A Study of Fuzzy BP-Subalgebras and Fuzzy BP-Ideals in BP-Algebras

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By

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DECLARATION

I declare that the dissertation entitled **A Study of Fuzzy BP-Subalgebras and Fuzzy BP-Ideals in BP-Algebras** submitted by me for the degree of Master of Philosophy (M.Phil.) is the record of work carried out by me during the period from August 2018 to July 2019 under the guidance of **Dr. P. Jeyalakshmi, M.Sc., M.Phil., Ph.D.,** Professor & Head, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for women, Coimbatore, and has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this University or any other University or other similar institution of Higher Learning.

R. Shob

Signature of the Candidate

CERTIFICATE

This is to certify that the dissertation entitled **A Study of Fuzzy BP-Subalgebras and Fuzzy BP-Ideals in BP-Algebras** submitted for the degree of **Master of Philosophy (M.Phil.)** by Shalini .R is the record of research work carried out by her during the period from August 2018 to July 2019 under my guidance and supervisior, and that this work has not formed the basis for the award of any Degree, Diploma, Associateship, Fellowship, Titles in this University or any other University or other similar institution of Higher Learning.

P. Jeyalakshmi 31.7.19

Signature of the Head of the Department

P. Jeyalakshmi 31.7.19

Signature of Supervisior

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CONTENT

INTRODUCTION

In 1966, Imai and Iseki [11] introduced new classes of abstract algebras, BCK-Algebras and BCI-Algebras. While studying the algebraic structures of BCK and BCI logic, many algebraic structures were studied by several authors, which are generalization of BCI and BCI-Algebras. In 1983, Hu and Li [10] introduced the notion of BCH-Algebras. In 1999, Neggers and Kim [24] introduced the notion of d-Algebras which is another generalization of BCK-Algebras. In 2013, Ahn and Han [1] introduced the notion of BP-Algebras.

In 1965, Zadeh [35] introduced the notion of fuzzy sets which give a complete picture of uncertainty in real physical world. In 1967 Goguen [9] extended the notion of fuzzy sets to the notion of L-Fuzzy sets where L is a Complete Lattice. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971, by Rosenfeld [29]. Xi [34] applied the concept of fuzzy sets to BCK-Algebras and got some results in 1991. In 1986 Atanassov [2] generalized the concept of fuzzy sets into Intuitionistic fuzzy sets.

This thesis is devoted to the study of Fuzzy BP-Subalgebras and Fuzzy BP-Ideals and Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras.

The following articles are chosen for our discussion:

1) **"ON BP-ALGEBRAS"** by Sun Shin Ahn and Jeong Soon Han [1].
2) **"f-DERIVATIONS ON BP-ALGEBRAS"** by N. Kandaraj and A. Arul Devi [16].
3) **"FUZZY ALGEBRAIC STRUCTURE IN BP-ALGEBRAS"** by Y.Christopher Jefferson. and M. Chandramouleeswaran [3].
4) **"FUZZY BP-IDEAL"** by Y.Christopher Jefferson and M. Chandramouleeswaran [4].
5) **"FUZZY T-IDEALS IN BP-ALGEBRAS"** by Y.Christopher Jefferson and M. Chandramouleeswaran [5].
6) **"ON INTUITIONISTIC FUZZY BP-IDEALS IN BP-ALGEBRAS"** by R. Shalini and P. Jeyalakshmi [30].

7) **"L-FUZZY BP-ALGEBRAS"** by Y.Christopher Jefferson and M. Chandramouleeswaran [6].
8) **"L-FUZZY BP-IDEAL"** by Y.Christopher Jefferson and M. Chandramouleeswaran [7].
9) **"ON INTUITIONISTIC L-FUZZY BP-IDEALS IN BP-ALGEBRAS"** by Y. Christopher Jefferson and M. Chandramouleeswaran [8].

This thesis is divided into five chapters.

The first chapter deals with the study of BP-Algebras and Quadratic BP-Algebras. In this chapter, the preliminaries of BP-Algebras and quadratic BP-Algebras are presented. Also the relations between BP-Algebra and several other algebras are discussed.

In this chapter, the following interesting results are discussed:

(i) Let (X, Δ, 0) be a abelian group. If $x * y = x \Delta y^{-1}$, then (X, $*$, 0) is a BP-Algebra.

(ii) If (X, $*$, 0) is a 0-commutative B-Algebra, then (X, $*$, 0) is a BP-Algebra.

(iii) Let X be a field with $|X| \geq 3$. Then every quadratic BP-Algebra (X, $*$, e) has of the form $x * y = x - y + e$, where x, y, z$\in$X.

(iv) Let X be a field with $|X| \geq 3$. Then every quadratic BP-Algebra X is a BCI-Algebra.

In Chapter 2, the notions of f-derivations, regular f-derivations, Composition of f-derivations on BP-Algebras and their properties are discussed.

In this chapter, the following main results are discussed:

(i) Every (r, l)-f-Derivation ((l, r)-f-Derivation) of a BP-Algebra is regular.

(ii) Let θ_f be a regular (r, l)-f-Derivation on a BP-Algebra X. Then f-ideal A on X is θ_f invariant.

(iii) Let X be a BP-Algebra and θ_f, θ'_f are the (l, r)-f-Derivations on X. Let $f^2 = f \circ f = f$, then ($\theta_f \circ \theta'_f$) is also a (l, r)-f-Derivation on X.

(iv) Let X be a BP-Algebra and θ_f, θ'_f are f-Derivations on X. Then $(f \circ \theta'_f) . (\theta_f \circ f) = (\theta_f \circ f) . (f \circ \theta'_f)$.

(v) Let X be a BP-Algebra and θ_f, θ'_f are (l, r)-f-derivations on X. Then $\theta_f \wedge \theta'_f$ is also a (l, r)-f-derivations on X.

Chapter 3 deals with the study of fuzzy Algebraic structure in BP-Algebras. In this chapter, properties of Fuzzy BP-Subalgebras are discussed and obtained some results.

In this chapter, the following important results are discussed:

(i) Any BP-Subalgebra of a BP-Algebra (X, $*$, 0) can be realized as a level subalgebra of some fuzzy BP-Subalgebra of X.

(ii) Let μ and λ be two fuzzy BP-Subalgebras of X with identical family of level BP-Subalgebras. If Im(μ) = $\{t_1, t_2, t_3,,,, t_n\}$ and Im(λ) = $\{s_1, s_2, s_3,,,, s_n\}$ where $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_n$ and $s_1 \geq s_2 \geq s_3 \geq \ldots\ldots \geq s_m$ then

   1) m = n
   2) U(μ, t_i) = U(λ, s_i) for i = 1,2,,,,,,n.
   3) If μ(x) = t_i, then λ(x) = s_i, $\forall$ x $\in$ X and i = 1,2,,,,,,n.

(iii) Two level BP-Subalgebras U(μ, s) and U(μ, t), (s < t) of a fuzzy BP-Subalgebra μ are equal if and only if there is no x $\in$ X such that s $\leq$ μ(x) < t.

(iv) If μ_1 and μ_2 are fuzzy BP-Subalgebras of X, then $\mu = \mu_1 \times \mu_2$ is a fuzzy BP-Subalgebra of X $\times$ X.

Chapter 4 deals with the study of Fuzzy BP-Ideals and Intuitionistic Fuzzy BP-Ideals in BP-Algebras. The properties of Fuzzy BP-Ideals, Fuzzy T-Ideals, Cartesian product of Fuzzy T-Ideals in BP-Algebras are discussed. Also we [30] introduced the notion of Intuitionistic fuzzy BP-Ideals in BP-Algebras and studied their properties.

In this chapter, the following interesting results are discussed.

(i) If μ is a fuzzy ideal of a BP-Algebra (X, $*$, 0) and $\mu_\alpha(x) = \min\{\alpha, \mu(x)\}\ \forall\, x \in X$ and $\alpha \in [0, 1]$, then $\mu_\alpha(x)$ is a fuzzy BP-Ideal of X.

(ii) A Fuzzy set μ of a BP-Algebra (X, $*$, 0) is a fuzzy BP-Ideal if and only if every nonempty level set of U(μ, s), s $\in$ Im(μ) is a BP-Ideal.

(iii) Let f : $X_1 \to X_2$ be an epimorphism of BP-Algebras. Let μ be a fuzzy set of X_2. If $f^{-1}(\mu)$ is a fuzzy BP-Ideal of X_1, then μ is a fuzzy BP-Ideal of X_2.

(iv) Let μ and λ be fuzzy sets in a BP-Algebra X such that $\mu \times \lambda$ is a fuzzy T-ideal of X$\times$X. Then

  (i) Either μ(0) $\geq$ μ(x) or λ(0) $\geq$ λ(x) for all x$\in$X.

  (ii) If μ(0) $\geq$ μ(x) for all x$\in$X, then either λ(0) $\geq$ μ(x) or λ(0) $\geq$ λ(x)

  (iii) If λ(0) $\geq$ λ(x) for all x$\in$X, then either μ(0) $\geq$ μ(x) or μ(0) $\geq$ λ(x)

  (iv) Either μ or λ is a fuzzy T-ideal of X.

Chapter 5 deals with the study of L-fuzzy BP-Ideals and Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras. The properties of L-Fuzzy BP-Subalgebras, L-fuzzy BP-Ideals, L-fuzzy T-ideals and Intuitionistic L-fuzzy BP-ideals in BP-Algebras are established.

The interesting results discussed in this chapter are given as follows:

(i) Let μ and λ be two L-Fuzzy Sub algebras of X with identical family of level sub algebras. Then Im(μ) = Im(λ) implies $\mu = \lambda$.

(ii) A L-fuzzy set μ of a BP-Algebra (X, $*$, 0) is a L-fuzzy BP-Ideal if and only if for any $\lambda \in$ L, U(μ, λ) = {x : x$\in$X, μ(x) $\geq$ λ} is an BP-Ideal of X where U(μ, λ) $\neq \phi$.

(iii) For any L-Fuzzy sets μ and λ of X, if $\lambda \times \mu$ is a L-Fuzzy BP-Ideal of X, then either λ or μ is a L-Fuzzy BP-Ideal of X.

(iv) An Intuitionistic L-Fuzzy set A of X is an Intuitionistic L-Fuzzy BP-Subalgebra of X if and only if the L-Fuzzy sets μ_A and $\overline{\nu}_A$ are L-Fuzzy BP-Subalgebras of X.

(v) Let f : X$\rightarrow$Y be an epimorphism of BP-algebra and let A = (μ_A, ν_A) be an Intuitionistic L-Fuzzy Set in Y. If A^f = (μ_A^f, ν_A^f) is an Intuitionistic L-fuzzy BP-ideal of X. Then A = (μ_A, ν_A) is an Intuitionistic L-Fuzzy BP-Ideal of Y.

REVIEW OF LITERATURE

In 1966, Imai and Iseki [11] introduced two classes of abstract algebras: BCK-Algebras and BCI-Algebras. It is known that the class of BCK-Algebras is a proper subclass of the class of BCI-Algebras. In 1983, Hu and Li [10] introduced a wide class of abstract algebras: BCH-Algebras. They have shown that the class of BCI-Algebras is a proper subclass of the class of BCH-Algebras. In 1999, Neggers and Kim [24] introduced the notion of d-Algebras which is another generalization of BCK-Algebras, and then they investigated several relations between d-Algebras and BCK-Algebras as well as some other interesting relations between d-Algebras and oriented digraphs. Also, in 2002 they introduced the notion of B-Algebras [25]. In 2006, Walendziak [32] obtained another axiomatization of B-Algebras. In 1998, Jun, Roh and Kim [13] introduced a new notion, called a BH-Algebras which is a generalization of BCH/BCI/BCK-Algebras. In 2007, Walendziak [33] introduced a notion, called an BF-Algebras. It was shown that a BF-Algebra is a generalizations of a B-Algebra. In 2006, Kim and Kye [19] introduced the notion of a quadratic BF-Algebra, and obtained that quadratic BF-Algebras, quadratic Q-Algebras, BG-Algebras and B-Algebras are equivalent notions on a field X with $|X| \geq 3$, and hence every quadratic BF-Algebra is a BCI-Algebra. In 2013 Ahn and Han [1] introduced the notion of BP-Algebras.

In 1965, Zadeh [35] introduced the theory of fuzzy sets. In 1967, Goguen [9] extended the notion of fuzzy sets to the notion of L-Fuzzy sets. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups in 1971, by Rosenfeld [29]. In 1991, Xi [34] applied the concept of fuzzy sets to BCK-Algebras and got some results.

In 1986, Atanassov [2] generalized the concept of fuzzy sets into Intuitionistic fuzzy sets. In 2000, Jun and Kim [14] studied Intuitionistic fuzzy ideals of BCK-Algebras. In 2010 Muralikrishna and Chandramouleeswaran [22, 23] studied Intuitionistic L-Fuzzy subalgebras of BG-Algebras and Cartesian product of Intuitionistic L-Fuzzy BF-Ideals in 2011.
Here we present abstract of some important articles on Various algebras related to this topic.

1. An introduction to the theory of BCK-algebras

K.Iseki and S.Tanaka (1978) [12]

In this article, the definition of BCK-algebras and its fundamental properties are studied.Various ideals in BCK-algebras are discussed in detailed manner. Also, the homomorphism properties on BCK-algebras are discussed.

2. On d-Algebras

J . Neggers and He E Sik Kim (1999) [24]

In this article, the authors introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras. Furthermore, they proved that the class of oriented digraphs corresponds in a simple way to the class of edge d-algebras and that arbitrary d-algebras also determine unique edge d-algebras in a natural manner.

3. Intuitionistic Fuzzy Ideals Of BCK-Algebras

Young Bae Jun And Kyung Ho Kim (2000) [14]

The authors considered the intuitionistic fuzzification of the concept of subalgebras and ideals in BCK-algebras, and investigated some of their properties. The authors introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of a BCK-algebra and investigated some related properties.

4. On quadratic B-algebras

Hee Kon Park and Hee Sik Kim (2001) [27]

In this paper, the authors introduced the notion of quadratic B-algebra which is a medial quasigroup, and obtained that every quadratic B-algebra on a field X with $|X| > 3$, is a BCI-algebra.

5. On Q-Algebras

Joseph Neggers, Sun Shin Ahn, And Hee Sik Kim (2001) [26]

In this article, the authors introduced the notion, of Q-algebra, which is a generalization of the idea of BCH/BCI /BCK-algebras and they generalized some theorems discussed in BCI-algebras. Moreover, they introduced the notion of "quadratic" Q-algebra, and proved that every quadratic Q-algebra $(X;*,e)$, $e \in X$, has a product of the form $x * y = x - y + e$, where $x, y \in X$ when X is a field with $|X| \geq 3$.

