A Study of Fuzzy β-Subalgebras and Fuzzy β-Ideals in β-Algebras

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By

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DECLARATION

DECLARATION

I declare that the dissertation entitled **"A Study of Fuzzy β-Subalgebras and Fuzzy β-Ideals in β-Algebras"** submitted by me for the degree of **Master of Philosophy (M.Phil.)** is the record of work carried out by me during the period from August 2016 to July 2017 under the guidance of **Dr. (Tmt.) P.Jeyalakshmi** and has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this University or any other similar institution of Higher Learning.

Signature of the Candidate

CERTIFICATE

CERTIFICATE FROM THE SUPERVISOR

I certify that the dissertation entitled **"A Study of Fuzzy β-Subalgebras and Fuzzy β-Ideals in β-Algebras "** submitted for the degree of **Master of Philosophy (M.Phil)** by **K.GAYATHRI** is the record of work carried out by her during the period from August 2016 to July 2017 under my guidance and supervision, and that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this University or any other similar institution of Higher Learning.

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INTRODUCTION

INTRODUCTION

In 1996, Imai and Iseki ([17],[18]) introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of BCI- algebras. In 2002, Neggers and Kim [26], introduced the notion of B- algebras which is another generalization of BCK-algebras. Also they introduced the notion of β-algebras [27] where two operations are coupled in such a way as to reflect the natural coupling, which exists between the usual group operation and its associated B-algebras. In 2012, Kim [20] investigated some properties of β-algebras.

The important point in the evaluation of the modern concept of uncertainty was the article by Zadeh [52],that introduced the theory of fuzzy sets. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1997, by Rosenfeld [36]. The concept of intuitionistic fuzzy set was introduced by Atanassov [10] in 1986, which is a generalization of the notion of fuzzy sets. Fuzzy sets give a degree of membership of an element in given set, while intuitionisic fuzzy sets give both a degree of membership and non-membership.

In 1991, Xi [49] applied the concept of fuzzy sets to BCK algebras and got some results. In 1993, Jun [19] applied it to BCI algebras. In 2013 the Abu Ayub Ansari and Chandramouleeswaran [1], introduced the concept of Fuzzy β-Subalgebras of β-algebras.

The aim of our thesis is to study the fuzzy β-subalgebras and fuzzy β-ideals in β-algebras.

The following articles are chosen for our discussion:

1) “Normal Fuzzy β-Subalgebras and β-algebras”, (2013) by Abu Ayub Ansari.M and Chandramouleeswaran. M [3].
2) “Fuzzy translations of fuzzy β-ideals of β-algebras”, (2014) by Abu Ayub Ansari.M and Chandramouleeswaran.M [4].
3) “L-fuzzy β-subalgebras of β-algebras”, (2014) by Rajam.K and Chandramouleeswaran.M [29].

4) "L-fuzzy level β-subalgebras of β-algebras", (2015) by Rajam.K and Chandramouleeswaran.M [30].
5) "L-fuzzy β-ideals of β-algebras", (2015) by Rajam.K and Chandramouleeswaran.M [31].
6) "L-fuzzy T-ideals of β-algebras", (2015) by Rajam.K and Chandramouleeswaran.M [32].
7) "On intuitionistic fuzzy β-subalgebras of β-algebras", (2013) by Sujatha. K, Chandramouleeswaran.M and Muralikrishna.P [44].
8) "Product on intuitionistic fuzzy β-subalgebras of β-algebras", (2014) by Sujatha.K, Muralikrishna.P and Chandramouleeswaran.M [45].
9) "Intuitionistic fuzzy α-translation on β-algebras", (2015) by Sujatha.K and Muralikrishna.P [47].

This thesis is divided into four chapters:

In chapter 1, the preliminary definitions and results on β-algebras and fuzzy sets are presented due to Neggers et al [27] and Zadeh [52]. The properties of level β-subalgebras and normal fuzzy β-subalgebras of a β-algebra are established.

The interesting results discussed in the chapter are given as follows:

1) Any β-subalgebras of β-algebra X can be realized as a level β-subalgebra for some fuzzy β-subalgebras of X.
2) Let X be a β-algebra. Let N = { γ / γ is a normal fuzzy β-subalgebra of X} and μ be the maximal in the poset (N,⊂). Then μ takes only the values 0 and 1.

In chapter 2, the notion of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy β-ideals of β-algebras are introduced and some of their properties are investigated.

In this chapter the following important results are discussed:

1) If μ be a fuzzy β-ideal of X then the fuzzy α-translation μ_α^T of μ is also a fuzzy β-ideal of X ∀ α∈[0.T] where T=1− sup{μ(x) / x ∈X}.

2) For α∈[0,T], let the fuzzy α-translation μ_α^T of a fuzzy set μ of X be a fuzzy β-ideal of X. Then every non-empty level subset $(\mu_\alpha^T)_t$={x ∈X /μ(x)≥ t−α}∀t∈ Im(μ) with t >α is a β-ideal of X.
3) Intersection of any two fuzzy β-ideal extension of a fuzzy β-ideal μ of X is a fuzzy β-ideal extension of μ.
4) Let μ be fuzzy set of X such that the fuzzy γ- multiplication μ_γ^M of μ is a fuzzy β- ideal of X for some γ ∈ [0,1] then μ is a fuzzy β- ideal of X.

Chapter 3 deals with a study of L-fuzzy β-subalgebras, L-fuzzy β-ideals and L-fuzzy T-ideals of β-algebras.

In the first two sections of this chapter, the notion of L-fuzzy β-subalgebras and L-fuzzy level β-subalgebras of a β-algebra are introduced and investigated some of their properties.

In the third section, the properties of L-fuzzy β-ideals of a β-algebra are discussed.

The last section of this chapter deals with the study of some properties of L-fuzzy T-ideals of a β-algebras.

Some interesting results discussed in this chapter are given as follows:

1) If A is a β–subalgebra of X, then the characteristic function χ_A is an L-fuzzy β–subalgebra of X.
2) An L-fuzzy set μ of a β-algebra of X is an L-fuzzy β–subalgebra iff the level subset μ_t of μ,∀t ∈ [0 ,1] is either empty or a β–subalgebra of X.
3) Let A is a subset of X. Define an L-Fuzzy set μ: X→L such that

   $$\mu(x)=\begin{cases} t_0 & \text{if } x \in A \\ t_1 & \text{if } x \notin A \end{cases}$$

   where $t_0, t_1 \in$ L with $t_0 > t_1$.Then μ is an L-fuzzy β–ideal of a β–algebra X if and only if A is a β–ideal of X .
4) Let μ_1 and μ_2 be two L-fuzzy T–ideals in a β–algebras $X_1 \times X_2$. Then the direct product $\mu_1 \times \mu_2$ is an L-fuzzy T–ideals in $X_1 \times X_2$.

The last chapter deals with the study of the intuitionistic fuzzy structures of β-algebras

In the first two sections of this chapter, the properties of intuitionistic fuzzy β-subalgebras and the product of intuitionistic fuzzy β-subalgebras of a β-algebra are discussed.

In the third section, the concept of an intuitionistic fuzzy α-translation for some α∈[0,1] on β-subalgebras of a β-algebras are studied and investigated some of their properties.

The interesting results discussed in this chapter are given as follows:

1) If A = $\{\langle x, \mu_A(x), \gamma_A(x) \rangle / x \in X\}$ is an IF β-sub algebra of X, then so is ΘA, where ΘA = $\{\langle x, \mu_A(x), \overline{\mu_A(x)} \rangle / x \in X\}$.
2) Let (X, $+$, $-$, 0) and (Y, $+$, $-$, 0) be two β-algebras and let f:X →Y be a endomorphism. If A is IF β-sub algebra of X, then f(A) defined by f(A) = $\{\langle x, (\mu_A)_f(x) = \mu_A(f(x)), (\gamma_A)_f(x) = \gamma_A(f(x)) \rangle / x \in X\}$ is IF β-subalgebra of Y.
3) If A_i= $\{ x \in X_i / \mu_{A_i}(x), \gamma_{A_i}(x)\}$ be an IF β-sub algebra of X_i respectively, for i=1,...,n, then $\prod_{i=1}^{n} A_i$ is an IF β-sub algebra of $\prod_{i=1}^{n} X_i$
4) Let X and Y be two β-algebras. Let A and B be two IF α-translation on β-sub algebras. Let f: X →Y be a homomorphism. If A is an IF α-translation on β-sub algebra of Y. Then f^{-1}(A) is a IF α-translation on β-sub algebra of X.

REVIEW OF LITERATURE

REVIEW OF LITERATURE

In 1996, Imai and Iseki [17] introduced two classes of algebras originated from the classical and non-classical propositional logic. These algebras are known as BCK and BCI algebras. It is known that the notion of BCI-algebra is a generalization of BCK-algebras in the sense that the class of BCK algebras is a proper subclass of the class of BCI-algebras [18]. In 2002, Neggers and Kim [26] introduced the notion of B-algebras, which is another generalization of BCK-algebras. Also they introduced the notion of β-algebras [27] where two operations are coupled in such a way as to reflect the natural coupling, which exists, between the usual group operation and its associated B-algebras, which is naturally defined by the group. In 2012, Young Hee Kim and Kenu Sook So, gave a new approach of β-algebras and proved some related results [20].

In 1965, Zadeh [52] introduced a new notion of fuzzy set, to evaluate the modern concept of uncertainty in real life. The notion of fuzzy sets is a generalization of the notion of crisp sets in which the boundaries are not crisp or sharp. In 1991, the study of fuzzy algebraic structures was initiated by Rosenfeld [35]. Goguen [16], generalized the notion of fuzzy sets into the notion of L-fuzzy sets. Atanasov [10] introduced the notion of Intuitionistic Fuzzy set in 1986, in which not only the membership value is considered but also non-membership values are considered.

In 1991, Xi [49] defined fuzzy set in BCK-algebras and investigated some properties. In 1993, Jun [19] applied it to BCI-algebras. In 2013 the Abu Ayub Ansari and Chandramouleeswaran [1], introduced the concept of Fuzzy β-Subalgebras of β-algebras. In 2014, Rajam and Chandramouleeswaran [29], introduced the notion of L-fuzzy β-subalgebras of β-algebras and investigated some of their properties. Also they introduced the notion of L-fuzzy β-ideals of a β-algebra.

Now we present the abstracts of some important articles on various algebras collected for our study.

1.Intuitionistic Fuzzy Ideals of BCK-Algebras

Young Bae Jun and Kyung Ho Kim, (2000) [51]

In this article, the authors considered the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras and investigated some of their properties. They introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of BCK-algebra and investigated some related properties.

2.On Fuzzy Subalgebras in B-Algebras

Sun Shin Ahn and Keumseong Bang, (2003) [9]

In this article, the authors classified the subalgebras by their family of level subalgebra in B-algebras.

3.Fuzzy Subalgebras of BG-Algebras

Sun Shin Ahn and Keumseong Bang, (2004) [8]

In this article, the authors classified the subalgebras by their family of level subalgebra in BG-algebras.

4.Interval Valued Fuzzy B-Algebras

A.Borumand Saeid, (2006) [36]

In this article, the notion of interval-valued fuzzy B-algebras (briefly i-v fuzzy B-algebras), the level and strong level B-subalgebras are introduced. Then some theorems which determined the relationship between these notions and B-subalgebras are proved. The image and inverse image of i-v fuzzy B-subalgebras are defined, and how the homomorphic images and inverse images of i-v fuzzy B-subalgebras become i-v fuzzy B-algebras are studied.

5.Intuitionistic Fuzzy Structure of B-Algebras

Y. H. Kim and T. E. Jeong, (2006) [21]

In this article, the authors defined intuitionistic fuzzy subalgebras of B-algebras which is related to several classes of algebras such as BCI/BCK algebras and obtained the some important results for the homomophic image and equivalence relations on IFS(X).

6.On Fuzzy Ideals of BCI-algebras

A.Kordi and A. Moussavi, (2007) [23]

In this article, the fuzzy p-idelas, fuzzy H-ideals and fuzzy BCI- positive implicative ideals of BCI-algebras are studied and related properties are investigated.

7.On BG-Algebras

Chang Bum Kim and Hee Sik Kim, (2008) [22]

In this article, the authors introduced the notion of BG-algebras which is a generalization of B-algebras. They constructed a BG-algebra from a non-empty set which is non-group derived. Moreover using the notion of normal subalgebra, they obtained several isomorphism theorems of BG-algebra and related properties.

8.Fuzzy Translations and Fuzzy Multiplications of BCK/BCI- Algebras

Kyoung Ja Lee, Young Bae Jun and Myung Im Doh, (2009) [24]

In this article, the authors investigated the fuzzy translations,(normalization, maximal) fuzzy extensions and fuzzy multiplications of fuzzy subalgebras in BCK/BCI-algebras are discussed. Relations among fuzzy translations (normalized, maximal) fuzzy extensions and fuzzy multiplications are investigated.

9.On Intuitionistic L-Fuzzy Subalgebras of BG-algebras

Chandramouleeswaran and Muralikrishna, (2010) [15]

In this article, the authors discussed the notions of Intuitionstic L-fuzzy Subalgebras of a BG-algebras and some of their basic properties.

10.Fuzzy BG-Ideals in BG-Algebra

Muthuraj,Sridharan and Sitharselvam, (2010) [25]

In this article, the authors introduced the concept of fuzzy BG- ideals in BG-algebra and discussed some of their properties.

11.Some Results on Intuitionistic Fuzzy Ideals in BCK-Algebras

B.Satyanarayana and R.Durga Prasad, (2011) [37]

In this article, the auothers gave some results on the intuitionistic fuzzy implicative ideals, intuitionstic fuzzy positive implicative ideals, intuitionstic fuzzy commutative ideals.

12.Intuitionistic Fuzzy B-Algebras

Jiayin peng, (2012) [28]

The notion of intuitionistic fuzzy B-algebras is introduced and their some properties are investigated. How to deal with the homomorphic image and inverse image of intuitionistic fuzzy ideals are B-algebras. The relations between a intuitionistic fuzzy B-algebras and a intuitionistic fuzzy B-algebra in the product B-algebras are given.

13.Fuzzy B-Subalgebras of B-algebra with Respect to t-norm

Tapan Senapati,Monoranjan Bhowmik and Madhumngal Pal, (2012) [38]

In this article, the authors applied the concept of t-norm T to fuzzy structure of B-algebras. The notion of a fuzzy B-subalgebra of B-algebras with respect to t-norm is introduced and several related properties are investigated. The direct product and T-product of T-fuzzy subalgebra of B-algebra are investigated.

14.Fuzzy Dot Sublgebras and Fuzzy Dot Ideals of B-Algebras

Tapan Senapati,Monoranjan Bhowmik and Madhumngal Pal, (2012) [39]

In this article, the notion of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras are introduced and investigated some of their properties. The homomorphic image and inverse image of fuzzy dot subalgebras and fuzzy dot ideals are studied. Also introduced the notion of fuzzy relations on the family of fuzzy dot subalgebras and fuzzy dot ideals of B-algebras and investigated some related properties.

15.Anti-Fuzzy β-Subalgebras of β-Algebras

Abu Ayub Ansari and Chandramouleeswaran,(2013) [2]

In this article, the authors introduced the notion of anti-fuzzy β-subalgebras of β-algebras and investigated some of their properties.

16.Intuitionistic L-fuzzy Ideals of BG-algebras

Monoranjan Bhowmik, Tapan Senapati and Madhumangal Pal, (2013) [13]

In this article, the authors applied the concept of an intuitionistic L-fuzzy sets to ideals and closed ideals of BG-algebra. The notion of intuitionistic L-fuzzy closed ideals of BG-algebras are introduced and some related properties are investigated. Also, the product of intuitionistic L-fuzzy BG-algebras are investigated.

17.Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

Monoranjan Bhowmik, Tapan Senapati and Madhumangal Pal, (2013) [14]

In this article, the concept of interval-valued fuzzy set is applied to ideals and closed ideals in B-algebras. The notion of an interval-valued fuzzy closed ideal of a

B-algebra is introduced and some related properties are investigated. Also, the product of interval-valued fuzzy B-algebra is investigated.

18.Fuzzy B- Ideals on B-Algebras

C.Yamini,S. Kailasayalli, (2014) [50]

In this article, the concept of B-ideals and fuzzy B-ideals are introduced. Homomorphism and anti homomorphism functions are satisfied while applying the fuzzy B-ideal concept. Fuzzy B-ideal is also applied in Cartesian product.

19.Fuzzy Dot β-Subalgebras of β-Algebras

Abu Ayub Ansari and Chandramouleeswaran, (2015) [5]

In this article, the authors introduced the notion of fuzzy dot β-subalgebras on β-algebra and investigated some of their properties.

20.Intuitionistic L-Fuzzy Strong β-Filter on β-Algebras

Sujatha, Muralikrishna and Chandramouleeswaran,(2015) [48]

In this article, the authors defined the notion of an intuitionistic L-fuzzy strong β-filter on β-algebras and investigated some of their properties and results.

21.Fuzzy Filters on β-Algebras

Sujatha, Chandramouleeswaran and Muralikrishna, (2015) [46]

In this article, the authors defined the notion on fuzzy filter on β-algebras and investigated some of their properties and results.

22.Translation of Intuitionistic Fuzzy B-Algebras

Tapan Senapati, (2015) [40]

In this article, the concept of intuitionistic fuzzy translation to intuitionistic fuzzy subalgebras and ideals in B-algebras are introduced with several related properties investigated. Examples are also given to illustrate results. The notion of intuitionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy subalgebras and ideals are introduced. Relationships are investigated between intuitionistic fuzzy translations, intutionistic fuzzy extensions and intuitionistic fuzzy multiplications of intuitionistic fuzzy subalgebras and ideals.

23.t-Intuitionistic Fuzzy Subalgebras of BG-Algebras

S. R. Barbhuiya, (2015) [11]

In this article, the notion of t-intuitionistic fuzzy subalgebras and t-intuitionistic fuzzy nomal subalgera of BG-algebras are introduced. Some theorems in t-intuitionistic fuzzy subalgebra and t-intuitionistic fuzzy normal subalgebra in BG-algebra are proved. The homomorphic image and inverse image are investigated in both t-intuitionistic fuzzy subalgebra and normal subalgebra..

24.Fuzzy Translation of Fuzzy Subalgebras in BG-algebras

Monoranjan Bhowmika and Tapan Senapatib, (2015) [12]

In this article, the concepts of fuzzy translation to fuzzy subalgebra in BG-algebras are introduced. The notion of fuzzy extensions and fuzzy multiplications of fuzzy subalgebras are introduced and several related properties are investigated. Also the relationships between fuzzy translations and fuzzy extensions of fuzzy subalgebras are investigated.

25.Cubic Subalgebras and Cubic Closed Ideals of B-algebras

Tapan Senapati, Chang Su Kim, Monoranjan Bhowmik and Madhumangal Pal,(2015) [41]

In this article, the concept of cubic set to subalgebras, ideals and closed ideals of B-algebras are introduced. Relations among cubic subalgebras with cubic ideals and cubic closed ideals of B-algebras are investigated. The homomorphic image and inverse image of cubic subalgebras, ideals are studied and some related properties are investigated. Also the product of cubic B-algebra are investigated.

26.Intuitionistic Fuzzy Dot β-Subalgebras of β-Algebras

Ramesh kumar and Gomathi Eswari, (2016) [34]

In this article, the authors introduced the notion of intuitionistic fuzzy dot β-subalgebra of β-algebras and investigated some of their properties.

27.Interval-Valued Intutionistic L-Fuzzy Strong β-Filter On β-Algebras

Ramesh and Satyanarayana, (2016) [33]

In this article, the authors defined the notion of an interval-valued intuitionistic L-fuzzy strong β-filters on β-algebra and investigated some of their properties.

28.Cubic Structure of BG-subalgebras of BG-Algebras

Tapan Senapati, (2016) [42]

In this article, the notion of cubic BG-subalgebras of BG-algebras are introduced. The homomorphic image and inverse image of cubic BG-subalgebras are studied and investigated some related properties.

29.Complete Ideal and n-Ideal of B-algebra

Habeeb Kreem Abdullah and Arkan Ajeal Atshan,(2017) [7]

In this article, the authors studied some types of ideals of B-algebras called complete ideal, closed complete ideal, n-ideal and closed n-ideal. In addition, they gave some propositions that explained some relationships between these ideals.

CHAPTER 1

Chapter-1

Fuzzy β-Subalgebras of β-Algebras

Section 1.1

Preliminaries on β-algebras and fuzzy sets

Definition [1.1.1]

A BCK-algebra (X, $*$, 0) is a non-empty ser X with a constant 0 and a binary operation $*$ satisfying the following axioms:

BCK 1. $((x * y) * (x * z)) * (z * y) = 0$

BCK 2. $(x * (x * y)) * y = 0$

BCK 3. $x * x = 0$

BCK 4. $x * y = 0$ and $y * x = 0 \implies x = y$

BCK 5. $0 * x = 0 \ \forall \ x , y , z \in$ X.

Definition [1.1.2]

A BCI-algebra (X, $*$, 0) is a non-empty ser X with a constant 0 and a binary operation $*$ satisfying the following axioms:

BCI 1. $((x * y) * (x * z)) * (z * y) = 0$

BCI 2. $(x * (x * y)) * y = 0$

BCI 3. $x * x = 0$

BCI 4. $x * y = 0$ and $y * x = 0 \implies x = y \ \forall \ x , y , z \in$ X.

Definition [1.1.3]

A B-algebra (X, $*$, 0) is a non-empty ser X with a constant 0 and a binary operation $*$ satisfying the following axioms:

B 1. $x * x = 0$

B 2. $x * 0 = x$

B 3. $(x * y) * z = x * (z * (0 * y))$ $\forall$ $x, y, z \in$ X.

