Generalisation of Continuity to Bitopological Spaces

By

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CONTENTS

INTRODUCTION

INTRODUCTION

The aim of this dissertation is to study the following two articles.

1. "SEMI OPEN SETS, SEMI CONTINUITY AND SEMI OPEN MAPPINGS IN BITOPOLOGICAL SPACES" by SHANTHA BOSE.

2. "PAIRWISE ALMOST CONTINUOUS MAP AND WEAKLY CONTINUOUS MAP IN BITOPOLOGICAL SPACES" by SHANTHA BOSE AND DIPTI SINHA.

In topological spaces continuity and open sets play an important role. Semi open sets, Semi - continuity, Semi-open mappings are generalisations of open sets, continuity and open mappings respectively. In 1963 N.Levine introduced the notions of semi open sets and semi-continuity in topological spaces and obtained a number of their properties. In 1969 N.Biswas introduced the concepts of semi-open mappings and investigated a few properties of such a mapping. In 1973 Takashi Noiri investigated a few other properties of semi-open sets and semi-open mappings. In 1963 J.C.Kelly introduced the concepts of bitopological spaces and in 1979 Shantha Bose introduced the concepts of Semi-open sets, Semi-continuity and Semi open mappings in bitopological spaces.

In the first chapter of this dissertation we study the generalisation of the following results to bitopological spaces.

(i) A set $A \subset X$ is semi-open if and only if $\overline{A} = \overline{A}^{\circ}$

(ii) If A is semi-open and $A \subset B \subset \overline{A}$, then B is semi open

(iii) Any continuous open map takes semi open sets to semi open sets.

(iv) Any semi continuous map of a semi continuous map is not semi continuous etc in bitopological spaces and finally we have the characterisation of semi-open mappings in bitopological spaces.

In chapter II two, more extensions of continuity namely almost continuity and weakly continuity are studied in bitopological spaces. This was introduced by Shantha Bose and Dipti Sinha in 1979.

Almost continuity is a generalisation of continuity introduced by M.K. Singal and A.R.Singal in 1968. In 1961 N.Levine introduced the concept of weakly continuous mappings in topological spaces and obtained a number of properties. Further properties of weakly continuous mappings are studied by M.K.Singal, A.R.Singal and T.Noiri. In this chapter necessary and sufficient conditions for a mapping to be pairwise almost continuous are studied and proved weakly continuous image of a pairwise connected space is pairwise connected and the image of a pairwise normal space under a closed pairwise almost continuous - surjective mapping is pairwise almost normal.

CHAPTER I

CHAPTER I

SEMI OPEN SETS, SEMI CONTINUITY AND SEMI OPEN MAPPINGS IN BITOPOLOGICAL SPACES

Section : 1.1

Semi Open Sets :

Definition : 1.1.1

Let (X,P,L) be a bitopological space. A is P semi open with respect to L if and only if there exists P open set $O \subset X$ such that $O \subset A \subset L$ cl O. Similarly $A \subset X$, is a L semi open set with respect to P if and only if there exists a L open set $O \subset X$ such that $O \subset A \subset P$ cl O.

A is semi open if and only if it is both P semi open with respect to L and L semi open with respect to P.

Theorem : 1.1.2

Let (X,P,L) be a bitopological space. Let $A \subset X$ then A is P semi open with respect to L if and only if L cl A = L cl(PInt. A) and A is L semi open with respect to P if and only if P cl. A = P cl. (L Int. A).

Proof :

Let A be P semi open with respect to L, then there exists a P open set O such that $O \subset A \subset L$ cl O.

Also $O \subset P$ Int. A

L cl $O \subset L$ cl (P Int. A)

$A \subset L$ cl (P Int. A)

L cl $A \subset L$ cl (P Int. A)........(1)

Also L cl (P Int. A) $\subset L$ cl A........(2)

From (1) and (2) L cl $A = L$ cl (P Int.A)

Conversely, we have,

P Int. $A \subset A \subset L$ cl $A = L$ cl (P Int. A)

Now put $O = P$ Int. A

Therefore $O \subset A \subset L$ cl O

Therefore A is P semi open with respect to L.

Similarly we can show that A is L semi open with respect to P if and only if P cl $A = P$ cl (L Int. A).

Corollary : 1.1.3

If A is P semi open set with respect to L and $A \neq \emptyset$ then P Int. $A \neq \emptyset$ where $A \subset X$ and (X,P,L) a bitopological space.

Proof :

Given $A \neq \emptyset \Rightarrow L$ cl $A \neq \emptyset$

Suppose P Int. $A = \emptyset$ then L cl (P Int. A) $= \emptyset$ which is contradiction to L cl $A = L$ cl (P Int. A)

Therefore P Int. $A \neq \emptyset$

Corollary : 1.1.4

Let A be P semi open with respect to L and $A \subset B$ then $A \subset L$ cl (P Int.. B).

Proof :

Given $A \subset B$

P Int. $A \subset P$ Int. B

L cl (P Int. A) $\subset L$ cl (P Int. B)

L cl $A \subset L$ cl (P Int. B)

Therefore $A \subset L$ cl $A \subset L$ cl (P Int. B)

Therefore $A \subset L$ cl (P Int. B)

Theorem : 1.1.5

Let $\{A_\alpha\}_{\alpha \varepsilon \Delta}$ be a collection of P semi open with respect to L in a bitopological space (X,P,L) then $\bigcup_{\alpha \varepsilon \Delta} A_\alpha$ is P semi open with respect to L.

Proof :

For each $\alpha \varepsilon \Delta$ there exists a P open set O_α such that

$O_\alpha \subset A_\alpha \subset L$ cl O_α

Then $\bigcup_{\alpha \varepsilon \Delta} O_\alpha \subset \bigcup_{\alpha \varepsilon \Delta} A_\alpha \subset \bigcup_{\alpha \varepsilon \Delta} L$ cl $O_\alpha = L$ cl $\bigcup_{\alpha \varepsilon \Delta} O_\alpha$

Put $O = \bigcup_{\alpha \varepsilon \Delta} O_\alpha$

Then O is P open and $O \subset \bigcup_{\alpha \varepsilon \Delta} A_\alpha \subset L$ cl O

Therefore $\bigcup_{\alpha \varepsilon \Delta} A_\alpha$ is P semi open with respect to L.

Theorem : 1.1.6

Let A be P semi open with respect to L in a bitopological space (X,P,L) and let $A \subset B \subset L$ cl A. Then B is P semi open with respect to L.

Proof :

Given A is P semi open with respect to L.

⇒ There exists a P open set $O \subset X$ such that $O \subset A \subset L$ cl O

Given $A \subset B \subset L$ cl A we have $O \subset B$

But L cl $A \subset L$ cl O

Therefore $B \subset L$ cl O

Hence $O \subset B \subset L$ cl O

Therefore B is P semi open with respect to L.

Remark : 1.1.7

A P open (L open) set is P semi open with respect to L (L semi open with respect to P). The converse is not true in general is shown in the following :

Example : 1.1. 8

Let X be the real plane and T the usual topology on X and (X, T_1) the half open rectangle space

$A = \{(x, y) / 0 \le x \le 1\} \cup (1,1)$ is

$0 \le y \le 1 \}$

T_1 semi open with respect to T but neither T_1 open nor T open.

Since $(0,1) \times (0,1) \subset [0,1) \times [0,1) \subset [0,1] \times [0,1]$

Since the point (1,1) is included, A_1 is not T_1 open and since the point (0,0) is not included A is T open.

Example : 1.1.9

Let X be the Real line and P, the usual toplogy on X and L the lower limit topology on X.

$A = \{ x/0 \leq x \leq 1\}$ is neither P open nor L open but is L semi open with respect to P.

Since A is closed it can be neither P open nor L open. Since $[0,1) \subset [0,1] \subset [0,1]$, A isisLsemi open with respect to P.

Note ;

Let us denote the class of P semi open sets with respect to L by P S.O $(X)_L$ and the class of all P open sets T_p.

Theorem :1.1.10

Let $\mathbf{B} = \{B_\alpha\}$ be a collection of sets in X such that (i) $T_p \subset$ B and (ii) $B \in \mathbf{B}$ and $B \subset D \subset L$ cl B then $D \in \mathbf{B}$ then P S.O $(X)_L \subset \mathbf{B}$.

Proof :

Let $A \in P\ S.O\ (X)_L \Rightarrow O \subset A \subset L\ \text{cl}\ O$ for some P open set $O \subset X$.

Given $T_p \subset \mathbf{B}$. Therefore we get $O \in \mathbf{B}$
Given that if $B \in \mathbf{B}$, $B \subset D \subset L\ \text{cl}\ B$ then $D \in \mathbf{B}$

Therefore we get $A \in \mathbf{B}$ by (ii)

Therefore $P.S.O.\ (X)_L\ \ \mathbf{B}$

Hence the theorem.

Theorem : 1.1.11

Let (X,P,L) be a bitopological space and let $A \subset Y \subset X$. If A is semi open in X, it is semi open in Y.

Proof :

A is semi open in X $\Rightarrow$ There exist an open set O such that $O \subset A \subset L\ \text{cl}_x\ O$

Since $O \subset A$ and $A \subset Y$ we have $O \subset Y$ thus $O = O \cap Y$

$$\begin{aligned} &\subset A \cap Y \\ &\subset Y \cap L\ \text{cl}_x \end{aligned}$$

or $O \subset A \subset L\text{cl}_y\ O$

Theorem : 1.1.12

Let (X,P,L) be a bitopolgical space.
Let $A \in P\ S.O(X)_L$. Then $A = O \cup B$ where (i) $O \in T_p$
(ii) $O \cap B = \emptyset$ (iii) B is L nowhere dense.

Proof :

$A \varepsilon$ P.S.O. $(X)_L \Rightarrow$ there exists a P open set O such that $O \subset A \subset L$ cl O.

But $A = O \cup (A \sim O)$ If we take $B = A \sim O$ then $A = O \cup B$

Therefore $A \subset L$ cl O

ie., $A \sim O \subset L$ cl $O \sim O$

$B \subset L$ cl $O \sim O$

Since O is open we get L cl $O \sim O$ is L nowhere dense and hence B is L nowhere dense. And (i) and (ii) are obvious.

The converse of the theorem is false is shown by the following example.

Example : 1.1.13

Let X = Reals and T_1, the lower limit topology on X and T_2 the usual topology on X.

Let $A = \{x / 0 \leq x < 1\} \cup \{2\}$. $\{x / 0 \leq x < 1\}$ is T_1 open and $\{2\}$ nowhere dense (i), (ii) and (iii) of the theorem 1.1.12 are true but $A \notin T_1$ S.O. $(x)_{T_2}$

For $\{x / 0 \leq x < 1\} \subset A$ but $A \not\subset \{2\}$.

Therefore $A \notin T_1$ S.O. $(x)_{T_2}$

The converse of the theorem 1.1.12 is not true even if pairwise connectedness is imposed upon A is shown by the following example.

Example : 1.1.14

Let (X,T,T_1) be the space where X is the real plane and T the usual topology on X and (X,T_1) the half open rectangle space.

Let $A = \{ (x,y) / 0 \le x < 1 \} \cup \{ (x,0) \quad 1 \le x \le 2 \}$

$0 \le y < 1 \}$

A is pairwise connected condition (i), (ii), (iii) of theorem 1.1.12 are true but $A \notin T_1$ S.O. $(X)_{T_2}$

For $\{ (x,y) / 0 \le x < 1 \} \subset A$

$0 \le y < 1 \}$

but $A \not\subset \{ (x,0) / 1 \le x \le \mu 2 \}$

Therefore $A \notin T_1$ S.O.$(X)_{T_2}$

Theorem : 1.1.15

Let (X,P,L) be a bitopological space and let $A = O \cup B$ where (i) $O \neq \emptyset$ is P open

(ii) A is pairwise connected

(iii) $B'^{\,P}$ the derived set of B with respect to the topology P is empty. Then A is P semi open with respect to L.