6. On B – algebras

J. Neggers and Hee Sik Kim (2002) [25]

In this article, the authors introduced and investigated a class of algebras which is related to several classes of algebras of interest such as BCK / BCI /BCH – algebras and which seems to have rather nice properties without being excessively complicated otherwise.

7. On derivations of BCI-algebras

Young Bae Jun And Xiao Long Xin (2004) [15]

The notion of left–right (resp. right–left) derivation of a BCI-algebra is introduced, and some related properties are investigated. Using the idea of regular derivation the authors gave characterizations of a *p*-semisimple BCI-algebra. They also give a condition for a derivation to be regular.

8. On 0-Commutative B-Algebras

Hee Sik Kim And Hong Goo Park (2005) [20]

In this paper, the authors showed that if X is a 0-commutative B-algebra, then $(x * a) * (y * b) = (b * a) * (y * x)$. Using this property they showed that the class of p-semisimple BCI-algebras is equivalent to the class of 0-commutative B-algebras.

9. On Quadratic BF-Algebras

Hee Sik Kim and Na Ri Kye (2006) [19]

In this article, the authors introduced the notion of a quadratic BF-algebra, and obtained that quadratic BF-algebras, quadratic Q-algebras, BG -algebras and B-algebras are equivalent notions on a field X with $|X| \geq 3$, and hence every quadratic BF-algebra is a BCI-algebra.

10. Some Axiomatizations Of B-Algebras

Andrzej Walendziak (2006) [32]

Some systems of axioms defining a B-algebra are given with a proof of the independence of the axioms. In this article, the author obtained a simplified axiomatization of commutative B-algebras.

11. On BF-Algebras

Andrzej Walendziak (2007) [33]

In this article, the author introduced the notion of BF-algebras, which is a generalization of B-algebras. They also introduced the notions of an ideal and a normal ideal in BF-algebras and investigated the properties and characterizations of them.

12. On BG-Algebras

Chang Bum Kim and Hee Sik Kim (2008) [18]

In this paper, the authors introduced the notion of BG-algebras which is a generalization of B-algebras. They constructed a BG-algebra from a non-empty set, which is non-group-derived. Moreover, using the notion of normal subalgebra, they obtained several isomorphism theorems of BG-algebra and related properties.

13. Fuzzy Translations and Fuzzy Multiplications of BCK / BCI–algebras

Kyoung Ja Lee, Young Bae Jun and Myung Im Doh (2009) [21]

Fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy sub algebras in BCK / BCI – algebras are discussed. Relations among fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications are investigated.

14. On Intuitionistic L-Fuzzy Subalgebras of BG-Algebras

M. Chandramouleeswaran and P. Muralikrishna (2010) [22]

In this article, the authors discussed the notions of Intuitionistic L-fuzzy subalgebras of a BG-algebras and some of their basic properties.

15. Fuzzy Subalgebras and Fuzzy T-ideals in TM-Algebras

Kandasamy Megalai and Angamuthu Tamilarasi (2011) [31]

In this article, the authors introduced the concepts of fuzzy subalgebras and fuzzy ideals in TM-algebras and investigated some of its properties.

16. Generalisation of Cartesian Product of Intuitionistic L−Fuzzy BF-Ideals

P. Muralikrishna and M. Chandramouleeswaran (2011) [23]

In this article, the authors introduced the notion of the generalization of Cartesian product on intuitionistic L-fuzzy ideals of BF-algebra and studied some simple but interesting properties.

17. On Left Derivations Of d-Algebras

N. Kandaraj, and M. Chandramouleeswaran (2012) [17]

In this article, the authors investigated some properties of left derivations of d-algebras.

18. L−Fuzzy T−Ideals of β−Algebras

K. Rajam and M. Chandramouleeswaran (2015) [28]

In this article, the authors introduced the notion of L− fuzzy T−ideals of β−algebras and investigated some of their properties.

CHAPTER 1

Preliminaries on BP-Algebras and quadratic BP-algebras

Section 1.1:

Preliminaries on BP- algebras

Definition 1.1.1: A BCK-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(BCK 1) $[(x*y) * (x*z)] * (z*y) = 0$

(BCK 2) $[x* (x* y)] * y = 0$

(BCK 3) $x* x = 0$

(BCK 4) $0* x = 0$

(BCK 5) $x * y = 0$ and $y* x = 0$ implies $x = y$, for any $x, y, z \in X$.

Definition 1.1.2: A BCI-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(BCI 1) $[(x*y) * (x*z)] * (z*y) = 0$

(BCI 2) $[x*(x*y)] * y = 0$

(BCI 3) $x*x = 0$

(BCI 4) $x * y = 0$ and $y* x = 0$ implies $x = y$, for any $x, y, z \in X$.

Definition 1.1.3: Let X be a BCI-algebra. Two elements x and y in X are said to be comparable if $x \leq y$ or $y \leq x$. Here $x \leq y$ if and only if $x*y = 0$. Also define $y*(y*x)$ by $x \wedge y$.

Definition 1.1.4: A d-algebra is a nonempty set X with a constant 0 and binary operation $*$ satisfying the following axioms:

(d 1) $x * x = 0$

(d 2) $0 * x = 0$

(d 3) $x * y = 0$ and $y * x = 0$ implies $x = y$, for all x, y $\in X$.

Definition 1.1.5: A B-algebra is a nonempty set X with a constant 0 and binary operation $*$ satisfying the following axioms:

(B 1) $x * x = 0$

(B 2) $x * 0 = x$

(B 3) $(x * y) * z = x * [z * (0 * y)]$, for all x, y, z $\in X$.

Definition 1.1.6: A B-algebra $(X, *, 0)$ is said to be a 0-commutative if for any x, y $\in X$, $x * (0 * y) = y * (0 * x)$.

Proposition 1.1.7: If $(X, *, 0)$ is a 0-commutative B-algebra, then we have the following properties: for any x, y, z, w $\in X$,

(i) $(x * z) * (y * w) = (w * z) * (y * x)$,

(ii) $(x * z) * (y * z) = x * y$,

(iii) $(z * y) * (z * x) = x * y$,

(iv) $(x * z) * y = (0 * z) * (y * x)$,

(v) $x * (y * z) = z * (y * x)$,

(vi) $(x * y) * z = (x * z) * y$,

(vii) $[(x * y) * (x * z)] * (z * y) = 0$,

(viii) $[x * (x * y)] * y = 0$,

(ix) $x * (x * y) = y$,

(x) The left cancellation law holds,

(i.e) $x * y = x * z$ implies $y = z$.

Definition 1.1.8: A BH-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(BH 1) $x * x = 0$,

(BH 2) $x * 0 = x$,

(BH 3) $x * y = 0$ and $y * x = 0$ implies $x = y$, for all x, y $\in$ X.

Definition 1.1.9: A BF-algebra is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(BF 1) $x * x = 0$

(BF 2) $x * 0 = x$

(BF 3) $0 * (x * y) = (y * x)$, for all x, y $\in$ X.

Definition 1.1.10 A BP-algebra $(X, *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following conditions:

(BP 1) $x * x = 0$

(BP 2) $x * (x * y) = y$

(BP 3) $(x * z) * (y * z) = x * y$, for all x, y, z $\in$ X.

Example 1.1.11: Let X = {0, a, b, c} be a set with the following table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X, *, 0)$ is a BP-algebra.

Example 1.1.12: Let X = {0, a, b, c} be a set with the following table:

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then $(X,*,0)$ is a BP-algebra.

Example 1.1.13: Let X = {0,1,2,3} be a set with the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X,*,0)$ is a BP-algebra.

Theorem 1.1.14: If $(X,*,0)$ is a BP-algebra, then the following hold: for any $x, y \in X$,

(i) $0*(0*x) = x$,

(ii) $0*(y*x) = x*y$,

(iii) $x*0 = x$,

(iv) $x*y = 0$ implies $y*x = 0$,

(v) $0*x = 0*y$ implies $x = y$,

(vi) $0*x = y$ implies $0*y = x$,

(vii) $0*x = x$ implies $x*y = y*x$.

Proof:

Let $(X, *, 0)$ be a BP-algebra

Then, **(BP 1)** $x * x = 0$

(BP 2) $x * (x * y) = y$

(BP 3) $(x * z) * (y * z) = x * y$, for all x, y, z $\in$ X.

(i) Put $x = 0$ and $y = x$ in (BP2), then $0 * (0 * x) = x$.

(ii) Using (BP3) and (BP1), we have
$x * y = (x * x) * (y * x) = 0 * (y * x)$.

(iii) Put $y = x$ in (BP2), then $x * (x * x) = x$.
It follows from (BP1) that $x * 0 = x$.

(iv) Let $x * y = 0$. By (ii), then $0 = 0 * 0 = 0 * (x * y) = y * x$. Thus $y * x = 0$.

(v) If $0 * x = 0 * y$, we have $0 * (0 * x) = 0 * (0 * y)$. It follows from (i) that $x = y$.

(vi) Let $0 * x = y$ using (i), we have $0 * y = 0 * (0 * x) = x$.

(vii) Let $0 * x = x$ by (ii), we have $x * y = 0 * (x * y) = y * x$.

Theorem 1.1.15: If $(X, *, 0)$ is a BP-algebra, then $(X, *, 0)$ is a BF-algebra.

Proof: Obviously theorem 1.1.14-(ii) and (iv) X is a BF-algebra.

The converse of theorem 1.1.15 does not hold in general.

Example 1.1.16: Let X = {0,1,2,3} be a set with the following table:

*	0	1	2	3
0	0	2	1	3
1	1	0	1	2
2	2	2	0	2
3	3	1	1	0

Then $(X, *, 0)$ is a BF-algebra, but not a BP-algebra, Because $(1*3)*(2*3) = 2*2 = 0 \neq 1 = 1*2$.

Definition 1.1.17: A BP-algebra $(X,*,0)$ is said to be 0-commutative if

$x*(0*y) = y*(0*x)$ for any $x, y \in X$.

Proposition 1.1.18: If $(X,*,0)$ is a 0-commutative BP-algebra, then the following hold: for any $x, y, z \in X$.

(i) $(x*z)*(y*z) = (z*y)*(z*x)$

(ii) $x*y = (0*y)*(0*x)$.

Proof: Let $(X,*,0)$ be a 0-commutative BP-algebra. Then $0*(y*x) = x*y$.

(i) By proposition 1.1.14-(ii), we have

$$\begin{aligned}(x*z)*(y*z) &= (x*z)*(0*(z*y)) \\ &= (z*y)*(0*(x*z)) \\ &= (z*y)*(z*x).\end{aligned}$$

(ii) Put $z = 0$ in (i). Then

$$(x*0)*(y*0) = (0*y)*(0*x)$$

$$\Rightarrow x*y = (0*y)*(0*x).$$

Note: Every abelian group can determine a BP-algebra.

Theorem 1.1.19: Let $(X, \Delta, 0)$ be a abelian group. If we define $x*y = x \Delta y^{-1}$, then $(X,*,0)$ is a BP-algebra.

Proof: For any $x \in X$, we have $x*x = x \Delta x^{-1} = 0$

Since X is abelian, we have

$$\begin{aligned}x*(x*y) = x*(x \Delta y^{-1}) &= x \Delta (x \Delta y^{-1})^{-1} \\ &= x \Delta y \Delta x^{-1} \\ &= (x \Delta x^{-1}) \Delta y = 0 \Delta y = y.\end{aligned}$$

Hence for any $x, y, z \in X$, we have

$$\begin{aligned}(x * y) * (z * y) &= (x \,\Delta\, y^{-1}) * (z \,\Delta\, y^{-1}) \\ &= (x \,\Delta\, y^{-1}) \,\Delta\, (z \,\Delta\, y^{-1})^{-1} \\ &= (x \,\Delta\, y^{-1}) \,\Delta\, (y \,\Delta\, z^{-1}) \\ &= x \,\Delta\, (y^{-1} \,\Delta\, y) \,\Delta\, z^{-1} \\ &= x \,\Delta\, z^{-1}\end{aligned}$$

$(x * y) * (z * y) = x * z$,

Proving the theorem.

Theorem 1.1.20: Let $(X, *, 0)$ be a BP-algebra then X is 0-commutative if and only if

$(0 * x) * (0 * y) = y * x$ for any $x, y \in X$.

Proof: Let $(X, *, 0)$ be a BP-algebra, then $0 * (0 * x) = x$.

Assume that $(0 * x) * (0 * y) = y * x$ for any $x, y \in X$, by 1.1.14-(ii), we have

$$\begin{aligned}x * (0 * y) &= (0 * (0 * x)) * (0 * y) \\ &= y * (0 * x).\end{aligned}$$

The conversely assume that $(X, *, 0)$ is a 0-commutative BP-algebra.

Then by proposition 1.1.18,

$(x * z) * (y * z) = (z * y) * (z * x)$.

Put $z = 0$ then,

$(x * 0) * (y * 0) = (0 * y) * (0 * x)$

$x * y = (0 * y) * (0 * x)$

Thus $(0 * x) * (0 * y) = y * x$.

Proposition 1.1.21: If $(X, *, 0)$ is a BP-algebra with $(x * y) * z = x * (z * y)$ for any $x, y, z \in X$, then $0 * x = x$ for any $x \in X$.

Proof: Let $x = z = 0$ in $(x * y) * z = x * (z * y)$

Then $(0 * y) * 0 = 0 * (0 * y)$

By theorem 1.1.14-(i) and (iii), we have $0 * y = y$.

Theorem 1.1.22: If $(X, *, 0)$ is a 0-commutative B-algebra, then $(X, *, 0)$ is a BP-algebra.

Proof: Let $(X, *, 0)$ is a 0-commutative BP-algebra.

Then (BP1) holds.

It follows from proposition 1.1.7-(ix) and (ii) that (BP2) and (BP3) hold.

Thus $(X, *, 0)$ is a BP-algebra.

Theorem 1.1.23: If $(X, *, 0)$ is a BP-algebra with $(x * y) * z = x * (z * y)$ for any $x, y, z \in X$, then $(X, *, 0)$ is a B-algebra.

Proof: Let $(X, *, 0)$ is a BP-algebra with $(x * y) * z = x * (z * y)$ for any $x, y, z \in X$.

Then $x * (z * (0 * y)) = x * ((z * y) * 0)$

$= x * (z * y)$

$= (x * y) * z$, for any $x, y, z \in X$.

Thus $(X, *, 0)$ is a B-algebra.

Note: The following example shows that B-algebra need not be a BP-algebra.

Example 1.1.24: Let $X = \{0,1,2,3\}$ be a set with the following table:

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	0	1
3	3	1	2	0

Then $(X, *, 0)$ is a B-algebra, but not a BP-algebra with $(x * y) * z = x * (z * y)$,

Since $0*(1*0) = 0*1 = 2 \neq 1 = 0*(0*1)$ and $(3*1) *2 = 1*2 = 3 \neq 0 = 3*3 = 3*(2*1)$.