Definition [1.1.4]

A β-algebra is a non-empty ser X with a constant 0 and two binary operations + and − satisfying the following axioms:

β 1. $x - x = 0$

β 2. $(0 - x) + x = 0$

β 3. $(x - y) - z = x - (z + y)$ $\forall$ $x, y, z \in$ X.

Example [1.1.5]

Let X = $\{0, 1, 2, 3\}$ be a set with constant 0 and two binary operations + and − defined on X with the cayley's table

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	1	0
3	3	2	0	1

−	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

Then (X, +, − ,0) is a β-algebra.

Definition [1.1.6]

A non-empty subset A of a β-algebra (X, +, − ,0) is called a β-subalgebra of X, if

1. $x + y \in$ A, $\forall$ $x, y \in$ A and

2. $x - y \in$ A, $\forall$ $x, y \in$ A.

Example [1.1.7]

Let X = $\{0, 1, 2, 3\}$ be a β-algebra with constant 0 and two binary operations + and − defined on X as follow

+	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

−	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then (X, +, − ,0) is a β-algebra.

Definition [1.1.8]

Let (X, +, − ,0) and (Y, +, − ,0) be two β-algebra. A mapping f : X→ Y is said to be a β-homomorphism if $f\ (x + y) = f\ (x) + f\ (y)$ and $f\ (x - y) = f\ (x) + f\ (y)$ ∀ x , y ∈ X.

Note:

In a β-homomorphism f(0) =f($0'$)

Definition [1.1.9]

Let X be a set of universal discourse. A fuzzy set μ in X is defined as a function μ: X → [0,1]. For each element x in X, μ(x) is called the membership value of x in X.

Definition [1.1.10]

If μ_1 and μ_2 are two fuzzy sets of X then intersection $\mu_1 \wedge \mu_2$ of μ_1 and μ_2 is defined as $(\mu_1 \wedge \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$.

In general $(\wedge\mu_i)(x) = \inf\{\mu_i(x) / i = 1, 2, 3, ...\}$.

Definition [1.1.11]

If μ_1 and μ_2 are two fuzzy sets of X then union $\mu_1 \vee \mu_2$ of μ_1 and μ_2 is defined as $(\mu_1 \vee \mu_2)(x) = \max\{\mu_1(x), \mu_2(x)\}$.

In general $(\vee\mu_i)(x) = \sup\{\mu_i(x) / i = 1, 2, 3, \ldots\}$.

Definition [1.1.12]

If μ_1 and μ_2 are two fuzzy sets of X then

(i). $\mu_1 \leq \mu_2$, if $\mu_1(x) \leq \mu_2(x)$

(ii). $\mu_1(x) \leq \mu_2(x)\ \forall\ x \in X$ then μ_2 is fuzzy extension of μ_1.

Definition [1.1.13]

Let μ be a fuzzy set of X. Then the complement of μ denoted by μ^c is defined as $\mu^c(x) = 1 - \mu(x)$.

Definition [1.1.14]

Let μ_1 and μ_2 are two fuzzy sets of X_1 and X_2 respectively. Then the direct product(Cartesian product) $\mu_1 \times \mu_2$ of μ_1 and μ_2 be the fuzzy set of $X_1 \times X_2$ defined as $\mu_1 \times \mu_2\ (x_1, x_2) = \min\{\mu_1(x), \mu_2(x)\}\ \forall\ (x_1, x_2) \in X_1 \times X_2$.

Definition [1.1.15]

Let μ be a fuzzy set of X. μ is said to have the supremum property if, for any subset A of X, there exist a $a_0 \in A$ such that $\mu(a_0) = \sup \mu(a)$.

Definition [1.1.16]

Let X be any non-empty set. A L-fuzzy set μ on X is defined as a function $\mu: X \rightarrow L$, where L is a complete lattice with glb 0 and lub 1.

Definition [1.1.17]

Let μ be a fuzzy set (L-fuzzy) in a set X. For $t \in [0,1]\{t \in L\}$, the set $\mu_t = \cup\ (\mu: t)$

$= \{x \in X / \mu(x) \geq t\}$ is called a level subset of μ. [or upper level subset of μ].

Proposition [1.1.18]

If $t_1 \leq t_2$, then $\mu_{t_2} \subseteq \mu_{t_1}$ where μ_{t_2} and μ_{t_1} are any two level subset of μ where μ be a fuzzy set on a set X.

Proof:

Let $x \in \mu_{t_2}$ Then $\mu(x) \geq t_2$.

But $t_2 \leq t_1$

$\therefore \mu(x) \geq t_1$

$\Longrightarrow x \in \mu_{t_1}$

$\mu_{t_2} \subseteq \mu_{t_1}$.

Section: 1.2

Level β-Subalgebrs of Fuzzy β-Subalgebras of β-Algebras

Definition [1.2.1]

Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-subalgebras of X if

1. $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}\ \forall\ x, y \in X$.
2. $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}\ \forall\ x, y \in X$.

Theorem [1.2.2]

A fuzzy set μ of a β-algebra X is a fuzzy β-subalgebra if and only if the level subset μ_t of μ ∀ t ∈ [0,1] is either empty or a β-subalgebras of X.

Proof:

Let μ be a β-subalgebra of X. For any t ∈ [0,1], assume that $\mu_t \neq \emptyset$

For any , $y \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$.

Now $\mu(x + y) \geq \min\{\mu(x), \mu(y)\} \geq t$, hence $x + y \in \mu_t$.

Also $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq t$, hence $x - y \in \mu_t$.

Therefore μ_t is a β-subalgebra of X.

Conversely, choose , y ∈ X such that μ(x) = t_1 and μ(y) = t_2, where t_1, t_2 ∈ [0,1]. Then $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$.

Assume $t_1 \leq t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$, by proposition [1.1.18]. Hence $y \in \mu_{t_1}$.

Since μ_{t_1} is a β-subalgebras of X. We have $(x + y) \in \mu_{t_1}$ and $x - y \in \mu_{t_1}$.

Thus μ($x + y$) ≥ t_1 = min {μ(x), μ(y)} and μ($x - y$) ≥ t_1 = min {μ(x), μ(y)}.

Therefore the fuzzy set μ of a β-algebra X is a fuzzy β-subalgebra of X.

This completes the proof.

Definition [1.2.3]

Let X be a β-algebra and μ be a fuzzy β-subalgebras of X. The β-subalgebras μ_t for t ∈ [0,1] with t≤ μ(0) are called a level β-subalgebras of μ.

Theorem [1.2.4]

Any β-subalgebra of β-algebra X can be realized as a level β-subalgebra for some fuzzy β-subalgebra of X.

Proof:

Let A be a β-subalgebras of a given β-algebra X and let μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

where t ∈ (0,1) is fixed. Then μ_t = A.

To prove : μ is a fuzzy β-subalgebra of X. Now for all , y ∈ X.

Case :(i) If x, y ∈ A, the $x + y$ and $x - y$ ∈ A.

Hence μ(x) = μ(y) =μ($x + y$) = t and μ($x - y$) =t.

Therefore, μ($x + y$) = t = min {t, t} ≥ min {μ(x), μ(y)}.

Also μ($x - y$) = t = min {t, t} ≥ min {μ(x), μ(y)}.

Case :(ii) If $x, y \notin$ A, Then μ(x) = μ(y) = 0.

μ($x + y$) ≥ 0 = min {0,0}= min {μ(x) . μ(y)}.

Similarly, μ($x - y$) ≥ 0 = min {0,0}= min {μ(x). μ(y)}.

Case :(iii) If $x \in$ A and $y \notin$ A, then μ(x) = t and μ(y) = 0, which implies that min{ μ(x), μ(y)}= min {t,0} = 0. But μ($x + y$) ≥ 0 and μ($x - y$) ≥ 0.

Therefore, μ($x + y$) ≥ min {μ(x), μ(y)} and μ($x - y$) ≥ min {μ(x), μ(y)}.

Case :(iv) When $x \notin$ A and $y \in$ A, interchange the roles of x and y in case (iii).

Thus in all cases, μ is a fuzzy β-subalgebra of X..

As a generalization of this theorem we prove the following theorem.

Theorem [1.2.5]

Let X be a β-algebra. Given any sequence $\{A_i\}$ of β-subalgebras of X such that $A_0 \subset A_1 \subset$ .....$\subset A_n$ = X, then there exists fuzzy β-subalgebras of X whose level β-subalgebras of X are exactly the β-subalgebras of the sequence $\{A_i\}$.

Proof:

Consider a set of numbers $t_0 > t_1 >$ .... $> t_n$, where each $t_i \in$ [0,1].

Let μ : X → [0,1] be a fuzzy set defined by μ(A_0) = t_0 and μ($A_i - A_{i-1}$) = t_i, 0 < i ≤ n.

Claim: μ is a fuzzy β-subalgebra of X. Now ∀ x , $y \in$ X.

Case :(i) If x , $y \in A_i - A_{i-1}$, then x , $y \in A_i \Longrightarrow x + y \in A_i$ and $x - y \in A_i$. Since A_i is a β-subalgebra of X.

Also , $y \in A_i - A_{i-1} \Longrightarrow$ μ(x) = t_i= μ(y) ⟹min {μ(x), μ(y)}= t_i.

Now $x + y \in A_i$ and $x - y \in A_i$

$\Longrightarrow x + y$ and $x - y \in A_i - A_{i-1}$ (or) $x + y$ and $x - y \in A_{i-1}$

$\Rightarrow$ μ(x + y) = μ($x - y$) = t_i (or) μ(x + y) = μ($x - y$) ≥ t_i.

Therefore μ($x - y$) ≥ t_i = min {μ(x), μ(y)} and μ($x + y$) ≥ t_i = min {μ(x), μ(y)}.

Case :(ii) For i > j $\Rightarrow t_j > t_i \Rightarrow A_j \subset A_i$.

If $x \in A_i - A_{i-1} \Rightarrow$ μ(x) = t_i and $y \in A_j - A_{j-1}$

$\Rightarrow$ μ(y) =$t_j > t_i \Rightarrow$ min {μ(x), μ(y)} = min { t_i , t_j }= t_i.

And $x \in A_i - A_{i-1} \Rightarrow x \in A_i$ and $y \in A_j - A_{j-1}$

$\Rightarrow y \in A_j \subset A_i \Rightarrow x , y \in A_i$.

Since A_i is a β-subalgebras of X, x + $y \in A_i$ and $x - y \in A_i$.

Therefore, μ($x - y$) ≥ t_i = min {μ(x), μ(y)} and μ(x + y) ≥ t_i = min {μ(x), μ(y)}.

In both cases, μ is a fuzzy β-subalgebra of X.

From the definition of μ, it follows that Im(μ) = {t_0, t_1, ... t_n}.

Hence μ_{t_0} = {$x \in$ X / μ(x) ≥ t_i}, for 0 ≤ i ≤ n are the level β-subalgebras of X.

Thus the sequence {μ_{t_i}} of level β-subalgebras of μ are of the form $\mu_{t_0} \subset \mu_{t_1} \subset$ .... μ_{t_n}= X.

Now μ_{t_0} = {$x \in$ X / μ(x) ≥ t_0}= A_0

To prove: μ_{t_i} = A_i for 0 < i ≤ n.

Clearly, $A_i \subseteq \mu_{t_i}$. If $x \in \mu_{t_i}$

If $x \in \mu_{t_i}$, then μ(x) ≥ t_i which implies μ(x) ∈ {t_1, t_2, ... t_n} and hence $x \in A_0$ or A_1 or .... or A_i. It follows that $x \in A_i$. Therefore μ_{t_i}= A for 0 ≤ i ≤ n.

Hence the level β-subalgebras of X are exactly the β-subalgebras of sequence {A_i }.

Theorem [1.2.6]

Let X be a β-algebra and μ be a fuzzy β-subalgebras of X and let $\mu_{t_1}$and μ_{t_2} be any two level subset of μ with $t_1 < t_2$, then μ_{t_1}= μ_{t_2} if and only if there is no $x \in$ X such that $t_1 \leq$ μ$(x) < t_2$.

Proof:

Assume that μ_{t_1}= μ_{t_2} and $t_1 < t_2$.

Suppose there is atleast one $x \in$ X, such that $t_1 \leq$ μ$(x) < t_2$, then μ$(x) \geq t_1$

$\Rightarrow x \in \mu_{t_1}$and μ$(x) < t_2 \Rightarrow$ μ$(x) \ngeq t_2 \Rightarrow x \notin \mu_{t_2}$ and therefore μ_{t_2} is a proper subset of μ_{t_1}. Which is a contradiction of the hypothesis.

Conversely if no $x \in$ X such that $t_1 \leq$ μ$(x) < t_2$ then we prove μ_{t_1} = μ_{t_2}.

Given $t_1 < t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$. Then $\mu_{t_2} \subseteq \mu_{t_1}$. Let $x \in \mu_{t_1} \Rightarrow$ μ$(x) \geq t_1$ but by the hypothsis no $x \in$ X so that $t_1 \leq$ μ$(x) < t_2 \Rightarrow$ μ$(x) \geq t_2 \Rightarrow x \in \mu_{t_2} \Rightarrow \mu_{t_1} \subset \mu_{t_2}$.

Therefore μ_{t_1} = μ_{t_2}.

Theorem [1.2.7]

Let the β-algebra X is a finite and μ be a fuzzy β-subalgebra of X with Im(μ) = $\{t_0,\ t_1, ...\ t_n\}$ then the family of β-subalgebras { μ_{t_i}, 1 ≤ i ≤ n} will be a the entire level β-subalgebras of μ in X.

Proof:

Given Im(μ) = $\{t_0,\ t_1, ...\ t_n\}$ and assume that $t_1 < t_2 < .... < t_n$.

Let t ∈ [0,1] and t ∉ Im(μ) and if t ≤ t_1 then μ_{t_1}= μ_t or if $t_{i-1} <$ t $< t_i$ then by above theorem μ_t = μ_{t_i} or if t > t_n then μ_t is empty and therefore μ_t is one of the member of the family of β-subalgebras {μ_{t_i}, 1 ≤ i ≤ n}. Hence {μ_{t_i}, 1 ≤ i ≤ n} will be the entire level β-subalgebras of X.

Corollary [1.2.8]

Let the β-algebra X is finite and μ be a fuzzy β-subalgebras of X with Im(μ) is finite, then for any t_i, t_j ∈ Im(μ), $\mu_{t_i} = \mu_{t_j}$ implies $t_i = t_j$.

Proof:

Let Im(μ) = { $t_1, t_2,$ .... t_n} and assume that $t_1 < t_2 <$ .... $< t_n$.

Suppose $t_i \neq t_j$. Now μ(x) ≥ $t_j > t_i$ which implies that $x \in \mu_{t_i}$.

Let x ∈ X such that $t_i <$ μ(x) $< t_j$ then $x \notin \mu_{t_j}$.

Therefore μ_{t_i} is a proper subset of μ_{t_j} which is a contradiction and hence $t_i = t_j$.

Theorem [1.2.9]

Let μ be a fuzzy β-subalgebra of β-algebra X with Im(μ) = {t_i / i ∈ Δ_i}, where Δ_i is any index set. Let τ = {μ_{t_i} / i ∈ Δ_i} be a family of level subalgebras of μ. Then the following are true.

i). There exist a unique t_0 ∈ Im(μ) such that $t_0 > t_i$, ∀ i ∈ Δ_i.

ii). $X_\mu = \mu_{t_0} = \wedge\, \mu_i$, i ∈ Δ_i where X_μ = { x ∈ X / μ(x) = μ(0)}.

iii). X = ∨ μ_i, i ∈ Δ_i.

iv). The level subalgebras in the family τ form an inclusion sequence.

v). If μ contains an infimum on all β-subalgebras of X, then τ contains all its level subalgebras.

Proof:

i). If t_0 = μ(0) and since μ(0) ≥ μ(x) ∀ x ∈ X, $t_0 > t_i$, ∀ i ∈ Δ_i.

Hence the proof.

ii). μ_{t_0} = { x ∈ X / μ(x) ≥ t_0}

= { x ∈ X / μ(x) = t_0} (since t_0 = μ(0) ≥ μ(x) ∀ x ∈ X).

={ x ∈ X / μ(x) = μ(0)}= X_{μ}.

Now ∀ i ∈ Δ_i, $t_i \leq t_0$. This implies that $\mu_{t_0} \subseteq \mu_{t_i}$ ∀ i ∈ Δ_i. Then, μ_{t_0}= ∧ μ_i, i∈ Δ_i.

iii). x ∈ X ⟹ μ(x) ∈ Im(μ) ⟹ μ(x) = t_i, for some i ∈ Δ_i.⟹ $x \in \mu_{t_i}$, for some i ∈ Δ_i

⟹$x \in \bigcup \mu_{t_i}$ ⟹ X ∈ ⋃ μ_{t_i} ⟹ X = ⋃ μ_{t_i}.

iv). Let i, j ∈ Δ_i. Since Δ_i is an indexing set, without loss of generating we can assume that i ≤ j. Then $t_i \leq t_j \Longrightarrow \mu_{t_j} \subseteq \mu_{t_i}$ ∀ i, j ∈ Δ_i.

Since this is true for any pair of i, j ∈ Δ_i, the family of level subalgebras form an inclusion sequence.

v). Let μ_t be a level β-subalgebra of μ where μ be a fuzzy β-subalgebras of X.

If t = t_i for some i ∈ Δ_i, then μ_{t_i} ∈τ. Hence the proof.

If t ≠ t_i, ∀ i ∈ Δ_i, then there is no x ∈ X such that μ(x) = t.

Let A = {x ∈ X / μ(x) > t}. It is clear that 0 ∈ A. Hence A ≠ ∅.

Let x, y ∈ A. Then μ(x) > t and μ(y) > t.

Therefore μ(x+ y) ≥ min {μ(x), μ(y)}≥ t and μ($x - y$) ≥ min {μ(x), μ(y)}≥ t.Which implies x+ y and $x - y$ ∈ A. Therefore A is a β-subalgebras of X.

By hypothesis μ contains the infimum on all β-subalgebras of X and hence on A.

Therefore there exists a y ∈ A such that μ(y) =inf {μ(x) / x ∈ A}.

which implies μ(y) ∈ Im(μ) ⟹ μ(y) = t_i, for i ∈ Δ_i⟹ $t_i \geq$ t.

But no x ∈ X such that t ≤ μ(x) < t_i ⟹ μ_t = μ_{t_i}.

Therefore τ contains all its level β-subalgebras. This completes the proof.

Section: 1.3

Normal Fuzzy β-Subalgebras of β-Algebras

Definition [1.3.1]

A fuzzy β-subalgebra μ of X is said to be a normal if there exists $x \in$ X such that μ(x) = 1.

Example [1.3.2]

Consider the β-algebra (X, $+$, $-$, 0) in Example[1.1.5] Define μ: X→ [0,1] such that

$$\mu(x) = \begin{cases} 1 & x = 0 \\ 0.5 & x = 1 \\ 0 & x = 2, 3 \end{cases}$$

Then μ is a normal fuzzy β-subalgebras of X.

However, in the β-algebra X in Example [1.1.8]

Define μ_1: X → [0,1]

$$\mu(x) = \begin{cases} 0.8 & x = 0 \\ 0.5 & x = 1 \\ 0.3 & x = 2, 3 \end{cases}$$

is not a normal fuzzy β-subalgebra of X.

Lemma [1.3.3]

A fuzzy β-subalgebra μ of X is normal if and only if μ(0) = 1.

Proof:

The lemma follows by definition of normal.

Theorem [1.3.4]

For any fuzzy β-subalgebra μ of X, we can generate a normal fuzzy β-subalgebra of X with contains μ.

Proof:

Let μ be a fuzzy β-subalgebra of X.

Define a fuzzy subset μ^+ of X as $\mu^+(x) = \mu(x) + \mu^c(0)$, $\forall\ x \in$ X.

Let $x, y \in$ X

Then $\mu^+(x + y) = \mu\ (x\ +\ y\ ) + \mu^c(0)$

$\geq \min\ \{\mu(x), \mu(y)\} + \mu^c(0)$

$= \min\ \{\mu(x) + \mu^c(0),\ \ \mu(y) + \mu^c(0)\}$

$= \min\ \{\mu^+(x), \mu^+(y)\}$

Similarly, $\mu^+(x - y\ ) \geq \min\ \{\mu^+(x), \mu^+(y)\}$. Also $\mu^+(0) = \mu(0) + \mu^c(0) = 1$.

Therefore μ^+ is a normal fuzzy β-subalgebra of X.

Clearly $\mu \subset \mu^+$. Thus μ^+ is a normal fuzzy β-subalgebra of X, which contains μ .

We call μ^+ as a normal fuzzy β-subalgebra of X generated by μ.

Corollary [1.3.5]

If μ it self is normal then $\mu = \mu^+$.

Corollary [1.3.6]

If μ is a fuzzy β-subalgebra of X then $(\mu^+)^+ = \mu^+$.

Theorem [1.3.7]

Let μ be a fuzzy β-subalgebra of X. If μ contains a normal fuzzy β-subalgebra of X generated by any other fuzzy β-subalgebra of X then μ is normal.

Proof:

Let γ be a fuzzy β-subalgebra of X. By theorem [1.3.4], γ^+ is normal fuzzy β-subalgebra of X.

Therefore by theorem [1.3.3], $\gamma^+(0) = 1$.

Let μ be a fuzzy β-subalgebra of X , such that $\gamma^+ \subset \mu$. $\Longrightarrow \mu(x) \geq \gamma^+(x)$) $\forall\ x \in$ X.

In particular $\mu(0) \geq \gamma^+(0) = 1$.

Hence μ is normal.

Theorem [1.3.8]

Let μ and γ be a normal fuzzy β-subalgebra of X. If μ ⊂ γ then $X_\mu \subset X_\gamma$.