Proof :

Given $A = O \cup B$ and $O \neq \emptyset$

To prove : $A \varepsilon$ P.S.O. $(X)_L$ ie., $O \subset A \subset L$ cl O

It is sufficient to show that $B \subset L$ cl O

Assume $B \not\subset L$ cl O $\Rightarrow B \cap L$ cl $O = \emptyset$

$= B \cap (X \sim L$ cl $O) \neq \emptyset$

If $B = B_1 \cup B_2$ where $B_1 \subset L$ cl O and $B_2 \subset (x \sim L$ cl O)

then $B_2 \neq \emptyset$ Now $A = (OUB_1) \cup B_2$ where $OUB_1 \neq \emptyset$ and $B_2 \neq \emptyset$. Since $B_1 \subset L$ cl O, $OUB_1 \subset O \cup L$ cl $O = L$ cl O. Since $B_2{}'^P \subset B'^P = \emptyset$ and also since $OUB_1 \subset L$ cl O. We get $((OUB_1) \cap (P$ cl $B_2)) \cup (L$ cl $(OUB_1) \cap B_2)) = \emptyset$ contradicting the connectedness of A.

Hence $B \subset L$ cl O therefore $A = OUB \subset O \cup L$ cl $O = L$ cl O.

Thus A is P semi open with respect to L.

Definition : 1.1.16

In bitopological spaces a map $f : (X,P,L) \longrightarrow (Y,P',L')$ is said to be an open map if and only if the induced maps, $f_1 : (X,P) \rightarrow (Y,P')$ and $f_2 : (X,L) \rightarrow (Y, L')$ are open maps.

Theorem : 1.1.17

Let $(X,P,L) \longrightarrow (Y,P',L')$ be a continuous open map. Let A be a P semi open set with respect to L, then f(A) is P' semi open with respect to L'.

Proof :

A is P open with respect to L $\Rightarrow$ there exists an P open set O such that $O \subset A \subset L$ cl O.

Now $f(O) \subset f(A) \subset f(L\ cl\ O)$

Since f is open map f(O) is P' open

Since f is continuous $f(L\ cl\ O) \subset L\ cl\ f(O)$

therefore f(A) is P' semi open with respect to L.

Remark : 1.1.18

The above theorem is not true if we drop openness and it is shown by the following example.

Example : 1.1.19

Let $X = X^* =$ Reals. Let T and T_1 be the usual topology and the lower limit topology on X respectively.

Let $f : (X,T,T_1) \longrightarrow (X^*, T, T_1)$ be defined as $f(x) = 1$ for every $x \varepsilon X$. f is continuous but f is not open. X is T semiopen with respect to T_1 in X but f(x) is not T semi open with respect to T_1 in X^*.

Here X is T semi open with respect to T_1, but $f(x) = \{1\}$ is not T semi open with respect to T_1 as there exist no T open set O such that $O \subset \{1\} \subset T_1\ cl\ O$

Theorem : 1.1.20

Let $\{(X_\alpha, P_\alpha, L_\alpha)/\alpha \in V\}$ be a family of bitopological spaces. Let (X,P,L) be the product space where $X = \Pi X_\alpha$ P(L) is generated by P'^{s}_{α} $(L_{\alpha}'^{s})$;$\alpha \varepsilon V$ and let $A = \Pi A_\alpha \times \Pi X_\beta$. Then A is P(L) is semi open with respect to L(P) in ΠX_α if and only if A_{α_j} is P_{α_j} (L_{α_j}) semi open with respect to L_{α_j} (P_{α_j}) for every j, j = 1,2,...n.

Proof :

Necessity :

Let A be P semi open with respect to L in ΠX_α .

Then by corollary 1.1.3 $\prod_{j=1}^{n}$ (P Int. A_{α_j}) $\times \prod_{\beta \neq \alpha} X_\beta = \emptyset$.

consider $\prod_{j=1}^{n} L_{\alpha_j}$ cl (P_{α_j} Int. A_{α_j}) $\times \prod_{\beta \neq \alpha_j} X_\beta$

= L cl ($\prod_{j=1}^{n} P_{\alpha_j}$ Int. A_{α_j}) $\times \prod_{\beta \neq \alpha_j} X_\beta$)

= L cl (P Int. A)

= L cl (A)

= $\prod_{j=1}^{n} L_{\alpha_j}$ cl A_{α_j} $\times \prod_{\beta \neq \alpha_j} X_\beta$

Thus we get L_{α_j} cl (P_{α_j} Int. A_{α_j}) = L_{α_j} cl A_{α_j}

for every j = 1,2,........n

Hence by theorem 1.1.2 we get A_{α_j} is P_{α_j} semi open with respect to L_{α_j}

Sufficiency :

Let A_{α_j} be P_{α_j} semi open with respect to L_{α_j} for every j = 1,2,....n. Then by corollary 1.1.3 $A_\alpha \neq \emptyset$

for every j =1,2,....n for A ≠ ∅

Hence by corollary 1.1.3 P_{α_j} Int $A_{\alpha_j} \neq \emptyset$ so that

$\prod_{j=1}^{n} P_{\alpha_j}$ Int. A_{α_j} $\times \prod_{\beta \neq \alpha_j} X_\beta \neq \emptyset$

Also, L cl (P Int. A) = $\prod_{j=1}^{n} L_{\alpha_j}$ cl (P_{α_j} Int. A_{α_j}) $\times \prod_{\beta \neq \alpha_j} X_\beta$

= $\prod_{j=1}^{n} L_{\alpha_j}$ cl (A_{α_j}) $\times \prod_{\beta \neq \alpha_j} X_\beta$

= L cl A

Hence by theorem 1.1.2 A is P semi open with respect to L. Similarly it can be proved that A is L semi open with respect to P if and only if A_{α_j} is L_{α_j} semi open with respect to P_{α_j} ; for every j = 1,2,......n.

Remark : 1.1.21

The above theorem need not be true in the ordinary topology

ie., If $X = X_1 \times X_2$, X_i being topological spaces and A A ϵ S.O.(X) then it is not true in general that A is a union of sets of the form $A_1 \times A_2$ where $A_1 \epsilon$ S.O.(X_1) and $A_2 \epsilon$ S.O(X_2) is shown in the following example.

Example :

Let $A = \{(x, y) / 0 < x < 1, 0 < y < 1\} \cup (1,1)$

Section : 1.2. Semi Continuity

Definition : 1.2.1

Let $f : (X,P,L) \longrightarrow (Y, P',L')$ be a single valued mapping. Then $f : (X,P,L) \longrightarrow (Y, P',L')$ is termed (PP',LL') semi continuous if and only if, for $O' \in P'$ (L') $f^{-1}(O')$ is P (L) semi open with respect to $L(P)$.

Note :

When we say that $f : (X,P,L) \rightarrow (Y, P',L')$ is semi continuous we mean that it is $(PP', LL',)$ semi continuous.

Remark : 1.2.2.

Continuity implies semi continuity is obvious but converse is not true is shown in the following example

Example : 1.2.3.

Let $X = X^* = \{ a, b, c, d \}$

$\Lambda_1 = \{ \emptyset, X, \{a\}, \{a, c\}\}$

$\Lambda_1' = \{ \emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$\Lambda_2 = \{\emptyset, X, \{b\}, \{b,d\}\}$

$\Lambda_2' = \{\emptyset, X, \{a\}, \{a,b,\}\{b\}, \{a,c\}, \{a,b,c\}\}$

Let $f_1 : (X, \Lambda_1', \Lambda_2') \longrightarrow (X^*, \Lambda'_1, \Lambda_2)$ be the identity mapping. f_1 is a semi continuous map but is not continuous.

We see the map $f : (X, \Lambda_1', \Lambda_2') \rightarrow (X^*, \Lambda_1, \Lambda_2)$ is semi continuous but it is not continuous as $\{a\} \in \Lambda_2'$ and $\{a\} \notin \Lambda_2$

Theorem : 1.2.4

Let $\{(X_\alpha, P_\alpha, L_\alpha) / \alpha \in \Lambda\}$ and $\{(Y_\alpha, P'_\alpha, L'_\alpha) / \alpha \in \Lambda\}$ be two arbitrary family of bitopological spaces with the same set of indices. Let $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ be a map for every $\alpha \in \Lambda$ Then the product map $f : (X,P,L) \longrightarrow (Y,P',L')$ (Where $X = \Pi X_\alpha$, $Y = \Pi Y_\alpha$) $P(L)$ is generated by P_{α_j}'s (L_{α_j}'s) and P' (L') is generated by P'_{α_j}'s (L'_{α_j}'s) defined by $f(x) = f((x_\alpha)) = (f_\alpha(x_\alpha))$ is semi continuous if and only if f_α is semi continuous for every $\alpha \in \Lambda$.

Proof :

Necessity :

Let f be a semi continuous map. Let $\alpha_0 \in \Lambda$ be arbitrary, chosen index and U_{α_0} an arbitrary P'_{α_0} open set in Y_{α_0}.

Since $U_{\alpha_0} \times \underset{\beta \neq \alpha_0}{\Pi} Y_\beta$ is P' open in ΠY_α,
$f^{-1}(U_{\alpha_0} \times \underset{\beta \neq \alpha_0}{\Pi} Y_\beta) = f^{-1}_{\alpha_0}(U_{\alpha_0}) \times \underset{\beta \neq \alpha_0}{\Pi} f^{-1}_\beta(\beta)$ is P semi open with respect to L.
Hence by theorem : 1.1.20 $f^{-1}_{\alpha_0}(U_{\alpha_0})$ is P_{α_0} semi open with respect to L_{α_0}
Similarly if U_{α_0} were an L' open set, $f^{-1}_{\alpha_0}(U_{\alpha_0})$ is L_{α_0} semi open with respect to P_{α_0}. Thus f_{α_0} is semi continuous and α_0 being an arbitrary index, f_α is semi continuous for every $\alpha \in \Lambda$.

Sufficiency :

Let f_α be semi continuous for every $\alpha \in \Lambda$. Let U be a basic P' open set in $\prod Y_\alpha$ ie., $U = \prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\beta \neq \alpha_j} Y_\beta$ Where U_{α_j} is P'_{α_j} open set in Y_{α_j} for every j, j = 1,2,....n

Now, $f^{-1}(U) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j}) \times \prod_{\beta \neq \alpha_j} f_\beta^{-1}(Y_\beta)$

$= \prod_{j=1}^{n} f_{\alpha_j}^{-1}(U_{\alpha_j}) \times \prod_{\beta \neq \alpha} X_\beta$

$f_{\alpha_j}^{-1}(U_{\alpha_j})$ is P_{α_j} semi open with respect to L_{α_j} for every j, j = 1,2,.........n (for f_{α_j} is semi continuous) and hence by theorem 1.1.20 $f^{-1}(U)$ is P semi open with respect to L in $\prod X_\alpha$. If O is any arbitrary P' open set in $\prod Y_\alpha$, then there exists a family $\{ U_\alpha / i \in I \}$ of P' basic open sets such that $O = \bigcup_{i \in I} U_i$

$$f^{-1}(O) = f^{-1}\left(\bigcup_{i \in I} U_i \right)$$

$$= \bigcup_{i \in I} f^{-1}(U_i)$$

Thus, $f^{-1}(O)$ being an arbitrary union of P semi open sets with respect to L is P semi open with respect to L by theorem 1.1.15 . Similarly if O is an L' open set is $\prod Y_\alpha$ $f^{-1}(O)$ is L semi open with respect to P. Thus f is a semi continuous map.

