\bigcup}_i e_{F_i} \tilde{\subseteq} \tilde{\bigcup}_e \tilde{\bigcup}_i (K^e_{Fi}, A) \tilde{\subseteq} (G, A).$

Since $\tilde{\bigcup}_e \tilde{\bigcup}_i (K^e_{Fi}, A)$ is soft open and $(G, A)^0$ is the largest soft open subset of (G, A),

we have $\tilde{\bigcup}_{e \in A} \{e_F : e_F$ is any soft interior point of (G, A) for $e \in A\} \tilde{\subseteq} (G, A)^0$.

Theorem : 2.28

Let (U, τ, A) be a soft topological space and let (G, A) be a soft set over U. Then for every soft interior point e_F of (G, A), $(e)_G = \tilde{\cup} e_F$ iff (G, A) is soft open.

Proof

If follows from the theorem 2.26 and 2.27.

Theorem : 2.29

Let (U, A, τ) be a soft topological space over X. Then the collection $\tau^{\alpha} = \{F(\alpha) : (F, A) \in \tau\}$ defines a topology on U for each $\alpha \in A$.

Theorem : 2.30

Let (X, A, τ) be a soft topological space and τ^{α} be the topologies on X as in Theorem 2.29.

Let $\tau^* = \{(G, A) \in S(X) : G(\alpha) \in \tau^{\alpha}$, for all $\alpha \in A\}$. Then τ^* is a soft topology on X with $(\tau^*)^{\alpha} = \tau^{\alpha}$, for all $\alpha \in A$.

Proof

Since $\tilde{\Phi}(\alpha) = \Phi \in \tau^{\alpha}$ for all $\alpha \in A \Rightarrow (\tilde{\Phi}, A) \in \tau^*$, and $\tilde{X}(\alpha) = X \in \tau^{\alpha}$, for all $\alpha \in A \Rightarrow (\tilde{X}, A) \in \tau^*$.

Now, let (F_1, A) and $(F_2, A) \in \tau^*$. Then $F_1(\alpha), F_2(\alpha) \in \tau^*, \forall\, \alpha \in A$.

Thus $F_1(\alpha) \cap F_2(\alpha) \in \tau^{\alpha}$, for all $\alpha \in A \Rightarrow (F_1 \tilde{\cap} F_2)(\alpha) \in \tau^{\alpha}$, for all $\alpha \in A$. Therefore, $(F_1, A) \tilde{\cap} (F_2, A) = (F_1 \tilde{\cap} F_2, A) \in \tau^*$.

Again, let $(F_i.A) \in \tau^*$, for all $i \in I$. Then $F_i(\alpha) \in \tau^{\alpha}$, for all $i \in I$, for all $\alpha \in A \Rightarrow \bigcup_{i \in I} F_i(\alpha) \in \tau^{\alpha}$, for all $\alpha \in A$.

So $(\underset{i\in I}{\tilde{\cup}} F_i)(\alpha) \in \tau^{\alpha}$, for all $\alpha \in A$. Hence $\underset{i\in I}{\tilde{\cup}} (F_i, A) \in \tau^*$. Therefore, τ^* is a soft topology on X.

Next let $U \in \tau^{\alpha}$. Then there exists $(F, A) \in \tau$ such that $U = F(\alpha)$. We construct $(G, A) \in S(X)$ such that $G(\alpha) = F(\alpha)$ and $G(\beta) = \Phi$, for all $\beta \neq \alpha$. Then $(G, A) \in \tau^*$ and $U = F(\alpha) = G(\alpha) \in (\tau^*)^{\alpha}$.

Therefore, $\tau^{\alpha} \subseteq (\tau^*)^{\alpha}$ (1)

Also, let $V \in (\tau^*)^{\alpha}$. Then there exists $(F, A) \in \tau^*$ such that $V = F(\alpha) \in \tau^{\alpha}$.

Therefore, $(\tau^*)^{\alpha} \subseteq \tau^{\alpha}$ (2)

Thus, from (1) and (2), we get $\tau^{\alpha} = (\tau^*)^{\alpha}$, for all $\alpha \in A$.

Remark : 2.31

It is to be noted that τ and τ^* of Theorem 2.30 may be different. This is shown by the following example.

Example : 2.32

Let $X = \{a, b\}$ and $A = \{\alpha, \beta\}$. Then all possible soft sets over X are given below :

$(\tilde{\Phi}, A) = \{\{\Phi\}, \{\Phi\}\}$

$(F_1, A) = \{\{\Phi\}, \{a\}\}$

$(F_2, A) = \{\{\Phi\}, \{b\}\}$

$(F_3, A) = \{\{\Phi\}, \{a, b\}\}$

$(F_4, A) = \{\{a\}, \{\Phi\}\}$

$(F_5, A) = \{\{a\}, \{a\}\}$

$(F_6, A) = \{\{a\}, \{b\}\}$

$(F_7, A) = \{\{a\}, \{a, b\}\}$

$(\tilde{X}, A) = \{\{a, b\}, \{a, b\}\}$

$(F_8, A) = \{\{b\}, \{\Phi\}\}$

$(F_9, A) = \{\{b\}, \{a\}\}$

$(F_{10}, A) = \{\{b\}, \{b\}\}$

$(F_{11}, A) = \{\{b\}, \{a, b\}\}$

$(F_{12}, A) = \{\{a, b\}, \{\Phi\}\}$

$(F_{13}, A) = \{\{a, b\}, \{a\}\}$

$(F_{14}, A) = \{\{a, b\}, \{b\}\}$

Let $\tau = \{(\tilde{\Phi}, A), (F_2, A), (F_6, A), (F_{11}, A), (\tilde{X}, A)\}$.

Then $\{X, A, \tau\}$ is a soft topological space such that $\tau^\alpha = \{\Phi, \{a\}, \{b\}, X\}$ and $\tau^\beta = \{\Phi, \{b\}, X\}$.

Now let us construct.

$\tau^* = \{(\tilde{\Phi}, A), (F_2, A), (F_3, A), (F_4, A), (F_6, A), (F_7, A), (F_8, A), (F_{10}, A), (F_{11}, A), (F_{12}, A), (F_{14}, A), (\tilde{X}, A)\}$.

Then $\tau \neq \tau^*$ but $(\tau^*)^\alpha = \tau^\alpha$, for all $\alpha \in A$.