Proof:

Since μ and γ are normal fuzzy β-subalgebra of X, μ(0) = 1 = γ(0) and $X_\mu \subset X_\gamma$.

Theorem [1.3.9]

Let μ be a fuzzy β-subalgebra of a β-algebra X. Let : [0,μ(0)] → [0,1] be an increasing function. Define a fuzzy set μ^f: X → [0,1] by $\mu^f(x) = f$ (μ(x)),) ∀ x ∈ X. Then

i). μ^f is a fuzzy β-subalgebra of X.

ii). If f (μ(0)) = 1, then μ^f is normal.

iii). If $f(a) \geq a$, a ∈ [0, μ(0)], then μ ⊂ μ^f.

Proof:

The result (i) follows by the definition of μ^f

ii). If f(μ(0)) = 1, then μ^f(0) = 1. Hence μ^f is normal.

iii). Let $f(a) \geq a$ ∀ a ∈ [0, μ(0)]. Then $\mu^f(x) = f(\mu(x)) \geq$ μ(x) ∀ x ∈ X.

Hence μ ⊂ μ^f.

Theorem [1.3.10]

Let X be a β-algebra. Let N = {γ /γ is a normal fuzzy β-subalgebra of X} and μ be the maximal in the poset (N, ⊂). Then μ takes only the values 0 and 1.

Proof:

Let μ ∈ N. Then μ is normal implies μ(0) = 1. Let x ∈ X such that μ(x) ≠ 1. Then it is enough to show that μ(x) = 0, ∀ x ∈ X. suppose there exists a x' ∈ X such that $0 < \mu(x') < 1$. Now define a fuzzy set γ: X ⟶ [0,1] by $\gamma(x) = 0.5(\mu(x) + \mu(x'))$, $\forall\, x \in X$. Then clearly $\gamma(x)$ is well defined. Also,

$$\gamma(x + y) = 0.5(\mu(x + y) + \mu(x'))$$

$$\geq 0.5\,\{\min(\mu(x), \mu(y)) + \mu(x')\}$$

$$= 0.5\,(\min\{\mu(x) + \mu(x'), \mu(y) + \mu(x')\})$$

$$= \min(0.5\,(\mu(x) + \mu(x')), 0.5(\mu(y)) + \mu(x'))\}$$

$$= \min\{\gamma(x), \gamma(y)\}$$

Similarly, $\gamma(x - y) \geq \min\{\gamma(x), \gamma(y)\} \Rightarrow \gamma$ is a fuzzy β-subalgebra of X.

Now,

$$\gamma^{+}(x) = \gamma(x) + \gamma^{c}(0)$$

$$= \gamma(x) + 1 - \gamma(0)$$

$$= 0.5\,(\mu(x) + \mu(x')) + 1 - 0.5\,(\mu(0) + \mu(x')) \text{ (since } \mu \text{ is normal)}$$

$$= 0.5\,(\mu(x) + 1)$$

Therefore $\gamma^{+}(0) = 0.5\,(\mu(x) + 1) = 1$. Hence γ^{+} is normal. Therefore $\gamma^{+} \in N$. Also $\gamma^{+}(x) > \mu(x)$, $\forall\, x \in X$.

This contradicts the fact that μ is maximal. Therefore $\mu(x) = 0$, $\forall\, x \in X$.

This completes the proof.

CHAPTER 2

CHAPTER 2

Fuzzy Translations and Fuzzy Multiplications of

Fuzzy β-Ideals of β-Algebras

Section: 2.1

Fuzzy Translations of Fuzzy β-Ideals of β-Algebras

Definition: 2.1.1

A non-empty subset I of a β-algebra (X,+, −,0) is called a β-ideal of X, if

βI 1. 0 ∈I

βI 2. $x + y$ ∈I, ∀ x,y∈ I, and

βI 3. If $x - y$ and $y \in$ I then $x \in$ I ∀ $x, y \in x$.

Example:2.1.2

Consider the β-algebra (X, +,−,0) in example 1.1.7

The subset I_1= $\{0,1,\}$ is a β-ideal of X.

But I_2= {0,1,3} is not a β-ideal of X,since 1+3 = 2 $\notin I_2$.

Definition: 2.1.3

Let μ be a fuzzy set in a β-algebra X. Then μ is called a fuzzy β-ideal of X if

FβI 1. μ(0)≥ $\mu(x)$ ∀ $x \in$ X

FβI 2. $\mu(x + y) \geq$ min $\{\mu(x),\ \mu(y)\}$ ∀ $x, y \in$ X and

FβI 3. $\mu(x) \geq$ min $\{\mu(x - y), \mu(y)\}$ ∀ $x, y \in$ X.

Example:2.1.4

Consider the β-algebra (X+,−,0) in example 1.1.7.The fuzzy set μ: X→ [0,1] such that

$\mu(x)$ =0.8 if x=0,2

$=0.5$ if $x=1,3$

is a fuzzy β-ideal of x.

Definition 2.1.5

Let μ be a fuzzy set of a β-algebra X and $\alpha \in$ [0.T], where T=1$-$ sup$\{\mu(x)|x\epsilon X\}$.

Then the fuzzy set $\mu_{\alpha=}^{T}$X → [0,1] is called a fuzzy α-translation of μ if

$\mu_\alpha^T(x)= \mu(x) + \alpha \ \forall \ x \in$X.

Example 2.1.6

For the fuzzy set μ of X in example[2.1.4] T= 0.2. Let α = 0.1∈ [0, 0.2].

Then the fuzzy set μ_α^T: X → [0,1] given by

$\mu_\alpha^T(x)$ = 0.9 if x= 0,2

= 0.6 if x= 1,3

is a fuzzy α-translation of μ.

Theorem 2.1.7

If μ be a fuzzy β-ideal of X then the fuzzy α-translation μ_α^T of μ is also a fuzzy β-ideal of X ∀ α∈[0.T].

Proof:

Let α ∈ [0,T] and for any $x \in$ X,

$$\mu_\alpha^T(0) = \mu(0) + \alpha$$

$$\geq \mu(x) + \alpha$$

$$= \mu_\alpha^T(x) \qquad (1)$$

For any $x, y \in$X, $\mu_\alpha^T(x + y) = \mu(x + y) + \alpha$

$$\geq \min\{\mu(x), \mu(y)\} + \alpha$$

$= \min\{\mu(x) + \alpha, \mu(y) + \alpha\}$

$= \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$ (2)

For $x, y \in X$, $\mu_\alpha^T(x) = \mu(x) + \alpha$

$\geq \min\{\mu(x-y), \mu(y)\} + \alpha$

$= \min\{\mu(x-y) + \alpha, \mu(y) + \alpha\}$

$= \min\{\mu_\alpha^T(x-y), \mu_\alpha^T(y)$ (3)

Hence ,by (1),(2),(3) μ_α^T of μ is a fuzzy β-ideal of X.

Remark 2.1.8

In general for any fuzzy set μ of X, the fuzzy α-translation μ_α^T of μ need not be a fuzzy β-ideal of X ∀α ∈[0,T], as shown by the following example.

Consider the β-algebra X in example1.1.7.The fuzzy set μ with μ(0)= 0.08, μ(1)= 0.5, μ(2)= μ(3)=0.6 and the α (=0.1)-translation μ_α^T with $\mu_\alpha^T(0)=0.9$, $\mu_\alpha^T(1)=0.6$, $\mu_\alpha^T(2)= \mu_\alpha^T(3)=0.7$ are not fuzzy β-ideals of X.

Theorem2.1.9

Let μ be a fuzzy set of X such that the fuzzy α-translation μ_α^T of μ is a fuzzy β-ideal of X for some α∈[0,T].Then μ is a fuzzy β-ideal of X.

Proof:

Assume that μ_α^T of μ is a fuzzy β-ideal of X for some α∈ [0,T]

Let $x, y \in X$.

i)$\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x) + \alpha$ which implies $\mu(0) \geq \mu(x)\ \forall\ x \in X$.

ii)$\mu(x+y) + \alpha = \mu_\alpha^T(x+y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$

$= \min\{\mu(x) + \alpha, \mu(y) + \alpha\} = \min\{\mu(x), \mu(y)\} + \alpha$

which implies $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$.

iii)$\mu(x) + \alpha = \mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(x-y), \mu_\alpha^T(y)\}$

= min{μ($x-y$)+α, μ(y) +α}

= min{μ($x-y$),μ(y)} +α

which implies μ(x) ≥ min{μ($x-y$),μ(y)}

Hence μ is a fuzzy β-ideal of X.

The following lemma shows that any fuzzy α-translation of a fuzzy β-ideal of X is a fuzzy β-sub algebra.

Theorem 2.1.10

Let α ∈ [0,T] and μ be a fuzzy β-ideal of X. Then the fuzzy α-translation μ_α^T of μ is a fuzzy β-sub algebra of X.

Proof:

Let α ∈ [0,T]

i. For any x, y ∈ X, $\mu_\alpha^T(x+y)$= μ($x+y$) +α

≥ min{μx),μ(y)} +α

= min{μ(x) +α,μ(y) +α}

= min{$\mu_\alpha^T(x)$, $\mu_\alpha^T(y)$}

ii. For any x, y ∈X, $\mu_\alpha^T(x-y)$ = μ($x-y$) +α

≥ min{μ(x),μ(y)} +α

= min{μ(x) +α, μ(y) +α}

= min{$\mu_\alpha^T(x)$, $\mu_\alpha^T(y)$}

Hence μ_α^T is a fuzzy β-sub algebra of X.

Theorem 2.1.11

For α ∈ [0,T], let the fuzzy α-translation μ_α^T of a fuzzy set μ of X be a fuzzy β-ideal of X. Then every non-empty level subset$(\mu_\alpha^T)_t$={x ∈X /μ(x) ≥ t−α}∀t∈ Im(μ) with t >α is a β-ideal of X.

Proof:

Assume that the fuzzy α-translation μ_α^T is a fuzzy β-ideal of X.

Let t ∈ Im(μ) be such that t > α.

Now $\mu(0) + \alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) + \alpha \geq t \ \forall\ x \in (\mu_\alpha^T)_t$

$\Rightarrow 0 \in (\mu_\alpha^T)_t$ (1)

Let $x, y \in (\mu_\alpha^T)_t \Rightarrow \mu(x) \geq t - \alpha$ and

$\mu(y) \geq t - \alpha \Rightarrow \mu(x) + \alpha \geq t$ and

$\mu(y) + \alpha \geq t \Rightarrow \mu_\alpha^T(x) \geq t$ and $\mu_\alpha^T(y) \geq t$.

Now $(\mu_\alpha^T)(x + y) \geq \min\{(\mu_\alpha^T)(x)(\mu_\alpha^T)(y)\} \geq \min\{t,t\} = t$

Which implies $\mu(x + y) + \alpha \geq t$

$\Rightarrow \mu(x + y) \geq t - \alpha \Rightarrow x + y \in (\mu_\alpha^T)_t$ (2)

Let $x, y \in X$ be such that $x - y$ and $y \in (\mu_\alpha^T)_t$

$\Rightarrow \mu(x - y) \geq t - \alpha$ and $\mu(y) \geq t - \alpha$

$\Rightarrow \mu(x - y) + \alpha \geq t$ and $\mu(y) + \alpha \geq t$

$\Rightarrow (\mu_\alpha^T)_t(x - y) \geq t$ and $((\mu_\alpha^T)_t(y) \geq t$

Hence $\mu(x) + \alpha = \mu_\alpha^T(x) \geq \min\{\mu_\alpha^T(x - y), \mu_\alpha^T(y)\} \geq \min\{t,t\} = t$

Therefore $x \in (\mu_\alpha^T)_t$ (3)

By (1), (2), and (3) $(\mu_\alpha^T)_t$ is a β-ideal of X.

Theorem 2.1.12

Inter section and union of any two fuzzy translations of a fuzzy β-ideal μ of X is also a fuzzy β-ideal of X.

Proof:

Let μ_α^T and $\mu_{\alpha'}^T$ be two fuzzy translation of a fuzzy β-ideal μ of X, where α, α'ε[0,T]. Assume that α ≤ α' By theorem [2.1.7] μ_α^T and $\mu_{\alpha'}^T$ are fuzzy β- ideal of X..

Now $(\mu_\alpha^T \wedge \mu_{\alpha'}^T)(x) = \min\{\mu_\alpha^T(x), \mu_{\alpha'}^T(x)\}$

$= \min\{\mu(x)+\alpha, \mu(x) + \alpha'\}$

$= \mu(x) + \alpha$

$= \mu_\alpha^T(x)$

Also $(\mu_\alpha^T \vee \mu_{\alpha'}^T)(x) = \max\{\mu_\alpha^T(x), \mu_{\alpha'}^T(x)\}$

$= \max\{\mu(x)+\alpha, \mu(x)+\alpha'\}$

$= \mu(x)+ \alpha'$

$= \mu_{\alpha'}^T$

Therefore $\mu_\alpha^T \wedge \mu_{\alpha'}^T$ and $\mu_\alpha^T \vee \mu_{\alpha'}^T$ are fuzzy β-ideals of X.

Theorem 2.1.13

Let μ and γ be two fuzzy β-ideals of X. Let T= min $\{T_\mu, T_\gamma\}$ where $T_\mu = 1 -$ sub $\{\mu(x) \ / x \in X\}$ and $T_\gamma = 1 -$ sup $\{\gamma(x) \ / x \in X\}$. Then the intersection of α- translation of μ and α'- translation of γ for some $\alpha, \alpha' \in [0, T]$ is a fuzzy β-ideal of X. But the union need not be a fuzzy β-ideal of X.

Proof:

Let μ and γ be two fuzzy β-ideals of X. Then by theorem [2.1.7] μ_α^T and $\gamma_{\alpha'}^T$ are fuzzy β-ideal of X. Then $\mu_\alpha^T \wedge \gamma_{\alpha'}^T$ is a fuzzy β-ideal of X.

Theorem 2.1.14

Let f: X → Y be an epimorphism between two β- algebras X and Y. And α ∈ [0,T].Then inverse image of any fuzzy β- ideal μ of Y is same as the α- translation of the inverse image of the fuzzy β-ideal μ.

Proof:

Let f: X →Y be an epimorphism between two β- algebras X and Y. And α ∈ [0,T]. Let μ be a fuzzy β-ideal of Y. Then by theorem[2.1.7] the α- translation μ_α^T is a fuzzy β-ideal of Y. Then $f^{-1}(\mu_\alpha^T)$ is a fuzzy β-ideal of X.

Also $f^{-1}(\mu_\alpha^T)(x) = \mu_\alpha^T(f(x)) = \mu(f(x)) + \alpha$

$= f^{-1}$(μ)(x)+α = $f^{-1}(\mu)^T_\alpha$ (x)

Hence $f^{-1}(\mu^T_\alpha)$ =$f^{-1}(\mu)^T_\alpha$.

Theorem 2.1.15

Let μ and γ be two fuzzy β-ideal of a β- algebra X. Let T= min {T_μ,T_γ}.Where T_μ=1−sup{μ(x)/ x ∈X} and T_γ=1−sup{$\gamma(x)$/ x ∈X} α∈[0,T]. Then the α- translation of Cartesian product μ × γ of μ and γ is a fuzzy β- ideal of X × X.

Proof:

Let μ and γ be two fuzzy β-ideals of a β-algebra X. Let α ∈ [0,T]. Now by theorem [2.1.7] $\mu^T_\alpha$and γ^T_α are fuzzy β- ideal of X. Then $\mu^T_\alpha \times \gamma^T_\alpha$ is a fuzzy β-ideal of X × X.

Also (μ × $\gamma)^T_\alpha$ (x, y) = (μ × γ) (x, y)+ α

= min {μ(x), $\gamma(y)$}+α

= min {μ(x) +α, $\gamma(y)$+α}

= min {$\mu^T_\alpha(x)$, $\gamma^T_\alpha(y)$}

= ($\mu^T_\alpha \times \gamma^T_\alpha$)(x, y) ∀ (x, y)∈ X× X

Hence $(\mu \times \gamma)^T_\alpha$ is a fuzzy β-ideal of X × X.

Definition 2.1.16

Let μ_1 and μ_2 be two fuzzy sets of X such that μ_2 is a fuzzy extension of μ_1. If μ_1 is a fuzzy β-ideal of X implies that μ_2 is called a fuzzy β-ideal of X, then μ_2 is called a fuzzy β-ideal extension of μ_1.

Example 2.1.17

Consider the β-algebra (X, $+, -$, 0) in example [1.1.7].The fuzzy sets μ_1 and μ_2 of X defined as below.

$$\mu_1(x) = \begin{cases} 0.8 \; if \; x = 0 \\ 0.3 \; if \; x = 1,3 \\ 0.5 \; if \; x = 2 \end{cases} \qquad \mu_2(x) = \begin{cases} 0.9 \; if \; x = 0 \\ 0.6 \; if \; x = 1 \\ 0.5 \; if \; x = 2,3 \end{cases}$$

are fuzzy β-ideal of X. clearly μ_2 is a fuzzy β-ideal extension of μ_1. But μ_2 is not a α- translation of μ_1.

Theorem 2.1.18

Intersection of any two fuzzy β-ideal extensions of a fuzzy β-ideal μ of X is a fuzzy β-ideal extension of μ.

Proof:

Let μ_1 and μ_2 be two fuzzy β-ideal extensions of a fuzzy β-ideal μ of X. Then $\mu_1(x) \geq \mu(x)$ and $\mu_2(x) \geq \mu(x)$ $\forall$ $x \in$ X. Now μ is fuzzy β-ideal of X which implies $\mu_1$and μ_2 are fuzzy β-ideal of X. Then, $\mu_1 \wedge \mu_2$ is also a fuzzy β-ideal of X.

Now $(\mu_1 \wedge \mu_2)(x) = \min \{\mu_1(x), \mu_2(x)\}$

$\geq \min \{\mu(x), \mu(x)\}$

$= \mu(x)$

Hence $\mu_1 \wedge \mu_2$ is a fuzzy β-ideal extension of μ.

Remark 2.1.19

For any two fuzzy sets μ_1 and μ_2 of X,

$(\mu_1 \wedge \mu_2)(x) = \min \{\mu_1(x), \mu_2(x)\} \leq \mu_1(x)$ and $\mu_2(x)$.

Hence $\mu_1(x)$ and $\mu_2(x)$ are fuzzy extension of $(\mu_1 \wedge \mu_2)$. But $\mu_1(x)$ and $\mu_2(x)$ need not be fuzzy β-ideal extension of $(\mu_1 \wedge \mu_2)$.

Remark 2.1.20

Union of any two fuzzy β-ideal extensions of fuzzy set of β-algebra need not be a fuzzy β-ideal extension. Consider the β-algebra X in the example [1.1.7].

Let the fuzzy sets μ, μ_1 and μ_2 be defined by

$$\mu(x) = \begin{cases} 0.7 \text{ } if \text{ } x = 0 \\ 0.3 \text{ } if \text{ } x = 1,3 \\ 0.5 \text{ } if \text{ } x = 2 \end{cases} \quad \mu_1(x) = \begin{cases} 0.8 \text{ } if \text{ } x = 0 \\ 0.6 \text{ } if \text{ } x = 1 \\ 0.5 \text{ } if \text{ } x = 2, 3 \end{cases} \quad \mu_2(x) = \begin{cases} 0.9 \text{ } if \text{ } x = 0 \\ 0.4 \text{ } if \text{ } x = 1,3 \\ 0.6 \text{ } if \text{ } x = 2 \end{cases}$$

Clearly μ_1 and μ_2 are fuzzy β-ideal extensions of the fuzzy β-ideal μ.

Now $(\mu_1 \vee \mu_2)(0)$=0.9 $(\mu_1 \vee \mu_2)(1)$ = $(\mu_1 \vee \mu_2)(2)$=0.6 and $(\mu_1 \vee \mu_2)(3)$=0.5.

Since, $(\mu_1 \vee \mu_2)(3) = 0.5$,

$\min\{(\mu_1 \vee \mu_2)(3-2), (\mu_1 \vee \mu_2)(2)\} = 0.6$,

the fuzzy set $\mu_1 \vee \mu_2$ is not a fuzzy β-ideal extension.

Theorem 2.1.21

Let μ be a fuzzy β-ideal of X. For ∀ α ∈[0,T] the fuzzy α- translation μ_α^T is a fuzzy β-ideal extension of μ itself.

Proof:

If μ be a fuzzy β-ideal of X then theorem [2.1.7] the fuzzy α- translation μ_α^T of μ is also a fuzzy β-ideal of X ∀α ∈ [0,T]. Also $\mu_\alpha^T(x) = \mu(x)+\alpha \geq \mu(x)\ \forall\ x \in X$.

Therefore μ_α^T is a fuzzy β-ideal extension of μ.

Theorem 2.1.22

Let μ be a fuzzy β-ideal of X. If $\alpha \geq \alpha'$ with $\alpha, \alpha' \in [0,T]$, then the fuzzy α- translation μ_α^T of μ is a fuzzy β-ideal extension of the fuzzy α'-translation $\mu_{\alpha'}^T$ of μ.

Proof:

Let μ be a fuzzy β-ideal of X. Then by theorem [2.1.7] the fuzzy α- translation μ_α^T and the fuzzy α'- translation $\mu_{\alpha'}^T$ are fuzzy β-ideal of X. Since $\alpha \geq \alpha'$ implies $\mu(x)+\alpha \geq \mu(x) + \alpha'\ \forall\ x \in X$.

Therefore $\mu_\alpha^T(x) \geq \mu_{\alpha'}^T(x) \forall\ x \in X$.

Hence μ_α^T is a fuzzy β-ideal extension of $\mu_{\alpha'}^T$.