Theorem : 1.2.5

Let (X,P,L), (X_1, P_1, L_1) and (X_2, P_2, L_2) be bitopolgical spaces and let $(X_1 \times X_2, P', L')$ be the product space where P'(L') is generated by $P_1(L_1)$ and $P_2(L_2)$. Let $h : X \longrightarrow (X_1 \times X_2)$ be a semi continuous map then

$f_i : X \longrightarrow X_i$, $i = 1,2$ are semi continuous map where $h(x) = (x_1, x_2)$ and $f_i(x) = x_i$, $i = 1,2$.

Proof :

Given $f_i(x) = x_i$, $x = f_i^{-1}(x_i)$

and $h(x) = (x_1, x_2)$, $x = h^{-1}(x_1, x_2)$

If O_1 is P_1 open, $f_1^{-1}(O_1) = h^{-1}(O_1 \times X_2)$

If O_2 is P_2 open, $f_2^{-1}(O_2) = h^{-1}(X_1 \times O_2)$

Therefore $f_i^{-1}(O_i)$ is P_i semi open with respect to L_i

Therefore $f_i : X \rightarrow X_i$, $i = 1,2$ are semi continuous map.

Remark : 1.2.6

A semi continuous function of a semi continuous function is not in general semi continuous is shown in the following example.

Example : 1.2. 7

Let $X = X_1 = X_2 = X_3 = \{a, b, c, d\}$

Let

$\Lambda_1 = \{\emptyset, X, \{a\}, \{a,c\}\}$

$\Lambda_2 = \{\emptyset, X, \{b\}, \{b,d\}\}$

$\Lambda'_1 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$\Lambda'_2 = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$

$\Lambda_1^* = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d,\}\}$

Λ_2^* = the discrete topology on X_3.

Let $f_1 : (X, \Lambda_1', \Lambda_2') \longrightarrow (X, \Lambda_1 \Lambda_2)$

$f_2 : (X, \Lambda_1^*, \Lambda_2^*) \rightarrow (X, \Lambda_1', \Lambda_2')$ be identity mappings. f_1 and f_2 are semi continuous mappings.

$f_1 f_2 : (X, \Lambda_1^*, \Lambda_2^*) \longrightarrow (X, \Lambda_1, \Lambda_2)$ is not semi continuous for $\{a, c\} \in \Lambda_1$ but $(f_1 f_2)^{-1} \{a, c\} = \{a, c\}$ is not Λ_1^* semi open with respect to Λ_2^*

Theorem : 1.2.8

Let (X, P, L) and (Y, P', L') be two bitopological spaces. If $f : X \longrightarrow Y$ is an open map and semi continuous map the inverse image $f^{-1}(B)$ of each $P'(L')$ semi open set with respect to $L'(P')$ is $P(L)$ semi open with respect to $L(P)$.

Proof :

Let B be an arbitrary P' semi open set with respect to L' in Y then there exist a P open set V in Y such that $V \subset B \subset L'$ cl V

Since f is open we have $f^{-1}(L' \text{ cl } V) \subset L$ cl $(f^{-1}(V))$ Since f is semi continuous $f^{-1}(V)$ is P semi open with respect to L and also $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(L' \text{ cl } V) \subset L$ cl $(f^{-1}(V))$ ie., $f^{-1}(V) \subset f^{-1}(B) \subset L$ cl $(f^{-1}(V))$

Thus $f^{-1}(B)$ is P semi open with respect to L by therem 1.1.6. Similarly if B, were an arbitrary L' semi open set with respect to P', $f^{-1}(B)$ is a L semi open set with respect to P.

Section : 1.3 Semi Open Mappings

Definition : 1.3.1

A mapping $f : (X,P,L) \longrightarrow (Y,P'L')$ is said to be (PP', LL') semi open mapping if and only if for each P open (L open) set U in X, f(U) is P' semi open with respect to L' (L' semi open with respect to P')

Remark : 1.3.2

An open map is a semi open map but not conversely is shown in the following example.

Example : 1.3.3

Let $X_1 = X_2 = \{a,b,c,d\}$

$\Lambda_1 = \{\emptyset, X \{a\}, \{a,c\}\}$

$\Lambda_2 = \{\emptyset, X, \{b\} \{b,d\}\}$

$\Lambda_1' = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$\Lambda_2' = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$

$\Lambda_1^* = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}\}$

Λ_2^* = the discrete topology on X_3.

Let $f : (X_1, \Lambda_1, \Lambda_2) \longrightarrow (X_2, \Lambda_1', \Lambda_2')$ be the identity map. Then f is a semi open map but f is not open as $\{a,c\}$ is Λ_1 open and $f(\{a,c\}) = \{a,c\}$ is not in Λ_1'

Theorem : 1.3.4

Let (X,P,L) and (Y,P',L') be two bitopological spaces. Let f map $(X,P,L) \longrightarrow (Y,P',L')$

1. f is semi open if and only if
   $f(P \text{ Int. } E) \subset L' \text{ cl } (P' \text{ Int. } f(E))$ and
   $f(L \text{ Int. } E) \subset P' \text{ cl } (L' \text{ Int. } f(E))$, where $E \subset X$

2. f is semi open if and only if for every $G \subset Y$,
   $P \text{ Int. } f^{-1}(G) \subset f^{-1}(L' \text{ cl } (P' \text{ Int } G))$ and
   $L \text{ Int. } f^{-1}(G) \subset f^{-1}(P' \text{ cl } (L' \text{ Int. } G))$

Proof :

Assume f is semi open mapping. ie., if for each P open set E in X, f(E) is P' semi open with respect to L', we have $L' \text{ cl } f(E) = L' \text{ cl } (P' \text{ Int } .f(E))$

$f(P \text{ Int } E) = f(E) \subset L' \text{ cl } f(E) = L' \text{ cl } (P' \text{ Int. } f(E))$

Therefore $f(P \text{ int } E) \subset L' \text{ cl } (P' \text{ Int. } f(E))$ - (1)

Similarly since f is semi open mapping ie., if for each L open set E in X, f(E) is L' semi open with respect to P', $P' \text{ cl } f(E) = P' \text{ cl } (L' \text{ Int. } f(E))$

$f(L \text{ Int. } E) = f(E) \subset P' \text{ cl } f(E) = P' \text{ cl } (L' \text{ Int. } f(E))$

Therefore $f(L \text{ Int. } E) \subset P' \text{ cl } (L' \text{ Int. } f(E))$ - (2)

2. Assume f is a semi open. Put $E = f^{-1}(G)$ in (1)

$f(P \text{ Int. } f^{-1}(G)) \subset L' \text{ cl } (P' \text{ Int. } f(f^{-1}(G)))$

$(P \text{ Int. } f^{-1}(G)) \subset f^{-1}(L' \text{ cl } (P' \text{ Int. } G))$

Put $E = f^{-1}(G)$ in (2)

$f(L \text{ Int. } f^{-1}(G)) \subset P' \text{ cl } (L' \text{ Int. } f(f^{-1}(G)))$

$L \text{ Int. } f^{-1}(G) \subset f^{-1} (L' \text{ cl } (P' \text{ Int. } G))$

By theorem 1.1.2 f is semi open.

Hence the theorem.

Remark : 1.3.5

The image of a semi open set under a semi open map is not necessarily semi open as shown by the following.

Example : 1.3.6

Let $X = X^* = \{a, b, c, d\}$

$\Lambda_1 = \{\emptyset, x, \{a\} \{a, c\}\}$

$\Lambda_2 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\} \{b, c\}, \{a, c\}, \{a, b, c\}\}$

$\Lambda_1^* = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}\}$

Λ_2^* = discrete topology on X^*.

The identity map $f : (X, \Lambda_1, \Lambda_2) \longrightarrow (X^*, \Lambda_1^*, \Lambda_2^*)$ is open and hence semi open.

The set $\{a, d\}$ is Λ_1 semi open with respect to Λ_2 but $f(\{a, d\}) = \{a, d\}$ is not Λ_1^* semi open with respect to Λ_2^*.

Theorem : 1.3.7

Suppose that $f : (X, P, L) \longrightarrow (X^*, P^*, L^*)$ be continuous and semi open. Let A be P semi open with respect to L. Then f(A) is P^* semi open with respect to L^*.

Proof :

Given A is P semi open with respect to L there exist a P open set such that $O \subset A \subset L$ cl O

$f(O) \subset f(A) \subset f(L$ cl $O) \subset L$ cl $f(O) = L^*$ cl $f(O)$,

f being continuous.

Since f is semi open, f(O) is P^* semi open with respect to L^*. Hence f(A) is P^* semi open with respect to L^* by theorem : 1.1.6

Remark : 1.3.8

If either continuity or semi openness is dropped from the theorem 1.3.7, the theorem fails is shown by the following example and by example 1.3.9 respectively.

Example :

If continuity is dropped in theorem 1.3.7

Let $X = X^* = \{a,b,c,d\}$

$\Lambda_1 = \{\emptyset, X, \{a\}, \{a,c\}\}$

$\Lambda_2 = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c,\}\}$

$\Lambda_1^* = \{\emptyset, X\{a\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Λ_2^* = discrete topology on X^*.

The identity map $f : (X, \Lambda_1, \Lambda_2) \longrightarrow (X^*, \Lambda_1^*, \Lambda_2^*)$ is open and hence semi open. $\{a,d\}$ is Λ_1 semi open with respect to Λ_2, but $f(\{a,d\}) = \{a,d\}$ is not Λ_1^* semi open with respect to Λ_2^*.

If semi openness is dropped from the theorem 1.3.7 the theorem fails as shown in the following example.

Example : 1.3.9

Let $X = X^* = R$. Let T be the usual topology on $X = X^* = R$. and T_1 be the lower limit topology on $X = X^* = R$. Let $f:(X,T,T_1) \longrightarrow (X^*,T,T_1)$ be defined as $f(x) = \frac{1}{2}$ for every $x \in X$. Then f is continuous but not semi open.

Here X is T semi open with respect to T_1 but $f(x) = \frac{1}{2}$ for every $x \in X$ is not T semi open with respect to T_1 as there exist no T open set O such that $O \subset \{\frac{1}{2}\} \subset T_1$ cl O.

Remark : 1.3.10

A semi open map of a semi open map is not semi open is shown by the following.

Example : 1.3.11

Let $X = X_1 = X_2 = X_3 = \{a,b,c,d\}$

$\Lambda_1 = \{\emptyset, X, \{a\}, \{a,c\}\}$

$\Lambda_1' = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$\Lambda_2 = \{\emptyset, X, \{b\}, \{b,d\}\}$

$\Lambda_2' = \{\emptyset, X, \{a,b\}, \{a,c\}, \{a\}, \{b,\} \{a,b,c\}\}$

$\Lambda_1^* = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{a,b,d\}\}$

Λ_2^* = discrete topology on X_3.

Let $f_1:(X_1,\Lambda_1,\Lambda_2) \longrightarrow (X_2, \Lambda_1', \Lambda_2')$ and

$f_2:(X_2, \Lambda_1' \Lambda_2') \longrightarrow (X_3, \Lambda_1^*, \Lambda_2^*)$ be identity mappings. Then f_1 and f_2 are semi open maps, but f_2f_1 is not semi open for $f_2f_1(\{a,c\}) = \{a,c\}$ is not Λ_1^* semi open with respect to Λ_2^*.