Theorem 1.1.25: If $(X,*,0)$ is a BP-algebra, then it is a BH-algebra.

Proof: Let $x*y = 0$ and $y*x = 0$ for any $x, y \in X$. Using theorem 1.1.14-(iii) and (BP2), we have $x = x*0 = x*(x*y) = y$.

Thus X is a BH-algebra.

The converse of theorem 1.1.25 need not be true in general.

Example 1.1.26: Let X = {0,1,2,3} be a set with the following table:

$*$	0	1	2	3
0	0	3	0	2
1	1	0	0	0
2	2	2	0	3
3	3	3	1	0

Thus $(X,*,0)$ is a BH-algebra, but not a BP-algebra.

Since $1*(1*2) = 1*0 = 1 \neq 2$.

Definition 1.1.27: Let $(X,*,0)$ and $(Y,*',0')$ be BP-algebra. A mapping $f : (X,*,0) \to (Y,*',0')$ is called a BP-homomorphism (or homomorphism of BP-algebra) if

$$f(x*y) = f(x) *' f(y), \text{ for any } x, y \in X. \text{ Then}$$

f is called

(i) A monomorphism if f is 1-1.

(ii) An epimorphism if f is onto.

(iii) If f is a homomorphism from X into itself, then f is called an endomorphism.

Note: If f is a BP-homomorphism then $f(0) = 0'$.

Definition 1.1.28: A nonempty subset A of a BP-algebra X is said to be a BP-subalgebra if $x*y \in A, \forall\ x, y \in A$.

Definition 1.1.29: A nonempty subset I of BP-algebra $(X,*,0)$ is said to be a BP-ideal of X if it satisfies the following conditions:

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$, $\forall$ $x, y \in I$.

Definition 1.1.30: A BP-ideal I of a BP-algebra is said to be closed if $0 * x \in I$, $\forall$ $x \in I$.

Definition 1.1.31: Let $(X,*,0)$ be a BP-algebra. A nonempty subset I of X is called a T-ideal of X if it satisfies the following conditions:

(i) $0 \in I$

(ii) $(x * y) * z \in I$ and $y \in I \Rightarrow x * z \in I$, $\forall$ $x, y, z \in X$.

Section 1.2:

A quadratic BP-algebras

Definition 1.2.1: Let X be a field with $|X| \geq 3$. An algebra $(X,*)$ is said to be quadratic if $x * y$ is defined by $x * y = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$, where $a_1, a_2, a_3, a_4, a_5, a_6 \in X$, for any $x, y \in X$.

Definition 1.2.2: A quadratic algebra $(X,*)$ is said to be a quadratic BP-algebra if for some fixed $e \in X$, it satisfies the conditions (BP1), (BP2) and (BP3).

Theorem 1.2.3: Let X be a field with $|X| \geq 3$. Then every quadratic BP-algebra $(X,*,e)$ has of the form $x * y = x - y + e$, where $x, y, z \in X$.

Proof: Define $x * y = Ax^2 + Bxy + Cy^2 + Dx + Ey + F$, where $A, B, C, D, E, F \in X$ and $x, y \in X$.

Consider, $e = x * x$

$= Ax^2 + Bx^2 + Cx^2 + Dx + Ex + F$

$= (A+B+C)x^2 + (D+E)x + F$

It follows that F = e, A+B+C = 0 = D+E,

i.e, D = -E.

Consider, x = $x * e$

$$= A x^2 + Bxe + Ce^2 + Dx + Ee + e.$$

It follows that A = 0, B e + D = 1 and $C e^2 + E e + e = 0$

Thus, B+C = 0, D = 1-Be.

Since D = -E, we have E = -1+Be.

We have the following more simpler form:

$x*y = B\,x\,y + C\,y^2 + D\,x + E\,y + e$

$= B\,x\,y + (-B)\,y^2 + (1-Be)\,x + (Be-1)\,y + e$

$= B\,(x\,y - y^2 - e\,x + e\,y) + (x - y + e)$

$x*y = B\,(x - y)(y - e) + (x - y + e)$.

By theorem 1.1.14-(ii), every BP-algebra satisfies the condition e $*$ (x$*$y) = y$*$x, for any x, y$\in$X.

Consider e $*$ (x$*$y) = B (e–(x$*$y)) ((x$*$y –e)+(e –(x$*$y)+e)

= B {(e-B(x-y) (y-e)-(x-y+e)) [B(x-y)(y-e)+(x-y+e)-e]}

+ [e-B(x-y) (y-e)-(x-y+e)+e]

= B {[-B(x-y) (y-e)-(x-y)] [B(x-y)(y-e)+(x-y)]}+[e-B(x-y)(y-e)-(x-y)]

$= -B\,[B(x-y)(y-e)+(x-y)]^2 - [B(x-y)(y-e)+(x-y)]+e$

$= -B\,(x-y)^2[B(y-e)+1]^2 - (x-y)[B(y-e)+1]+e$

Since y$*$x = B (y-x)(x-e)+(y-x+e) = B (y-x)(x-e)+(y-x)+e,

We obtain, $-B\,(x-y)^2[B(y-e)+1]^2 - (x-y)[B(y-e)+1]+e = B\,(y-x)(x-e)+(y-x)+e$.

If we let x = e in the above identity,

Then we have,

$-B\,(e-y)^2[B(y-e)+1]^2 - (e-y)[B(y-e)+1]+e = (y-e)+e.$

It follows that B = 0.

Hence C = 0, E = -1 and D = 1. Thus x * y = x-y+e.

It is easy to check that this binary operation satisfies (BP2) and (BP3).

Example 1.2.4:

(i) Let K = GF(p^n) be a Galois field. Define $x * y = (x - y) + e,\ e \in K$. T hen (K, *,e) is a quadratic BP-Algebra.

CHAPTER 2

f-Derivations on BP-algebras

Section 2.1:

Properties of f-derivations on BP-algebras

In this chapter, assume that f be an endomorphism of a BP-algebra $(X, *, 0)$ unless otherwise specified.

Definition 2.1.1: Let X be an BP-algebra. By a left-right f-derivation [(l, r)-f-derivation] on X, we mean a self map θ_f on X satisfies the identity

$$\theta_f(x * y) = (\theta_f(x) * f(y)) \wedge (f(x)\ \theta_f(y)), \quad \text{for all } x, y \in X.$$

If θ_f satisfies the identity

$$\theta_f(x * y) = (f(x) * \theta_f(y)) \wedge (\theta_f(x) * f(y)), \quad \text{for all } x, y \in X.$$

Then it is said that θ_f is a right-left f-derivation [(r, l)-f-derivation] of X. If θ_f is both an (r, l) and an (l, r)-f-derivation, then θ_f is said to be a f-derivation.

Example 2.1.2: Let X = {0,1,2,3} be a BP-algebra with the following cayley table:

*	0	1	2	3
0	0	3	2	1
1	1	0	3	2
2	2	1	0	3
3	3	2	1	0

1) Define an endomorphism f of X by f(0) = 0, f(1) = 3, f(2) = 2, f(3) = 1 and a self map $\theta_f : X \to X$ by $\theta_f(0) = 1$, $\theta_f(1) = 0$, $\theta_f(2) = 3$, $\theta_f(3) = 2$.

   Then θ_f is a (l, r)-f-derivation of X.

2) Define an endomorphism f of X by f(0) = 0, f(1) = 3, f(2) = 2, f(3) = 1 and a self map θ_f : X→X by $\theta_f(0) = 2$, $\theta_f(1) = 1$, $\theta_f(2) = 0$, $\theta_f(3) = 3$.

   Then θ_f is a f-derivation of X.

Definition 2.1.3: An f-derivation θ_f on a BP-algebra X is said to be regular if $\theta_f(0) = 0$.

Proposition 2.1.4: Every (r, l)-f- derivation ((r, l)-f-derivation) of a BP-algebra is regular.

Proof: Let X be a BP-algebra and θ_f be a (r, l)-f-derivation on X.

Then for all x $\in$X,

$$\begin{aligned}\theta_f(0) &= \theta_f(x * x)\\ &= (f(x) * \theta_f(x)) \wedge (\theta_f(x) * f(x))\\ &= (\theta_f(x) * f(x)) * [(\theta_f(x) * f(x)) * (f(x) * \theta_f(x))]\\ &= f(x) * \theta_f(x)\\ &= 0.\end{aligned}$$

Let θ_f be a (l, r)-f-derivation on X.

Then for all x $\in$X, we have

$$\begin{aligned}\theta_f(0) &= \theta_f(x * x)\\ &= (\theta_f(x) * f(x)) \wedge (f(x) * \theta_f(x))\\ &= (f(x) * \theta_f(x)) * [(f(x) * \theta_f(x)) * (\theta_f(x) * f(x))]\\ &= \theta_f(x) * f(x)\\ &= 0.\end{aligned}$$

The following result gives a necessary and sufficient condition for the derivation θ_f to be regular.

Proposition 2.1.5: Let θ_f be a self map of a BP-algebra on X, then the following hold:

(i) If θ_f is an (l, r)-f-derivation on X, then $\theta_f(x) = \theta_f(x) \wedge f(x)$ for all $x \in X$ if and only if $\theta_f(0) = 0$.

(ii) If θ_f is an (r, l)-f-derivation on X, then $\theta_f(x) = f(x) \wedge \theta_f(x)$ for all $x \in X$ if and only if $\theta_f(0) = 0$.

Proof: Obvious.

Proposition 2.1.6: Let θ_f be a (l, r)-f-derivation on a BP-algebra X.
Then $\theta_f(a) = \theta_f(0) * (0 * f(a))$, for all $a \in X$.

Proof: Let θ_f be a (l, r)-f-derivation on a BP-algebra X.

Now,

$$\theta_f(a) = \theta_f(0 * (0 * a) \quad (\text{since } 0 * (0 * x) = x)$$

$$= (\theta_f(0) * f(0 * a)) \wedge (f(0) * \theta_f(0 * a))$$

$$= (f(0) * \theta_f(0 * a)) * ((f(0) * \theta_f(0 * a)) * (\theta_f(0) * f(0 * a)))$$

$$= \theta_f(0) * f(0 * a)$$

$$= \theta_f(0) * (f(0) * f(a))$$

$$= \theta_f(0) * (0 * f(a))$$

Proposition 2.1.7: Let θ_f be a self map on a BP-algebra X and θ_f be an (r, l)-f-derivation on X. Then $\theta_f(x) = f(x)$, for all $x \in X$ if and only if $\theta_f(0) = 0$.

Proof: Let θ_f be a (r, l)-f-derivation on X.

Assume that $\theta_f(0) = 0$.

Now, $\theta_f(x) = \theta_f(x * 0)$ (since $x * 0 = x$)

$$= (f(x) * \theta_f(0)) \wedge (\theta_f(x) * f(0))$$

$$= (\theta_f(x) * f(0)) * (\theta_f(x) * f(0)) * (f(x) * \theta_f(0))$$

$= f(x) * \theta_f(0)$

$= f(x)$

Conversely, assume that $\theta_f(x) = f(x)$.

Now $\theta_f(0) = \theta_f(x * x)$

$= (f(x) * \theta_f(x)) \wedge (\theta_f(x) * f(x))$

$= (\theta_f(x) * f(x)) * [(\theta_f(x) * f(x)) * (f(x) * \theta_f(x))]$

$= f(x) * \theta_f(x)$

$= f(x) * f(x)$ (since $\theta_f(x) = f(x)$)

$= 0.$

Definition 2.1.8: An ideal A on a BP-algebra X is said to be an f-ideal if f(A) $\subseteq$ A.

Example 2.1.9: Let X = {0,1,2,3,} be a BP-algebra with the following cayley table:

Consider the ideal A = {0,3} of X.

*	0	1	2	3
0	0	2	1	3
1	1	0	3	2
2	2	3	1	0
3	3	1	2	0

If $\theta_f : X \rightarrow X$ is defined by $\theta_f(0) = 0$, $\theta_f(1) = 2$, $\theta_f(2) = 1$, $\theta_f(3) = 3$ and define an endomorphism f of X by $\theta_f(x) = f(x)$.

Since f(0) = 0, f(3) = 3, f(A) $\subseteq$ A proving that A is an f-ideal on X.

Definition 2.1.10: Let θ_f be a self map of a BP-algebra X. An f-ideal on X is said to be θ_f-invariant if $\theta_f(A) \subseteq$ A.

Example 2.1.11: Example (2.1.9), $\theta_f(0) \in$ A and $\theta_f(3) = 3 \in$ A.

Hence θ_f(A) $\subseteq$ A, showing that A is θ_f-invariant.

Theorem 2.1.12: Let θ_f be a regular (r, l)-f-derivation on a BP-algebra X. Then f-ideal A on X is θ_f-invariant.

Proof: Let θ_f be a regular (r, l)-f-derivation on X.

Now,
$$\begin{aligned}
\theta_f(x) &= \theta_f(x * 0) \\
&= (f(x) * \theta_f(0)) \wedge (\theta_f(x) * f(0)) \\
&= (f(x) * 0) \wedge (\theta_f(x) * 0) \\
&= f(x) \wedge \theta_f(x) \\
&= \theta_f(x) * (\theta_f(x) * f(x)) \\
&= f(x), \quad \forall\ x \in X.
\end{aligned}$$

Let y $\in \theta_f$(A) then y = θ_f(x) for some x $\in$ A.

It follows that
$$\begin{aligned}
y * f(x) &= \theta_f(x) * f(x) \\
&= 0 \in A.
\end{aligned}$$

Since x $\in$ A, then f(x) $\in$ f(A) $\subseteq$ A as A is an f-ideal.

It follows that y $\in$ A since A is an ideal on X.

Hence θ_f(A) $\subseteq$ A.

Thus A is θ_f-invariant.

Section 2.2:

Composition of f-derivations on BP-algebras

Definition 2.2.1: Let X be a BP-algebra and θ_f, θ'_f be two self maps on X. Define $\theta_f \circ \theta'_f$: X→X as $(\theta_f \circ \theta'_f)(x) = \theta_f(\theta'_f(x))$ for all $x \in X$.

Proposition 2.2.2: Let X be a BP-algebra and θ_f, θ'_f are the (l, r)-f-derivations on X. Let $f^2 = f \circ f = f$, then $(\theta_f \circ \theta'_f)$ is also a (l, r)-f-derivation on X.