Theorem 2.1.23

Let μ be a fuzzy β-ideal of X and α' ∈[0,T]. For every fuzzy β-ideal extension γ of the fuzzy α'- translation $\mu_{\alpha'}^{T}$ of μ, there exist α∈[0,T] such that α ≥ α' and γ is the fuzzy β-ideal extension of the fuzzy α- translation μ_{α}^{T} of μ.

Proof:

Let μ be fuzzy β-ideal of X and α' ∈[0,T]. Then by theorem [2.1.7] the fuzzy α'– translation $\mu_{\alpha'}^{T}$ is a fuzzy β-ideal of X. Let γ be a fuzzy β-ideal extension of $\mu_{\alpha'}^{T}$.

Therefore $\gamma(x) \geq \mu_{\alpha'}^{T}(x)$ ∀ x ∈X.

Then choose α = $\alpha' + \min_{x\in X}\{\gamma(x) - \mu_{\alpha'}^{T}(x)\}$

Clearly α ∈ [0,T] such that α ≥ α'. Then μ_{α}^{T} is a fuzzy α- translation μ and

$\gamma(x) \geq \mu_{\alpha}^{T}(x)$.

Hence γ is also a fuzzy β-ideal extension of the fuzzy μ_{α}^{T}.

Section 2.2

Fuzzy Multiplications of Fuzzy β- ideals of β- Algebras

Definition 2.2.1

Let μ be a fuzzy set of a β- algebra X and γ ∈[0,1]. Then the fuzzy set μ_{γ}^{M}:X → [*0,1*] is called a fuzzy γ- multiplication of μ if μ_{γ}^{M}= μ(x).γ ∀ x ∈X.

Example 2.2.2

For the fuzzy β- ideal μ of X in example [2.1.4] the fuzzy set μ_{γ}^{M}: X→[0,1] such that

$$\mu_{\gamma}^{M}(x) = \begin{cases} 0.08 \ if \ x = 0,2 \\ 0.05 \ if \ x = 1,3 \end{cases}$$

Where γ= 0.1 ∈[0,1] is a fuzzy γ- multiplication of μ.

Lemma 2.2.3

If a fuzzy set μ: X→ [0,1] is a constant, then μ is a fuzzy β- ideal of X.

Proof:

If μ(x) is constant for all x ∈X, then μ is a fuzzy β- ideal of X.

Corollary 2.2.4

Let μ be a fuzzy set of X. If γ=0, then the fuzzy γ- multiplication μ_γ^M of μ is a fuzzy β- ideal of X.

Proof:

If γ= 0 then $\mu_\gamma^M(x)$ = μ(x) . γ = 0 for all x ∈X.

⟹μ_γ^M is a fuzzy β- ideal of X.

Corollary 2.2.5

Let μ be a fuzzy set of X. If $\gamma = \frac{1}{\mu(x)}$, ($\mu(x)$ >0) ∀ x ∈X, then the fuzzy γ- multiplication μ_γ^M of μ is a fuzzy β- ideal of X.

Proof:

Now $\mu_\gamma^M(x) = \mu(x).\gamma = \frac{\mu(x)}{\mu(x)}$ = 1∀ x ∈X.

Then μ_γ^M is a fuzzy β- ideal of X.

Theorem 2.2.6

If μ be a fuzzy β-ideal of X then the fuzzy γ- multiplication $\mu_\gamma^M$of μ is also a fuzzy β- ideal of X ∀ γ ∈[0,1]

Proof:

Let γ ∈[0,1] and for any x ∈X,

$$\mu_\gamma^M(0) = \mu(0).\gamma \geq \mu(x).\gamma = \mu_\gamma^M(x). \quad (1)$$

For any x, y ∈X, $\mu_\gamma^M(x+y)$= μ$(x+y).\gamma$

$\geq$ min {μ(x), μ(y)}. γ

= min {μ(x). γ, μ(y). γ}

= min {$\mu_\gamma^M(x)$, $\mu_\gamma^M(y)$} (2)

For any x, y ∈X, $\mu_\gamma^M(x)$= μ(x). γ

$\geq$ min {μ($x - y$), μ(y)}. γ

= min {μ($x - y$). γ, μ(y). γ}

= min {$\mu_\gamma^M(x - y)$, $\mu_\gamma^M(y)$}. (3)

Hence by (1), (2) and (3) μ_γ^M of μ is a fuzzy β- ideal of X.

Theorem 2.2.7

Let μ be fuzzy set of X such that the fuzzy γ- multiplication μ_γ^M of μ is a fuzzy β- ideal of X for some $\gamma \in [0,1]$ then μ is a fuzzy β- ideal of X.

Proof:

Assume that the μ_γ^M of μ is a fuzzy β- ideal of X for some $\gamma \in [0,1]$

Let $x, y \in$ X

μ(0). γ = μ_γ^M (0) $\geq$ $\mu_\gamma^M(x)$= μ(x). γ

which implies μ(0) $\geq$ μ(x) $\forall$ $x \in$ X. (1)

μ($x + y$).γ = μ_γ^M ($x + y$)

$\geq$ min {$\mu_\gamma^M(x)$, $\mu_\gamma^M(y)$}

= min {$\mu(x).\gamma, \mu(y).\gamma$}

= min {μ(x), μ(y)}. γ

which implies μ($x + y$) $\geq$ min {μ(x), μ(y)} (2)

μ(x). γ = $\mu_\gamma^M(x)$ $\geq$ min {$\mu_\gamma^M(x - y)$, $\mu_\gamma^M(y)$}

$= \min \{\mu(x-y).\gamma, \mu(y).\gamma\}$

$= \min \{\mu(x-y), \mu(y)\}.\gamma$

Which implies $\mu(x) \geq \min \{\mu(x-y), \mu(y)\}$ (3)

Hence by (1), (2) and (3) μ is a fuzzy β- ideal of X.

Theorem 2.2.8

Let $\gamma \in [0,1]$ and μ be a fuzzy β- ideal of X. Then the fuzzy γ- multiplication μ_γ^M of μ is a fuzzy β- subalgebra of X.

Proof:

Let $\gamma \in [0,1]$

For any $x, y \in X$, $\mu_\gamma^M(x+y) = \mu(x+y).\gamma$

$\geq \min \{\mu(x), \mu(y)\}.\gamma$

$= \min \{\mu(x).\gamma, \mu(y).\gamma\}$

$= \min \{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$ (1)

For any $x, y \in X$, $\mu_\gamma^M(x-y) = \mu(x-y).\gamma$

$\geq \min \{\mu(x), \mu(y)\}.\gamma$

$= \min \{\mu(x).\gamma, \mu(y).\gamma\}$

$= \min \{\mu_\gamma^M(x), \mu_\gamma^M(y)\}$ (2)

Hence by (1) and (2) μ_γ^M is a fuzzy β- subalgebra of X.

Theorem 2.2.9

Let μ be a fuzzy set of X, α ∈ [0,T] and γ ∈ [0,1]. If μ_γ^M is a fuzzy β- ideal of X, then the fuzzy α-translation μ_α^T is a fuzzy β- ideal extension of μ_γ^M

Proof:

Let α ∈ [0,T], γ ∈ (0,1] and μ_γ^M is a fuzzy β- ideal of X. Then by theorem [2.2.7] the fuzzy set μ is a fuzzy β- ideal of X. And by theorem [2.1.7] the α- translation μ_α^T is a fuzzy β- ideal of X.

$$\text{Now } \mu_\alpha^T(x) = \mu(x)+\alpha \geq \mu(x)$$

$$\geq \mu(x).\gamma$$

$$= \mu_\gamma^M$$

Hence μ_α^T is a fuzzy β- ideal extension of μ_γ^M.

Remark 2.2.10

If γ=0 then by theorem [2.2.4] the fuzzy γ- multiplication $\mu_\gamma^M$of μ is fuzzy β- ideal of X. also μ_α^T is a fuzzy extension of μ_γ^M but not a fuzzy β- ideal extension. Since μ_α^T need not be fuzzy β- ideal of X.

CHAPTER 3

CHAPTER 3

L-Fuzzy β-Subalgebras, L-Fuzzy β-Ideals and L-Fuzzy T-Ideals of β-algebras

Section: 3.1

L-Fuzzy β-Subalgebras of β-Algebras

Definition: 3.1.1

Let μ be a L-fuzzy set in a β–algebra X. Then μ is called an L-fuzzy β-subalgebra of X if

i) $\mu(x+y) \geq \mu(x) \wedge \mu(y)\ \forall\ x, y \in X$.

ii) $\mu(x-y) \geq \mu(x) \wedge \mu(y)\ \forall\ x, y \in X$.

Example: 3.1.2

Consider the β–algebra(X,+ , −,0) in example 1.1,5.

Define μ: X→L such that

$$\mu(x)=\begin{cases} 1\ if\ x=0 \\ s\ if\ x=1 \\ 0\ if\ x=2,3 \end{cases}$$

where 0 ≤ s ≤1. Then μ is an L-fuzzy β-algebra in X.

Example: 3.1.3

Consider the β-algebra(X,+,−,0) in example 1.1,5.

Define μ: X→L such that

$$\mu(x)=\begin{cases} t_3\ if\ x=0 \\ t_2\ if\ x=1,2 \\ t_1\ if\ x=3 \end{cases}$$

where $0 \leq t_1 \leq t_2 \leq t_3 \leq 1$.Then μ is not an L-fuzzy β–subalgebra in X. For, $\mu(0-1)=\mu(3)=t_1<t_2=t_3 \wedge t_2=\mu(0) \wedge \mu(1)$, for 0,1 ∈ X.

Lemma: 3.1.4

If μ_1 and μ_2 be two L-fuzzy β–subalgebras of X then $\mu_1 \wedge \mu_2$ is also an L-fuzzy β–subalgebra of X.

Proof:

For , y ∈ X

$(\mu_1 \wedge \mu_2)(x+y) = \mu_1(x+y) \wedge \mu_2(x+y)$

$\geq (\mu_1(x) \wedge \mu_1(y)) \wedge (\mu_2(x) \wedge \mu_2(y))$

$\geq (\mu_1(x) \wedge \mu_2(x)) \wedge (\mu_1(y) \wedge \mu_2(y)$

$= (\mu_1 \wedge \mu_2)(x) \wedge (\mu_1 \wedge \mu_2)(y)$

Similarly for x , y ∈ X,

$(\mu_1 \wedge \mu_2)(x-y) = \mu_1(x-y) \wedge \mu_2(x-y)$

$\geq (\mu_1(x) \wedge \mu_1(y)) \wedge (\mu_2(x) \wedge \mu_2(y))$

$\geq (\mu_1(x) \wedge \mu_2(x)) \wedge (\mu_1(y) \wedge \mu_2(y))$

$= (\mu_1 \wedge \mu_2)(x) \wedge (\mu_1 \wedge \mu_2)(y)$

Therefore, $\mu_1 \wedge \mu_2$ is an L-fuzzy β–subalgebras of X.

Corollary: 3.1.5

If $\{\mu_i$/ i=1,2,3….} be a family of L-fuzzy β–subalgebras of X, then $(\wedge \mu_i)$ is also an L-fuzzy β–subalgebras of X.

Lemma: 3.1.6

Let X be a β-algebra of X and μ be an L-fuzzy β–subalgebras of X. Then

1. $\mu(x) \leq \mu(0)$, $\forall x \in X$

2. $\mu(x) \leq \mu(x^*)$, $\forall x \in X$,where $x^*=0-x$

Proof:

For any x ∈X ,μ(0) = μ($x - x$) ≥ μ(x) ∧ μ(x) = μ(x) and

μ(x^*) = μ($0 - x$) ≥μ(0) ∧ μ(x) = μ(x)

and μ(0) = μ($x^* - x^*$) ≥ μ(x^*) ∧ μ(x^*) = μ(x^*)

Hence μ(x) ≤ μ(x^*) ≤ μ(0).

Theorem: 3.1.7

Let μ be an L-fuzzy β–subalgebra of β–algebra X.

Then the set X_μ={x ∈ X / μ(x)=μ(0)} is a β–subalgebra of X.

Proof:

Let $x, y \in X_\mu$.Then by definition μ(x)= (0) , μ(y) = μ(0).

Now

μ($x + y$) ≥ μ(x) ∧ μ(y) = μ(0) ∧ μ(0)= μ(0) (1)

Since x, y ∈ X and X is a of β–algebra , $x + y$ ∈ X. Therefore, by lemma [3.1.6]

μ(0) ≥ μ($x + y$) (2)

Combining equations (1) and (2), we get μ($x + y$) = μ(0)

This shows that $x + y \in X_\mu$.

μ($x - y$) ≥ μ(x) ∧ μ(y) =μ(0) ∧ μ(0)= μ(0) (3)

Since x, y ∈ X and X is a of β–algebra , $x - y$ ∈ X. Therefore by lemma [3.1.6]

μ(0) ≥ μ($x - y$) (4)

Combining equations (3) and (4) we get, μ($x - y$) = μ(0)

This shows that $x + y \in X_\mu$.

Hence $x + y, x - y \in X_\mu$. Therefore X_μ is β–algebra of X.

Lemma: 3.1.8

Let μ and γ be an L-fuzzy β–subalgebras of β–algebrsa of X. If $\mu \subset \gamma$ and μ(0) = γ(0) , then $X_\mu \subset X_\gamma$.

Proof:

Assume thatμ⊂ γ and μ(0) = γ(0)

If $x \in X_\mu$ then $\mu(x) = \mu(0)$.

Since μ⊂ γ , $\mu(x) \leq \gamma(x)\ \forall x \in X$.

Now

$$\gamma(x) \geq \mu(x) = \mu(0) = \gamma(0) \qquad (5)$$

Now γ is an L-fuzzy β–subalgebra of β–algebra of X.

Therefore by lemma [3.1.6]

$$\gamma(0) \geq \gamma(x)\ \forall x \in X. \qquad (6)$$

From equations (5) and (6) we obtain γ(0) ≥ γ(x) ≥ γ(0) , this proves that

$\gamma(x) = \gamma(0)\ \forall x \in X$.

Therefore $x \in X_\gamma$.

Theorem: 3.1.9

Let μ be an L-fuzzy β–subalgebra of β–algebra of X and let f: $[0, \mu(0)] \rightarrow [0,1]$ be an increasing function. Define an L-fuzzyset μ^f: X → L by $\mu^f(x) = f(\mu(x))$ $\forall\, x \in X$. Then μ^f is an L-fuzzy β–subalgebra of X.

Proof:

Let $x, y \in X$. Then

$\mu^f(x + y) = f(\mu(x + y))$

$\geq f(\mu(x) \wedge \mu(y))$

$= f(\mu(x)) \wedge f(\mu(y))$

$= \mu^f(x) \wedge \mu^f(y)$.

Therefore ,$\mu^f(x+y) \geq \mu^f(x) \wedge \mu^f(y)$.

$\mu^f(x-y) = f(\mu(x-y))$

$\geq f(\mu(x) \wedge \mu(y))$

$= f(\mu(x)) \wedge f(\mu(y))$

$= \mu^f(x) \wedge \mu^f(y)$.

Therefore ,$\mu^f(x-y) \geq \mu^f(x) \wedge \mu^f(y)$. Hence μ^f is an L-fuzzy β–subalgebra of X.

Theorem: 3.1.10

If A is a β–subalgebra of X , then the characteristic function χ_A is an L-fuzzy β–subalgebra of X.

Proof:

Let $x, y \in X$

Case (i): Let $x, y \in X$

Since A is a β–subalgebraof X , $x+y$, $x-y \in A$. Hence $\chi_A(x)$ =1 , $\chi_A(y)$ = 1 and therefore $\chi_A(x+y)$=1 and χ_A $(x-y)$=1.

Then,

$\chi_A(x+y) \geq \chi_A(x) \wedge \chi_A(\text{y})$, $\forall\, x, y \in X$.

and $\chi_A(x-y) \geq \chi_A(x) \wedge \chi_A(y)$, $\forall\, x, y \in X$.

Thus, χ_A is an L-fuzzy β–subalgebra of X.

Case (ii): Assume that both $x, y \notin$ A.Then $\chi_A(x)$ =0 = $\chi_A(y)$.Then for $\forall\, x, y \in X$.

$\chi_A(x+y) \geq 0 = 0 \wedge 0 = \chi_A(x) \wedge \chi_A(y)$, $\forall\, x, y \in X$.

and $\chi_A(x-y) \geq 0 = 0 \wedge 0 = \chi_A(x) \wedge \chi_A(y)$, $\forall\, x, y \in X$.

χ_A is an L-fuzzy β–subalgebraof X.

Case(iii): Assume that $x \in$ A and $y \notin$ A. Then $\chi_A(x)$ =1 , $\chi_A(y)$ = 0.

Then

$\chi_A(x+y) \geq 1 = 0 \wedge 1 = 1 \wedge 0 = \chi_A(x) \wedge \chi_A(y), \forall\, x, y \in X.$

and $\chi_A(x-y) \geq 1 = 0 \wedge 1 = 1 \wedge 0 = \chi_A(x) \wedge \chi_A(y), \forall\, x, y \in X.$

χ_A is an L-fuzzy β–subalgebra of X.

Case(iv): Interchanging he roles of x and y in case (iii) we can prove that χ_A is an L-fuzzy β–subalgebra of X when $x \notin$ A and $y \in$ A.

Thus the theorem is completely proved. The converse of the above theorem is also true.

Corollary: 3.1.11

Let A be any subset of a β–algebras of X. If any characteristic function χ_A of A is an L-fuzzy β–subalgebraof X, the A is a β–subalgebraof X.

Section 3.2

L-Fuzzy Level β-Subalgebras of β-Algebras

Definition: 3.2.1

Let X be a β–algebra and μ be an L-fuzzy β–subalgebra of X. The β–subalgebra of μ_t ,for t∈ [0 ,1] with t ≤ μ(0) , are called a level of β–algebra of μ.

Theorem: 3.2.2

An L-fuzzy set μ of a β-algebra of X is an L-fuzzy β–subalgebra iff the level subset $\mu_t$of μ,∀t ∈ [0 ,1] is either empty or a β–subalgebra of X.

Proof:

Let μ of a β–algebra of X. For any t ∈ [0 ,1], assume that $\mu_t \neq \emptyset$.

For any $x, y \in \mu_t$, μ$(x+y) \geq$ μ$(x) \wedge$ μ$(y) \geq$ t. Hence $x+y \in \mu_t$.

Also,$\mu(x-y) \geq \mu(x) \wedge \mu(y) \geq t$. hence $x-y \in \mu_t$. Therefore μ_t is an β–subalgebra of X.

Now choose $x, y \in X$, such that $\mu(x)=t_1$ and $\mu(y)=t_2$, $t_1, t_2 \in [0,1]$.

Then $x \in \mu_{t_1}, y \in \mu_{t_2}$.

If $t_1 \leq t_2$, then $\mu_{t_1} \subseteq \mu_{t_2} \Rightarrow y \in \mu_{t_1}$.

Since μ_{t_1} is a β–subalgebra of X , we have $x+y \in \mu_{t_1}$ and $x-y \in \mu_{t_1}$.

Thus $\mu(x+y) \geq t_1 = \mu(x) \wedge \mu(y)$ and $\mu(x-y) \geq t_1 = \mu(x) \wedge \mu(y)$ proving that L-Fuzzy set μ of X is an L-Fuzzy β–subalgebra of X.

Theorem: 3.2.3

Any β–subalgebra of a β–algebra of X can be realized as a level β–subalgebra for some L-Fuzzy β–subalgebra of X.

Proof:

Let A be a β–subalgebra of X and let μ be an L-Fuzzy set in X defined by

$$\mu(x)= \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad t \in (0,1) \text{ is fixed.}$$

Then μ_t=A . Let $x, y \in X$

Case(i):

If $x, y \in A$ then $x+y$ and $x-y \in A$,

Hence $\mu(x) = \mu(y) = \mu(x+y)$ and $\mu(x-y) = t$.

Therefore

$\mu(x+y) = t = t \wedge t \geq \mu(x) \wedge \mu(y)$ and

$\mu(x-y) = t = t \wedge t \geq \mu(x) \wedge \mu(y)$.

Case(ii):

If $x, y \in A$ then $\mu(x) = \mu(y) = 0$.

Then $\mu(x + y) \geq 0 = 0 \wedge 0 = \mu(x) \wedge \mu(y)$ and

$\mu(x - y) \geq 0 = 0 \wedge 0 = \mu(x) \wedge \mu(y)$

Case(iii):

If $x \in A$ and $y \notin A$, then $\mu(x)$= t and $\mu(y)$=0

This implies that $\mu(x) \wedge \mu(y) = t \wedge 0 = 0$

But $\mu(x + y) \geq 0$ and therefore $\mu(x + y) \geq \mu(x) \wedge \mu(y)$,

Also $\mu(x - y) \geq 0$ and therefore $\mu(x - y) \geq \mu(x) \wedge \mu(y)$.

Case(iv):

If $x \notin A$ and $y \in A$, we can interchange the roles of x and y in case(iii).Thus in all cases, μ is an a β–subalgebras of X.

As a generalization of this theorem we prove the following theorem.

Theorem: 3.2.4

Let X be a β–algebra. Given any sequence $\{A_i\}$ of β–subalgebra of X such that $A_0 \subset A_1 \subset A_2 \subset \ldots.\ A_n = X$, then there exists an L-fuzzy β–subalgebra $\{\mu_{t_i}\}$ of X whose level β–subalgebras are exactly the β–subalgebras of $\{A_i\}$ of X.

Theorem: 3.2.5

Let X be a β–algebra and μ be a L-Fuzzy β–subalgebra of X and let μ_{t_1} and μ_{t_2} be any two level subset of μ with $t_1 \leq t_2$, then $\mu_{t_1} = \mu_{t_2}$. If and only if there is no $x \in X$ such that $t_1 \leq \mu(x) \leq t_2$.