Theorem : 1.3.12

Let $f_1 : (X_1, P_1\ L_1) \longrightarrow (X_2, P_2, L_2)$ and

$f_2 : (X_2, P_2, L_2) \longrightarrow (X_3, P_3, L_3)$ be semi open mappings. and let f_2 be continuous. Then $f_2 f_1$ is semi open.

Proof :

f_1 is a semi open mapping $\Rightarrow$ for each P_1 open set U in X_1 , $f_1(U)$ is P_2 semi open with respect to L_2.
f_2 is a semi open mapping $\Rightarrow f_2\ (f_1(U))$ is P_3 semi open with respect to L_3. Therefore $f_2 f_1 : (X_1, P_1, L_1) \longrightarrow (X_3, P_3, L_3)$ is a semi open mapping.

Theorem : 1.3.13

Let$\{(X_\alpha , P_\alpha , L_\alpha) / \alpha \in A\}$ and $\{(Y_\alpha , P'_\alpha , L'_\alpha) / \alpha \in A\}$ be two arbitrary families of bitopological spaces with the same set of indices. Let $f_\alpha : X_\alpha \longrightarrow Y_\alpha$ be a mapping for every $\alpha \in A$ and surjective for all but atmost a finite number of indices. Then the product map $f(\Pi X_\alpha = X,P,L) \quad (\Pi Y_\alpha = Y,P',L')$ where P (P') and L(L') are generated by P_α's (P'_α's) and L_α's(L'_α's) respectively, defined by $f(x) = f((x_\alpha)) = (f_\alpha\ (x_\alpha))$ is semi open if and only if f_α is semi open for every $\alpha \in A$.

Proof :

Necessity :

Let $\alpha_0 \in A$ be an arbitrary chosen index and U_{α_0} be an arbitrary P_{α_0} open set in X_{α_0}.

Since $U_{\alpha_0} \times \prod_{\beta\neq\alpha_0} X_\beta$ is P open in $\prod X_\alpha$,
$f(U_{\alpha_0} \times \prod_{\beta\neq\alpha_0} X_\beta) = f_{\alpha_0}(U_{\alpha_0}) \times \prod_{\beta\neq\alpha_0} f_\beta (X_\beta)$ is P' semi open set with respect to L' in $\prod X_\alpha$.

Since f_α is surjective for all but finite number of indices from the theorem : 1.1.20 it follows that $f_{\alpha_0} (U_{\alpha_0})$ is P'_{α_0} semi open with respect to P'_{α_0}. Thus f_{α_0} is semi open map.

Similarly if U_{α_0} is an arbitrary L_{α_0} open set in X_{α_0}, $f_{\alpha_0}(U_{\alpha_0})$ is L'_{α_0} semi open with respect to P'_{α_0} .

Hence f_{α_0} is semi open map. Since $\alpha_0 \varepsilon A$ is arbitrary, f_α is semi open for every $\alpha \varepsilon A$.

Sufficiency :

Let U be a basic open set in $\prod X_\alpha$

ie., $U = \prod_{j=1}^{n} U_{\alpha_j} \times \prod_{\beta\neq\alpha_j} X_\beta$ where U_{α_j} is P_{α_j} open for every j = 1,2....n

Then we have, $f(U) = \prod_{j=1}^{n} f_{\alpha_j} (U_{\alpha_j}) \times \prod_{\beta\neq\alpha_j} f_\beta (X_\beta)$

Since all but finitely many f_α 's are surjective, all but almost a finitely many $f_\beta (X_\beta) = Y_\beta$.

Therefore $f(U) = \prod_{j=1}^{n} f_{\alpha_j} (U_{\alpha_j}) \times \prod_{\beta\neq\alpha} Y_\beta$

Also $f_{\alpha_j} (U_{\alpha_j})$ is P'_{α_j} semi open with respect to L'_α for every j = 1,2....n

Hence by theorem : 1.1.20 f(U) is a P' semi open with respect to L'. Let O be an arbitrary P open set in $\prod X_\alpha$.

Then there exist a family $\{U_i / i \in I\}$ of basic open sets such that $O = \bigcup_{i \in I} U_i$ Then $f(O) = \bigcup_{i \in I} f(U_i)$ is P' semi open with respect to L' by theorem : 1.1.20

Similarly if O were an arbitrary L open set, f(O) is L' semi open with respect to P'. Thus f is a semi open map.

Theorem : 1.3.14

Let (X,P,L) and (Y,P'L') be two bitopological spaces. Let f map X into Y. Then the following statements are equivalent.

(i) f is a semi open mapping

(ii) for every $A \subset X$, $f(P \text{ Int. } A) \subset f(A)_{o\, p'(L)}$

$f(L \text{ Int. } A) \subset f(A)_{oL'(P')}$

(iii) for every $x \in X$ and each P neighbourhood (L neighbourhood) U of x in X there exist a P' semi neighbourhood (L' semi neighbourhood) V of f (x) with respect to L'(P') in Y such that $V \subset f(U)$

Proof :

(i) ⇒ (ii) follows from the fact that $f(A)_{op'(L')}$ $(f(A_{O\, L'(P')})$ is the largest the P' semi open set with respect to L' (L' semi open set with respect to P') contained in f(A). f being semi open map f(P Int. A) is P' semi open with respect to L' (f(L Int. A) is L' semi open with respect to P').

(ii) ⇒ (iii) Let $x \in X$ and U be a P - neighbourhood of x. Then there exist a P open set O such that $x \in O \subset U$.

By hypothesis, we have $f(O) = f(P\ \mathrm{Int.}\ O) \subset f(O)_{\mathrm{o}P'(L')}$

Hence $f(O) = f(O)_{\mathrm{o}\ P'(L')}$

$f(O)_{\mathrm{o}\ P'(L')}$ being a union of P' semi open set with respect to L' is P' semi open with respect to L'. Thus $f(O) = V$ (say) is P' semi open with respect to L'.

Since $x \in O \subset U$, $f(x) \in f(O) \subset f\ (U)$ we have $f(x) \in V \subset f\ (U)$

Similarly if U were an L neighbourhood of x, there exist a L' semi open set with respect to P' (say) V such that $f(x) \in V \subset f(U)$.

(iii) ⇒ (i) Let U be an arbitrary P open set in X. By hypothesis, for every $y \in f\ (U)$ there exist a semi open neighbourhood with respect to L' (say) V_y such that $V_y \subset f(U)$ ie., there exist a P' semi open set with respect to L' (say) A_y such that $y \in A_y \subset V_y$.

$f(U) = \cup \{ A_y / y \in f(U) \}$ being the union of P' semi open sets with respect to L' is P' semi open with respect to L'.

Similarly if U is an arbitrary L open in X, f(U) is L' semi open with respect to P'.

Hence f is a semi open map.

Theorem : 1.3.15

Let (X,P,L) and (Y,P',L') be bitopological space. A necessary and sufficient condition for a map $f : X \longrightarrow Y$ to be semi open is that $f^{-1}\ (B_{-P'(L')}) \subset P\ \ \mathrm{cl}\ f^{-1}(B)$ and $f^{-1}(B_{-L'(P')}) \subset L\ \ \mathrm{cl}\ f^{-1}\ (B)$ for every $B \subset Y$.

Proof :

Let f be a semi open map.

Let $B \subset Y$ and $U = X \sim P\ \mathrm{cl}\ (f^{-1}(B))$, U is P open in X.

Since f is a semi open map, f(U) is P' semi open with respect to L'. ie., $f(U) = f(X \sim P\ \mathrm{cl}\ (f^{-1}(B)))$

$$= f(X) \sim P\ \mathrm{cl}\ f(f^{-1}(B))$$

$$= Y \sim P\ \mathrm{cl}\ (B)$$

$$Y \sim f(U) = Y \sim (Y \sim P\ \mathrm{cl}\ B)$$

$$= P\ \mathrm{cl}\ B$$

Therefore $Y \sim f(U)$ is P' semi closed with respect to L'

Also $B \subset P\ \mathrm{cl}\ B$ ie., $B \subset Y \sim f(U)$ therefore $B_{-P'(L')} \subset Y \sim f\ (U)$

Therefore $B_{-P'(L')}$ is the smallest semi closed set containing B.

Therefore $f^{-1}(B_{-P'(L')}) \subset f^{-1}(Y \sim f(U))$

$$\subset f^{-1}(Y) \sim f^{-1}f(U)$$

$$\subset X \sim U = P\ \mathrm{cl}\ (f^{-1}(B)).$$

Similarly we can show that,

$$f^{-1}(B_{-L'(P')}) \subset f^{-1}(Y \sim f(U)) = P\ \mathrm{cl}\ f^{-1}(B)$$

Conversely, let U be an arbitrary P open set in X.

Put $B = Y \sim f\ (U)$ by hypothesis we have

$$f(U) \cap B_{-P'(L')} = f(U \cap f^{-1}(B_{-P'(L')}))$$

$$\subset f(U \cap P\ \mathrm{cl}\ f^{-1}(B)).$$

Since U is P open, $U \cap P\mathrm{cl}\ (f^{-1}(B)) = P\ \mathrm{cl}\ (U \cap f^{-1}(B))$

Since $U = X \sim P\ \mathrm{cl}\ f^{-1}(B)$, $U \cap f^{-1}(B) = \emptyset$

Thus $f(U) \cap B = \emptyset$, $B_{-P'(L')} \subset Y \sim f(U) = B$

B is a P' semi closed with respect to L' in Y and hence $f(U) = Y \sim B$ is a P' semi open set in Y with respect to L'. Similarly if U is a L open set in Y, it can be shown that f(U) is L' semi open with respect to P' in Y. Thus f is a semi open map.

Theorem : 1.3.16

Let (X,P,L), (Y,P',L'), (Z,P^*,L^*) be three bitopological spaces. $f:(X,P,L) \longrightarrow (Y,P',L')$ and $g : (Y,P',L') \longrightarrow (Z,P^*,L^*)$ be two maps.

Let $g \circ f$ be a semi open map

(i) If f is continuous and surjective, g is a semi open map

(ii) If g is open, semi continuous and injective f is a semi open map.

Proof :

(i) Let U be an arbitrary P' open (L' open) set in Y. f being continuous, $f^{-1}(U)$ is a P open (L open) set. $g \circ f$ being semi open, $g \circ f\ (f^{-1}(U))$ is P^* semi open with respect to L^* in Z (L^* semi open with respect to P^*)
f being surjective, $g \circ f\ (f^{-1}(U)) = g\ (U)$ is P^* semi open with respect to L^* in Z (L^* semi open with respect to P^* in Z) Therefore g is a semi open map.

(ii) Let U be an arbitrary P open set in X.
$g \circ f\ (U)$ is P^* semi open with respect to L^*.
Since g is open and semi continuous $g^{-1}(g \circ f)(U)$ is P' semi open with respect to L' in Y and also g is injective.
Therefore $g^{-1}(g \circ f)(U) = f(U)$.
Similarly if U were an arbitrary L open set in X, $f(U)$ is L' semi open with respect to P'.
Thus f is a semi open map.

CHAPTER II

PAIRWISE ALMOST CONTINUOUS MAP AND WEAKLY CONTINUOUS MAP IN BITOPOLOGICAL SPACES

In Chapter two, pairwise almost continuous maps and weakly continuous maps in bitopological spaces introduced by Shantha Bose and Dipti Sinha in their article "Pairwise almost continuous map and weakly continuous map in bitopological spaces"- are studied.

Similar to ordinary topological spaces here also continuity implies almost continuity and almost continuity implies weakly continuity. But the reverse concepts need not be true. Here a number of properties of almost continuous maps and weakly continuous maps are studied.