Proof: Let X be a BP-algebra, and θ_f, θ'_f are the (l, r)-f-derivations on X.

$$
\begin{aligned}
(\theta_f \circ \theta'_f)(x*y) &= \theta_f(\theta'_f(x*y)) \\
&= \theta_f[(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))] \\
&= \theta_f[(f(x) * \theta'_f(y)) * (f(x) * \theta'_f(y)) * (\theta'_f(x) * f(y))] \\
&= \theta_f(\theta'_f(x) * f(y)) \quad [\text{since } y*(y*x) = x] \\
&= (\theta_f(\theta'_f(x) * f^2(y)) \wedge (f(\theta'_f(x)) * \theta_f(f(y)) \\
&= \theta_f(\theta'_f(x) * f^2(y)). \\
&= (\theta_f(\theta'_f(x) * f(y)) \\
&= (f(x) * \theta_f(\theta'_f(y)) * [(f(x) * \theta_f(\theta'_f(y)] * [(\theta_f(\theta'_f(x)) * f(y)] \\
&= (f(x) * (\theta_f \circ \theta'_f)(y) * [(f(x) * (\theta_f \circ \theta'_f)(y)) * (\theta_f \circ \theta'_f(x) * f(y)] \\
&= ((\theta_f \circ \theta'_f)(x) * f(y)) \wedge (f(x) * (\theta_f \circ \theta'_f)(y))
\end{aligned}
$$

Which implies that $(\theta_f \circ \theta'_f)$ is a (l, r)-f-derivation on X.

Proposition 2.2.3: Let X be a BP-algebra θ_f and θ'_f are the (r, l)-f-derivations on X such that $f^2 = f \circ f = f$. Then $\theta_f \circ \theta'_f$ is also a (r, l)-f-derivation on X.

Proof: Similar to the proof of above propositions.

Theorem 2.2.4: Let X be a BP-algebra and θ_f, θ'_f be two f-derivations on X such that $f^2 = f$. Then $\theta_f \circ \theta'_f$ is also a f-derivation on X.

Proof: Obvious by above propositions.

The following proposition shows that the composition of f-derivations is commutative.

Proposition 2.2.5: Let X be a BP-algebra and θ_f, θ'_f be two f-derivations on X such that $f \circ \theta_f = \theta_f \circ f$, $\theta'_f \circ f = f \circ \theta'_f$. Then $\theta_f \circ \theta'_f = \theta'_f \circ \theta_f$.

Proof: Let X be a BP-algebra and θ_f, θ'_f be two f-derivations on X.

Since θ'_f is a (l, r)-f-derivation on X, then for all x, y $\in$X.

$$(\theta_f \circ \theta'_f)(x*y) = \theta_f(\theta'_f(x*y))$$

$$= \theta_f[(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))]$$

$$= \theta_f(\theta'_f(x) * f(y)).$$

But θ_f is a (r, l)-f-derivation on X.

$$(\theta_f \circ \theta'_f)(x*y) = \theta_f((\theta'_f(x) * f(y))$$

$$= (f(\theta'_f(x)) * \theta_f(f(y)) \wedge (\theta_f(\theta'_f(x) * f^2(y))$$

$$= (f(\theta'_f(x)) * \theta_f(f(y))$$

$$= (f \circ \theta'_f)(x) * (\theta_f \circ f)(y)$$

Thus we have for all x, y $\in$X,

$$(\theta_f \circ \theta'_f)(x*y) = (f \circ \theta'_f)(x) * (\theta_f \circ f)(y)) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Also since θ_f is a (r, l)-f-derivation on X, then for all x, y$\in$ X,

$$(\theta'_f \circ \theta_f)(x*y) = \theta'_f(\theta_f(x*y))$$

$$= \theta'_f[(f(x) * \theta_f(y)) \wedge (\theta_f(x) * f(y))]$$

$= \theta'_f \, ((f(x) * \theta_f(y))$

But θ'_f is a (l, r)-f-derivation on X,

$(\theta'_f \circ \theta_f)\,(x * y) = [\theta'_f(f(x) * f(\theta_f(y))] \wedge (f^2(x) * \theta'_f(\theta_f(y))]$

$= (\theta'_f(f(x)) * f(\theta_f(y))$

$= (\theta'_f \circ f)(x) * (f \circ \theta_f)(y)$

$= (f \circ \theta'_f)(x) * (\theta_f \circ f)(y)$

Thus we have for all x, y $\in$ X,

$(\theta'_f \circ \theta_f)\,(x * y) = (f \circ \theta'_f)(x) * (\theta_f \circ f)(y)$ ……………………(2)

From (1) and (2) we get for all x, y $\in$ X,

$(\theta_f \circ \theta'_f)\,(x * y) = (\theta'_f \circ \theta_f)\,(x * y)$.

By putting y = 0 we get for all x, y $\in$ X,

$(\theta_f \circ \theta'_f)\,(x) = (\theta'_f \circ \theta_f)\,(x)$.

Which implies that $(\theta_f \circ \theta'_f) = (\theta'_f \circ \theta_f)$.

Definition 2.2.6: Let (X, $*$,0) be a BP-algebra and θ_f, θ'_f be two self maps on X. Define $\theta_f . \theta'_f : X \to X$ as $(\theta_f . \theta'_f)\,(x) = \theta_f(x) * \theta'_f(x)$ for all $x \in X$.

Proposition 2.2.7: Let X be a BP-algebra and θ_f, θ'_f are f-derivations on X. Then $(f \circ \theta'_f) . (\theta_f \circ f) = (\theta_f \circ f) . (f \circ \theta'_f)$.

Proof: Let X be a BP-algebra and θ_f, θ'_f be two derivations on X.

Since θ'_f is a (l, r)-f-derivation on X. Then for all x, y$\in$ X.

$(\theta_f \circ \theta'_f)\,(x * y) = \theta_f\,(\theta'_f\,(x * y))$

$= \theta_f\,[(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))]$

$= \theta_f\ (\theta_f{}'\ (x) * f(y))$

But θ_f is a (l, r)-f-derivation on X.

$\theta_f\ (\theta'_f(x) * f(y)) = (f\,(\theta'_f(x)) * \theta_f(f(y))) \wedge (\theta_f(\theta'_f(x) * f^2(y))$

$= (f\,(\theta'_f(x) * \theta_f(f(y))$

$= (f \circ \theta'_f)\,(x) * (\theta_f \circ f)(y)$

$(\theta_f \circ \theta'_f)(x * y) = (f \circ \theta'_f)(x) * (\theta_f \circ f)(y)$, for all x, y $\in$ X...........(1)

Also we have that θ'_f is a (r, l)-f-derivation on X, then for all x, y $\in$ X.

$(\theta_f \circ \theta'_f)\,(x * y) = \theta_f\ (\theta'_f(x * y))$

$= \theta_f\,[(f(x) * \theta'_f(y)) \wedge (\theta'_f(x) * f(y))]$

$= \theta_f\,(f(x) * \theta'_f(y))$

But θ_f is a (l, r)-f-derivation on X.

$\theta_f\,(f(x) * \theta'_f(y)) = (\theta_f\,(f(x)) * f(\theta'_f(y)) \wedge (f^2(x) * \theta_f\,(\theta'_f(y))]$

$= (\theta_f\,(f(x)) * f\,(\theta'_f(y))$

$(\theta_f \circ \theta'_f\ )\,(x * y) = (\theta_f \circ f\,)(x) * (f \circ \theta'_f)(y)$, for all x, y $\in$ X...........(2)

From (1) and (2) we get for all x $\in$ X.

(By putting y = x)

$(f \circ \theta'_f)(x) * (\theta_f \circ f)(x) = (\theta_f \circ f)(x) * (f \circ \theta'_f)(x)$

$((f \circ \theta'_f)\,.\,(\theta_f \circ f))(x) = ((\theta_f \circ f)\ .\,(f \circ \theta'_f))(x)$

Which implies that

$(f \circ \theta'_f)\,.\,(\theta_f \circ f) = (\theta_f \circ f)\ .\,(f \circ \theta'_f)$

Notation: $Der_f\,(X)$ denotes the set of all f-derivation on X.

Definition 2.2.8: Let θ_f, $\theta'_f \in Der_f\,(X)$. Define the binary operation $\wedge$ as

$(\theta_f \wedge \theta'_f)(x) = \theta_f\,(x) \wedge \theta'_f\,(x)$.

Proposition 2.2.9: Let X be a BP-algebra and θ_f, θ'_f are (l, r)-f-derivation on X. Then $\theta_f \wedge \theta'_f$ is also a (l, r)-f-derivation on X.

Proof: Let X be a BP-algebra and θ_f, θ'_f are (l, r)-f-derivation on X.

We have,

$$(\theta_f \wedge \theta'_f)(x * y) = \theta_f(x * y) \wedge \theta'_f(x * y)$$

$$= \{(\theta_f(x) * f(y)) \wedge (f(x) * \theta_f(y))\} \wedge \{(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))\}$$

$$= (\theta_f(x) * f(y)) \wedge (\theta'_f(x) * f(y))\}$$

$$= \theta_f(x) * f(y)$$

$$= (\theta'_f(x) * (\theta'_f(x) * \theta_f(x))) * f(y)$$

$$= (\theta_f(x) \wedge \theta'_f(x)) * f(y)$$

$$= (\theta_f \wedge \theta'_f)(x) * f(y)$$

$$= (f(x) * (\theta_f \wedge \theta'_f)(y)) * (f(x) * (\theta_f \wedge \theta'_f)(y)) * ((\theta_f \wedge \theta'_f)(x) * f(y))$$

$$= ((\theta_f \wedge \theta'_f)(x) * f(y)) \wedge (f(x) * (\theta_f \wedge \theta'_f)(y))$$

This shows that $(\theta_f \wedge \theta'_f)$ is a (l, r)-f-derivation on X.

Hence the proof.

Proposition 2.2.10: Let X be a BP-algebra and θ_f, θ'_f are (r, l)-f-derivation on X. Then $\theta_f \wedge \theta'_f$ is also a (r, l)-f-derivation on X.

Proof: Analogously as above proof follows.

Theorem 2.2.11: If θ_f, $\theta'_f \in Der_f(X)$, $\theta_f \wedge \theta'_f \in Der_f(X)$.

Also $(\theta_f \wedge (\theta'_f \wedge \theta''_f))(x * y) = ((\theta_f \wedge \theta'_f) \wedge \theta''_f)(x * y)$.

Proof: If θ_f, $\theta'_f \in Der_f(X)$, then θ_f is both a (l, r) and a (r, l) derivation. Similarly θ'_f is both a (l, r) and a (r, l) derivation.

By proposition (2.2.9) and (2.2.10), $\theta_f \wedge \theta'_f$ is both a (l, r) and a (r, l) derivation.

Hence $\theta_f \wedge \theta'_f \in Der_f(X)$.

To shows the associativity, choose $\theta_f, \theta'_f, \theta''_f \in Der_f(X)$.

$$
\begin{aligned}
((\theta_f \wedge \theta'_f) \wedge \theta''_f)(x*y) &= (\theta_f \wedge \theta'_f)(x*y) \wedge (\theta''_f)(x*y) \\
&= ((\theta''_f)(x*y)) * ((\theta''_f)(x*y)) * ((\theta_f \wedge \theta'_f)(x*y)) \\
&= (\theta_f \wedge \theta'_f)(x*y) \\
&= \theta_f(x*y) \wedge \theta'_f(x*y) \\
&= [(\theta_f(x) * f(y)) \wedge (f(x) * \theta_f(y))] \wedge [(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))] \\
&= (\theta_f(x) * f(y)) \wedge (\theta'_f(x) * f(y)) \\
&= (\theta_f(x) * f(y)).
\end{aligned}
$$

Also,

$$
\begin{aligned}
(\theta_f \wedge (\theta'_f \wedge \theta''_f)(x*y)) &= (\theta_f)(x*y) \wedge (\theta'_f \wedge \theta''_f)(x*y) \\
&= \theta_f(x*y) \wedge [\theta'_f(x*y) \wedge \theta''_f(x*y)] \\
&= \theta_f(x*y) \wedge (\theta'_f)(x*y) \\
&= [(\theta_f(x) * f(y)) \wedge (f(x) * \theta_f(y))] \wedge [(\theta'_f(x) * f(y)) \wedge (f(x) * \theta'_f(y))] \\
&= (\theta_f(x) * f(y)) \wedge (\theta'_f(x) * f(y)) \\
&= (\theta_f(x) * f(y)).
\end{aligned}
$$

This shows that,

$$((\theta_f \wedge \theta'_f) \wedge \theta''_f)(x*y) = (\theta_f \wedge (\theta'_f \wedge \theta''_f)(x*y)$$

$$\Rightarrow ((\theta_f \wedge \theta'_f) \wedge \theta''_f) = (\theta_f \wedge (\theta'_f \wedge \theta''_f)).$$

We conclude that $Der_f(X)$ is closed under the binary composition $\wedge$ defined in (2.2.8) which is also associative.

Theorem 2.2.12: $Der_f(X)$ is a semigroup under the binary composition $\wedge$.

Proof: Obvious by above theorem.

CHAPTER 3

Fuzzy Algebraic Structure in BP-Algebras

Section 3.1:

Preliminaries on Fuzzy Sets

Definition 3.1.1: Let X be any arbitrary set. Let I = [0, 1] be the unit interval. Any function $\mu : X \to [0,1]$ is called a fuzzy set on X. The collection of all fuzzy sets defined on X is denoted by I^X.

Definition 3.1.2: Let μ and γ be fuzzy sets on X, then

(i) $\mu = \nu \Leftrightarrow \mu(x) = \nu(x)$, $\forall$ x $\in$X,

(ii) $\mu \le \nu \Leftrightarrow \mu(x) \le \nu(x)$, $\forall$ x $\in$X,

(iii) $(\mu \cap \nu)(x) \Leftrightarrow \min\{\mu(x), \nu(x) / x \in X\}$,

(iv) $(\mu \cup \nu)(x) \Leftrightarrow \max\{\mu(x), \nu(x) / x \in X\}$,

(v) $\mu^C(x) \Leftrightarrow 1 - \mu(x)$, x $\in$X.

Definition 3.1.3: For a family of fuzzy sets $\{\mu_\lambda\}_{\lambda \in \Lambda}$, the union $\cup$ and the intersection $\cap$ are defined by

$$(\underset{\lambda \in \Lambda}{\cup} \mu_\lambda)(x) = \sup_{\lambda \in \Lambda} \{\mu_\lambda(x) / x \in X\}$$
$$(\underset{\lambda \in \Lambda}{\cap} \mu_\lambda)(x) = \inf_{\lambda \in \Lambda} \{\mu_\lambda(x) / x \in X\}.$$

Definition 3.1.4: The symbol O_X or ϕ will be used to denote an empty fuzzy set O_X is defined as O_X(x) = 0 for all x $\in$X and I_X or X denotes the whole fuzzy set where I_X is defined as I_X(x) = 1 $\forall$ x $\in$X.