Proof:

Assume that $\mu_{t_1} = \mu_{t_2}$ and $t_1 \leq t_2$.

suppose that there exists atleast one $x \in X$ such that $t_1 \leq \mu(x) \leq t_2$ and then

$\mu(x) \geq t_1 \Longrightarrow x \in \mu_{t_1}$.

This implies $x \notin \mu_{t_2}$ and therefore μ_{t_2} is a proper subset of μ_{t_1} a contradiction.

Conversely, assume that no $x \in X$ such that $t_1 \leq \mu(x) \leq t_2$.

Let $x \in \mu_{t_1} \Rightarrow \mu(x) \geq t_1$.

But by the hypothesis no $x \in X$ exists so that $t_1 \leq \mu(x) \leq t_2$.

Hence $\mu(x) \geq t_2$

$\Rightarrow x \in \mu_{t_1} \Rightarrow \mu_{t_1} \subset \mu_{t_2}$.

Therefore $\mu_{t_1} = \mu_{t_2}$.

Theorem: 3.2.6

Let the β–algebra X be finite and μ be a L-Fuzzy β–subalgebra of X with Im(μ)=$\{t_1, t_2 \ldots . . t_n\}$ then the family of β–subalgebras $\{\mu_{t_i}, 0 \leq i \leq n\}$ will be the entire level β–subalgebras of μ∈ X.

Proof:

Given Im(μ) = $\{t_1, t_2 ... t_n\}$ and assume that $t_1 \leq t_2 \leq t_3 \ldots \leq t_n$.

Let t ∈ [0,1] and t ∉ Im(μ) and if $t \leq t_1$ then $\mu_{t_1} = \mu_t$ or if $t_{i-1} \leq t \leq t_i$,then by theorem [3.1.15] $\mu_t = \mu_{t_i}$(or) if $t \geq t_n$ then μ_t is empty and therefore μ_t is one of the member of the family of β–subalgebras $\{\mu_{t_i}, 1 \leq i \leq n\}$, proving that $\{\mu_{t_i}, 1 \leq i \leq n\}$ will be the entire level of β–subalgebras of X.

Corollary [3.2.7]

Let the β–algebra X be finite and μ be a L-Fuzzy β–subalgebra of X with Im(μ) is finite ,then for any t_i , t_j ∈ Im(μ), $\mu_{t_i} = \mu_{t_j}$ and $t_i = t_j$.

Proof:

Let Im(μ) = $\{t_1, t_2 \ldots \ldots \ldots . . t_n\}$ and assume $t_1 \leq t_2 \leq t_3 \leq \ldots \ldots . . \leq t_n$.

Suppose that $t_i \neq t_j$. Now $\mu(x) \geq t_j \geq t_i$ which implies that $x \in \mu_{t_i}$.

Let $x \in X$ such that $t_i \leq \mu(x) \leq t_j$ then $x \notin \mu_{t_j}$.

Therefore μ_{t_i} is a proper subset of μ_{t_j} , a contradiction.

Theorem: 3.2.8

Let μ be a L-Fuzzy β–subalgebra of β–algebra X with $Im(\mu)=\{t_i / i\in \Delta_i\}$, where Δ_i is an index set. Let $\tau= \{\mu_{t_i}/i\in \Delta_i\}$ be a family level β–subalgebras of μ.Then the following are true.

1. There exists a unique $t_0 \in Im(\mu)$ such that $t_0 \geq t_i \forall\, i\in \Delta_i$.
2. $X_\mu = \mu_{t_0}= \wedge\, \mu_i\ ,i\in \Delta_i$.
3. $X=\vee\mu_i\ ,i\in \Delta_i$.
4. The level subalgebras in the family τ- from an inclusion sequence.
5. If μ contains an infimum on all β-subalgebra of X , then τ- contains all its level β-subalgebras of X

Proof:

Proof follows by the above discussions.

Section: 3.3

L-Fuzzy β-Ideals of β-Algebras

Definition: 3.3.1

Let μ be a L-fuzzy set in a β–algebra X. Then μ is called an L-fuzzy β-ideal of X if

1.$\mu(0) \geq \mu(x)\ \forall x \in X$.

2. $\mu(x + y) \geq \mu(x) \wedge \mu(y)\ \forall\ x, y \in X$ and

3. $\mu(x) \geq \mu(x - y) \wedge \mu(y)\ \forall\ x, y \in X$.

Example: 3.3.2

In example [1.1.5] of β–algebra X ,define the fuzzy set μ_1: X→L such that

$$\mu_1(x) = \begin{cases} t_3 \ if\ x = 0 \\ t_2\ if\ x = 1 \\ t_1\ if\ x = 2,3 \end{cases}$$

where $0 \le t_1 \le t_2 \le t_3 \le 1, t_1, t_2, t_3 \in L$. Then μ_1 is an L-fuzzy β–ideal of X.

The fuzzy set $\mu_2: X \rightarrow L$ such that

$$\mu_2(x) = \begin{cases} t_3 \; if \; x = 0 \\ t_2 \; if \; x = 1,3 \\ t_1 \; if \; x = 2 \end{cases}$$

where $0 \le t_1 \le t_2 \le t_3 \le 1, t_1, t_2, t_3 \in L$ is also an L-fuzzy β–ideal of X.

Lemma: 3.3.3

Intersection of two L-fuzzy β–ideals of a β–algebra is again a L-fuzzy β–ideal.

Proof:

Let $\mu_1$and $\mu_2$be two L-fuzzy β–ideals of a β–algebraof X.

For $x, y \in X$,

$(\mu_1 \wedge \mu_2)(0) = \mu_1(0) \wedge \mu_2(0)$

$\ge (\mu_1(x) \wedge \mu_2(x))$

$\ge (\mu_1 \wedge \mu_2)(x)$. (1)

$(\mu_1 \wedge \mu_2)(x + y) = \mu_1(x + y) \wedge \mu_2(x + y)$

$\ge (\mu_1(x) \wedge \mu_1(y)) \wedge (\mu_2(x) \wedge \mu_2(y))$

$\ge (\mu_1(x) \wedge \mu_2(x)) \wedge (\mu_1(y) \wedge \mu_2(y))$

$= (\mu_1 \wedge \mu_2)(x) \wedge (\mu_1 \wedge \mu_2)(y)$ (2)

$(\mu_1 \wedge \mu_2)(x) = \mu_1(x) \wedge \mu_2(x)$

$= (\mu_1(x - y) \wedge \mu_1(y)) \wedge (\mu_2(x - y) \wedge \mu_2(y))$

$= (\mu_1(x - y) \wedge \mu_2(x - y)) \wedge (\mu_1(y) \wedge \mu_2(y))$

$= (\mu_1 \wedge \mu_2)(x - y)) \wedge (\mu_1 \wedge \mu_2)(y)$ (3)

Hence $\mu_1 \wedge \mu_2$be two L-fuzzy β–ideals of X.

The above result can be generalized as

Corollary: 3.3.4

The intersection of any collection of L-fuzzy β–ideals of a β–algebra is again a L-fuzzy β–ideal.

Remark:

Union of two L-fuzzy β–ideals of X need not be a L-fuzzy β–ideals of X . In example [3.3.2] the fuzzy sets μ_1 and μ_2 are L-fuzzy β–ideals of X. But $\mu_1 \vee \mu_2$ with $(\mu_1 \vee \mu_2)(0)=t_3$, $(\mu_1 \vee \mu_2)(1)=(\mu_1 \vee \mu_2)(2)=t_2$, $(\mu_1 \vee \mu_2)(3)=t_1$, is not an L-fuzzy β–ideals of X.

Theorem: 3.3.5

Let μ be a L-Fuzzy β–ideal of β–algebra X. If $x \leq z + y$ then

$\mu(x) \geq \mu(z) \wedge \mu(y)$

Proof:

$\forall\, x, y, z \in$ X.

$\mu(x) \geq \mu(x - y) \wedge \mu(y)$

$\geq ((\mu(x - y) - z)) \wedge \mu(z)) \wedge \mu(y)$

$= (\mu(x - (z + y)) \wedge \mu(z)) \wedge \mu(y)$

$= (\mu(0) \wedge \mu(z)) \wedge \mu(y)$

$= \mu(z) \wedge \mu(y)$

Lemma: 3.3.6

Let μ be an L-fuzzy β–ideal of a β–algebra X. If $x \leq y$ then $\mu(x) \geq \mu(y)$.

Proof:

For $x, y \in$ X , $x \leq y \Rightarrow x - y = 0$ then

$\mu(x) \geq \mu(x - y) \wedge \mu(y)$

$= \mu(0) \wedge \mu(y)$

Hence the theorem.

Theorem[3.3.7]

Let A is a subset of X. Define an L-Fuzzy set μ: X→L such that

$$\mu(x)=\begin{cases} t_0 & \text{if } x \in A \\ t_1 & \text{if } x \notin A \end{cases}$$

where $t_0, t_1 \in L$ with $t_0 > t_1$.Then μ is an L-fuzzy β–ideal of a β–algebra X if and only if A is a β–ideal of X .

Proof:

Assume that μ is an L-fuzzy β–ideal of X.

If $x \in A$, $\mu(0) \geq t_0$ and if $x \notin A$, $\mu(0) \geq t_1$

Since $t_0 > t_1$, $\mu(0) \geq t_0 > t_1$

$\Rightarrow \mu(0) = t_0$

$\Rightarrow 0 \in A$ (1)

For $x, y \in A$, $\Rightarrow \mu(x) = t_0 = \mu(y)$

Now $\mu(x + y) \geq \mu(x) \wedge \mu(y)$

$= t_0 \wedge t_0$

$= t_0$

Therefore $\mu(x + y) = t_0$

$\Rightarrow x + y \in A$ (2)

For $x, y \in X$,

If $x - y \in A$ and $y \in A$ then

Now $\mu(x) \geq \mu(x - y) \wedge \mu(y)$

$= t_0 \wedge t_0 = t_0$

$\Rightarrow \mu(x) = t_0$

$\Rightarrow x \in A$ (3)

Hence by (1),(2) and (3) A is β–ideal of X.

Conversely , suppose A is a β–ideal of X.

Now $0 \in A \Rightarrow \mu(0) = t_0$

Also $\forall x \in X$, $Im(\mu) = t_0, t_1$ and $t_0 > t_1$.

$\Rightarrow \mu(0) \geq \mu(x)$ (4)

For $x, y \in A, \Rightarrow x + y \in A$

$\Rightarrow \mu(x) = \mu(y) = \mu(x + y) = t_0$

$= t_0 \wedge t_0$

$= \mu(x) \wedge \mu(y)$

Hence $\mu(x + y) \geq \mu(x) \wedge \mu(y)$ (5)

For $x \in X$, If $x - y \in A$ and $y \in A \Rightarrow x \in A$

$\Rightarrow \mu(x) = t_0 = t_0 \wedge t_0$

$= \mu(x - y) \wedge \mu(y)$

And for some $x \in X$ if $x - y \notin A$ and $y \notin A$

$\Rightarrow x \in A$ (or) $x \notin A$

$\Rightarrow \mu(x) = t_0 (or) t_1$

$\geq t_1 = t_1 \wedge t_1 = \mu(x - y) \wedge \mu(y)$.

Hence $\mu(x) \geq \mu(x - y) \wedge \mu(y)$ (6)

By (4),(5) and (6) μ is an L-fuzzy β–ideal of X.

Result:

For any non-empty set A in a β–algebra X , the characteristic function χ_A of A is an L-fuzzy β–ideal of X.

Theorem [3.3.8]

An L-fuzzy set μ is a an L-fuzzy β-ideal of X if and only if the non empty level subset μ_t is a β-ideal of X, ∀ t∈ L.

Proof:

Assume that μ is an L-fuzzy β–ideal of X.

Now $\mu(0) \geq \mu(x)\ \forall\ x \in X$

$\Longrightarrow \mu(0) \geq t$ for any $\forall\ t \in L$

$\Longrightarrow 0 \in \mu_t, \forall\ t \in L$ (1)

For any $\forall\ t \in L, \mu_t \neq \emptyset$

For $, y \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$

Now $\mu(x+y) \geq \mu(x) \wedge \mu(y) \geq t$.

Hence $x + y \in \mu_t$ (2)

Let $x, y \in X$ be such that $x - y, y \in \mu_t$.

Then we have $\mu(x-y) \geq t$ and $\mu(y) \geq t$

Now $\mu(x) \geq \mu(x-y) \wedge \mu(y)$

$\geq t \wedge t \Longrightarrow x \in \mu_t$ (3)

Hence by (1),(2) and (3) μ_t is a β–ideal of X.

Conversely, assume that each non-empty level subset μ_t is a β–ideal of X, for some L-fuzzy set μ of X.

Claim: μ is an L-fuzzy β–ideal of X.

For any $x \in X$,let $\mu(x)$=t. Then μ_t is a β–ideal of X$\Longrightarrow 0 \in \mu_t$

$\Longrightarrow \mu(0) \geq \mu(x)\ \forall\ x \in X$ (4)

Choose $x, y \in X$, such that $\mu(x) = t_1$ and $\mu(y) = t_2$, where $t_1, t_2 \in L$.

Then $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$

Assume $t_1 \leq t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$, hence $y \in \mu_{t_1}$

Since μ_t is a β–ideal of X, we have $x + y \in \mu_{t_1}$

Thus $\mu(x + y) \geq t_1 = \mu(x) \wedge \mu(y)$. (5)

Suppose that there exits $x, y \in X$ such that $\mu(x) \leq \mu(x - y) \wedge \mu(y)$.

Let $t' = \dfrac{\{\mu(x - y) + (\mu(x) \wedge \mu(y))\}}{2}$

Clearly $t' \in L$.

Then $\mu(x - y) \geq t'$ and $\mu(y) \geq t'$

$\Longrightarrow x - y \in \mu_{t'}$ and $\Longrightarrow y \in \mu_{t'}$.

However, $\mu(x) \leq t' \wedge t' = t'$

$\Longrightarrow x \notin \mu_{t'}$ which contradicts the facts that $\mu_{t'}$ is a β–ideal of X.

Hence for $x, y \in X$, $\mu(x) \geq \mu(x - y) \wedge \mu(y)$ (6)

Therefore by (4),(5) and (6) μ is a fuzzy β–ideal of X.

Corollary[3.3.9]

If μ is an L-fuzzy β–ideal of X ,then the set $X_{\mu(y)} = \{x \in X | \mu(x) = \mu(y)\}$ is a β–ideal for any $y \in X$.

Corollary[3.3.10]

If μ is an L-fuzzy β–ideal of X ,then the set $X_{\mu} = \{x \in X | \mu(x) = \mu(0)\}$ is a β–ideal of X.

Theorem [3.3.11]

If every L-fuzzy β–ideal μ of a β–algebra X is such that $|\mathrm{Im}(\mu)| < \infty$, then every descending chain of β–ideals of X, terminates after a finite stage.

Proof:

Let μ be any L-fuzzy β–ideal of X such that $|Im(\mu)| < \infty$.

Suppose a strictly descending chain $X=A_0 \supset A_1 \supset A_2 \supset \ldots A_n \ldots$ of β–ideals of X does not terminate after a finite stage. Then define a fuzzy set μ of X by

$$\mu(x)=\begin{cases} \frac{n}{n+1} & \text{if } x \in A_n - A_{n-1} \\ 1 & \text{if } x \in \cap A_n \end{cases}$$

(a) Every β–ideal A_n contains $0 \Longrightarrow 0 \in \bigcap A_n \Longrightarrow \mu(0) \geq \mu(x)\ \forall\ x \in X$.

(b)Let $x, y \in X$

If $x + y \in \bigcap A_n$, then $\mu(x + y)=1 \geq \mu(x) \wedge \mu(y)$.

If $x + y \notin \bigcap A_n$. Assume that $x + y \in A_t - A_{t+1}$ for some $t \geq 1$.

Then atleast one of the x or y does not lie in $\bigcap A_n$. (since if both x and y lies in $\bigcap A_n$ then $x + y \in \bigcap A_n$ which implies $x + y \in \bigcap A_n$, $\forall n$)

Let A_m be a maximal fuzzy β–ideal of X such that x or $y \in A_m - A_{m+1}$.

Hence both x and $y \in A_m$.

If $t \geq m$ then $A_m \subseteq A_{t+1} \subset A_t$ implies $x + y \in A_{t+1}$,which contradicts to our assumption. Hence $m \geq t$.

Now $\mu(x + y) = \frac{t}{t+1} \geq \frac{m}{m+1} = \mu(x) \wedge \mu(y)$.

(c) Let $x, y \in X$.

case(i): If $x - y \in A_t - A_{t+1}$, and $y \in A_k - A_{k+1}$ for some $t, k \in N \cup \{0\}$.

Let $t \geq k$. Then $x - y$ and $y \in A_t \Longrightarrow x \in A_t$(since A_t is a closed β–ideal in X).

Hence $\mu(x + y) \geq \frac{t}{t+1} = \mu(x) \wedge \mu(y)$.

case(ii): If $x - y \in \bigcap A_n$ and $y \in \bigcap A_n$,then $x \in \bigcap A_n$,

$\Longrightarrow \mu(x) \geq 1 = \mu(x - y) \wedge \mu(y)$.

case(iii): If $x - y \notin \bigcap A_n$ and If $y \in \cap A_n$,then $x - y \in A_k - A_{k+1}$ for some $k \in N \cup \{0\}$. Then $x \in \cap A_n$.

Hence $\mu(x) \geq \frac{k}{k+1} = \mu(x-y) \wedge \mu(y)$.

case(iv): If $x - y \in \cap A_n$ and If $y \notin \cap A_n$,then $y \in A_k - A_{k+1}$ for some $k \in N \cup \{0\}$. Then $\in \cap A_k$. Hence $\mu(x) \geq \frac{k}{k+1} = \mu(x-y) \wedge \mu(y)$.

Therefore μ is an L-fuzzy β–ideal μ of a X. clearly $|Im(\mu)| = \infty$. Which is a contradiction to our assumption.

Therefore every descending chain of β–ideal X terminates after a finite stage. This completes the proof.

Theorem [3.3.12]

Let X= $A_0 \subset A_1 \subset A_2 \subset \ldots A_n \subset \ldots$ be a strictly ascending sequence of β–ideals of X and (t_n) be a strictly descending sequence (0,1). Let μ be a L-fuzzy set defined by

$$\mu(x) = \begin{cases} 0 \text{ if } x \notin A_n \ \forall\, n \in N \\ t_n \text{if } x \in A_n - A_{n-1} \ \forall\, n \in N \end{cases}$$

where $A_0 = \emptyset$.Then μ is an L-fuzzy β–ideal of a X.

Proof:

(a) Let $A = \cup A_n$. Then A is a L-fuzzy β–ideal of a X.

Clearly $\mu(0) = t_1 \geq \mu(x), \forall\, x \in X$.

(b) **Case(i):** If $x, y \in A_n - A_{n-1}$

$\Rightarrow x, y \in A_n \Rightarrow x + y \in A_n$,Since A_n is aβ–ideal of a X.

Also $x, y \in A_n - A_{n-1} \Rightarrow \mu(x) = t_n = \mu(y)$

$\Rightarrow \mu(x) \wedge \mu(y) = t_n$

Now $x + y \in A_n$

$\Rightarrow x + y \in A_n - A_{n-1}$ or $x + y \in A_{n-1} \Rightarrow \mu(x+y) = t_n$

Hence $\mu(x+y) \geq t_n$

Therefore $\mu(x+y) \geq t_n = \mu(x) \wedge \mu(y)$.

Case(ii): For $i > j \Rightarrow t_j > t_i \Rightarrow A_j \subset A_i$.

If $x \in A_i - A_{i-1} \Rightarrow \mu(x) = t_i$ and $y \in A_j - A_{j-1}$

$\Rightarrow \mu(y) = t_j > t_i \Rightarrow \mu(x) \wedge \mu(y)$.

$= t_i \wedge t_j = t_i$

And $x \in A_i - A_{i-1} \Rightarrow x \in A_i$ and $y \in A_j - A_{j-1}$.

$\Rightarrow y \in A_j \subset A_i \Rightarrow x, y \in A_i$ then $x + y \in A_i$.

Since it is a β–ideal of X. Therefore $\mu(x + y) \geq t_i = \mu(x) \wedge \mu(y)$.

(c)Let $x, y \in X$

Case(i): If $x \notin A$

$\Rightarrow x - y \notin A$ or $y \notin A$

$\Rightarrow \mu(x) = 0 = \mu(x - y) \wedge \mu(y)$

Case(ii): If $x \in A$

$\Rightarrow x \in A_n - A_{n-1}$ for some n

$\Rightarrow x \in A_n$ and $x \notin A_{n-1}$

$\Rightarrow x - y \notin A_{n-1}$ or $y \notin A_{n-1}$

Hence $\mu(x - y) \leq t_n$ or $\mu(y) \leq t_n$

Therefore $\mu(x) = t_n \geq \mu(x - y) \wedge \mu(y)$.

Hence μ is a L-fuzzy β–ideal of a X.

Section 3.4

L-fuzzy T-Ideals of β-Algebras

Definition[3.4.1]

A non empty subset I of β–algebra of $(X, +, -, 0)$ is called T-ideal of X if the following conditions are satisfied.

1.$0 \in I$

2.$(x + y) + z \in I$ and $y \in I \Rightarrow (x + z) \in I$ and

3.$(x - y) - z \in I$ and $y \in I \Rightarrow (x - y) \in I \ \forall \ x, y, z \in I$.

Example:[3.4.2]

Let X={0,1,2,3,4,5} be a set with constant 0 and two binary operations + and – defined by cayely tables:-

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	3	1
3	3	4	5	0	1	2
4	4	2	1	2	5	0
5	5	3	3	1	0	4

-	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	0	4	5	2	3
2	2	5	0	4	1	3
3	3	4	5	0	1	2
4	4	3	1	2	0	5
5	5	2	3	1	4	0

Then (X,+,−,0} is a β–algebra.