CHAPTER - 2

PAIRWISE ALMOST CONTINUOUS MAP AND WEAKLY CONTINUOUS MAP IN BITOPOLOGICAL SPACES

Section : 2.1

Almost continuity in Bitopological spaces.

Definition : 2.1.1

Let (X, P, L) be a bitopological space.

A set $A \subset X$ is said to be P-regularly open with respect to L if $A = P \text{ Int } (L \text{ cl } A)$

Definition : 2.1.2

A set $A \subset X$ is P-regularly closed with respect to L if $A = P \text{ cl } (L \text{ Int. } A)$

A set $A \subset X$ is regularly open (closed) if it is both P-regularly open (closed) with respect to L and L-regularly open (closed) with respect to P.

Note :

A is regularly open if and only if $(X \sim A)$ is regularly closed.

Definition : 2.1.3

Let $f : (X,P,L) \longrightarrow (Y,P',L')$ be a mapping. Then f is said to be PP' almost continuous with respect to L at a point $x \in X$ if and only if for every P' neighbourhood M of f (X) there exists a P neighbourhood N of x such that,

$f(N) \subset P' \text{ Int } (L' \text{ cl } M)$

Definition : 2.1.4

A mapping $f : (X,P,L) \longrightarrow (Y, P', L')$ is said to be pairwise almost continuous at $x \in X$ if it is PP' almost continuous with respect to L at x and LL' almost continuous with respect to P at x.

We say that f is pairwise almost continuous in X if it is pairwise almost continuousat $x \in X$ for all $x \in X$.

Remark : 2.1.5

If f is continuous, f is pairwise almost continuous but converse is not true is shown in the following example.

Example : 2.1.6

Let $X = X_1 = X_2 = \{a, b, c, d\}$

$P = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}\}$

$L = \{\emptyset, X, \{a\}, \{b\}, \{a,b\} \{a,c,\} \{a,b,c\}\}$

$P' = \{\emptyset, X, \{a\} \{a,c\}, \{a,c,d\}\}$

$L' = \{\emptyset, X, \{b\}, \{b,d\}\}$

Let $f : (X_1, P, L,) \longrightarrow (X_2, P', L')$ be the identity mapping. Then f is pairwise almost continuous but not continuous.

The set $\{a,c\} \in P'$

and $f^{-1}(\{a,c\}) = \{a,c\} \notin P$

Thus f is not continuous.

Theorem : 2.1.7

For a map $f : (X,P,L,) \longrightarrow (Y,P',L')$ the following are equivalent:

(a) f is PP' almost continuous with respect to L.

(b) Inverse image of every P' regularly open set with respect to L' is a P open subset of X.

(c) Inverse image of every P' regularly closed set with respect to L' is a P closed subset of X.

(d) For each $x \in X$ and each P' regularly open neighbourhood M of f(x), there exists a P neighbourhood N of x such that $f(N) \subset M$.

(e) $f^{-1}(A) \subset P$ Int. $\{f^{-1}$ (P' Int (L' cl A))$\}$ for every P' open subset A of Y.

(f) P cl f^{-1}(P' cl (L' Int B) $\subset f^{-1}(B)$ for every P' closed subset B of Y.

(g) For any point $x \in X$ and for any net $\{x_\lambda / \lambda \in D\}$ which converges to x, the net $\{f(x_\lambda)/\lambda \in D\}$ is eventually in each P' regularly open set with respect to L' containing f (x).

Proof :

(a) ⇒ (b)

Let M be a P' regularly open set of Y with respect to L'

Consider $f^{-1}(M)$ Let $x \in f^{-1}(M)$ then $f(x) \in M$

Since f is PP' almost continuous with respect to L there exists a P open set N such that $x \in N$ and $f(N) \subset P'$ Int (L'clM)

Since M is P' regularly open set of Y with respect to L', P' Int $(L'$ cl $M) = M$

Therefore $x \varepsilon f(N) \subset M$

ie, $x \varepsilon N \subset f^{-1}(M)$

Hence $f^{-1}(M)$ is a P neighbourhood of x which is an arbitrary point of $f^{-1}(M)$

Hence $f^{-1}(M)$ is a P open set.

(b) $\Rightarrow$ (c) :

Let M be any P' regularly closed set of Y with respect to L'.

Then $Y \approx M$ is a P' regularly open set of Y with respect to L'. (b) $\Rightarrow$ that $f^{-1}(Y \sim M)$ is a P open set of X.

ie, $f^{-1}(Y) \sim f^{-1}(M)$ is a P open set of X.

Since $f : (X,P,L) \longrightarrow (Y,P',L')$, $X \sim f^{-1}(M)$ is a P open set of X. Hence $f^{-1}(M)$ is a P closed subset of X.

(c) $\Rightarrow$ (d) :

Let M be a P' regularly open subset of Y with respect to L'. Then $M = P'$ Int $(L'$ cl $M)$.

Therefore $(Y \sim M)$ is a P' regularly closed subset of Y with respect to L'.

(c) $\Rightarrow$ $f^{-1}(Y \sim M)$ is a P closed subset of X.

$f^{-1}(Y) \sim f^{-1}(M)$ is a P closed subset of X.

$X \sim f^{-1}(M)$ is a P closed subset of X.

$f^{-1}(M)$ is a P open subset of X.

Also $x \varepsilon f^{-1}(M) = N$ (say).

Then N is a P neighbourhood of x such that $f(N) \subset M$.

(d) ⇒ (e) :

Let A be a P' open subset of Y. Then $P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ A)$ is a P' regularly open neighbourhood of $f(x)$ with respect to L' in Y.

(d) ⇒ there exists a P open neighbourhood N of x such that $f(N) \subset P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ A)$

$x \in N \subset f^{-1}\ (P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ A))$.

Thus $x \in P\ \mathrm{Int}\ (f^{-1}(p'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ A)))$

Therefore $f^{-1}(A) \subset P\ \mathrm{Int}\ (f^{-1}(P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ A)))$.

(e) ⇒ (f) :

Let B be a P' closed subset of Y. Then $(Y \sim B)$ is a P' open subset of Y.

(e) ⇒ $f^{-1}\ (Y \sim B) \subset P\ \mathrm{Int}\ (f^{-1}\ (P\ \mathrm{Int}\ (L'\ \mathrm{cl}\ (Y \sim B))))$

Now $P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ (Y \sim B)) = Y \sim P'\mathrm{cl}\ (Y \sim L'\ \mathrm{cl}\ (Y \sim B))$
Since $P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ (Y \sim B))$ is a P' open subset A of Y.
Since $Y \sim B$ is P' open subset of Y, $Y \sim (L'\ \mathrm{cl}\ (Y \sim B))$ is a P' closed subset of Y.

$\Rightarrow$ $Y \sim (P'\ \mathrm{cl}\ (Y \sim L'\ \mathrm{cl}\ (Y \sim B)))$ is a P' open subset of Y.

$P'\ \mathrm{Int}\ (L'\ \mathrm{cl}\ (Y \sim B)) = Y \sim P'\ \mathrm{cl}\ (Y \sim (Y \sim L'\ \mathrm{Int}\ B))$

If $x \in Y \sim L'\ \mathrm{Int}\ B$

if and only if $x \notin L'\ \mathrm{Int}\ B$

if and only if $x \notin B$

if and only if $x \notin L'\mathrm{cl}\ B$

if and only if $x \in Y \sim L'\ \mathrm{cl}\ B$

if and only if $x \in L'\ \mathrm{cl}\ (Y \sim B)$.

There fore we have $Y \sim L' \text{ Int } B = L' \text{ cl } (Y \sim B)$

Therefore $P' \text{ Int } (L' \text{ cl } (Y \sim B)) = Y \sim P' \text{ cl } (L' \text{ Int } B)$

$$f^{-1}(P' \text{ Int } (L' \text{ cl } (Y \sim B))) = f^{-1}(Y \sim P' \text{ cl } (L' \text{ Int } B))$$
$$= f^{-1}(Y) \sim f^{-1} (P' \text{ cl } (L' \text{ Int } B))$$
$$= x \sim f^{-1} (P' \text{ cl } (L' \text{ Int } B))$$

Therefore $x \sim f^{-1} ((B) \subset P \text{ Int } (X \sim f^{-1} (P' \text{ cl } (L' ,\text{Int } B)))$

Therefore $f^{-1} (B) \supset x \sim P \text{ Int } (f^{-1}(P' \text{ cl } (L' \text{ Int } B)))$

Therefore $(f^{-1}(P' \text{ cl } (L' \text{ Int } B))) \subset f^{-1}(B)$.

(f) $\Rightarrow$ (g) :

Let $\{x_\lambda\}_{\lambda \epsilon D}$ be a net converging to x in the P-topology. Consider the net $\{f(x_\lambda)\}_{\lambda \epsilon D}$. Let N be a P'-regularly open set with respect to L' containing f(x). Then $(Y \sim N)$ is a P' closed.

Therefore (f) $\Rightarrow$ $P \text{ cl } (f^{-1} (P' \text{ cl}(L' \text{ Int } (Y \sim N)))) \subset f^{-1} (Y \sim N)$

Since $(Y \sim N)$ is P' regularly closed with respect to L'

$Y \sim N = P' \text{ cl } (L' \text{ Int } (Y \sim N))$

Therefore $P \text{ cl } (f^{-1} (Y \sim N) \subset f^{-1} (Y \sim N)$

Therefore $f^{-1} (Y \sim N)$ is P closed

ie, $f^{-1} (Y) \sim f^{-1} (N)$ is P closed.

Thus $f^{-1} (N)$ is P open.

Also $\{x_\lambda\}_{\lambda \epsilon D}$ converges to x in the P- topology. Hence there exists $\lambda_0 \epsilon D$ such that for every $\lambda \geq \lambda_0$ $x \epsilon f^{-1}(N)$

ie., for every $\lambda \geq \lambda_0$, $f (x_\lambda) \epsilon N$.

ie., the net $\{f(x_\lambda)\}_{\lambda \epsilon D}$ is eventually in N, any P' regularly open subset of Y with respect to L'.

(g) $\Rightarrow$ (a) :

Let f be not PP' almost continuous with respect to L. Therefore there exists $x \in X$ such that f is not PP' almost continuous with respect to L at x.

ie, there exists a P' open set N containing f(x) such that for all P open sets U containing x, $f(U) \cap (Y \sim (P' \text{ Int } (L' \text{ cl } A))) \neq \emptyset$

Since $f(U) \not\subset P' \text{ Int } (L' \text{ cl } A)$

ie., since $f(U) \cap P' \text{ Int } (L' \text{ cl } A) = \emptyset$

Therefore $U \cap f^{-1}(Y \sim (P' \text{ Int } (L' \text{ cl } A))) \neq \emptyset$

ie., there exists $x_u \in U$ such that $x_U \in f^{-1}(Y \sim (P' \text{ Int } (L' \text{cl } A)))$

For all P open sets U containing x if

$\mathcal{U}$ = {all P open sets containing x }

For each $U \in \mathcal{U}$, choose a point x_U belonging to $U \cap f^{-1}(Y \sim P' \text{ Int } (L' \text{ cl } A))$

Then $\{x_U / U \in \mathcal{U}\}$ is a net converging to x and is such tha no $f(x_U)$ is in $P' \text{ Int } (L' \text{ cl } A))$.

Thus $\{f(x_U)\} / U \in \mathcal{U}\}$ is not eventually in the P' regularly open set with respect to L'

ie., $\{f(x_U)\} / U \in \mathcal{U}\}$ is not eventually in $P' \text{ Int } (L' \text{ cl } A)$ which is a contradiction.