Definition 3.1.5: The constant fuzzy set denoted by α is defined as α(x) = α, $\forall$ x $\in$X.

Definition 3.1.6: Any subset μ of X can be identified with a fuzzy set χ_μ the characteristic function of μ. The function χ_μ: X→[0, 1] is defined by,

χ_μ(x) = 1 if x $\in \mu$

= 0 if x $\notin \mu$.

Definition 3.1.7: Let μ be a fuzzy set on X. The set { x $\in$X / μ(x) ≥ 0} is called the support of μ and is denoted by supp μ. If μ takes only the values 0, 1 then μ is called a crisp set in X.

Definition 3.1.8: Let μ be a fuzzy set of a set X. For a fixed t $\in$ [0, 1], the set μ_t = U(μ, t) = { x $\in$X / μ(x) $\geq$ t} is called upper level subset (or level subset) of μ.

Definition 3.1.9: A fuzzy set μ in X is said to have the sup property if for any subset $T \subset X$ there exists $x_0 \in X$ such that $\mu(x_0) = \sup_{t \in T} \mu(t)$.

Definition 3.1.10: Let h be a mapping from X into Y.

i) Let μ be a fuzzy set in X. Then the image of μ under h, denoted by h(μ) is the fuzzy set in Y is defined by

$$(h(\mu))(y) = \begin{cases} \sup_{z \in h^{-1}(y)} \mu(z) & \text{if } h^{-1}(y) = \{x \mid h(x) = y\} \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

ii) Let μ be a fuzzy set in Y. The inverse image (or pre-image) of μ under h, denoted by $h^{-1}(\mu)$ is the fuzzy set in X is defined by $(h^{-1}(\mu))(x) = \mu(h(x))$ for all $x \in X$.

Definition 3.1.11: For any two fuzzy sets λ and μ of a set X, their Cartesian product is defined to be the set $\lambda \times \mu$: X×X→[0, 1] where ($\lambda \times \mu$)(x, y) = min{ λ (x), μ (y)}, $\forall$ x, y$\in$X.

Section 3.2:

Properties of Fuzzy BP-Subalgebras

Definition 3.2.1: A fuzzy set μ of a BP-algebra $(X,*,0)$ is called a fuzzy BP-Subalgebra of X if, for all x, y$\in$X the following condition is satisfied

$$\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$$

Example 3.2.2: Let X = {0, a, b, c} be a set with the following table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then $(X,*,0)$ is a BP-algebra.

Define μ as $\mu(x) = \begin{cases} .8 \text{ if } x=0 \\ .5 \text{ if } x=b \\ .4 \text{ if } x=a,c \end{cases}$

Then μ is a fuzzy BP-Subalgebra of X.

Theorem 3.2.3: Intersection of any two fuzzy BP-Subalgebra of X is again a fuzzy BP-Subalgebra.

Lemma 3.2.4: Let (X, $*$,0) be a BP-Subalgebra. Let μ be a fuzzy BP-subalgebra of X. Let $\alpha \in [0, 1]$. Then

(i) U(μ, α) is either ϕ or a BP-subalgebra of X.

(ii) $\mu(0) \geq \mu(x)$ for all x $\in$X.

Proof: For any $\alpha \in [0, 1]$, assume that U(μ,α) is nonempty.

Let x, y $\in$U(μ,α). Therefore $\mu(x) \geq \alpha$, $\mu(y) \geq \alpha$.

Since $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} \geq \min\{\alpha, \alpha\} = \alpha$.

$x*y \in$U(μ,α). Therefore U(μ,α) is a BP-Subalgebra.

Also, $\mu(0) = \mu\{x*x\}$

$\geq \min\{\mu(x), \mu(x)\}$

$= \mu(x)$.

Since $x*x = 0$, $\forall$ x $\in$X.

Thus $\mu(0) \geq \mu(x)$, $\forall$ x $\in$X.

Lemma 3.2.5: A fuzzy set μ of a BP-subalgebra X is a fuzzy BP-subalgebra if and only if for all t $\in [0, 1]$, the level set of μ, U(μ,t) is either empty or a BP-Subalgebra of X.

Proof: Assume that the level subset of μ in X, U(μ,t) $\neq \phi$.

Then for any x, y$\in$ U(μ,t), $\mu(x) \geq t$, $\mu(y) \geq t$.

Now, $\mu(x*y) \geq \min\{\mu(x), \mu(y)\} \geq t$, which implies $x * y \in$U(μ,t) and hence U(μ,t) is a BP-Subalgebra of X.

Conversely assume that U(μ,t) is a BP-Subalgebra of X.

Take $t = \min\{\mu(x), \mu(y)\}$ for any x, y $\in$X.

Implies $x * y \in$ U(μ,t)

Hence $\mu(x*y) \geq t = \min\{\mu(x), \mu(y)\}$.

Thus proving that μ is a fuzzy BP-Subalgebra of X.

Lemma 3.2.6: Any BP-Subalgebra of a BP-algebra $(X, *, 0)$ can be realized as a level subalgebra of some fuzzy BP-Subalgebra of X.

Proof: Let A be a BP-Subalgebra of X and μ be a fuzzy set in X defined by

$$\mu(x) = \begin{cases} t & if\ x \in A \\ 0 & 0therwise \end{cases}$$

Where $t \in [0, 1]$

If $x, y \in A$ then $x * y \in A$.

Therefore $\mu(x) = \mu(y) = \mu(x * y) = t$ and

$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$

If both $x, y \notin A$.

Then $0 = \mu(x * y) \geq \min\{\mu x), \mu(y)\}$

If at most one of $x, y \in A$, then also we have $x * y \notin A$.

Hence atleast one of $\mu(x)$ or $\mu(y)$ is equal to 0.

Therefore, $\mu(x * y) = 0 \geq \min\{\mu(x), \mu(y)\}$

This shows that A is a level subalgebra of X corresponding to the fuzzy BP-Subalgebra μ of X.

Theorem 3.2.7: Let A be a subset of X. Then the characteristic function X_A is a fuzzy BP-Subalgebra of X if and only if A is a BP-Subalgebra of X.

Proof: Proof follows by above two lemmas.

Theorem 3.2.8: Let μ be a fuzzy BP-Subalgebra of (X, $*$,0) with finite image. If U(μ ,s) = U(μ ,t) for some s, t$\in$Im(μ), then s = t.

Proof: Let μ be a fuzzy BP-Subalgebra of X with finite image such that

U(μ , s) = U(μ ,t) for some s, t$\in$Im(μ)

Now, μ is a fuzzy algebra of X shows that U(μ ,s) is a BP-subalgebra.

Therefore, if x, y $\in$ U(μ ,t) = U(μ ,s) then μ(x) $\geq$ t and μ(y) $\geq$ t.

Also, x, y $\in$ U(μ ,t) = U(μ ,s) and U(μ ,s) is a BP-Subalgebra shows that x $*$y $\in$ U(μ ,s).

This shows that μ(x $*$y) $\geq$ min{ μ(x), μ(y)} $\geq$ s.

Thus we have, μ(x $*$y) $\geq$s as well as μ(x $*$y) $\geq$ s as well as μ(x$*$y) $\geq$t.

Whenever x, y$\in$ U(μ , t) = U(μ , s).

Similarly, we can prove that, μ(x$*$y) $\geq$ s as well as μ(x$*$y) $\geq$ t

Whenever x, y$\in$U(μ , s) = U(μ , t).

This proves that s = t.

Lemma 3.2.9: Let μ and λ be two fuzzy BP-Subalgebras of X with identical family of level BP-Subalgebras. If Im(μ) = { t_1, t_2, t_3,,,, t_n } and Im(λ) = { s_1, s_2, s_3,,,, s_n } where $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_n$ and $s_1 \geq s_2 \geq s_3 \geq \ldots\ldots \geq s_m$ then

(i) m = n

(ii) U(μ , t_i) = U(λ , s_i) for I = 1,2,,,,,,n.

(iii) If μ(x) = s_i , then λ(x) = s_i , $\forall$ x $\in$X and i = 1,2,,,,,,n.

Proof: Let μ and λ be two fuzzy BP-Subalgebras of X with identical family of level BP-Subalgebras F(μ) = F(λ).

Let Im(μ) = $\{t_1, t_2, t_3,,,, t_n\}$ where $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_n$ ....(1) and

Im(λ) = $\{s_1, s_2, s_3,,,, s_n\}$ where $s_1 \geq s_2 \geq s_3 \geq \ldots.. \geq s_m$ ....(2)

Therefore, $U(\mu, t_1) \subseteq U(\mu, t_2) \subseteq U(\mu, t_3) \subseteq \ldots\ldots\ldots \subseteq U(\mu, t_n) = X$.............(3)

And $U(\lambda, s_1) \subseteq U(\lambda, s_2) \subseteq U(\lambda, s_3) \subseteq \ldots\ldots\ldots \subseteq U(\lambda, s_m) = X$ ....(4)

Also F(μ) = {U(μ, t_i): 1 $\leq$ i $\leq$ n},

F(λ) = {U(λ, s_i): 1 $\leq$ j $\leq$ m}

Assume m $\neq$ n.

Then, m $\geq$ n of n $\geq$ m.

Let m $\geq$ n.

Then U(μ, t_i) = U(λ, s_i), i =1, 2,......n.

This shows that both t_i and $s_i \in$ Im(μ).

For i $>$ n we observe that $t_i \notin$ Im(μ) and hence, U(μ, t_i) $\neq$ U(λ, s_i) where i = n+1,n+2,...m

Let n $\geq$ m. Then U(μ, t_i)=U(λ, s_i) where i = 1,2,......m.

This shows that both t_i and $s_i \in$ Im(λ).

For j $>$ m we observe that $s_j \notin$ Im(μ) and hence, U(μ, t_i) $\neq$ U(λ, s_i) where i = m+1,m+2,.....n.

From (3) and (4) implies $t_i \neq s_i$, $\forall$ i =1, 2,......,n.

Hence we can find some i such that U(μ, t_i) $\neq$ U(λ, s_i).

This contradicts that F(μ) = F(λ).

Hence we conclude that m = n.

Since μ and λ have identical family of level subalgebras, we have

U(μ, t_i) = U(λ, s_i), i =1, 2,……n.

Hence (ii) is obtained.

(iii) Follows from (i) and (ii)

Let μ(x) = t_i, implies λ(x) = s_i, for i = 1,2,….n.

Theorem 3.2.10: Let μ and λ be two fuzzy subalgebras of X with identical family of level subalgebras. Then Im(μ) = Im(λ) implies $\mu = \lambda$.

Proof: Let μ and λ be two fuzzy subalgebras of X with identical family of level subalgebras. Let

Im(μ) = Im(λ) = $\{s_1, s_2, s_3,,,, s_n\}$, where $s_1 \geq s_2 \geq s_3 \geq \ldots.. \geq s_n$

By lemma [3.2.9]

x $\in$ X, there exists s_i such that $\mu(x) = s_i = \lambda(x)$

Thus $\mu(x) = \lambda(x)$ $\forall$ x $\in$ X, proving that $\mu = \lambda$.

Theorem 3.2.11: Two level BP-Subalgebras U(μ,s) and U(μ,t), (s < t) of a fuzzy BP-Subalgebra μ are equal if and only if there is no x $\in$ X such that s $\leq \mu$(x) < t.

Proof: Let U(μ,s) and U(μ,t) be two level BP-Subalgebras of fuzzy BP-Subalgebra μ of X.

Suppose that U(μ,s) = U(μ,t) for some s < t.

Suppose there is one x $\in$ X such that s $\leq \mu$(x)< t.

Then, μ(x) $\geq$ s and μ(x) < t.

That is, x$\in$U(μ,s) and x$\notin$ U(μ,t).

This contradicts to U(μ,s) = U(μ,t).

Conversely, assume that there is no x$\in$X such that s $\leq \mu$(x) $<$ t.

Such that s $\leq \mu$(x) $<$ t.

Suppose, U(μ,s) $\neq$ U(μ,t).

For, , x$\in$U(μ,t) $\Rightarrow$ μ(x) $\geq$ t > s.

$\Rightarrow$ μ(x) > s $\Rightarrow$, x$\in$U(μ,s).

Since U(μ,s) $\neq$ U(μ,t).

Choose, U(μ,s) $\not\subset$ U(μ,t).

Hence there is an , x$\in$U(μ,s) and x$\notin$ U(μ,t)

$\Rightarrow$ μ(x) $\geq$ s and μ(x) $<$ t.

Thus there exists an element x $\in$X such that s $\leq \mu$(x) $<$ t, thus contradicting our hypothesis.

Hence U(μ,s) = U(μ,t).

Theorem 3.2.12: If μ_1 and μ_2 are fuzzy BP-Subalgebras of X, then $\mu = \mu_1 \times \mu_2$ is a fuzzy BP-Subalgebra of X×X.

Proof: For any (x_1, x_2) and $(y_1, y_2) \in$ X×X we have,

$$\begin{aligned}\mu[(x_1,x_2)*(y_1,y_2)] &= \mu(x_1 * y_1,\ x_2 * y_2)\\ &= (\mu_1 \times \mu_2)[x_1 * y_1,\ x_2 * y_2]\\ &= \min\{\mu_1(x_1 * y_1),\ \mu_2(x_2 * y_2)\}\\ &\geq \min\{\min(\mu_1(x_1),\ \mu_1(y_1)), \min(\mu_2(x_2), \mu_2(y_2))\}\end{aligned}$$

$= \min\{\min(\mu_1(x_1), (\mu_2(x_2)), \min(\mu_1(y_1), \mu_2(y_2))\}$

$= \min\{(\mu_1 \times \mu_2)(x_1, x_2), (\mu_1 \times \mu_2)(y_1, y_2)\}$

$= \min\{\mu(x_1, x_2),\ \mu(y_1, y_2)\}$

Hence $\mu = \mu_1 \times \mu_2$ is a fuzzy BP-Subalgebra of X×X.

Lemma 3.2.13: Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two BP-algebras. Let f : $X_1 \rightarrow X_2$ be an epimorphism. If σ is fuzzy BP-Subalgebra of X_2, then $f^{-1}(\sigma)$ is a fuzzy BP-Subalgebra of X_1.

Alternatively, we have epimorphic pre image of a fuzzy BP-Subalgebra is a fuzzy BP-Subalgebra.

Proof: $(f^{-1}(\sigma))(x *_1 y) = \sigma(f(x *_1 y))$

$= \sigma(f(x) *_2 f(y))$ (since f is an epimorphism)

$\geq \min\{\sigma(f(x), \sigma(f(y)\}$ (since σ is a fuzzy BP-Subalgebra)

$= \min[(f^{-1}(\sigma))(x), (f^{-1}(\sigma))(x))]$ $\forall$ x, y $\in$ X

Thus $f^{-1}(\sigma)$ is a fuzzy BP-Subalgebra of X_1.