I_1 ={0,2} is a T-ideal of X while I_2 ={0,4} is not a T-ideal of X for (0+4)+4 =4+4 = 5 $\notin I_2$.

Definition [3.4.3]

Let μ be an L-fuzzy set in a β–algebra of X. Then μ is called an L-fuzzy T-ideal of X if,

1.$\mu(0) \geq \mu(x)$

2.$\mu(x + z) \geq \mu((x + y) + z) \wedge \mu(y)$ and

3. $\mu(x - y) \geq \mu((x - y) - z) \wedge \mu(y) \forall\, x, y \in X$.

Example: [3.4.4]

In the β-algebra X of example 3.4.2 the L-fuzzy set μ_1: X→ L defined by

$$\mu_1(x) = \begin{cases} t_5 \text{ if } x = 0 \\ t_4 \text{ if } x = 1 \\ t_3 \text{ if } x = 2 \\ t_2 \text{ if } x = 3 \\ t_1 \text{ if } x = 4,5 \end{cases}$$

where $0 \leq t_1 < t_2 < t_3 < t_4 < t_5 \leq 1$,

$t_1, t_2, t_3 t_4, t_5 \in L$ is an L-fuzzy T-ideal of X.

Lemma: [3.4.5]

Let μ be an L-fuzzy T-ideal of a β–algebra X. If $x \leq y$ then $\mu(x) \geq \mu(y)$.

Proof:

For $x, y \in X$, $x \leq y \Rightarrow x - y = 0$ Then

$\mu(x) = \mu(x - 0) \geq \mu((x - 0) - 0) \wedge \mu(y)$

$= \mu(0 - 0) \wedge \mu(y)$

$= \mu(0) \wedge \mu(y)$

$= \mu(y)$.

Hence $\mu(x) \geq \mu(y)$.

Theorem[3.4.6]

Let A be a subset of X. Define an L-Fuzzy set μ: X→L such that

$$\mu(x) = \begin{cases} t_0 \text{ if } x \in A \\ t_1 \text{ if } x \notin A \end{cases}$$

where $t_0, t_1 \in L$ with $t_0 > t_1$.Then μ is an L-fuzzy T–ideal of a β–algebra X if and only if A is a T–ideal of X .

Proof:

Assume that μ is an L-fuzzy T–ideal of X.

If $x \in$ A, $\mu(0) \geq t_0$

and if $x \notin$ A, $\mu(0) \geq t_1$, Since $t_0 > t_1$

$\Rightarrow \mu(0) \geq t_0 > t_1$

$\Rightarrow \mu(0) = t_0 \Rightarrow 0 \in$ A

For $x, y, z \in$ A

$\Rightarrow \mu(x) = t_0$, $\mu(y) = t_0$ and $\mu(z) = t_0$

$\mu(x+z) \geq \mu((x+y)+z) \wedge \mu(y)$

$\geq (\mu(x+y) \wedge \mu(z)) \wedge \mu(y)$

$= (((\mu(x) \wedge \mu(y)) \wedge \mu(z))) \wedge \mu(y)$

$= t_0 \wedge t_0 \wedge t_0$

$= t_0$

Therefore $\mu(x+z) \geq t_0 \Rightarrow x+z \in$ A.

For $x, y, z \in$ A, $\Rightarrow \mu((x-y)-z) = t_0$, $\mu(y) = t_0$.

Now $\mu(x-y) \geq \mu((x-y)-z) \wedge \mu(y) = t_0 \wedge t_0 \wedge t_0$

$\Rightarrow \mu(x-y) = t_0 \Rightarrow x-z \in$ A.

Hence A is T–ideal of X.

Conversely, suppose A is a T–ideal of X.

Now $0 \in$ A $\Rightarrow \mu(0) = t_0$

Also $\forall x \in$ X , Im(μ) ={ t_0,t_1 }and $t_0 > t_1$

$\Rightarrow \mu(0) \geq \mu(x)$.

since A is T–ideal of X,$\forall x, y, z \in$ S,($x+y$)+$z \in$ A and $y \in$ A.

$\Rightarrow x+z, x-z \in$ A.

then $\mu(x+z) = t_0 \geq \mu((x+z)+z) \wedge \mu(y)$

similarly,$\mu(x-z) \geq \mu((x-y)-z) \wedge \mu(y)$.

Hence μ is an L-fuzzy T–ideal of X.

Theorem[3.4.7]

An L-fuzzy set μ is a T–ideal of a X if and only if the non empty level subset μ_t is a T–ideal of X ,∀t ∈ L.

Proof:

Assume that μ is an L-fuzzy T–ideal of X.

Now $\mu(0) \geq \mu(x)\ \forall\ x \in X$

$\Rightarrow \mu(0) \geq t$ for any ∀ t ∈ L.

$\Rightarrow 0 \in \mu_t, \forall\ t \in L$.

For any∀ t ∈ L,$\mu_t \neq \emptyset$

For $x, y, z \in \mu_t$, we have

$\mu((x+y)+z) \geq t$ and $\mu(y) \geq t$

$\Rightarrow x+z \in \mu_t$

Also we have $\mu((x-y)-z) \geq t$ and $\mu(y) \geq t$

Hence $\mu((x-y)-z) \geq \mu((x-y)-z) \wedge \mu(y)$

$\geq t \wedge t = t$

$\Rightarrow x-z \in \mu_t$

Hence $\mu_t$is a T–ideal of X.

Conversely ,

Assume that each non-empty level subset $\mu_t$of a fuzzy subset μ of X is a T–ideal of X.

Claim: μ is an L-fuzzy T–ideal of X.

For any $x \in X$, let$\mu(x)$=t.

Since μ_t is a T–ideal of X,$0 \in \mu_t$

$\Rightarrow \mu(0) \geq \mu(x)\ \forall\ x \in X$

Choose $x, y, z \in X$, such that $\mu((x+y)+z) = t_1$ and $\mu(y) = t_2$, where $t_1, t_2 \in L$.

Then $x + z \in \mu_{t_1}$and $y \in \mu_{t_2}$

Assume $t_1 \leq t_2$. Then $\mu_{t_2} \subseteq \mu_{t_1}$, hence $y \in \mu_{t_1}$.

Since μ_t is a T–ideal of X.
we have $x + z \in \mu_{t_1}$

Thus $\mu(x+z) \geq t_1 = \mu((x+y)+z) \wedge \mu(y)$.

Similarly, $\mu(x-y)-z) \geq \mu((x+y)+z) \wedge \mu(y)$.
Therefore μ is an L-fuzzy T–ideal of X.

Theorem [3.4.8]

Let $\mu_1$and μ_2 be two L-fuzzy T–ideals in a β–algebras $X_1 \times X_2$. Then the direct product $\mu_1 \times \mu_2$is an L-fuzzy T–ideals in $X_1 \times X_2$.

Proof:

For $(x , y) \in X_1 \times X_2$,we have

$(\mu_1 \times \mu_2)(0,0) = \mu_1(0) \wedge \mu_2(0)$

$\geq \mu_1(x) \wedge \mu_2(y)$

$\geq (\mu_1 \times \mu_2)(x, y)$.

Let (x_1, x_2),(y_1, y_1) and $(z_1, z_1) \in X_1 \times X_2$.

Then$(\mu_1 \times \mu_2)((x_1, x_2) + (z_1, z_2))$

$=(\mu_1 \times \mu_2)((x_1 + z_1),(x_2 + z_2))$

$= (\mu_1(x_1 + z_1) \wedge \mu_2(x_2 + z_2)$

$\geq (\mu_1((x_1 + y_1)z_1) \wedge \mu_1(y_1) \wedge (\mu_2((x_2 + y_2) + z_2) \wedge \mu_2(y_2)$

$= (\mu_1((((x_1 + y_1)z_1) \wedge \mu_2((x_2 + y_2) + z_2)) \wedge (\mu_1(y_1) \wedge \mu_2(y_2))$

$= (\mu_1 \times \mu_2)(((x_1 + y_1)z_1, ((x_2 + y_2) + z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$

$= (\mu_1 \times \mu_2)(((x_1 + y_1),((x_2 + y_2)))+(z_1 + z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$

$=(\mu_1 \times \mu_2) (((x_1, x_2),(y_1, y_2))) + (z_1, z_2)) \wedge (\mu_1 \times \mu_2)(y_1, y_2)$

Similarly $(\mu_1 \times \mu_2)$ $((x_1, x_2) - (z_1, z_2)) \geq (\mu_1 \times \mu_2)$ $(((x_1, x_2),(y_1, y_2))) + (z_1, z_2)) \wedge$ $(\mu_1 \times \mu_2)(y_1, y_2)$

Hence $\mu_1 \times \mu_2$ is an L-fuzzy T–ideal of a β–algebra in $X_1 \times X_2$.

Theorem [3.4.9]

Let μ_1 and μ_2 be two fuzzy sets in a β–algebra X such that $\mu_1 \times \mu_2$ is an L-fuzzy T–ideal of $X_1 \times X_2$. Then

1. Either $\mu_1(0) \geq \mu_1(x)$ (or) $\mu_2(0) \geq \mu_2(x) \forall x \in X$
2. If $\mu_1(0) \geq \mu_1(x)$, $\forall x \in X$ then either $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.
3. If $\mu_2(0) \geq \mu_2(x)$, $\forall x \in X$ then either $\mu_1(0) \geq \mu_1(x)$ or $\mu_1(0) \geq \mu_2(x)$.
4. Either μ_1 or μ_2 is an L-fuzzy T–ideal of X.

Proof:

Let $\mu_1 \times \mu_2$ is an L-fuzzy T–ideal of $X_1 \times X_2$.

Suppose that $\mu_1(0) < \mu_1(x)$ and $\mu_2(0) < \mu_2(y)$ for some $x, y \in X$. Then

$(\mu_1 \times \mu_2)\ (x, y) = \mu_1(x) \wedge \mu_2(y) \geq \mu_1(0) \wedge \mu_2(0) = (\mu_1 \times \mu_2)\ (0,0)$

This contradiction yields that either $\mu_1(0) \geq \mu_1(x)$ (or) $\mu_2(0) \geq \mu_2(x)$ $\forall x \in X$.

Given $\mu_1(0) \geq \mu_1(x)$, $\forall x \in X$ and assume that there exist $x, y \in X$ such that
$\mu_2(0) < \mu_1(x)$ and $\mu_2(0) < \mu_2(y) \forall x, y \in X$.
Now $\mu_2(0) < \mu_1(x) \leq \mu_1(0) \Rightarrow \mu_2(0) < \mu_1(0)$.
Then $(\mu_1 \times \mu_2)\ (0,0) = \mu_1(0) \wedge \mu_2(0) = \mu_2(0)$.

$(\mu_1 \times \mu_2)\ (x, y) = \mu_1(x) \wedge \mu_2(y)$

$\geq \mu_2(0) \wedge \mu_2(0) = \mu_2(0)$

$\geq(\mu_1 \times \mu_2) (0,0)$.

Thus $(\mu_1 \times \mu_2) (x, y) \geq (\mu_1 \times \mu_2) (0,0)$

which is contradiction.

Hence if $\mu_1(0) \geq \mu_1(x)$, $\forall\, x \in X$ then either $\mu_2(0) \geq \mu_1(x)$ or $\mu_2(0) \geq \mu_2(x)$.

Similarly we can prove that if $\mu_2(0) \geq \mu_2(x)$, $\forall\, x \in X$ then either $\mu_1(0) \geq \mu_1(x)$ or $\mu_1(0) \geq \mu_2(x)$.

First we can prove that μ_2 is a L-fuzzy T–ideal of X.

Assume that $\mu_2(0) \geq \mu_2(x)$, $\forall\, x \in X$.
Then it follows that either $\mu_1(0) \geq \mu_1(x)$ (or) $\mu_1(0) \geq \mu_2(x)$.

If $\mu_1(0) \geq \mu_2(x)$ for any $x \in X$.

Then,

$\mu_2(x) \geq \mu_1(0) \wedge \mu_2(x)$

$= (\mu_1 \times \mu_2) (0, x)$

$\mu_2(x + z) \geq \mu_1(0) \wedge \mu_2(x + z)$

$= (\mu_1 \times \mu_2) (0, x + z)$

$= (\mu_1 \times \mu_2) (0+0, x + z)$

$= (\mu_1 \times \mu_2) ((0, x)+(0, z))$

$\geq (\mu_1 \times \mu_2) ((0, x)+(0, y))+(0, z)) \wedge (\mu_1 \times \mu_2) (0, y)$

$= (\mu_1 \times \mu_2) (((0+0), (x + y))+(0, z)) \wedge (\mu_1 \times \mu_2) (0, y)$

$= (\mu_1 \times \mu_2) (((0+0),+0, (x + y) + z)) \wedge (\mu_1 \times \mu_2) (0, y)$

$= (\mu_1 \times \mu_2) ((0, (x + y) + z)) \wedge (\mu_1 \times \mu_2) (0,y)$

$= \mu_2((x + y) + z) \wedge \mu_2(y)$

Similarly we can prove that $\mu_2(x - z) \geq \mu_1((x - y) - z)) \wedge \mu_2(y)$.
Hence μ_2 is an L-fuzzy T–ideal of X. Similarly we can prove that μ_1 is an L-fuzzy T–ideal of X.

Theorem [3.4.10]

Let f: X→X be an endomorphism on a β–algebra. Let μ be an L-fuzzy T–ideal of X. Define a fuzzy set μ_f : X→ [0.1] defined by $\mu_f(x)=\mu(f(x))$, $\forall\, x \in X$. Then μ_f is an L-fuzzy T–ideal of X.

Proof:

Let $x \in X$. Then $\mu_f(x) = \mu(f(x)) \leq \mu(0) = \mu(f(0)) = \mu_f(0)$

Let $x, y \in X$.

$\mu_f\,(x + z) = \mu(f(x + z)) = \mu(f(x) + f(z))$

$\geq \mu((f(x) + f(y)) + f(z)) \wedge \mu(f(y))$

$= \mu((f(x + y)) + f(z)) \wedge \mu(f(y))$

$= \mu(f((x + y) + z)) \wedge \mu(f(y))$

$= \mu_f\,((x + y) + z)) \wedge \mu_f(y)$.

Similarly we can prove that

$\mu_f\,(x - z) = \mu_f\,((x - y) - z)) \wedge \mu_f(y)$.

Hence μ_f is an L-fuzzy T–ideal of X.

CHAPTER 4

Chapter-4

Intuitionistic Fuzzy β-Subalgebras and

α-Translations of β-Algebras.

Section 4.1

On Intuitionistic Fuzzy β-Subalgebras of β-Algebras

Definition [4.1.1]

An intuitionistic fuzzy set (IFS) in a non empty set X is defined by

A = $\{\langle\, x, \mu_A(x), \gamma_A(\, x) \,\rangle \,/\, \ x \in$ X$\}$

Where μ_A: X → [0,1] is a membership function of A and γ_A: X → [0,1] is a non membership function of A satisfying $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in$ X.

Definition [4.1.2]

Let (X, $+$, $-$, 0) be a β-algebra. Then A = $\{\langle\, x, \mu_A(x), \gamma_A(\, x) \,\rangle \,/\, x \in$ X$\}$ is called an intuitionistic fuzzy (IF) β-subalgebra of X, if following conditions are satisfied:

IFβS 1. $\mu_A(x + y) \geq \min \{ \mu_A(x), \mu_A(y)\}$ and

$\gamma_A(x + y) \leq \max \{ \gamma_A(x), \gamma_A(y)\}$.

IFβS 2. $\mu_A(x - y) \geq \min \{ \mu_A(x), \mu_A(y)\}$ and

$\gamma_A(x - y) \leq \max \{ \gamma_A(x), \gamma_A(y)\}$

where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in$ X.

Example [4.1.3]

Let X = {0, 1, 2, 3} be a set with constant 0 and binary operation $+$ and $-$ are defined on X by the following cayley's tables.

+	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

−	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

Then (X, +, −, 0) is a β-algebra.

Now define the IF subset of X.

$$\mu_A(x) = \begin{cases} .8\,, x = 0 \\ .5\,, x = 1,2 \\ .3, x = 3 \end{cases}$$ and

$$\gamma_A(x) = \begin{cases} .1, x = 0 \\ .5, x = 1,2 \\ .4, x = 3 \end{cases}$$

is a IF β-sub algebra of X.

Then the intuitionistic fuzzy set A = {⟨ x, $\mu_A(x)$, $\gamma_A(x)$ ⟩ / x ∈ X} is an intuitionistic fuzzy β-sub algebra of X.

Definition [4.1.4]

Let (X, +, −, 0) and (Y, +, −, 0) be two β-algebras.

Let A = {⟨ x, $\mu_A(x)$, $\gamma_A(x)$ ⟩ / x ∈ X} and B = {⟨ y, $\mu_B(y)$, $\gamma_B(y)$ ⟩ / y ∈ Y} be IF sets in X and Y. The Cartesian product of A and B, denoted by, A×B is defined as follows:

$$A\times B = \{\langle (\mu_A \times \mu_B)(x, y), (\gamma_A \times \gamma_B)(x, y)\rangle / x \in X \times Y\}$$

where $(\mu_A \times \mu_B)$: X×Y → [0,1] is given by $(\mu_A \times \mu_B)(x, y) = \min\{\mu_A(x), \mu_B(y)\}$ and $(\gamma_A \times \gamma_B)$: X × Y →[0,1] is given by $(\gamma_A \times \gamma_B)(x, y) = \max\{\gamma_A(x), \gamma_B(y)\}$ for all $x \in$ X and $y \in$ Y.

Theorem [4.1.5]

If A and B are two IF β-subalgebras of X, then A ∧ B is also an IF β-subalgebras of X.

Proof:

Let $x, y \in$ X,

$(\mu_A \wedge \mu_B)(x + y) = \min\{\mu_A(x + y), \mu_B(x + y)\}$

$\geq \min(\min\{\mu_A(x), \mu_A(y)\}, \min\{\mu_B(x), \mu_B(y)\})$

$= \min(\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\})$

$= \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(y)\}$.

$(\gamma_A \wedge \gamma_B)(x + y) = \max\{\gamma_A(x + y), \gamma_B(x + y)\}$

$\leq \max(\max\{\gamma_A(x), \gamma_A(y)\}, \max\{\gamma_B(x), \gamma_B(y)\})$

$= \max(\max\{\gamma_A(x), \gamma_B(x)\}, \max\{\gamma_A(y), \gamma_B(y)\})$

$= \max\{(\gamma_A \wedge \gamma_B)(x), (\gamma_A \wedge \gamma_B)(y)\}$.

Similarly, $(\mu_A \wedge \mu_B)(x - y) \geq \min\{(\mu_A \wedge \mu_B)(x), (\mu_A \wedge \mu_B)(y)\}$ and

$(\gamma_A \wedge \gamma_B)(x + y) \geq \max\{(\gamma_A \wedge \gamma_B)(x), (\gamma_A \wedge \gamma_B)(y)\}$.

Hence the proof.

Note:

The above theorem can be extended for any family of IF β-subalgebra of X as follows:

“Let $\{A_i$/ i∈I$\}$ be an arbitrary family of IF β-subalgebra of X. Then $\wedge A_i$ is also a IF β-sub algebra of X.”

Theorem [4.1.7]

Let X be a β-algebra. Let A = {⟨ x, μ(x), $\gamma(x)$ ⟩ / x ∈ X} be an IF β-subalgebra of X. Then

i). μ(x) ≤ μ(0) and $\gamma(x) \geq \gamma(0)$.

ii). μ(x) ≤ μ(0−x) ≤ μ(0) and $\gamma(x) \geq \gamma(0-x) \geq \gamma(0)$.

Proof:

For any x ∈ X,

To prove (i) μ(0) = μ($x - x$) ≥ min{ μ(x), μ(x)} = μ(x)

Similarly, $\gamma(0) = \gamma(x - x) \leq$ max{ $\gamma(x)$, $\gamma(x)$} = $\gamma(x)$

To prove (ii) μ(0−x) ≥ min { μ(0), μ(x)} = μ(x).

Hence, μ(x) ≤ μ(0−x) ≤ μ(0) and $\gamma(0-x) \leq$ max { $\gamma(0)$, $\gamma(x)$} = $\gamma(x)$.

Hence $\gamma(x) \geq \gamma(0-x) \geq \gamma(0)$.

Theorem [4.1.8]

Let A be an IF β-subalgebra of β-algebra X. Let X_A = { x ∈ X / $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)$}. Then X_A is also a β-subalgebra of X.

Proof:

Let A be a IF β-subalgebra of a β-algebra X.

Let $x, y \in X_A$, then $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = \gamma_A(0)$.

$\mu_A(y) = \mu_A(0)$ and $\gamma_A(y) = \gamma_A(0)$

For , $\mu_A(x + y) \geq$ min{ $\mu_A(x)$, $\mu_A(y)$}

= min{ $\mu_A(0)$, $\mu_A(0)$}

= $\mu_A(0)$

$\mu_A(x + y) \geq \mu_A(0)$

Now, $\mu_A(0) = \mu_A(0+0) \geq \min\{ \mu_A(0), \mu_A(0) \}$

$= \min\{ \mu_A(x), \mu_A(y)$

$= \mu_A(x + y)$

$\Longrightarrow \mu_A(x + y) = \mu_A(0)$

For, $\gamma_A(x + y) \leq \max\{ \gamma_A(x), \gamma_A(y)\}$

$= \max\{ \gamma_A(0), \gamma_A(0)\}$

$= \gamma_A(0)$

$\gamma_A(x + y) \leq \gamma_A(0)$

Now, $\gamma_A(0) = \gamma_A(0 + 0)$

$\leq \max\{ \gamma_A(0), \gamma_A(0)\}$

$= \max\{ \gamma_A(x), \gamma_A(y)\} = \gamma_A(x + y)$

$\Longrightarrow \gamma_A(x + y) = \gamma_A(0)$.