Therefore f is PP' almost continuous with respect to L.

Theorem : 2.1.8

Let f be a PP' open, PP' continuous mapping of(X,P,L) into (Y, P', L') and if g is a mapping of (Y, P', L') into (Z,P",L") then $g \circ f$ is PP' almost continuous with respect to L if and only if g is P' P" almost continuous with respect to L'.

Proof :

Let $g \circ f$ be PP'' almost continuous with respect to L. Let A be a P'' regularly open with respect to L'' subset of Z. Then $(g \circ f)^{-1}$ A is P open.

Since $g \circ f$ is PP'' almost continuous with respect to L. ie., $f^{-1}(g^{-1}(A))$ is P open in X.
Since f is a PP' open mapping, therefore $f(f^{-1}(g^{-1}(A)))$ is P' open.
ie., $g^{-1}(A)$ is P' open in Y and consequently g is P' P'' almost continuous with respect to L'. Conversely assume g is an P' P'' almost continuous mapping with respect to L'.

Let A be a P'' regularly open subset of Z with respect to L. Then by the theorem 2.1.7 (b) g^{-1} (A) is p' open. Now $(g \circ f)^{-1} A = f^{-1} (g^{-1}(A))$

Since f is PP' continuous and $g^{-1}(A)$ is P' open, therefore $f^{-1}(g^{-1}A)$ is P open.

Thus $g \circ f$ is PP'' almost continuous with respect to L.

Theorem : 2.1.9

Let $f : (X,P,L) \longrightarrow (Y, P', L')$ and $x \in X$.
If there exists a P neighbourhood N of x such that f/N is P/NP' almost continuous with respect to L/N. Then f is PP' almost continuous with respect to L at x.

Proof :

Let U be a P' regularly open subset of Y with respect to L' containing f(x).

Since f/N is P/NP' almost continuous with respect to L/N by the theorem 2.1.7(d), there exists a P open set V such that $x \in N \cap V$ and $f(N \cap V) \subset U$.

Since $(N \cap V)$ is a P neighbourhood of x, f is PP' almost continuous with respect to L at x.

Corollary : 2.1.10

Let $f : (X,P,L) \longrightarrow (Y,P',L')$ be a mapping. Let $\{G_\lambda : \lambda \in \Lambda\}$ be a P open cover of X. If for each $\lambda \in \Lambda$, f/G_λ is PP' almost continuous with respect to L. Then f is PP' almost continuous with respect to L.

Proof :

Let x be an arbitrary point in G_λ for some $\lambda \in \Lambda$

Since f/G_λ is PP' almost continuous with repect to L by theorem 2.1.9 f is continuous at x.

Since x is arbitrary f is almost continuous at every $x \in G_\lambda$

Hence f is PP' almost continuous with respect to L.

Therem 2.1.11

Let $f : (X,P,L) \longrightarrow (Y,P',L',)$ be a map of x into Y and $X = X, UX_2$ where X_1 and X_2 are P - closed and f/X_1 and f/X_2 are PP' almost continuous with respect to L. Then f is PP' almost continuous with respect to L.

Proof :

Let A be a P' regularly open subset of Y with respect to L'. Since f/X_1 and f/X_2 are both PP' almost continuous with respect to L, by the theorem 2.1.7(b) we have $(f/X_1)^{-1}(A)$ and $(f/X_2)^{-1}$ (A) are both P closed in X_1 and X_2 respectively.
Since X_1 and X_2 are P closed subsets of X $(f/X_1)^{-1}$ (A) and $(f/X_2)^{-1}(A)$ are also P closed subsets of X.
Also $f^{-1}(A) = (f/X_1)^{-1}(A) \cup (f/X_2)^{-1}(A)$.

Thus f^{-1} (A) is the union of two P closed sets and is therefore P closed.

Hence f is PP' almost continuous with respect to L.

Theorem : 2.1.12

If f is a mapping of (X, P, L) into (Y, P', L') and $X = X_1 \cup X_2$ and if f/X_1 and f/X_2 are both PP' almost continuous with respect to L at a point $x \in X_1 \cap X_2$ then f is PP' almost continuous with respect to L at x.

Proof :

Let U be any P' regularly open subset of Y with respect to L' containing f(x).

Since $x \in X_1 \cap X_2$ and f/X_1, f/X_2 are both PP' almost continuous with respect to L at x by the theorem 2.1.7(d) there exist P open sets V_1 and V_2 such that $x \in X_1 \cap V_1$ and $f(x_1 \cap V_1) \subset U$ and $x \in X_2 \cap V_2$ and $f(x_2 \cap V_2) \subset U$.

Since $X = X_1 U X_2$

$$
\begin{aligned}
V_1 \cap V_2 &= V_1 \cap V_2 \cap X \\
&= V_1 \cap V_2 \cap (X_1 \text{ U } X_2) \\
&= (V_1 \cap V_2 \cap X_1) \text{ U } (V_1 \cap V_2 \cap X_2) \\
f(V_1 \cap V_2) &= f(V_1 \quad V_2 \cap X_1) \text{ U } f(V_1 \cap V_2 \cap X_2) \\
&\subset f(V_1 \cap X_1) \text{ U } f(V_2 \cap X_2) \\
&\subset U.
\end{aligned}
$$

Thus $V_1 \cap V_2 = V$ is a P open set containing x such that $f(V) \subset U$

Therefore f is PP' almost continuous with respect to L at x.

Theorem : 2.1.13

Let $f_\alpha : (X_\alpha, P_\alpha, L_\alpha) \longrightarrow (X'_\alpha, P'_\alpha, L'_\alpha)$ be $P_\alpha \; P'_\alpha$ almost continuous mapping with respect to L_α for each $\alpha \in I$ and let $f : (\prod_{\alpha \in I} X_\alpha, P, L) \longrightarrow (\prod_{\alpha \in I} X'_\alpha, P', L')$ where P, L are product topologies generated by P_α 's and L_α 's respectively on $\prod_{\alpha \in I} X_\alpha$ and P', L' are product topologies generated by P'_α 's and L'_α 's respectively on $\prod_{\alpha \in I} X'_\alpha$, be defined by setting $f(x_\alpha) = (f_\alpha(x_\alpha))$ for each $(x_\alpha) \in \prod_{\alpha \in I} X_\alpha$. Then f is PP' almost continuous with respect to L.

Proof :

Let $(x_\alpha) \epsilon \prod_{\alpha \epsilon I} X_\alpha$ and let O' be a P' regularly open set with respect to L' in $\prod_{\alpha \epsilon I} X'_\alpha$ containing $f(X_\alpha)$. Then there is a member $\prod_{\alpha \epsilon I} O'_\alpha$ of the defining P' base of the product topology on $\prod_{\alpha \epsilon I} X'_\alpha$ such that $f(x_\alpha) \epsilon \prod_{\alpha \epsilon I} O'_\alpha \subset O'$ where $O'_\alpha = X'_\alpha$ for all $\alpha \epsilon I$ except for a finite number of indices α_i $i = 1,2,\ldots n$ (say) and $O'_{\alpha i}$ is a $P_{\alpha i}$ open subset of $X_{\alpha i}$, $i = 1,2,\ldots n$.

Now since O' is P' regularly open set with respect to L' therefore $P'\,\mathrm{Int}\,(L'\,\mathrm{cl}\,(\prod_{\alpha \epsilon I} O'_\alpha)) \subset P'\mathrm{Int}\,(L'\,\mathrm{cl}\,O') = O'$.

Thus for each $_{\alpha i}$, $f_{\alpha i}\,(x_{\alpha i})\ \ O'_{\alpha i} \subset L'_{\alpha i}\ \mathrm{cl}(O'_{\alpha i}))$

Now $P'_{\alpha i}\,\mathrm{Int}\,O'_{\alpha i} = O'_{\alpha i} \subset P'_{\alpha i}\,\mathrm{Int}\,(L'_{\alpha i}\,\mathrm{cl}\,(O'_{\alpha i}))$

Therefore $f_{\alpha i}\,(x_{\alpha i})\ \ P'_{\alpha i}\,\mathrm{Int}\,(L'_{\alpha i}\,\mathrm{cl}\,(O'_{\alpha i}))$ and $f_{\alpha i}$ being $P_{\alpha i}\,P'_{\alpha i}$ almost continuous with respect to $_i L_{\alpha i}$ there is a $P_{\alpha i}$ open subset $U_{\alpha i}$ of $X_{\alpha i}$ such that $x_{\alpha i}\ \ U_{\alpha i}$ and

$$f_{\alpha i}\,(x_{\alpha i})\ \ f_{\alpha i}(U_{\alpha i}) \subset P'_{\alpha i}\,\mathrm{Int}\,(L_{i \alpha i}\,\mathrm{cl}\,(O'_{\alpha i})).$$

Thus $\prod_{\alpha \epsilon I} U_\alpha$ where $U_\alpha = X$ where $\alpha \neq$ where $\alpha \neq$ $_{\alpha i}$, $i=1,2\ldots n$ is a P open set containing (x_α).

Therefore $f\,(\prod_{\alpha \epsilon I} U_\alpha)\ \ O'$.

Hence f is PP' almost continuous with respect to L.

Theorem : 2.1.14

Let $h : (X,P,L) \longrightarrow (\prod_{\alpha \epsilon I} X_\alpha, P', L')$ where P' and L' are the product topologies on $\prod_{\alpha \epsilon I} X_\alpha$ generated respectively by the topologies P_α's and L_α's on X_α be a PP' almost continuous with respect to L. For each $\alpha \epsilon I$, define

$f_\alpha : (X,P,L) \longrightarrow (X_\alpha, P_\alpha, L_\alpha)$ by setting $f_\alpha(x) = (h(x)_\alpha)$

Then f_α is PP almost continuous with respect to L.

Proof :

Let P_α denote the projection of $\prod_{\alpha \epsilon I} X_\alpha$ into X_α.

Then $P_\alpha \circ h = f_\alpha$, for each α. Now p is P' P_α open and $P'P_\alpha$ continuous for each α . and h is PP' almost continuous with respect to L.

Therefore By theorem 2.1.8 $P_\alpha \circ h$ is PP_α almost continuous with respect to L.

ie., f_α is PP_α almost continuous with respect to L.

Definition : 2.1.15

Let (X,P,L) be a bitopological space. Let $A \subset X$. Let $x \epsilon A \subset X$. Then x is called a P boundary point of A is $x \notin$ P Int. A.

Theorem : 2.1.16

Let $f : (X,P,L) \longrightarrow (Y, P', L')$

The set of all points of X at which f is not PP' almost continuous with respect to L is the union of the boundaries of the inverse images of P' regularly open subsets of Y with respect to L'.

Proof :

Assume f is not PP' almost continuous with respect to L at a point $x \epsilon X$.

Then by the theorem 2.1.7(d) there exists a P' regularly open set with respect to L'(say) V such that $f(x) \epsilon V$ and for every P open set U containing x,

$f(U) \not\subset V$

ie $f(U) \cap V = \emptyset$

ie $f(U) \cap (Y \sim V) \neq \emptyset$

Thus for each P open set containing x, we have

$U \cap f^{-1}(Y \sim V) \neq \emptyset$

ie $U \cap (X \sim f^{-1}(V)) \neq \emptyset$

Therefore x cannot be a P interior point of $f^{-1}(V)$

ie $x \notin P\ \mathrm{Int}(f^{-1}(v))$

But $x \in f^{-1}(V)$

Hence x is a point in the P boundary of $f^{-1}(V)$.

Now let x belongs to the boundary of $f^{-1}(G)$ for some P' regularly open subset G with respect to L' of Y. Then $f(x) \in G$.