CHAPTER 4

Fuzzy BP-ideals and Intuitionistic Fuzzy BP-Ideals in BP-algebras

Section 4.1:

Fuzzy BP-ideal in BP-Algebras

Definition 4.1.1: Let X be a BP-algebra. A fuzzy set μ of X is said to be a fuzzy BP-ideal of X if it satisfies the following conditions: $\forall$ x, y$\in$X.

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$.

Example 4.1.2: Let (X = {0,1,2,3}, $*$,0) be a BP-algebra with the following cayley table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define μ : X$\rightarrow$[0,1] by

$$\mu(x) = \begin{cases} 0.9 \text{ } if \text{ } x = 0 \\ 0.6 \text{ } if \text{ } x = 2 \\ 0.3 \text{ } if \text{ } x = 1,3 \end{cases}$$

Therefore μ is a fuzzy BP-ideal of the BP-algebra X.

Proposition 4.1.3: Intersection of two fuzzy BP-ideals of X is again a fuzzy BP-ideal of X.

Proof: Let μ and ψ be any two fuzzy BP-ideal of X.

$$
\begin{aligned}
(\mu \cap \psi)(0) &= (\mu \cap \psi)(x * x) \\
&\geq \min\{\mu(x * x), \psi(x * x)\} \\
&\geq \min\{\min\{\mu(x), \mu(x)\}, \min\{\psi(x), \psi(x)\}\} \\
&= \min\{\min\{\mu(x), \psi(x)\}, \min\{\mu(x), \psi(x)\}\} \\
&= \min\{(\mu \cap \psi)(x), (\mu \cap \psi)(x)\} \\
&= \{(\mu \cap \psi)(x)\}
\end{aligned}
$$

Therefore $(\mu \cap \psi)(0) \geq (\mu \cap \psi)(x)$

$$
\begin{aligned}
(\mu \cap \psi)(x) &= \min\{\mu(x), \psi(x)\} \\
&= \min\{\min(\mu(x * y), \mu(y)), \min(\psi(x * y), \psi(y))\} \\
&= \min\{\min(\mu(x * y), \psi(x * y)), \min(\mu(y), \psi(y))\} \\
&= \min\{(\mu \cap \psi)(x * y), (\mu \cap \psi)(y)\}, \text{ for all } x, y \in X.
\end{aligned}
$$

Hence $\mu \cap \psi$ is a fuzzy BP-ideal of X.

Proposition 4.1.4: If μ is a fuzzy BP-ideal of a BP-algebra $(X, *, 0)$, then $\forall$ x, y $\in$ X.

(i) μ is order reversing. ie, $x \leq y$ implies $\mu(x) \geq \mu(y)$

(ii) $\mu(x * (x * y)) \geq \mu(y)$.

Proof: Since μ is a fuzzy BP-ideal of X.

Let $x \leq y \Rightarrow x * y = 0$

$$\Rightarrow \mu(x * y) = \mu(0)$$

Therefore $\mu(x * y) = \mu(0) \geq \mu(x)$

$\geq \min\{\mu(x*y), \mu(y)\}$

$\geq \min\{\mu(0), \mu(y)\}$

$\mu(x) \geq \mu(y).$

By definition of BP-algebra $x*(x*y) = y$.

$(x*(x*y)) * y = y* y$

$\Rightarrow (x* (x*y)) * y = 0$

$x*(x*y) \leq y.$

By (i) μ is order reversing.

Therefore $\mu(x*(x*y)) \geq \mu(y)\ \forall\ x, y \in X.$

Proposition 4.1.5: If μ is a fuzzy ideal of a BP-algebra $(X, *, 0)$ and

$$\mu_\alpha(x) = \min\{\alpha, \mu(x)\}$$

$\forall x \in X$ and $\alpha \in [0,1]$, then $\mu_\alpha(x)$ is a fuzzy BP-ideal of X.

Proof: Let μ be a fuzzy ideal of the BP-algebra $(X, *, 0)$ and $\alpha \in [0,1]$.

Therefore $\mu(0) \geq \mu(x)\ \forall x \in X.$

Now, $\mu_\alpha(x)(0) = \min\{\alpha, \mu(0)\}$

$\geq \min\{\alpha, \mu(x)\}$

$= \mu_\alpha(x) \quad \forall x \in X.$

Also, μ is a fuzzy ideal of X shows that

$\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \quad \forall x, y \in X.$

$\mu_\alpha(x) = \min\{\alpha, \mu(x)\}$

$\geq \min\{\alpha, \min(\mu(x*y), \mu(y))\}$

$= \min\{\min(\alpha, \mu(x*y)), \min(\alpha, \mu(y))\}$

$= \min\{\mu_\alpha(x)(x*y), \mu_\alpha(x)(y)\}.$

$\Rightarrow \mu_\alpha(x)$ is a fuzzy ideal of X.

Since this is true for all $\alpha \in [0, 1]$, $\mu_\alpha(x)$ is a fuzzy BP-ideal of X for all $\alpha \in [0, 1]$.

Theorem 4.1.6: A fuzzy set of a BP-algebra (X, $*$,0) is a fuzzy BP-ideal if and only if for any $\lambda \in [0, 1]$, U(μ, λ) = {x : x$\in$X, μ(x) $\geq \lambda$} is an ideal of X where U(μ, λ) $\neq \phi$

Proof: Suppose μ is a fuzzy ideal of X and U(μ, λ) $\neq \phi$ for $\lambda \in [0,1]$.

Let x $\in$ U(μ, λ), μ(x) $\geq \lambda$.

By definition of fuzzy BP-ideal.

We have $\mu(0) \geq \mu(x) \geq \lambda$.

Thus 0$\in$U(μ, λ).

Suppose x$*$y $\in$U(μ, λ) and y$\in$U(μ, λ).

Therefore, $\mu(x*y) \geq \lambda$ and $\mu(y) \geq \lambda$.

By definition, we have $\mu(x) \geq \min\{\mu(x*y), \mu(y)\} \geq \lambda$.

x$\in$U(μ, λ).

Hence (μ, λ) is an BP-ideal of X.

Conversely, suppose that for each $\lambda \in [0,1]$, U(μ, λ) is either empty or an ideal of X.

For any x$\in$X, let $\mu(x) = \lambda$.

Then x$\in$U(μ, λ).

Since U(μ, λ)$\neq \phi$ is an ideal of X.

We have 0$\in$U(μ, λ) and hence $\mu(0) \geq \lambda = \mu(x)$.

Thus $\mu(0) \geq \mu(x)$ $\forall$ x$\in$X.

Assume $\mu(x) \geq \min\{\mu(x*y), \mu(y)\}$ $\forall$ x, y$\in$X is not true.

Then there exists x_0, $y_0 \in$ X such that

$\mu(x_0) \leq \min\{\mu(x_0 * y_0), \mu(y_0)\}$

$\Rightarrow \mu(x_0) < \lambda_0 < \min\{\mu(x_0 * y_0), \mu(y_0)\}$

We have $x_0 * y_0$, $y_0 \in U(\mu, \lambda_0)$ and $U(\mu, \lambda_0) \neq \phi$.

But $U(\mu, \lambda_0)$ is an ideal of X.

So $x_0 \in U(\mu, \lambda_0)$ by the definition of BP-ideal, $\mu(x_0) \geq \lambda_0$,

contradicting ($\mu(0) \geq \mu(x)$, $\forall x \in X$).

Therefore $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

Theorem 4.1.7: A fuzzy set μ of a BP-algebra (X, $*$,0) is a fuzzy BP-ideal if and only if every nonempty level subset $U(\mu, s)$ of μ, $s \in Im(\mu)$ is a BP-ideal.

Proof: Let μ be a fuzzy BP-ideal.

Claim: $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal.

Since $U(\mu, s) \neq \phi$ there exist $x \in U(\mu, s)$.

Such that $\mu(x) \geq s$.

Since μ is a fuzzy BP-ideal, $\mu(0) \geq \mu(x)$ $\forall x \in X$.

Hence for this $x \in U(\mu, s)$, $\mu(0) \geq s$ which shows that $0 \in U(\mu, s)$.

Now, for any x, $y \in X$.

Assume that $x * y \in U(\mu, s)$ and $y \in U(\mu, s)$.

$x * y \in U(\mu, s) \Rightarrow \mu(x * y) \geq s$

Also $y \in U(\mu, s) \Rightarrow \mu(y) \geq s$

Therefore $\mu(x * y) \geq s$ and $\mu(y) \geq s$.

$\Rightarrow \min\{\mu(x * y), \mu(y)\} \geq s$.

Since μ is a fuzzy BP-ideal,

$\mu(x) \geq \min\{\mu(x * y), \mu(y)\} \geq s$.

This implies $x \in U(\mu, s)$.

This proves that $U(\mu, s)$ is a BP-ideal of X.

Conversely, let $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal of X.

Claim: μ is a fuzzy BP-ideal.

Let $x, y \in X$.

For any $s \in Im(\mu)$, let $s = \min\{\mu(x * y), \mu(y)\}$.

Therefore, $\mu(x * y) \geq s$ and $\mu(y) \geq s$.

This shows that $x * y \in U(\mu, s)$.

Since $U(\mu, s)$ is a BP-ideal we have $x \in U(\mu, s)$.

This proves that $\mu(x) \geq s = \min\{\mu(x * y), \mu(y)\}$

This shows that μ is a fuzzy BP-ideal of X.

Theorem 4.1.8: Let μ be a fuzzy BP-ideal of BP-algebra X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in U(\mu, t)$ but $x \notin U(\mu, s)$, $\forall s > t$.

Proof: Let μ be a fuzzy BP-ideal of X and let $x \in X$**.**

Assume $\mu(x) = t$. So that $x \in U(\mu, t)$.

If possible, let $x \in U(\mu, s)$ for $s > t$.

Then μ(x) $\geq$ s >t.

This contradicts the fact that μ(x) = 1,concludes that x $\notin$ U(μ, s), $\forall$ s >t.

Conversely, let x $\in$ U(μ, t) but x $\notin$ U(μ, s), $\forall$ s >t.

x $\in$ U(μ, t) $\Rightarrow$ μ(x) $\geq$ t.

Since x $\notin$ U(μ, s) $\forall$ s >t, and hence μ(x) = t.

Theorem 4.1.9: Let X be a BP-algebra. Let λ and μ be the fuzzy BP-ideals of X. Then $\lambda \times \mu$ is a fuzzy BP-ideal of X×X.

Proof: Let X be a BP-algebra and let λ and μ be the fuzzy BP-ideals of X. For any (x, y) $\in$ X×X.

$$(\lambda \times \mu)(0, 0) = \min\{\lambda(0), \mu(0)\}$$
$$\geq \min\{\lambda(x), \mu(x)\}$$
$$= (\lambda \times \mu)(x).$$

Let (x_1, x_2) and $(y_1, y_2) \in$ X×X, x = (x_1, x_2) and y = (y_1, y_2).

$$(\lambda \times \mu)(x) = (\lambda \times \mu)(x_1, x_2)$$
$$= \min\{\lambda(x_1), \mu(x_2)\}$$
$$\geq \min\{\min(\lambda(x_1 * y_1), \lambda(y_1)), \min(\mu(x_2 * y_2), \mu(y_2))\}$$
$$= \min\{\min(\lambda(x_1 * y_1), \mu(x_2 * y_2)), \min(\lambda(y_1), \mu(y_2))\}$$
$$= \min\{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2), (\lambda \times \mu)(y_1, y_2)\}$$
$$= \min\{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)), (\lambda \times \mu)(y_1, y_2)\}$$
$$= \min\{(\lambda \times \mu)(x, y), (\lambda \times \mu)(y)\}$$

Thus $\lambda \times \mu$ is a fuzzy BP-ideal of X×X.

Theorem 4.1.10: For any two fuzzy sets λ and μ of X, if $\lambda \times \mu$ is a fuzzy BP-ideal of X×X, then either λ or μ is a fuzzy BP-ideal of X.

Proof: Let λ and μ be fuzzy sets of X such that $\lambda \times \mu$ is a fuzzy BP-ideal of X.

$(\lambda \times \mu)(0, 0) \geq (\lambda \times \mu)(x, y)\ \forall$ x, y$\in$X$\times$X.

Assume λ(x) $>\lambda$(0) and μ(y)$>\mu$(0) for some x, y$\in$X.

Then $(\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\}$

$> \min\{\lambda(0), \mu(0)\}$

$= (\lambda \times \mu)(0, 0)$ which is contradiction.

Thus $\lambda(x) \geq \lambda(0)$ or $\mu(0) > \mu(x)$, $\forall$ x, y$\in$X.

Let x = (x_1, x_2) and y = $(y_1, y_2) \in$ X$\times$X.

$(\lambda \times \mu)(x) \geq \min\{(\lambda \times \mu)(x * y), (\lambda \times \mu)(y)\}$

$= \min\{(\lambda \times \mu)(x_1 * y_1,\ x_2 * y_2), (\lambda \times \mu)(y_1, y_2)\}$

$= \min\{\min(\lambda(x_1 * y_1),\ \mu(x_2 * y_2)), \min(\lambda(y_1),\ \mu(y_2))\}$

$\Rightarrow \min\{\lambda(x_1),\ \mu(x_2)\} \geq \min\{\min(\lambda(x_1 * y_1), \lambda(y_1)), \min(\mu(x_2 * y_2), \mu(y_2))\}$.

$\Rightarrow$ either $\lambda(x_1) \geq \min\{\lambda(x_1 * y_1),\ \lambda(y_1)\}$

Or $\mu(x_2) \geq \min\{\mu(x_2 * y_2), \mu(y_2)\}$

$\Rightarrow \lambda$ or μ is a fuzzy ideal of X.

Theorem 4.1.11: Let λ and μ be fuzzy BP-ideals of $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ respectively. Then $\lambda \times \mu$ is a fuzzy BP-ideal of $(X_1 \times X_2, *, 0)$.

Proof: Let λ be a fuzzy BP-ideal of X_1.

Let μ be a fuzzy BP-ideal of X_2.

Claim: $\lambda \times \mu$ is fuzzy BP-ideals of $X_1 \times X_2$.

For any (x, y) $\in X_1 \times X_2$.

$(\lambda \times \mu)(0, 0) = \min\{\lambda(0),\ \mu(0)\}$

$\geq \min\{\lambda(x),\ \mu(y)\}$

$= (\lambda \times \mu)(x, y)$.