Therefore $x + y \in X_A$, $x, y \in$ X.

Similarly, $-y \in X_A$.

Thus, X_A is a β-subalgebra of X

Theorem [4.1.9]

Let A be an IF β-subalgebra of X. Let $X_A = \{ x \in X / \mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = 1 - \gamma_A(0)\}$. Then X_A is a β-subalgebra of X.

Proof:

Let $x, y \in X_A$, then $\mu_A(x) = \mu_A(0)$ and $\gamma_A(x) = 1 - \gamma_A(0)$.

$\mu_A(y) = \mu_A(0)$ and $\gamma_A(y) = 1 - \gamma_A(0)$.

$\mu_A(x + y) \geq \min\{ \mu_A(x), \mu_A(y)\}$

$= \min\{ \mu_A(0), \mu_A(0)\}$

$=\mu_A(0)$

$\mu_A(x+y) \geq \mu_A(0)$.

Now, $\mu_A(0) = \mu_A(0+0) \geq \min\{ \mu_A(0), \mu_A(0) \}$

$= \min\{ \mu_A(x), \mu_A(y) = \mu_A(x+y)$

$\Longrightarrow \mu_A(x+y) = \mu_A(0)$

$\gamma_A(x+y) \leq \max\{ \gamma_A(x), \gamma_A(y)\}$

$= \max\{1-\gamma_A(0), 1-\gamma_A(0)\}$

$= 1-\gamma_A(0)$

$\gamma_A(x+y) \leq 1-\gamma_A(0)$.

Now, $1-\gamma_A(0) = 1- \gamma_A(0-0)$

$\leq 1-\max\{ \gamma_A(0), \gamma_A(0)\}$

$= \min\{1-\gamma_A(0), 1-\gamma_A(0)\}$

$\leq \max\{1-\gamma_A(0), 1-\gamma_A(0)\}$

$= \max\{ \gamma_A(x), \gamma_A(y)\}$

$= \gamma_A(x+y)$

$\Longrightarrow \gamma_A(x+y) = 1-\gamma_A(0)$.

Therefore $x+y \in X_A$.

Similarly, $x-y \in X_A$.

Hence, X_A is a β-subalgebra of X.

Theorem [4.1.10]

Let A and B be two IF β-subalgebras of X and Y respectively. Then A × B is also an IF- β-subalgebra of X × Y.

Proof:

A = $\{\langle\, x, \mu_A(x), \gamma_A(\,x)\,\rangle\, /\;\, x \in$ X}

B = $\{\langle y, \mu_B(y), \gamma_B(\,y)\,\rangle\, /\;\, y \in$ Y} be IF β-subalgebras in X and Y.

Take $x = (x_1\,, x_2)\,, y = (y_1\,, y_2) \in$ X × Y

$(\mu_A \times \mu_B)(\,x + y) = (\mu_A \times \mu_B)\,((x_1\,, x_2) + (y_1\,, y_2)\,)$

$= (\mu_A \times \mu_B)\,((x_1 + y_1)\,, (\,x_2 + y_2))$

$= \min\,\{\mu_A\,(x_1 + y_1),\; \mu_B\,(\,x_2 + y_2)\}$

$\geq \min\,\{\min\,\{\mu_A\,(x_1), \mu_A\,(y_1)\}, \min\,\{\mu_B\,(\,x_2), \mu_B\,(y_2)\}\}.$

$= \min\,\{\min\,\{\mu_A\,(x_1), \mu_B\,(\,x_2)\,\}, \min\,\{\mu_A\,(y_1), \mu_B\,(y_2)\}\}.$

$= \min\,\{(\mu_A \times \mu_B)\,(x_1, x_2), (\mu_A \times \mu_B)\,(y_1, y_2)\}$

$= \min\,\{(\mu_A \times \mu_B)\,(x), (\mu_A \times \mu_B)\,(y)\}.$

Similarly, $(\mu_A \times \mu_B)(\,x - y) \geq \min\,\{(\mu_A \times \mu_B)\,(x), (\mu_A \times \mu_B)\,(y)\}.$

Also, $(\gamma_A \times \gamma_B)(\,x + y) = (\gamma_A \times \gamma_B)\,((x_1\,, x_2) + (y_1\,, y_2)\,)$

$= (\gamma_A \times \gamma_B)\,((x_1 + y_1)\,, (\,x_2 + y_2))$

$= \max\,\{\gamma_A\,(x_1 + y_1),\; \gamma_B\,(\,x_2 + y_2)\}$

$\leq \max\,\{\max\,\{\gamma_A\,(x_1), \gamma_A\,(y_1)\}, \max\,\{\gamma_B\,(\,x_2), \gamma_B\,(y_2)\}\}.$

$= \max\,\{\max\,\{\gamma_A\,(x_1), \gamma_B\,(\,x_2)\,\}, \max\,\{\gamma_A\,(y_1), \gamma_B\,(y_2)\}\}.$

$= \max\,\{(\gamma_A \times \gamma_B)\,(x_1, x_2), (\gamma_A \times \gamma_B)\,(y_1, y_2)\}$

$= \max\,\{(\gamma_A \times \gamma_B)\,(x), (\gamma_A \times \gamma_B)\,(y)\}.$

Similarly, $(\gamma_A \times \gamma_B)(\,x - y) \leq \max\,\{(\gamma_A \times \gamma_B)\,(x), (\gamma_A \times \gamma_B)\,(y)\}.$

Hence the Cartesian product of A×B is also an IF β-subalgebra of X × Y.

Theorem [4.1.11]

If A = {⟨ x, $\mu_A(x)$, $\gamma_A(x)$ ⟩ / x ∈ X} is an IF β-subalgebra of X, then so is ΘA, where ΘA = {⟨ x, $\mu_A(x)$, $\overline{\mu_A(x)}$ ⟩ / x ∈ X}.

Proof:

Let A be an IF β-subalgebra of X.

Let x, y ∈ X.

$\overline{\mu}_A(x + y) = 1 - \mu_A(x + y)$

$\geq 1 - \min\{\mu_A(x), \mu_A(y)\}$.

$\geq \max\{1 - \mu_A(x), 1 - \mu_A(y)\}$

$= \min\{1 - \mu_A(x), 1 - \mu_A(y)\}$

$= \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$.

Similarly, $\overline{\mu}_A(x - y) \geq \min\{\overline{\mu}_A(x), \overline{\mu}_A(y)\}$.

Hence ΘA is an IF β-subalgebra of X.

Definition [4.1.12]

Let f: X → Y be a function. Let A and B be two IF β-sub algebra of X and Y. Then the inverse image of B under f is defined as f^{-1}(B) ={⟨x, $f^{-1}(\mu_B(x))$, $f^{-1}(\gamma_B(x))$⟩ / x ∈ X} such that $f^{-1}(\mu_B(x)) = \mu_B(f(x)$ and $f^{-1}(\gamma_B(x)) = \gamma_B(f(x))$.

Theorem [4.1.13]

Let (X, +, −, 0) and (Y, +, −, 0) be two β-algebras and let f: X →Y be a homomorphism. If A is an IF β-subalgebra of Y, then f^{-1}(A) is an IF β-subalgebra of X.

Proof:

Let A be an IF β-subalgebra of Y. For x, y ∈ X

$f^{-1}(\mu_A(x+y))$ = μ_A(f($x+y$))

= μ_A(f(x) + f(y))

≥ min {μ_A(f(x), μ_A(f(y))}

= min {$f^{-1}(\mu_A(x)$, $f^{-1}(\mu_A(y))$}

$f^{-1}(\gamma_A(x+y))$ = γ_A(f($x+y$))

= γ_A(f(x) + f(y))

≤ max {γ_A(f(x), γ_A(f(y))}

= max {$f^{-1}(\gamma_A(x)$, $f^{-1}(\gamma_A(y))$}.

Analogously, $\mu_A(x-y)$ ≥ min{$f^{-1}(\mu_A(x)$,$f^{-1}(\mu_A(y))$} and

$\gamma_A(x-y)$ ≤ max {$f^{-1}(\gamma_A(x)$, $f^{-1}(\gamma_A(y))$}.

Hence f^{-1} is a IF β-subalgebra of X.

Theorem [4.1.14]

Let (X, +, −, 0) and (Y, +, −, 0) be two β-algebras and let f: X →Y be an endomorphism. If A is IF β-subalgebra of X, then f(A) defined by f(A) = {⟨ x, $(\mu_A)_f(x) = \mu_A(f(x)$, $(\gamma_A)_f(x) = \gamma_A(f\ (x)$ ⟩ / x ∈ X }. Then f(A) is IF β-subalgebra of Y.

Proof:

Let x, y ∈ X.

$(\mu_A)_f\ (x+y))$ = μ_A(f($x+y$)) = $\mu_A(f(x)+f(y))$

≥ min {$\mu_A(f(x)$, $\mu_A(f(y)$}

= min {$(\mu_A)_f(x)$, $(\mu_A)_f(y)$}.

Similarly, $(\mu_A)_f(x-y))$ ≥ min {$\mu_A)_f(x)$, $(\mu_A)_f(y)$}.

$(\gamma_A)_f\ (x+y))$ = γ_A(f($x+y$)) = $\gamma_A(f(x)+f(y))$

$\leq$ max $\{\gamma_A(f(x), \gamma_A(f(y))\}$

= max $\{\gamma_A)_f(x), (\gamma_A)_f(y)\}$.

Similarly, $(\gamma_A)_f(x - y)) \leq$ max $\{\gamma_A)_f(x),(\gamma_A)_f(y)\}$.

Hence f(A) is an IF β-sub algebra of Y.

Section 4.2

Product on Intuitionistic Fuzzy β-Subalgebras of β-Algebras

Definition [4.2.1]

Let μ_A be a fuzzy set in a β-algebra X. Then μ_A is called an Anti fuzzy β-subalgebra of X, if

i). $\mu_A(x + y) \leq$ max $\{ \mu_A(x), \mu_A(y)\}$ and

ii).$\mu_A(x - y) \leq$ max $\{ \mu_A(x), \mu_A(y)\}$.

Definition [4.2.2]

Let A = $\{\langle x, \mu_A(x), \gamma_A(x) \rangle / \ x \in$ X$\}$ be a IFs of a β-algebra X. Then A is called an Anti Intuitionistic Fuzzy (AIF) β-subalgebra of X, if

AIFβS1) $\mu_A(x + y) \leq$ max $\{ \mu_A(x), \mu_A(y)\}$ and

$\gamma_A(x + y) \geq$ min $\{ \gamma_A(x), \gamma_A(y)\}$

AIFβS2) $\mu_A(x - y) \leq$ max $\{ \mu_A(x), \mu_A(y)\}$ and

$\gamma_A(x - y) \geq$ min $\{ \gamma_A(x), \gamma_A(y)\}$.

Where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in$ X.

Definition [4.2.3]

Let (X, +, −, 0) and (Y, +, −, 0) be two β-algebras. Let A = $\{\langle x, \mu_A(x), \gamma_A(x) \rangle / \ x \in$ X$\}$ and B = $\{\langle y, \mu_B(y), \gamma_B(y) \rangle / \ y \in$ Y$\}$ be AIF sets in X and Y. The Cartesian product of A and B, denoted by, A×B is defined to be the set

A×B = $\{\langle (\mu_A \times \mu_B)(x, y), (\gamma_A \times \gamma_B)(x, y)\rangle / x \in$ X × Y$\}$

where $(\mu_A \times \mu_B)$: X×Y → [0,1] is given by $(\mu_A \times \mu_B)(x, y)$ = max { $\mu_A(x), \mu_B(y)$} and $(\gamma_A \times \gamma_B)$: X × Y →[0,1] is given by $(\gamma_A \times \gamma_B)(x, y)$ = min{$\gamma_A(x), \gamma_B(y)$} for all x ∈ X and y ∈ Y.

Theorem [4.2.4]

Let A = {x ∈ X / $\mu_A(x), \gamma_A(x)$} and B = {y ∈ Y / $\mu_B(y), \gamma_B(y)$} be any two IF β-subalgebras of X and Y. Then $(\mu_A \times \mu_B)(x, y)$ and $(\overline{\gamma_A \times \gamma_B})(x, y)$ are fuzzy β-subalgebras X×Y.

Proof:

Let A×B be an IF β-sub algebras of X×Y.

Clearly, $(\mu_A \times \mu_B)(x, y)$ = min { $\mu_A(x), \mu_B(y)$} is fuzzy β-subalgebras of X×Y.

We have, $(\gamma_A \times \gamma_B)(x, y)$ = max {$\gamma_A(x), \gamma_B(y)$}

$\Rightarrow 1 - (\gamma_A \times \gamma_B)(x, y) = 1 - (\max\{\gamma_A(x), \gamma_B(y)\})$

$\Rightarrow (\overline{\gamma_A \times \gamma_B})(x, y) = \min\{(1 - \gamma_A(x)), (1 - \gamma_B(y))\}$

$\Rightarrow (\overline{\gamma_A \times \gamma_B})(x, y) = \min\{\overline{\gamma_A(x)}, \overline{\gamma_B(y)}\}$

Hence $(\mu_A \times \mu_B)(x, y)$ and $(\overline{\gamma_A \times \gamma_B})(x, y)$ are fuzzy β-subalgebras of X×Y.

Theorem [4.2.5]

If A×B is an IF β-subalgebra of X×X, then either A or B is an IF β-subalgebra of X×X.

Proof:

Since A×B is an IF β-subalgebra of X×X.

Take (x_1, y_1) and (x_2, y_2) ∈ X×X.

Then $(\mu_A \times \mu_B)\{(x_1, y_1) + (x_2, y_2)\} \geq \min\{(\mu_A \times \mu_B)(x_1, y_1), (\mu_A \times \mu_B)(x_2, y_2)\}$.

Put $x_1 = x_2 = 0$, we get

$(\mu_A \times \mu_B)\{(0, y_1) + (0, y_2)\} \geq \min\{(\mu_A \times \mu_B)(0, y_1), (\mu_A \times \mu_B)(0, y_2)\}$.

We have, $(\mu_A \times \mu_B)((0, 0) + (y_1, y_2)) \geq \min\{(\mu_A \times \mu_B)(0, y_1), (\mu_A \times \mu_B)(0, y_2)\}$.

So, $\mu_B(y_1 + y_2) \geq \min\{\mu_B(y_1), \mu_B(y_2)\}$.

Similarly, $\gamma_B(y_1 + y_2) \leq \max\{\gamma_B(y_1), \gamma_B(y_2)\}$ and also,

$$\mu_B(y_1 - y_2) \geq \min\{\mu_B(y_1), \mu_B(y_2)\}$$

$$\gamma_B(y_1 - y_2) \leq \max\{\gamma_B(y_1), \gamma_B(y_2)\}.$$

Hence B is a IF β-subalgebra of X.

Theorem [4.2.6]

In general, $A_1 \times A_2 \times \ldots \times A_n$ is an IF β-subalgebra of $X_1 \times X_2 \times \ldots \times X_n$ then any one of them A_i is an IF β-subalgebra of $X_1 \times X_2 \times \ldots \times X_n$.

Proof:

Using the above theorem

$$(\mu_{A_1 \times A_2 \times \ldots \times A_n})[(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)] = \{\mu_{A_1 \times A_2 \times \ldots \times A_n}((x_1 + y_1), \ldots, (x_n + y_n))\}$$

$$= \min\{\mu_{A_1}(x_1 + y_1), \ldots, \mu_{A_n}(x_n + y_n)\}$$

$$\geq \min\{\min\{\mu_{A_1}(x_1), \mu_{A_1}(y_1)\}, \ldots, \min\{\mu_{A_n}(x_n), \mu_{A_n}(y_n)\}.$$

Put $x_i = y_i = 0$, where i= 1, ... , n−1. So we get

$\mu_{A_n}(x_n + y_n) \geq \min\{\mu_{A_n}(x_n), \mu_{A_n}(y_n)\}$.

Similarly, $\gamma_{A_n}(x_n + y_n) \leq \max\{\gamma_{A_n}(x_n), \gamma_{A_n}(y_n)\}$.And also,

$\mu_{A_n}(x_n - y_n) \geq \min\{\mu_{A_n}(x_n), \mu_{A_n}(y_n)\}$, $\gamma_{A_n}(x_n - y_n) \leq \max\{\gamma_{A_n}(x_n), \gamma_{A_n}(y_n)\}$.

Hence A_n is IF β-subalgebra of X.

Theorem [4.2.7]

Let A and B be two AIF β-subalgebra of X and Y respectively. Then A×B is also an AIF β-subalgebras of X×Y.

Proof:

A = { x ∈ X / $\mu_A(x)$, $\gamma_A(x)$} and

B = { y ∈ Y/ $\mu_B(y)$, $\gamma_B(y)$} be AIF β-subalgebras in X and Y.

Take $x = (x_1, x_2)$, $y = (y_1, y_2)$ ∈ X×Y.

$(\mu_A \times \mu_B)(x + y) = (\mu_A \times \mu_B)\ ((x_1, x_2)+(y_1, y_2))$

$= (\mu_A \times \mu_B(x_1 + y_1), (x_2 + y_2))$

$= \max\ \{\mu_A(x_1 + y_1), \mu_B((x_2 + y_2)\}$

$\leq \max\ \{\max\ \{\mu_A(x_1), \mu_A(y_1)\}, \max\{\mu_B(x_2), \mu_B(y_2)\}\}.$

$= \max\ \{\max\ \{\mu_A(x_1),\mu_A(y_1)\}, \max\{ \mu_B(x_2), \mu_B(y_2)\}\}.$

$= \max\ \{(\mu_A \times \mu_B)\ (x_1, x_2), (\mu_A \times \mu_B)\ (y_1, y_2)\}$

$= \max\ \{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(y)\}.$

Similarly, $(\mu_A \times \mu_B)(x - y) \leq \max\ \{(\mu_A \times \mu_B)(x), (\mu_A \times \mu_B)(y)\}$ and also,

$(\gamma_A \times \gamma_B)(x + y) = (\gamma_A \times \gamma_B)\ ((x_1, x_2)+(y_1, y_2))$

$= (\gamma_A \times \gamma_B)((x_1 + y_1), (x_2 + y_2))$

$= \min\{\gamma_A(x_1 + y_1), \gamma_B((x_2 + y_2)\}$

$\geq \min\{\min\ \{\gamma_A(x_1), \gamma_A(y_1)\}, \min\{\gamma_B(x_2), \gamma_B(y_2)\}\}.$

$= \min\{\min\ \{\gamma_A(x_1), \gamma_B(x_2)\}, \min\{\gamma_A(y_1), \gamma_B(y_2)\}\}.$

$= \min\{(\gamma_A \times \gamma_B)\ (x_1, x_2), (\gamma_A \times \gamma_B)\ (y_1, y_2)\}$

$= \min\{(\gamma_A \times \gamma_B)(x), (\gamma_A \times \gamma_B)(y)\}.$

Similarly, we provide that $(\gamma_A \times \gamma_B)(x - y) \geq \min\{(\gamma_A \times \gamma_B)(x), (\gamma_A \times \gamma_B)(y)\}$.

Hence the Cartesian product of A×B is also an AIF β-subalgebra of X×Y.

Corollary [4.2.8]

Let A and B be two AIF β-subalgebras of X and Y respectively. Then the $\overline{A \times B}$ is also an IF β-subalgebra of X×Y.

Proof:

By the above theorem, A×B is an AIF β-subalgebra of X×Y. This implies that $\overline{A \times B}$ is an IF β-subalgebras of X×Y.

Theorem [4.2.9]

Let A = $\{x \in X / \mu_A(x), \gamma_A(x)\}$ and B = $\{y \in Y / \mu_B(y), \gamma_B(y)\}$ be any two IF β-subalgebras of X and Y. Then $(\overline{\mu_A \times \mu_B})(x, y)$ and $(\gamma_A \times \gamma_B)(x, y)$ are anti fuzzy β-subalgebras of X×Y.

Proof:

It can be proved as the theorem[4.2.7].

Definition [4.2.10]

Let A = $\{\langle x, \mu_A(x), \gamma_A(x) \rangle / x \in X\}$ be an IFS defined on β-algebra X. Then ΘA = $\{\langle x, \mu_A(x), \overline{\mu_A(x)} \rangle\}$ and ◊A = $\{\langle x, \overline{\gamma_A(x)}, \gamma_A(x) \rangle\}$ in X.

Theorem [4.2.11]

Let A=$\{x \in X / \mu_A(x), \gamma_A(x)\}$ and B=$\{y \in Y / \mu_B(y), \gamma_B(y)\}$ be IF β-subalgebras X and Y. Then

Θ(A×B) = $\{\langle (x, y), (\mu_A \times \mu_B)(x, y), (\overline{\mu_A \times \mu_B})(x, y) \rangle\}$ and

◊(A×B) = $\{\langle (x, y), (\overline{\gamma_A \times \gamma_B})(x, y), (\gamma_A \times \gamma_B)(x, y) \rangle\}$ are IF β-subalgebra of X×Y.

Proof:

Let A and B be IF β-subalgebra of X andY. Then $x, y \in$ X×Y.

i). Θ(A×B) = {⟨ (x, y),($\mu_A \times \mu_B)(x, y)$,($\overline{\mu_A \times \mu_B}$)(x, y)⟩}

clearly, $(\mu_A \times \mu_B)$ is also an IF β-subalgebra of X×Y.

Now, $(\mu_A \times \mu_B)(x, y)$ = min $\{\mu_A(x), \mu_B(y)\}$.