If f is PP' almost continuous with respect to L at x by the theorem 2.1.7(d) there exists a P open set U such that $x \in U$ and $f(U) \subset G$.

Thus $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(G)$

Therefore x is a P interior point of $f^{-1}(G)$ which is a contradiction to x belonging to boundary of $f^{-1}(G)$. Hence f is not PP' almost continuous at x with respect to L.

Definition : 2.1.17

A bitopological space (X, P, L) is said to be P almost regular with respect to L if for each P regularly closed set A with respect to L and each point $x \notin A$ there exist disjoint sets U and V such that U is P open, V is L open and $x \in U$, $A \subset V$.

Definition : 2.1.18

Let (X, P, L) be a bitopological space. Let $x, y \in X$ such that $x \neq y$. Then (X, P, L) is said to be a pairwise Urysohn space if there exists a P open set U and a L open set V such that $x \in U$, $y \in V$ and $L\ \mathrm{cl}\ U \cap P\ \mathrm{cl}\ V = \emptyset$.

Theorem : 2.1.19

If f is a PP' almost continuous map with respect to L of a regular space (X, P, L) with respect to L onto a bitopological space (Y, P', L') and f is (PP', LL') closed such that $f^{-1}(y)$ is P compact with respect to L for all $y \in Y$, then (Y, P', L) is P' almost regular with respect to L'.

Proof :

Let A be a P' regularly closed with respect to L'.

Then $A = P'\ \mathrm{cl}\ (L'\ \mathrm{Int.}\ A)$

Suppose $y \notin A$ then $f^{-1}(y) \cap f^{-1}(A) = \emptyset$

Since f is PP' almost continuous with respect to L by theorem 2.1.7(c) we have $f^{-1}(A)$ is P closed. Also $f^{-1}(y)$ is P compact with respect to L.

Since (X, P, L) is P regular with respect to L by the definition of P almost regular, there exist a P open set G and a L open set H such that

$$f^{-1}(A) \subset H, \; f^{-1}(y) \subset G \text{ and } G \cap H = \emptyset$$

Now let $P_1 = \{ z / f^{-1}(z) \subset G \}$ and $Q = \{ z / f^{-1}(z) \subset H \}$

Obviously $A \subset Q$, $y \in P_1$, $P_1 \cap Q = \emptyset$

Also $Y \sim P_1 = f(X \sim G)$ and $Y \sim Q = f(X \sim H)$

Since f is (PP', LL') closed, G is P open

$= f(X \sim G)$ is P' closed

$= Y \sim P_1$ is P' closed

and H is L open $= f(X \sim H)$ is L' closed

$= Y \sim Q$ is L' closed.

ie P_1 is P' open and Q is L' open such that $y \in P_1$, $A \subset Q$, $P_1 \cap Q = \emptyset$

Therefore (Y, P', L') is P' almost regular with respect to L'.

Definition : 2.1.20

Let (X, P, L) be a bitopological space. X is said to be pairwise almost normal if given disjoint sets A and B such that A is P regularly closed with respect to L and B is L regularly closed set with respect to P, there exist a L open set G and P open set H such that $A \subset G$, $B \subset H$ and $G \cap H = \emptyset$.

Remark : 2.1.21

A pairwise normal (regular) space is pairwise almost normal (regular) but not conversely is shown in the following example.

Example : 2.1.22

Let $X = \{a, b, c, d\}$

$P = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$

$L = \{\emptyset, X, \{a\}, \{b\}, \{a, d\}, \{a, b\}, \{a, b, d\}\}$

Then (X, P, L) is a pairwise almost normal space but not a pairwise normal space.

Definition : 2.1.23

A space (X, P, L) is said to be pairwise normal if given a P closed set A and a P closed set B with $A \cap B = \emptyset$ there exist a L open U and P open set V such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$

P closed sets $= \{ X, \emptyset, \{b, c, d\}, \{a, c, d\}, \{b, d\} \{c, d\}, \{d\} \}$

L closed sets $= \{X, \emptyset, \{b, c, d\}, \}\{a, c, d\}, \{c, d\}, \{b, c\}, \{c\} \}$

with $d \cap c = \emptyset$, then by the definition of pairwise normal there exist a L open set $\{a, d\}$, $\{a, b, d\}$ and a P open set $\{a, c\}$, $\{a, b, c\}$

such that $\{a, d\} \cap \{a, c\} \neq \emptyset$

$\{a, b, d\} \cap \{a, b, c\} \neq \emptyset$

$\{a, d\} \cap \{a, b, c\} \neq \emptyset$

$\{a, c\} \cap \{a, b, d\} \neq \emptyset$

Thus (X, P, L) is not a pairwise normal space.

Theorem : 2.1.24

If f is a closed pairwise almost continuous map of a pairwise normal space (X, P, L) onto a space (Y, P', L') then (Y, P', L') is pairwise almost normal.

Proof :

Let A be a P' regularly closed set with respect to L' and B be a L' regularly closed set with respect to P' and $A \cap B = \emptyset$

Since f is pairwise almost continuous, $f^{-1}(A)$ is P closed and $f^{-1}(B)$ is L closed.

Since (X, P, L) is pairwise normal there exist disjoint sets G and H such that G is P open, H is L open and $f^{-1}(A) \subset H$ and $f^{-1}(B) \subset G$.

Now if we define $P_1 = \{z/f^{-1}(z) \subset G\}$ and $Q = \{z/f^{-1}(z) \subset H\}$ the closeness of f implies that P_1 is P' open, Q is L' open and $A \subset Q$, $B \subset P_1$ and $P_1 \cap Q = \emptyset$. Thus (Y, P', L') is pairwise almost normal.

Section 2.2 Weakly Continuous Map

Definition : 2.2.1

Let f be a map from $(X, P, L) \longrightarrow (Y, P', L')$. Then f is said to be PP' weakly continuous with respect to L if for each point $x \in X$ and each P'neighbourhood V of f(x) there exists a P neighbourhood U of x such that $f(U) \subset L'$ cl V.

Note :

f is $(X, P, L) \longrightarrow (Y, P', L')$ is pairwise weakly continuous if it is PP' weakly continuous with respect to L and LL' weakly continuous with respect to P.

Theorem : 2.2.2

If $f : (X, P, L) \longrightarrow (Y, P', L')$ is a PP' weakly continuous map with respect to L and is also a PP' open map then f is PP' almost continuous with respect to L.

Proof :

Let $x \in X$ and M be any P' neighbourhood of f(x).

Since f is PP' weakly continuous with respect to L there exist a P neighbourhood N of x such that $f(N) \subset L'$ cl M.

Since f is PP' open, f(N) is P' open.

Hence $f(N) \subset P'$ Int (L' cl M)

Therefore f is PP' almost continuous with respect to L.

Remark : 2.2.3

Every PP' almost continuous map with respect to L is PP' weakly continuous with respect to L. The converse is not true is shown in the following example.

Example : 2.2.4

Let X = the set of all real numbers.

$P = \{\emptyset, X,$ Complements of all countable subsets of X $\}$

$L = \{\emptyset, X,$ the sets whose complements are countable$\}$

$Y = \{a, b, c\}$

$P' = \{\emptyset, Y, \{c\}, \{b\}, \{b, c\}\}$

$L' = \{\emptyset, Y, \{a\}, \{c\}, \{a, c\}\}$

Let $f : (X, P, L) \quad (Y, P', L')$ be defined by

$f(x) = \{a$ if x is rational $\}$

$\{b$ if x is irrational$\}$

Then f is PP' weakly continuous with respect to L but not PP' almost continuous with respect to L.

When x is irrational, the neighbourhood of b is $\{b, c\}$

$P'\text{ Int }(L'\text{ cl }\{b, c\})) = \{b, c\}$

when x is rational, the neighbourhood of a is $\{a, b\}$

$f(\{a, b\}) = \{a, b\}$

$f(\{a, b\}) \not\subset P'\text{ Int }(L'\text{ cl}(\{b, c\}))$

ie $\{a, b\} \not\subset \{b, c\}$

Therefore f is not PP' almost continuous with respect to L.

Definition : 2.2.5

A bitopological space (X, P, L) is said to be P semi regular with respect to L if for each point x of the space and each P open set U containing x there is a P open set V such that $x \in V \subset P \text{ Int } (L \text{ cl } V) \subset U$.

Theorem : 2.2.6

If f is a PP' almost continuous map with respect to L of a bitopological space (X, P, L) into a space (Y, P', L') which is P semi regular with respect to L' then f is PP' continuous.

Proof :

Let $x \in X$ and let A be a P' open set containing f(x). Since Y is P' semi regular with respect to L' there exist a P' open set V such that $f(x) \in V \subset P' \text{ Int}(L' \text{ cl } V) \subset A$

Since f is PP' almost continuous with respect to L and since A is a P' open set containing f(x) there is a P open subset U of X containing x such that $f(x) \in f(U) \subset P' \text{ Int}(L' \text{ cl } V)$.

Therefore U is an P open set containing x such that $f(U) \subset A$

Therefore $f(x) \in f(U) \subset A$.

ie $\quad x \in U \subset f^{-1}(A)$

Thus f is PP' continuous at x. Since x is arbitrary it follows that f is PP' continuous.

Definition : 2.2.7

$A \subset (X, P, L)$ is P compact with respect to L if for every P open covering of A, there exist a L open finite finer covering.

Definition : 2.2.8

$A \subset (X, P, L)$ is a P almost compact with respect to L if for every P open covering of A, there exist a P open finite subfamily whose L closures cover A.

Theorem : 2.2.9

If f is a PP' weakly continuous map with respect to L and also one to one map of (X, P, L) onto (Y, P', L') and if X a compact with respect to L and Y is Urysohn, f is PL' open.

Proof :

Let A be a P open set then $X \sim A$ is a P closed subset of X which is a P compact set with respect to L.

Claim 1 :

$f(X \sim A)$ is P' almost compact with respect to L'.

Let $\{V_\alpha \,/\, \alpha \in \Lambda\}$ be a P' open covering of $f(X \sim A)$ since f is PP' weakly continuous with respect to L for each V_α there exist a P open set U_α such that $f(U_\alpha) \subset L'$ cl (V_α).

Now $\{U_\alpha \,/\, \alpha \in \Lambda\}$ is a P open cover of $(X \sim A)$ which is a P compact set with respect to L.

Therefore there exist a L open finite finer covering (say)

$\{W_{\alpha i} / \alpha_i \in \Lambda\}$ i = 1,2,.......n such that

$f(W_{\alpha_i}) \subset f(U_{\alpha i}) \subset L'$ cl V

Now $X \sim A \subset \bigcup_{i=1}^{n} W_{\alpha_i}$

Therefore $f(X \sim A)$ $f(\bigcup_{i=1}^{n} W_{\alpha i}) = \bigcup_{i=1}^{n} f(W_{\alpha i})$

$\subset \bigcup_{i=1}^{n} L'$ cl $(V_{\alpha i})$

Now $\{V_{\alpha_i} / \alpha_i \in \Lambda\}$ i = 1,2,........n is a P' open family whose L' closures cover $f(X \sim A)$.

Hence $f(X \sim A)$ is P' almost compact with respect to L'. Since f is one to one, $f(X \sim A) = Y \sim f(A)$.

Now Y is Urysohn and $Y \sim f(A)$ is P' almost compact with respect to L' since $f(X \sim A)$ is P' almost compact with respect to L'.

Claim 2 :

$Y \sim f(A)$ is L' closed

Let $t \notin y \sim f(A) = B$ (say). Therefore $t \neq y$ for $y \in B$.