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$.

$$(\lambda \times \mu)(x_1, x_2) = \min\{\lambda(x_1),\ \mu(x_2)\}$$

$$\geq \min\{\min(\lambda(x_1 *_1 y_1),\ \lambda(y_1)), \min(\mu(x_2 *_2 y_2), \mu(y_2))\}$$

$$= \min\{\min(\lambda(x_1 *_1 y_1),\ \mu(x_2 *_2 y_2)), \min(\lambda(y_1), \mu(y_2))\}$$

$$= \min\{(\lambda \times \mu)(x_1 *_1 y_1,\ x_2 *_2 y_2),\ (\lambda \times \mu)(y_1, y_2)\}$$

$$= \min\{(\lambda \times \mu)((x_1,\ x_2) * (y_1,\ y_2)),\ (\lambda \times \mu)(y_1, y_2)\}$$

Thus $\lambda \times \mu$ is a fuzzy BP-ideal of $X_1 \times X_2$.

Theorem 4.1.12: Let f : $(X_1, *_1, 0_1) \rightarrow (X_2, *_2, 0_2)$ be an epimorphism of BP-algebras. Let μ be a fuzzy set of X_2. If $f^{-1}(\mu)$ is a fuzzy BP-ideal of X_1, then μ is a fuzzy BP-ideal of X_2.

Proof: Let f : $X_1 \rightarrow X_2$ be an epimorphism of BP-algebras. Let μ be a fuzzy set of X_2.

Let $f^{-1}(\mu)$ is a fuzzy BP-ideal of X_1.

Claim: μ is a fuzzy BP-ideal of X_2.

$$\mu(0_2) = \mu(\mathrm{f}(0_1)) \geq f^{-1}(\mu(x_1))$$

$$= \mu(\mathrm{f}(x_1)) = \mu(x_2).$$

Let $x_2,\ y_2 \in X_2$. Since f is an epimorphism, $x_1,\ y_1 \in X_1$ such that f(x_1) = x_2 and f(y_1) = y_2.

i.e, $x_1 = f^{-1}(x_2)$ and $y_1 = f^{-1}(y_2)$.

$$\mu(x_2) = \mu(\mathrm{f}(x_1))$$

$$= f^{-1}(\mu(x_1))$$

$$\geq \min\{f^{-1}(\mu(x_1 *_1 y_1)),\ f^{-1}(\mu(y_1))\}$$

$$= \min\{\mu(\mathrm{f}(x_1 *_1 y_1),\ \mu(\mathrm{f}(y_1))\}$$

$$= \min\{\mu((\mathrm{f}(x_1) *_2 \mathrm{f}(y_1)),\ \mu(\mathrm{f}(y_1))\}$$

$$= \min\{\mu(x_2 *_2 y_2),\ \mu(y_2)\}$$

Therefore μ is a fuzzy BP-ideal of X_2.

Theorem 4.1.13: Inverse image of fuzzy BP-ideal is again a fuzzy BP-ideal.

Proof: Let f : $(X_1, *_1, 0_1) \rightarrow (X_2, *_2, 0_2)$ be an epimorphism.

Let σ be fuzzy BP-ideal of X_2.

To prove: $f^{-1}(\sigma)$ is a fuzzy BP-ideal of X_1. Let x, y $\in X_1$.

$$(f^{-1}(\sigma))(x) = \sigma(f(x))$$
$$\geq \min\{\sigma(f(x) *_2 f(y)),\ \sigma(f(y))\}$$
$$= \min\{\sigma(f(x *_1 y),\ \sigma(f(y))\} \quad \text{(since f is epimorphism)}$$
$$= \min\{(f^{-1}(\sigma))(x *_1 y),\ (f^{-1}(\sigma))(y)\}.$$

Thus $f^{-1}(\sigma)$ is a fuzzy BP-ideal of X_1.

Section 4.2:

Fuzzy T-ideals in BP-algebras

Definition 4.2.1: A fuzzy set μ in a BP-algebra X is called a fuzzy T-ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x * z) \geq \min\{\mu((x * y) * z),\ \mu(y)\},\ \forall$ x, y, z $\in$ X.

Theorem 4.2.2: Every fuzzy T-ideal μ of a BP-algebra X is order reversing, that is if $x \leq y$ then $\mu(x) \geq \mu(y)$, $\forall$ x, y $\in$ X.

Proof: Let x, y $\in$ X such that $x \leq y$.

Therefore $x * y = 0$.

$$\mu(x) = \mu(x * 0)$$
$$\geq \min\{\mu((x * y) * 0),\ \mu(y)\}$$
$$= \min\{\mu(0 * 0),\ \mu(y)\}$$
$$= \min\{\mu(0),\ \mu(y)\} = \mu(x) \geq \mu(y).$$

Theorem 4.2.3: A fuzzy set μ in a BP-algebra X is a fuzzy T-ideal if and only if it is a fuzzy BP-ideal μ of X.

Proof: Let μ be a fuzzy T-ideal T-ideal of X, $\mu(0) \geq \mu(x)$.

$\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}, \forall x, y, z \in X.$

Put z = 0, then

We get $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$

Hence μ is a fuzzy BP-ideal of X.

Conversely, μ is a fuzzy BP-ideal of X.

Then, $\mu(x * z) \geq \min\{\mu((x * z) * y), \mu(y)\}, \forall x, y, z \in X.$

$= \min\{\mu((x * y) * z), \mu(y)\}, \forall x, y, z \in X,$

Which proves the result.

Theorem 4.2.4: Let μ be a fuzzy set in a BP-algebra X and let $t \in Im(\mu)$. Then μ is a fuzzy T-ideal of X if and only if the level subset $\mu t = \{x \in X / \mu(x) \geq t\}$ is a T-ideal of X, which is called a level T-ideal of μ.

Proof: Assume that μ is a fuzzy T-ideal of X.

Clearly, $0 \in \mu_t$

Let $(x * y) * z \in \mu_t$ and $y \in \mu_t$

Then $\mu((x * y) * z) \geq t$ and $\mu(y) \geq t$.

Now, $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$

$\geq \{t, t\} = t.$

Hence μ_t is T-ideal of X.

Conversely, let μ_t be T-ideal of X for any t$\in$[0,1].

Suppose, assume that there exist some $x_0 \in$X.

Such that $\mu(0) < \mu(x_0)$.

Take S = ½ [$\mu(0) + \mu(x_0)$]

$\Rightarrow$ S $< \mu(x_0)$ and $0 \leq \mu(0) <$ s < 1.

$\Rightarrow$ $x_0 \in \mu_s$ and $0 \notin \mu_s$, which is a contradiction, since μ_s is a T-ideal of X.

Therefore, $\mu(0) \geq \mu(x)$, $\forall$ x$\in$X.

Assume that x_0, y_0, $z_0 \in$X.

Such that,

$\mu(x_0 * z_0) \geq \min\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$

Let s = ½ [$\mu(x_0 * z_0) + \min\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$]

$\Rightarrow$ S $> \mu(x_0 * z_0)$

And S $< \min\{\mu((x_0 * y_0) * z_0), \mu(y_0)\}$

$\Rightarrow$ S $> \mu(x_0 * z_0)$, S $< \{\mu((x_0 * y_0) * z_0)\}$ and S $< \mu(y_0)$.

$\Rightarrow$ $x_0 * z_0 \notin \mu_s$, which is a contrsdiction, since μ_s is a T-ideal of X.

Therefore, $\mu(x * z) \geq \min\{\mu((x * y) * z), \mu(y)\}$ $\forall$ x, y, z$\in$X.

Definition 4.2.5: Let f : X$\rightarrow$X be an endomorphism and μ is a fuzzy set in a BP-algebra X. Define a new fuzzy set μ_f in X by $\mu_f(x) = \mu(f(x))$, for all x in X.

Theorem 4.2.6: Let f be an endomorphism of a BP-algebra X. If μ is a fuzzy T-ideal of X, then so is μ_f.

Proof: μ_f (x) = μ (f(x)) $\leq$ μ (0).

Let x, y, z$\in$X,

Therefore μ_f (x$*$z) = μ (f(x$*$z))

= μ (f(x) $*$ f(z))

$\geq$ min{(f(x) $*$ f(y)), μ (f(y))}

= min{ μ (f(x$*$y) $*$ f(z)), μ (f(y))}

= min{ μ (f(x$*$y) $*$ z), μ (f(y))}

= min{ μ_f ((x$*$y) $*$ z), μ_f (y)}

Hence μ_f is a fuzzy T-ideal of X.

Cartesian Product of fuzzy T-ideals in BP-algebras

Theorem 4.2.7: If μ and λ are fuzzy T-ideal in a BP-algebra of X, then $\mu \times \lambda$ is a fuzzy T-ideal in X×X.

Proof: For any (x, y) $\in$ X×X we have,

($\mu \times \lambda$) (0, 0) = min{ μ (0), λ (0)}

$\geq$min{ μ (x), λ (y)}

= ($\mu \times \lambda$) (x, y)

Let (x_1, x_2), (y_1, y_2) and $(z_1, z_2)$$\in$X×X.

$(\mu \times \lambda)[((x_1, x_2)*(z_1, z_2)] = (\mu \times \lambda)[(x_1 * z_1, x_2 * z_2)]$

$= \min\{\mu(x_1 * z_1), \lambda(x_2 * z_2)]$

$\geq \min\{\min\{\mu((x_1 * y_1)* z_1), \mu(y_1)\}, \min\{\lambda((x_2 * y_2)* z_2), \lambda(y_2)\}\}$

$= \min\{\min\{\mu((x_1 * y_1)* z_1), \lambda((x_2 * y_2)* z_2)\}, \min\{\mu(y_1), \lambda(y_2)\}\}$

$= \min\{(\mu \times \lambda)[((x_1 * y_1)* z_1), ((x_2 * y_2)* z_2)], (\mu \times \lambda)(y_1, y_2)\}$

$= \min\{(\mu \times \lambda)[(x_1 * y_1,\ x_1 * y_2) * (z_1,\ z_2)],\ (\mu \times \lambda)(y_1, y_2)\}$

$= \min\{(\mu \times \lambda)[((x_1,\ x_2) * (y_1,\ y_2)) * (z_1,\ z_2)],\ (\mu \times \lambda)(y_1, y_2)\}$

Hence $\mu \times \lambda$ is a fuzzy T-ideal in X×X.

Section 4.3:

Intuitionistic Fuzzy BP-Ideals in BP-Algebras

Definition 4.3.1: An Intuitionistic Fuzzy Set (IFS) A in a non-empty set X is defined as an object of the form $A = \{< x, \mu_A(x), \nu_A(x) > / x \in X\}$ (or briefly A = (μ_A, ν_A)) where μ_A : X → [0,1] is the degree of membership and ν_A : X → [0,1] is the degree of non membership of the element x ∈ X satisfying $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 4.3.2: Let A = $\{< x, \mu_A(x), \nu_A(x) > / x \in X\}$ and B = $\{< x, \mu_B(x), \nu_B(x) > / x \in X\}$ be two Intuitionistic Fuzzy Sets of the set X, then define

$(i)\ \ \overline{A} = \{< x, \nu_A(x), \mu_A(x) > / x \in X\}$

$(ii)\ A \subseteq B \text{ iff } \mu_A(x) \le \mu_B(x) \text{ and } \nu_A(x) \ge \nu_B(x)\ \forall x \in X$

$(iii)\ A = B \text{ iff } \mu_A(x) = \mu_B(x) \text{ and } \nu_A(x) = \nu_B(x)\ \forall x \in X$

$(iv)\ A \cup B = \{< x, \max(\mu_A(x), \mu_B(x)), \min(\nu_A(x), \nu_B(x)) / x \in X\}$

$(v)\ \ A \cap B = \{< x, \min(\mu_A(x), \mu_B(x)), \max(\nu_A(x), \nu_B(x)) / x \in X\}$

Definition 4.3.3: An Intuitionistic Fuzzy set A = $\{< x, \mu_A(x), \nu_A(x) > / x \in X\}$ in a BP-Algebra X is said to be an Intuitionistic Fuzzy BP-Subalgebra of X if

1. $\mu_A(x * y) \ge \min\{\mu_A(x), \mu_A(y)\}$
2. $\nu_A(x * y) \le \max\{\nu_A(x), \nu_A(y)\}\ \forall\ x, y \in X$

Example 4.3.4: Consider the BP-Algebra X = {0, 1, 2, 3} with the following cayley's table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1

3	3	2	1	0

Define,

$$\mu_A(x) = \begin{cases} 0.6 & if\ x \neq 2 \\ 0.1 & if\ x = 2 \end{cases}$$

and

$$\nu_A(x) = \begin{cases} 0.2 & if\ x \neq 2 \\ 0.8 & if\ x = 2 \end{cases}$$

Then Intuitionistic Fuzzy set $A = \{< x,\ \mu_A(x),\ \nu_A(x) > / x \in X\}$ of X is an Intuitionistic Fuzzy BP-Subalgebra of X.

Lemma 4.3.5: In an Intuitionistic fuzzy BP-subalgebra A of X we have

(i) $\mu_A(0) \geq \mu_A(x)$

(ii) $\nu_A(0) \leq \nu_A(x)\ \forall x \in X$.

Proof: Let A be an Intuitionistic Fuzzy BP-Subalgebra of X. Then $\forall x \in X$,

$\mu_A(0) = \mu_A(x * x) \geq \min\{\mu_A(x),\ \mu_A(x)\} = \mu_A(x)$.

Similarly,

$\nu_A(0) = \nu_A(x * x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)$.

Theorem 4.3.6: Intersection of any two Intuitionistic fuzzy BP-Subalgebras of X is again an Intuitionistic fuzzy BP-Subalgebra of X.

Proof. Let A = $\{< x, \mu_A(x), \nu_A(x) > / x \in X\}$ and

B = $\{< x, \mu_B(x), \nu_B(x) > / x \in X\}$ be Intuitionistic fuzzy BP-Subalgebras of X and let C = A ∩ B. For all x, y ∈ X,

$$\mu_C(x * y) = \mu_{A\cap B}(x * y)$$

$$= \min\{\mu_A(x * y), \mu_B(x * y)\}$$

$$\geq \min\{min(\mu_A(x), \mu_A(y)), min(\mu_B(x), \mu_B(y))\}$$

$$= \min\{min(\mu_A(x), \mu_B(x)), min(\mu_A(y), \mu_B(y))\}$$

$$= min\ \{\mu_C(x), \mu_C(y)\}$$

$$v_A(x * y) = v_{A\cap B}(x * y)$$

$$= \max\{v_A(x * y), v_B(x * y)\}$$

$$\leq max\ \{\max((v_A(x), v_A(y)), \max(v_B(x), v_B(y))\}$$

$$= \max\{\max((v_A(x), v_B(x)), \max(v_A(y), v_B(y))\}$$

$$= \max\ \{v_c(x), v_c(y)\}$$

Hence, C is an Intuitionistic fuzzy BP-Subalgebra of X.