$1-(\mu_A \times \mu_B)(x, y) = 1 - \{\min \{\mu_A(x), \mu_B(y)\}\}$

$(\overline{\mu_A \times \mu_B})(x, y) = \max \{(1 - \overline{\mu_A(x)}), (1 - \overline{\mu_B(y)})\}$

$= \max \{\mu_A(x), \mu_B(y)\}$.

ii). ◊(A×B) = {⟨ (x, y), ($\overline{\gamma_A \times \gamma_B}$)(x, y), ($\gamma_A \times \gamma_B)(x, y)$⟩}

clearly, $(\gamma_A \times \gamma_B)$ is also an IF β-sub algebra of X×Y.

Now, $(\gamma_A \times \gamma_B)(x, y)$ = max $\{\gamma_A(x), \gamma_B(y)\}$.

$1-(\gamma_A \times \gamma_B)(x, y) = \min \{(1 - \gamma_A(x)), (1 - \gamma_B(y))\}$

$(\overline{\gamma_A \times \gamma_B})(x, y) = \min \{\overline{\gamma_A(x)}), \overline{\gamma_B(y)}\}$

Hence Θ(A×B) and ◊(A×B) are IF β-subalgebra of X×Y.

Result:

(i). Let A = { $x \in$ X / $\mu_A(x)$, $\gamma_A(x)$} and B = { $y \in$ Y/ $\mu_B(y)$, $\gamma_B(y)$} be AIF β-subalgebra of X and Y. Then Θ($\overline{A \times B}$) and ◊($\overline{A \times B}$) are AIF β-subalgebra of X×Y.

(ii). Let A_i= { $x \in X_i$ / $\mu_{A_i}(x)$, $\gamma_{A_i}(x)$} be IF β-subalgebra of X_i, i=1,...,n.

Then $\prod_{i=1}^{n} A_i$ is called direct product of finite IF β-subalgebra of $\prod_{i=1}^{n} X_i$, if

$\prod_{i=1}^{n} \mu_{A_i} (x_i, y_i)$ = min { $\prod_{i=1}^{n} \mu_{A_i} (x_i)$, $\prod_{i=1}^{n} \mu_{A_i} (y_i)$} and

$\prod_{i=1}^{n}\gamma_{A_i}(x_i, y_i)$ = max { $\prod_{i=1}^{n}\gamma_{A_i}(x_i)$, $\prod_{i=1}^{n}\gamma_{A_i}(y_i)$}.

Definition [4.2.12]

Let A_i= {⟨ x , $\mu_{A_i}(x)$, $\gamma_{A_i}(x)$ ⟩/x ∈ X_i} be n IF β-subalgebra of X_i, i=1,...,n.

Then Θ($\prod_{i=1}^{n}A_i$) and ◊($\prod_{i=1}^{n}A_i$) is called direct product of finite IF β-subalgebra, if

$$\Theta(\prod_{i=1}^{n}A_i) = \langle x_i / \prod_{i=1}^{n}\mu_{A_i} \ / \ \prod_{i=1}^{n}\overline{\mu_{A_i}} \rangle \text{ and } \Diamond(\prod_{i=1}^{n}A_i) = \langle x_i / \prod_{i=1}^{n}\overline{\gamma_{A_i}} \ / \prod_{i=1}^{n}\gamma_{A_i} \rangle.$$

Theorem [4.2.13]

If A_i= { x ∈ X_i/ $\mu_{A_i}(x)$, $\gamma_{A_i}(x)$} be an IF β-subalgebra of X_i respectively, for i=1,...,n. Then $\prod_{i=1}^{n}A_i$ is an IF β-subalgebra of $\prod_{i=1}^{n}X_i$

Proof:

Let A_i= { x ∈ X_i/ $\mu_{A_i}(x)$, $\gamma_{A_i}(x)$} be an IF β-subalgebra of X_i.

Let $(x_1, x_2,..., x_n)$ and $(y_1, y_2,..., y_n) \in \prod_{i=1}^{n}X_i$.

Then, $\prod_{i=1}^{n}\mu_{A_i}((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n))= \prod_{i=1}^{n}\mu_{A_i}(x_1 + y_1,...., x_n + y_n)$

$= \min\{\mu_{A_1}(x_1 + y_1),..., \mu_{A_n}(x_n + y_n)\}$

$\geq \min\{\min(\mu_{A_1}(x_1), \mu_{A_1}(y_1)),..., \min(\mu_{A_n}(x_n), \mu_{A_n}(y_n))\}$

$= \min\{\min(\mu_{A_1}(x_1),..., \mu_{A_n}(x_n)\}\min\{(\mu_{A_1}(y_1),...,\mu_{A_n}(y_n)\}$

$= \min\{\prod_{i=1}^{n}\mu_{A_i}(x_i), \prod_{i=1}^{n}\mu_{A_i}(y_i)\}$ and

$\prod_{i=1}^{n}\gamma_{A_i}((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n))= \prod_{i=1}^{n}\gamma_{A_i}(x_1 + y_1,...., x_n + y_n)$

$= \max \{\gamma_{A_1}(x_1 + y_1),..., \gamma_{A_n}(x_n + y_n)\}$

$\leq \max\{\max(\gamma_{A_1}(x_1), \gamma_{A_1}(y_1)),..., \max(\gamma_{A_n}(x_n), \gamma_{A_n}(y_n))\}$

$= \max\{\max(\gamma_{A_1}(x_1),..., \gamma_{A_n}(x_n)\}\max\{(\gamma_{A_1}(y_1),...,\gamma_{A_n}(y_n)\}$

$= \max\{\prod_{i=1}^{n}\gamma_{A_i}(x_i), \prod_{i=1}^{n}\gamma_{A_i}(y_i)\}$

Similarly, we can prove that

$$\prod_{i=1}^{n}\mu_{A_i}((x_1, x_2,..., x_n) - (y_1, y_2,..., y_n)) \geq \min\{\prod_{i=1}^{n}\mu_{A_i}(x_i), \prod_{i=1}^{n}\mu_{A_i}(y_i)\}$$

$$\prod_{i=1}^{n}\gamma_{A_i}((x_1, x_2,..., x_n) -(y_1, y_2,..., y_n)) \geq \max\{\prod_{i=1}^{n}\gamma_{A_i}(x_i), \prod_{i=1}^{n}\gamma_{A_i}(y_i)\}.$$

Hence $\prod_{i=1}^{n}A_i$ is an IF β-sub algebra of $\prod_{i=1}^{n}X_i$.

Result:

Let $A_i = \{ x \in X_i / \mu_{A_i}(x), \gamma_{A_i}(x)\}$ be an AIF β-subalgebra of X_i respectively, for i= 1,...,n. Then $\prod_{i=1}^{n}A_i$ is an AIF β-subalgebra of $\prod_{i=1}^{n}X_i$.

Theorem [4.2.14]

Let $A_i = \{ x \in X_i / \mu_{A_i}(x), \gamma_{A_i}(x)\}$ be an IF β-subalgebra of X_i, where i=1,...,n. Then $\prod_{i=1}^{n}\overline{A_i}$ is an AIF β-subalgebra of $\prod_{i=1}^{n}X_i$

Proof:

By using the above theorem,

$$\prod_{i=1}^{n}\mu_{A_i}((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n) \geq \min\{\prod_{i=1}^{n}\mu_{A_i}(x_i), \prod_{i=1}^{n}\mu_{A_i}(y_i)\}$$

$$1- \{\prod_{i=1}^{n}\mu_{A_i}((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\} \leq 1- \{\min\{\prod_{i=1}^{n}\mu_{A_i}(x_i), \prod_{i=1}^{n}\mu_{A_i}(y_i)\}\}$$

$$\prod_{i=1}^{n}\overline{\mu_{A_i}}\ \{(x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\}\leq \max\ \{1-\prod_{i=1}^{n}\mu_{A_i}\ (x_i),\ 1-\ \prod_{i=1}^{n}\mu_{A_i}\ (y_i)\}\}$$

$$\prod_{i=1}^{n}\overline{\mu_{A_i}}\ \{(x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\}\leq \max\ \{\prod_{i=`1}^{n}\overline{\mu_{A_i}}\ (x_i), \prod_{i=`1}^{n}\overline{\mu_{A_i}}\ (y_i)\}\}\ \text{and}$$

$$\prod_{i=1}^{n}\gamma_{A_i}\ ((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\leq \max\{\prod_{i=1}^{n}\gamma_{A_i}\ (x_i),\ \prod_{i=1}^{n}\gamma_{A_i}\ (y_i)\}$$

$$1-\{\ \prod_{i=1}^{n}\gamma_{A_i}\ ((x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\}\geq 1-\{\ \max\{\prod_{i=1}^{n}\gamma_{A_i}\ (x_i),\ \prod_{i=1}^{n}\gamma_{A_i}\ (y_i)\}\}$$

$$\prod_{i=1}^{n}\overline{\gamma_{A_i}}\ \{(x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\}\geq \min\ \{1-\prod_{i=1}^{n}\gamma_{A_i}\ (x_i),\ 1-\ \prod_{i=1}^{n}\gamma_{A_i}\ (y_i)\}\}$$

$$\prod_{i=1}^{n}\overline{\gamma_{A_i}}\ \{(x_1, x_2,..., x_n) + (y_1, y_2,..., y_n)\}\geq \min\ \{\prod_{i=`1}^{n}\overline{\gamma_{A_i}}\ (x_i),\ \prod_{i=`1}^{n}\overline{\gamma_{A_i}}\ (y_i)\}\}$$

Hence $\prod_{i=1}^{n}\overline{A_i}$ is also an AIF β-subalgebra of $\prod_{i=1}^{n}X_i$

The following results can be proved easily

Result:

(i). Let A_i={ x ∈ X_i/ $\mu_{A_i}(x)$, $\gamma_{A_i}(x)$} be an AIF β-subalgebras of X_i respectively, for i= 1,...,n. Then $\prod_{i=1}^{n}\overline{A_i}$ is an IF β-sub algebra of $\prod_{i=1}^{n}X_i$

(ii). Let A_i = {⟨ $x, \mu_{A_i}(x)$, $\gamma_{A_i}(x)$⟩ /x ∈ X_i} be an IF β-subalgebras of X_i. Then Θ($\prod_{i=1}^{n}A_i$) and ◊($\prod_{i=1}^{n}A_i$) are an IF β-subalgebras of $\prod_{i=1}^{n}X_i$

Section 4.3

Intuitionistic Fuzzy α-Translation on β-Algebras

Definition [4.3.1]

Let A = { $x \in$ X / $\mu_A(x)$, $\gamma_A(x)$} be an intuitionistic fuzzy set of an β-algebra X and α ∈ (0, T) where T= [0,1 – sup{ $\mu_A(x)$+ $\gamma_B(x)$} / ∀ $x \in$ X }]. A mapping $A^T{}_{\alpha} = \{(\mu_A)^T_\alpha, (\gamma_A)^T_\alpha\}$ where $(\mu_A)^T_\alpha$: X → [0,1] and $(\gamma_A)^T_\alpha$: X → [0,1] is called an intuitionistic fuzzy α-translation of A, if it satisfies the following conditions:

i). $(\mu_A)^T_\alpha (x) = \mu_A(x) + \alpha$

ii). $(\gamma_A)^T_\alpha(x) = \gamma_A(x) - \alpha$ ∀ $x \in$ X.

Definition [4.3.2]

Let X be an β-algebra. Let A be an intuitionistic fuzzy set of X and α ∈ [0, T] where T= [0,1 – sup{ $\mu_A(x)$+ $\gamma_A(x)$} / ∀ $x \in$ X }]. A mapping $A^T_\alpha = \{(\mu_A)^T_\alpha, (\gamma_A)^T_\alpha\}$ where $(\mu_A)^T_\alpha$: X → [0,1] and $(\gamma_A)^T_\alpha$: X → [0,1] is called an intuitionistic fuzzy α-translation on β-subalgebra of A, if it satisfies the following conditions:

i). $(\mu_A)^T_\alpha (x+y) = \mu_A(x+y) + \alpha$ and $(\mu_A)^T_\alpha (x-y) = \mu_A(x-y) + \alpha$

ii). $(\gamma_A)^T_\alpha(x+y) = \gamma_A(x+y) - \alpha$ and $(\gamma_A)^T_\alpha(x-y) = \gamma_A(x-y) - \alpha$ ∀ $x \in$ X.

Example [4.3.3]

Consider β-algebra X in Example[1.1.5].

Define A = { $x \in$ X / $\mu_A(x)$, $\gamma_A(x)$} where

$\mu_A(x) = \begin{cases} 0.7 & x = 0,1 \\ 0.6 & otherwise \end{cases}$ and

$\gamma_A(x) = \begin{cases} 0.1 & x = 0,1 \\ 0.3 & otherwise \end{cases}$

Let α = 0.05, then

$A^T_\alpha = \{(\mu_A)^T_\alpha(x), (\gamma_A)^T_\alpha(x)\}$ where

$(\mu_A)^T_\alpha(x) = \begin{cases} 0.12 \; x = 0,1 \\ 0.11 \; otherwise \end{cases}$ and

$(\gamma_A)^T_\alpha(x) = \begin{cases} 0.05 \; x = 0,1 \\ 0.25 \; otherwise \end{cases}$

Hence $(A)^T_\alpha$ is an intuitionistic fuzzy α-translation on β-subalgebra of X.

Theorem [4.3.4]

If A is an IF β-subalgebra of a β-algebra X, then A is an intuitionsitic fuzzy α-translation of X for all α ∈[0,1 – sup{ $\mu_A(x)$+ $\gamma_A(x)$}.

Proof:

Let x, y ∈ X and α ∈[0,1 – sup{ $\mu_A(x)$+ $\gamma_A(x)$}.

Then,

$\mu_A(x+y) \geq$ min { $\mu_A(x)$, $\mu_A(y)$} and $\gamma_A(x+y) \leq$ max { $\gamma_A(x)$, $\gamma_A(y)$}

Now, $(\mu_A)^T_\alpha(x+y) = \mu_A(x+y) + \alpha$

$\geq$ min { $\mu_A(x)$, $\mu_A(y)$} + α

= min { $\mu_A(x)$ + α, $\mu_A(y)$ + α}

= min {$(\mu_A)^T_\alpha$ (x), $(\mu_A)^T_\alpha(y)$}

Similarly, $(\mu_A)^T_\alpha$ $(x-y) \geq$ min {$(\mu_A)^T_\alpha$ (x), $(\mu_A)^T_\alpha$ (y)} and

$(\gamma_A)^T_\alpha$ $(x+y) = \gamma_A(x+y) - \alpha$

$\leq$ max { $\gamma_A(x)$, $\gamma_A(y)$} − α

= max { $\gamma_A(x)$ + α, $\gamma_A(y)$− α}

= max {$(\gamma_A)^T_\alpha$ (x), $(\gamma_A)^T_\alpha(y)$}

Similarly, $(\gamma_A)^T_\alpha(x-y) \leq$ max {$(\gamma_A)^T_\alpha$ (x), $(\gamma_A)^T_\alpha(y)$}.

Hence A is IF α-translation on β-subalgebra of X.

The following results can be proved easily

Result:

(i). If A_α^T = $\{(\mu_A)_\alpha^T, (\gamma_A)_\alpha^T\}$ is an intuitionistic fuzzy α-translation on β-subalgebra of X, then A is an IF β-subalgebra of X.

(ii). If A_α^T is an intuitionistic fuzzy α-translation on β-sub algebra of β-algebra X, then

i). $(\mu_A)_\alpha^T\,(x) \le (\mu_A)_\alpha^T\,(0)$

ii). $(\gamma_A)_\alpha^T(x) \ge (\gamma_A)_\alpha^T(0)$.

(iii). If A_α^T is an intuitionistic fuzzy α-translation on β-subalgebra of β-algebra X, then

i). $(\mu_A)_\alpha^T(x) \le (\mu_A)_\alpha^T\,(x-0)$

ii). $(\gamma_A)_\alpha^T(x) \ge (\gamma_A)_\alpha^T(x-0)$.

Definition [4.3.5]

Let f: X →Y be a function. Let A and B be two IF α-translation on β-subalgebras in X and Y respectively. Then inverse image of B under f is defined by f^{-1}(B) = $\{f^{-1}(\mu_B)_\alpha^T(x), f^{-1}(\gamma_B)_\alpha^T(x)\ /x \in$ X} suchthat

$$f^{-1}(\mu_B)_\alpha^T(x) = \mu_B(f(x)+\alpha) \text{ and } f^{-1}(\gamma_B)_\alpha^T(x) = \gamma_B(f(x)-\alpha).$$

Theorem [4.3.6]

Let X and Y be two β-algebras. Let A and B be two IF α-translation on β-subalgebras. Let f: X →Y be a homomorphism. If A is an IF α-translation on β-subalgebra of Y. Then f^{-1}(A) is a IF α-translation on β-subalgebra of X.

Proof:

Let A be an IF α-translation on β-subalgebra of Y and $x, y \in$ Y.

Then, $f^{-1}(\mu_A)_\alpha^T(x+y) = f^{-1}(\mu_A)(x+y)+\alpha$

$=\mu_A\,(f(x+y))+\alpha$

$=\mu_A(f(x)+f(y))+\alpha$

$\geq \min \{ \mu_A(f(x) + \alpha), \mu_A(f(y) + \alpha)\}$

$= \min \{f^{-1}(\mu_A)_\alpha^T(x), f^{-1}(\mu_A)_\alpha^T(y)\}$

Also, $f^{-1}(\mu_A)_\alpha^T(x - y) \geq \min \{f^{-1}(\mu_A)_\alpha^T(x), f^{-1}(\mu_A)_\alpha^T(y)\}$.

Similarly, $f^{-1}(\gamma_A)_\alpha^T(x + y) \leq \max\{f^{-1}(\gamma_A)_\alpha^T(x), f^{-1}(\gamma_A)_\alpha^T(y)\}$ and

$f^{-1}(\gamma_A)_\alpha^T(x - y) \leq \max\{f^{-1}(\gamma_A)_\alpha^T(x), f^{-1}(\gamma_A)_\alpha^T(y)\}$.

Hence f^{-1}(A) is an IF α-translation on β-subalgebra of X.

Theorem [4.3.7]

Let X and Y be two β-algebras. Let A and B be two IF α-translation on β-subalgebras. Let f: X →Y be a epimorphism. If A is an IF α-translation on β-subalgebras of X. Then f(A) is IF α-translation on β-subalgebra of Y.

That is $f(\mu_B)_\alpha^T(x) = \mu_B(f(x) + \alpha)$ and $f(\gamma_B)_\alpha^T(x) = \gamma_B(f(x) - \alpha)$.

Proof:

Let A be an IF α-translation on β-sub algebra of Y and $x, y \in$ Y.

Then, $f(\mu_A)_\alpha^T(x + y) = f(\mu_A(x + y)) + \alpha$

$= \mu_A(f(x + y)) + \alpha$

$= \mu_A(f(x) + f(y)) + \alpha$

$\geq \min \{ \mu_A(f(x) + \alpha), \mu_A(f(y) + \alpha)\}$

$= \min \{f(\mu_A)_\alpha^T(x), f(\mu_A)_\alpha^T(y)\}$

Also, $f(\mu_A)_\alpha^T(x - y) \geq \min \{f(\mu_A)_\alpha^T(x), f(\mu_A)_\alpha^T(y)\}$.

Similarly, $f(\gamma_A)_\alpha^T(x + y) \leq \max\{f(\gamma_A)_\alpha^T(x), f(\gamma_A)_\alpha^T(y)\}$ and

$f(\gamma_A)_\alpha^T(x - y) \leq \max\{f(\gamma_A)_\alpha^T(x), f(\gamma_A)_\alpha^T(y)\}$.

Hence f (A) is an IF α-translation on β-sub algebra of Y.

SUMMARY AND CONCLUSION

SUMMARY AND CONCLUSION

The notion of BCK-algebras was proposed by Imai and Iseki [17] in 1966. In the same year, Iseki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. Since then numerous mathematical papers have been written investigating the algebraic properties of the BCK/BCI-algebras. After that, many researchers considered the publication of ideals and subalgebras in BCK/BCI-algebras. In 2002, Neggers and Kim [26] introduced the notion of B-algebras, which is another generalization of BCK-algebras. Also they introduced the notion of β-algebras[27] where two operations are coupled in such a way as to reflect the natural coupling, which exists, between the usual group operation and its associated B-algebras, which is naturally defined by the group.

The concept of fuzzy sets was introduced initially by Zadeh [52] in 1965, several researchers explored on the generalization of the notion of fuzzy sets. In 1986, the concept of intuitionistic fuzzy set was first published by Atanassov [10], as a generalization of the notion of fuzzy sets.

In this thesis, we have made an attempt to give a discussion about the Fuzzy β-Subalgebras and Fuzzy β-Ideals in β-Algebras.

In chapter 1, the preliminary definitions and results on β-algebras and fuzzy sets are presented due to Neggers et al [27] and Zadeh[52]. The properties of level β-subalgebras and normal fuzzy β-subalgebras of a β-algebra are established due to Abu Ayub Ansari and Chandramouleeswaran.[3].

In chapter 2, the notion of fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy β-ideals of β-algebras are introduced and some of their properties are investigated due to Abu Ayub Ansari and Chandramouleeswaran.[4].

In chapter 3, the properties of L-fuzzy β-subalgebras, L-fuzzy β-ideals and L-fuzzy T-ideals of a β-algebras are discussed due to Rajam and Chandramouleeswaran.[29,30,31,32].

In the last chapter, the properties of intuitionistic fuzzy β-subalgebras and the product of intuitionistic fuzzy β-subalgebras of a β-algebra are discussed. Also the concept of an intuitionistic fuzzy α-translation for some $\alpha \in [0,1]$ on β-subalgebras of a β-algebras are studied and investigated some of their properties due to Sujatha et al. [44,45,47].

We hope that, a deep study of Fuzzy β-Subalgebras and Fuzzy β-Ideals in β-Algebras can be extended to some other algebraic structures. So it provides a lot of scope for further research.

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