Since Y is Urysohn there exists a L' open set U_{ty} and a P' open set V_{yt} such that P cl $U_{ty} \cap L'$ cl $V_{yt} = \emptyset$.

$\{ V_{yt} / y \in B \}$ is a P' open covering of B, a P' almost compact set with respect to L' there exists a P' open finite subfamily $\{V_{yt} / y_i \in B, \}$ i = 1,2,......n whose L' closures cover B.

ie, $B \subset \bigcup_{i=1}^{n} L'$ cl $V_{y_i t}$ if $U = \bigcap_{i=1}^{n} U_{y_i t}$, $U \cap B = \emptyset$.

Also U is L' open, $t \neq L'$ cl B ($= Y \sim f(A)$).

Therefore $Y \sim f(A)$ is L' closed

Therefore $f(A)$ is L' open

Hence f is a PL' open map.

Lemma : 2.2.10

A map $f : (X, P, L) \longrightarrow (Y, P', L')$ is PP' weakly continuous with respect to L if and only if for each P' open set V in Y, $f^{-1}(V) \subset P \text{ Int } (f^{-1}(L' \text{ cl } V))$

Proof :

Let f be a PP' weakly continuous map with respect to L. Let V be a P' open set in Y.

Consider $f^{-1}(V)$ Let $x \in f^{-1}(V)$ then $f(x) \in V$. Therefore there exists a P open set O containing x such that $f(O) \subset L' \text{ cl } V$.

Therefore $x \in O \subset f^{-1}(L' \text{ cl } V)$

Therefore $x \in P \text{ Int } (f^{-1}(L' \text{ cl } V))$

Therefore $f^{-1}(V) \subset P \text{ Int}(f^{-1}(L' \text{ cl } V))$.

Conversely let $f^{-1}(V) \subset P \text{ Int}(f^{-1}(L' \text{ cl } V))$

Take $U = P \text{ Int } (f^{-1}(L' \text{ cl } V))$

Consider $P \text{ Int } (f^{-1}(L' \text{ cl } V)) \subset f^{-1}(L' \text{ cl } V)$

$$U \subset f^{-1}(L' \text{ cl } V)$$

$$f(U) \subset L' \text{ cl } V.$$

Thus f is PP' weakly continuous with respect to L.

Theorem : 2.2.11

Let $(X,P,L) \longrightarrow (Y, P', L')$ be a map and $g:X \longrightarrow X \times Y$ be the graph of f given by $g(x) = (x,f(x)$ for every $x \in X$. Then $g:X \longrightarrow X \times Y$ is PP^* weakly continuous with respect to P. (Here $(X \times Y, P^*, L^*)$ is the product space where P^* is generated by P and P' and L^* is generated by L and L'^*) if and only if $f:X \longrightarrow Y$ is PP' weakly continuous with respect to L.

Proof :

Let g be PP^* weakly continuous with respect to L. Let $x \in X$ and $V \subset Y$ be any P' open set containing f(x). Then $X \times V$ is a P^* open set in $X \times Y$ containing g(x).

Since g is PP^* weakly continuous with respect to L there exist a P open set $U \subset X$ containing x such that

$g(U) \subset L^*$ cl $(X \times V)$

$\subset X \times L'$ cl V.

$(U, f(U)) \subset X \times L'$ cl V

$f(U) \subset L'$ cl V

Therefore f is PP' weakly continuous with respect to L.

Conversely, let f be PP' weakly continuous with respect to L. Let $x \in X$ and W be any P^* open set in $X \times Y$ containing g(x).

Then there exist a P open set $U \subset X$ and P' open set $V \subset Y$ such that $g(x) = (x,f(x)$ U $x_{\epsilon}V \subset W$.

Since f is PP' weakly continuous with respect to L then there exist a P open set $O \subset X$ containing x such that $O \subset U$ and $f(O) \subset L'$ cl V.

Therefore $g(O) = (O, f(O))$

$$\subset U \times L' \text{ cl } V$$
$$\subset L^{*} \text{ cl } (U \times V)$$
$$\subset L^{*} \text{ cl } W$$

Thus g is PP^{*} weakly continuous with respect to L.

Theorem : 2.2.12

Let (X,P,L) be a pairwise connected space and let f be a pairwise weakly continuous map of X onto (Y,P',L') then (Y, P', L') is pairwise connected.

Proof :

Let Y be not connected then there is a seperation U, V of disjoint non empty open subsets of Y whose union is Y. ie., $Y = A \cup B$ where A is P' open and B is L' open and $A \cap B = \emptyset$

Also A is L' closed and B is P' closed. Since f is surjective, $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$

Since f is pairwise weakly continuous by the lemma 2.2.10, $f^{-1}(A) \subset P$ Int $(f^{-1}(L'$ cl $A))$ and $f^{-1}(B) \subset L$ Int $(f^{-1}(P'$ cl $B))$

Since A is L' closed, L' cl $A = A$ and B is P' closed, P' cl $B = B$.

Therefore $f^{-1}(A) \subset P$ Int $(f^{-1}(A))$ and

$f^{-1}(B) \subset L$ Int $(f^{-1}(B))$

$\Rightarrow$ $f^{-1}(A)$ is P open and $f^{-1}(B)$ is L open. Also $X = f^{-1}(A) \cup f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \emptyset$

Therefore X is not pairwise connected which is a contradiction to our assumption. Hence (Y,P',L') is pairwise connected.

Theorem : 2.2.13

Let $f : (X,P,L) \longrightarrow (Y, P', L')$ be a LL' weakly continuous map with respect to P then $(L \text{ cl } (f^{-1}(V)) \subset f^{-1}(L' \text{cl} V))$ for every P' open $V \subset Y$.

Proof :

Let $x \in L$ cl $(f^{-1}(V))$ and $x \notin f^{-1}(L'$ cl $V))$

$\Rightarrow$ $f(x) \notin L'$ cl V.

Then there exists a L' open set W containing $f(x)$ such that $W \cap V = \emptyset$. . Also V is P' open.

Therefore $V \cap P'$ cl $W = \emptyset$

Since f is LL' weakly continuous with respect to P and W is a L' open set containing $f(x)$ there exists a L open set U such that $f(U) \subset P'$ cl W

Therefore $f(U) \cap V = \emptyset$

Also $x \in L$ cl $(f^{-1}(V))$ and U is a open set containing x

$\Rightarrow$ $U \cap f^{-1}(V) \neq \emptyset$ ie $f(U) \cap V \neq \emptyset$ which is a contradiction.

Therefore $L \text{ cl } (f^{-1}(V)) \subset f^{-1} (L' \text{ cl } V)$ for every P' open $V \subset Y$.

Theorem : 2.2.14

If (Y, P', L') is a pairwise Urysohn space and $f : (X,P,L) \longrightarrow (Y, P', L')$ is a pairwise weakly continuous injection then (X,P,L) is a pairwise Hausdorff space.

Proof :

Let x_1, d_2, be any two distinct points of X. Sincre f is injective $f(x_1) = f(x_2)$ since (Y, P', L') is pairwise Urysohn space there exist a P' open set U and L' open set V such that $f(x_1) \in U$ and $f(x_2) \in V$ and $L' \text{ cl } U \quad P' \text{ cl } V = 0$.
Therefore $P \text{ Int } f^{-1}(L' \text{ cl } U) \quad L \text{ Int } f^{-1}(P' \text{ cl } V) = 0$
Since f is a pairwise weakly continuous by the lemma 2.2.10 we get $x_1 \in f^{-1}(U) \subset P \text{ Int } (f^{-1}(L' \text{ cl } U))$ and

$x_2 \in f^{-1}(V) \subset L \text{ Int } (f^{-1}(P' \text{ cl } V))$.

Therefore (X,P,L) is a pairwise Hausdorff space.

Theorem : 2.2.15

Let $A \subset X$ and let $f : X \longrightarrow A$ be a PP' weakly continuous retraction of X onto A with respect to L. (P' and L' are topologies A induced respectively by P and L). If (X,P,L) is pairwise Hausdorff then A is closed in the upper bound topology of P and L.

Proof :

Assume that A is not closed in the upper bound topology of P and L (denoted by **T** topology).

Let $x \in \mathbf{T}$ cl $A - A$. Then $x \neq A \Rightarrow f(x) \neq x$.

Also since X is pairwise Hausdorff, there exists a P open set U and a L open set V such that $f(x) \in U$ and $x \in V$ and

$U \cap V = \emptyset$.

Also $V \cap L$ cl $U = \emptyset$.

Now let W be any P open set containing x then $V \cap W$ is a $\mathbf{T}$ open set containing x and $x \in \mathcal{T}$ cl A.

Therefore $(V \cap W) \cap A \neq \emptyset$

Therefore exists y such that $y \in (W \cap V) \cap A$.

Now $y \in A \Rightarrow f(y) = y$.

Also $f(y) \in V \Rightarrow f(y) \notin L$ cl U and $y \in W$.

Therefore $f(W) \not\subset P$ cl U.

Therefore $f(W) \not\subset L$ cl $(U \cap A)$

$= f(W) \not\subset L'$ cl $(U \cap A)$.

$U \cap A$ is a P' open set and W is any arbitrary P open set and $f(W) \not\subset L'$ cl $(U \cap A)$. This contradicts the fact that f is PP' weakly continuous map with respect to L.

Hence A is $\mathbf{T}$ closed. ie., A is closed in the upper bound topology of P and L.

Theorem : 2.2.16

Let f_1 be a PP' weakly continuous map with respect to L and f_2 be a LL' weakly continuous map with respect to P from a bitopological space (X,P,L) into a Urysohn space (Y, P', L') then $\{x \in X / f_1(x) = f_2(x)\}$ is closed in the upper bound topology of P and L.

Proof ;

Let **T** denote the upper bound topology of P and L.

Suppose A denote the set $\{x \in X / f_1(x) \neq f_2(x)\}$. Let $x \in X - A \Rightarrow f_1(x) \neq f_2(x)$. Since Y is a Urysohn space there exists a P' open set U and a L' open set V such that $f_1(x) \in U$ and $f_2(x) \in V$, L' cl U $\cap$ P' cl V = $\emptyset$.

Since f_1 is PP' weakly continuous with respect to L by lemma 2.2.10 we have $x \in f_1^{-1}(U) \subset P$ Int $(f_1^{-1}(L'$ cl U) and also since f_2 is LL' weakly continuous with respect to P by lemma 2.2.10 we have,

$x \in f_2^{-1}(V) \subset L$ Int $(f_2^{-1}(P'$ cl V))

Let O = P Int $(f_1^{-1}$ (L' cl U)) $\cap$ L Int $(f_2^{-1}$(P' cl V)) then O is **T** open. Also O $\subset$ X - A.

For if $x \in O$ then,

$x \in P$ Int $(f_1^{-1}$ (L' cl U)) $\Rightarrow f_1(x) \in$ L' cl U.

$x \in L$ Int $(f_1^{-1}$ (P' cl V)) $\Rightarrow f_2(x) \in$ P' cl V.

Since L' cl U $\cap$ P' cl V = $\emptyset$, $f_1(x) \neq f_2(x) \Rightarrow x \in X - A$.

Therefore $x \in O \subset X - A$ and O is open

Therefore X - A is $\mathcal{T}$ open So A is $\mathcal{T}$ closed.

Thus A is closed in the upper bound topology of P and L.

Corollary : 2.2.17

Let f_1 be a PP' weakly continuous map with respect to L and f_2 be a LL' weakly continuous map with respect to P of (X,P,L) onto a Urysohn space. If B is **T** dense in X ($\mathcal{T}$ is the upper bound topology of P and L) and $f_1 = f_2$ on B then $f_1 = f_2$.

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