We can generalize the above result as follows:

Theorem 4.3.7: Intersection of any family of Intuitionistic fuzzy BP-Subalgebras of X is again an Intuitionistic fuzzy BP-Subalgebra of X..

Theorem 4.3.8: An Intuitionistic Fuzzy set A of X is an Intuitionistic Fuzzy BP-Subalgebra of X if and only if the fuzzy sets μ_A *and* $\bar{\nu}_A$= 1-ν_A are fuzzy BP-Subalgebras of X.

Proof: Let A = $\{x, \mu_A(x), v_A(x) \mid x \in X\}$ be an Intuitionistic fuzzy BP-Subalgebra of X. Clearly, μ_A is a fuzzy BP-Subalgebra of X. For all x, y $\in$ X,

$$\bar{v}_A(x * y) = 1 - v_A(x * y)$$

$$\geq 1 - \max\ [v_A(x), v_A(y)]$$

$$= \min\{(1\text{-}v_A(x)), (1\text{-}v_A(y))\}$$

$$= \min\ \{\bar{\nu}_A(x), \bar{\nu}_A(y)\}$$

Hence $\bar{\nu}_A$ is a fuzzy BP-Subalgebra of X.

Conversely, assume μ_A and $\overline{\nu}_A$ are fuzzy BP-Subalgebra of X.

Then, $\mu_A(x \,\square\, y) \geq \min\{\mu_A(x), \mu_A(y)\}$ *and*

$\bar{\nu}_A(x \,\square\, y) \geq \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\}$ □ x, y □ X.

Hence, to prove that A $= \{x, \mu_A(x), \nu_A(x) \,/x\, \square \text{ X}\}$ is an Intuitionistic fuzzy BP-subalgebra of X, it is enough to prove that $\nu_A(x * y) \leq \max\{\ \nu_A(x), \nu_A(y)\}\ \forall x, y \in X$.

Since $\overline{\nu_A}$ is fuzzy BP-Subalgebra of X,

$$\overline{\nu_A}(x * y) \geq \min\{\overline{\nu_A}(x), \overline{\nu_A}(y)\}$$

$$1 - \nu_A(x * y) \geq \min\{(1 - \nu_A(x)), (1 - \nu_A(y))\}$$

$$= 1 - \max\{\nu_A(x), \nu_A(y)\}$$

That is, $\mathrm{v_A}(\mathrm{x} * \mathrm{y}) \leq \max\{\mathrm{v_A}(\mathrm{x}), \mathrm{v_A}(\mathrm{y})\}\ \forall \mathrm{x}, \mathrm{y} \in \mathrm{X}$.

This completes the Proof.

Intuitionistic Fuzzy BP-Ideals:

Definition 4.3.9: A n Intuitionistic Fuzzy set A of a BP-Algebra X is said to be an Intuitionistic Fuzzy BP-ideal of X if,

(*i*) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$

(*ii*) $\mu_A(x) \geq \min\{\mu_A(x \,\square\, y), \mu_A(y)\}$

(*iii*) $\nu_A(x) \leq \max\{\nu_A(x \,\square\, y), \nu_A(y)\}$ for all x, y □ X.

Example 4.3.10: Let X = {0, l, m} be a BP-algebra with Cayley's table given by

□	0	l	m
0	0	m	l
l	l	0	m
m	m	l	0

Define $\mu_A(0) = 1$ and $\mu_A(l) = \mu_A(m) = t$, and $v_A(0) = 0$ and $v_A(0) = 0$ and $v_A(l) = v_A(m) = s$, where t, s (0,1) and $s + t \leq 1$.

Then (μ_A, v_A) is an Intuitionistic fuzzy BP-ideal of X.

Lemma 4.3.11: Let $A = (\mu_A, \nu_A)$ in X be an Intuitionistic fuzzy BP- ideal of X

If $x * y \leq z$ then,

$\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z)\}$

$\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$

Proof: Let $x, y, z \in X$ such that $x * y \leq z$

Then $(x * y) * z = 0$ and thus

$$\begin{aligned}\mu_A(x) &\geq \min\{\mu_A(x*y), \mu_A(y)\}\\ &\geq \min\{\min\{\mu_A((x*y)*z), \mu_A(z)\}, \mu_A(y)\}\\ &= \min\{\min\{\mu_A(0), \mu_A(z)\}, \mu_A(y)\}\\ &= \min\{\mu_A(y), \mu_A(z)\}\end{aligned}$$

Similarly, $\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$

Lemma 4.3.12: Let $A = (\mu_A, \nu_A)$ in X be an Intuitionistic fuzzy BP-ideal of X.

If $x \leq y$ then :

$\mu_A(x) \geq \mu_A(y)$, $\nu_A(x) \leq \nu_A(y)$ that is, μ_A is order reserving and ν_A is order preserving.

Proof : Let x, y, z X be such that x y = 0

$$\begin{aligned}\mu_A(x) &\geq \min\{\mu_A(x*y), \mu_A(y)\}\\ &= \min\{\mu_A(0), \mu_A(y)\}\\ &= \mu_A(y)\end{aligned}$$

$$\begin{aligned}\nu_A(x) &\leq \max\{\nu_A(x*y), \nu_A(y)\}\\ &= \max\{\nu_A(0), \nu_A(y)\}\\ &= \nu_A(y)\end{aligned}$$

Definition 4.3.13: Let $f : X \to Y$ be a BP-Homomorphism of BP-Algebras. For any Intuitionistic Fuzzy Set $A = (\mu_A, \nu_A)$ in Y, we define a new Intuitionistic Fuzzy Set

$A^f = (\mu_A^f, \nu_A^f)$ in X by $\mu_A^f(x) = \mu_A(f(x))$, $\nu_A^f(x) = \nu_A(f(x))$ for all x, y $\in$ X.

Theorem 4.3.14: Let $f : (X, *, 0) \to (Y, *', 0')$ be a homomorphism of BP-algebras. If an Intuitionistic Fuzzy Set $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy BP-ideal of Y, then an Intuitionistic Fuzzy Set $A^f = (\mu_A^f, \nu_A^f)$ in X is an Intuitionistic fuzzy BP-Ideal of X.

Proof: We have that $\mu_A^f(x) = \mu_A(f(x))$

$$\le \mu_A(0')$$

$$= \mu_A(f(0))$$

$$= \mu_A^f(0)$$

$$\nu_A^f(x) = \nu_A(f(x))$$

$$\ge \nu_A(0')$$

$$= \nu_A(f(0))$$

$$= \nu_A^f(0) \quad \forall x \in X.$$

Let x, y $\in$ X, Then

$$\min\{\mu_A^f(x * y), \mu_A^f(y)\} = \min\{\mu_A(f(x * y)), \mu_A(f(y))\}$$

$$= \min\{\mu_A(f(x) *' f(y)), \mu_A(f(y)\}$$

$$\le \mu_A(f(x))$$

$$= \mu_A^f(x)$$

$$\max\{\nu_A^f(x * y), \nu_A^f(y)\} = \max\{\nu_A(f(x * y)), \nu_A(f(y))\}$$

$$= \max\{\nu_A(f(x) *' f(y)), \nu_A(f(y)\}$$

$$\ge \nu_A(f(x))$$

$$= \nu_A^f(x)$$

Hence $A^f = (\mu_A^f, \nu_A^f)$ is an Intuitionistic fuzzy BP-Ideal of X.

Theorem 4.3.15: Let $f : (X, *, 0) \to (Y, *', 0')$ be an epimorphism of BP-algebra and let $A = (\mu_A, \nu_A)$ be an Intuitionistic Fuzzy set in Y. If $A^f = (\mu_A^f, \nu_A^f)$ is an Intuitionistic fuzzy BP-ideal of X. Then $A = (\mu_A, \nu_A)$ is an Intuitionistic Fuzzy BP-Ideal of Y.

Proof: For any x $\in$ Y there exist a $\in$ X such that f (a) = x.

Then,

$\mu_A(x) = \mu_A(f(a)) = \mu_A^f(a) \leq \mu_A^f(0) = \mu_A(f(0)) = \mu_A(0')$

$\nu_A(x) = \nu_A(f(a)) = \nu_A^f(a) \geq \nu_A^f(0) = \nu_A(f(0)) = \nu_A(0')$

Let x, y Y. Then f (a) = x and f(b) = y for some a, b X.

Thus

$$\begin{aligned}
\mu_A(x) &= \mu_A(f(a)) \\
&= \mu_A^f(a) \\
&\geq \min\{\mu_A^f(a*b), \mu_A^f(b)\} \\
&= \min\{\mu_A(f(a*b)), \ \mu_A(f(b))\} \\
&= \min\{\mu_A(f(a)*' f(b)), \mu_A(f(b))\} \\
&= \min\{\mu_A(x*' y), \mu_A(y)\}
\end{aligned}$$

$$\begin{aligned}
\nu_A(x) &= \nu_A(f(a)) \\
&= \nu_A^f(a) \\
&\leq \max\{\nu_A^f(a*b), \nu_A^f(b)\} \\
&= \max\{\nu_A(f(a*b)), \ \nu_A(f(b))\} \\
&= \max\{\nu_A(f(a)*' f(b)), \nu_A(f(b))\} \\
&= \max\{\nu_A(x*' y), \nu_A(y)\}
\end{aligned}$$

Hence $A = (\mu_A, \nu_A)$ is an Intuitionistic Fuzzy BP-Ideal of Y.

CHAPTER 5

L-Fuzzy BP-Ideals and Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras

Section 5.1:

L-Fuzzy BP-Subalgebra in BP-Algebras

Definition 5.1.1: A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound.

Definition 5.1.2: A lattice L is called a complete lattice if every subset A={ a_α } has a sup denoted by $\vee a_\alpha$ and inf denoted by $\wedge a_\alpha$ where $0 \equiv \wedge a_\alpha$ is the least element of L and $1 \equiv \vee a_\alpha$ is the greatest element of L $\ni 0 \leq a$ and $1 \geq a$ for every $a \in$ L.

Definition 5.1.3: Let X be a nonempty set and L:(L, $\leq$) be a complete lattice with least element 0 and greatest element 1. A L-Fuzzy set μ of X is a function μ : X$\rightarrow$L.

Definition 5.1.4: Let μ and λ be the L-fuzzy set in a set X. The Cartesian product $\lambda \times \mu$: X$\times$X$\rightarrow$L is defined by,

$$(\lambda \times \mu)(x, y) = \{\lambda(x) \wedge \mu(y)\} \ \forall \ x, y \in X.$$

Definition 5.1.5: A L-Fuzzy set μ of a BP-Algebra (X, $*$,0) is called a L-Fuzzy BP-subalgebra of X if, for all x, y $\in$ X. The following condition is satisfied,

$$\mu(x * y) \geq \mu(x) \wedge \mu(y)$$

Example 5.1.6: Let X = {0,a,b,c} be a set with the following table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Then (X, *, 0) is a BP-Algebra.

Define μ : X→L by $\mu(x) = \begin{cases} 1 & if \quad x = 0 \\ t_1 & if \quad x = b \\ t_2 & if \quad x = a \\ 0 & if \quad x = c \end{cases}$

t_1, t_2 $\in$ L and inf L $\leq t_1 \leq t_2 \leq$ sup L

Then μ is a L-Fuzzy BP-Algebra of X.

Theorem 5.1.7: Intersection of any two L-Fuzzy BP-Subalgebras of X is again a L-Fuzzy BP-Subalgebra.

Proof: Obvious.

Definition 5.1.8: Let μ be any L-Fuzzy set of a BP-Algebra $(X,*,0)$ and let t $\in$ L. The set $U(\mu,t)=\{x\in X:\mu(x)\geq t\}$ is called a level subset of L-fuzzzy set μ of X.

Lemma 5.1.9: Let $(X,*,0)$ be a BP-algebra. Let L-fuzzy set μ be a L-Fuzzy BP-Subalgbra of X. Then

1. $\mu(0)\geq\mu(x)$ for all x $\in$ X.

Proof:

Since x $*$ x = 0 for all x $\in$ X,

$$\begin{aligned}\mu(0)&=\mu(x*x)\\&\geq\mu(x)\wedge\mu(x)\\&=\mu(x)\end{aligned}$$

$\forall$ x $\in$ X.

Lemma 5.1.10: A L-Fuzzy set μ of a BP-algebra X is a L-Fuzzy BP-Subalgebra if and only if for all t $\in$ L, the level set of $U(\mu,t)$ of a L-fuzzy set μ is either empty or a BP-Subalgebra of X.

Proof: Let a L-fuzzy set μ of a BP-algebra X is a L-fuzzy BP-Subalgebra of X. Let t $\in$ L.

Assume that the level subset of μ in X, U(μ,t) of L-fuzzy set μ in X is not empty.

Then for any x, y $\in$ U(μ,t), μ(x) $\geq$ t, μ(y) $\geq$ t.

Now, μ(x $*$y) $\geq$ {μ(x) $\wedge$ μ(y)} $\geq$ t, which implies x $*$y $\in$ U(μ,t) and hence U(μ,t) is a BP-Subalgebra of X.

Conversely assume that U(μ,t) is a BP-Subalgebra of X.

Take t ={μ(x) $\wedge$ μ(y)} for any x, y $\in$ X.

Implies $x * y \in U(\mu, t)$

Hence $\mu(x * y) \geq t = \{\mu(x) \wedge \mu(y)\}$.

Thus proving that μ is a L-fuzzy BP-Subalgebra of X.

Lemma 5.1.11: (i)Any BP-Subalgebra of a BP-algebra $(X, *, 0)$ can be realized as a level subalgebra of some fuzzy BP-Subalgebra of X.

(ii)Let A be a subset of X. Then the characteristic function X_A is a fuzzy BP-Subalgebra of X if and only if A is a BP-Subalgebra of X.

Proof: The above lemma can be proved as in the case of fuzzy BP-subalgbera.

Theorem 5.1.12: Let μ be a L-fuzzy BP-Subalgebra of $(X, *, 0)$ with finite image. If $U(\mu, s) = U(\mu, t)$ for some $s, t \in Im(\mu)$, $\wedge$ in L then s = t.

Proof: Let μ be a L-fuzzy BP-Subalgebra of X with finite image such that

$U(\mu, s) = U(\mu, t)$ for some $s, t \in Im(\mu)$ in L.

Now, μ is a L-fuzzy algebra of X shows that $U(\mu, s)$ is a BP-subalgebra.

Therefore, if $x, y \in U(\mu, t) = U(\mu, s)$ then $\mu(x) \geq t$ and $\mu(y) \geq t$.

Also, $x, y \in U(\mu, t) = U(\mu, s)$ and $U(\mu, s)$ is a BP-Subalgebra shows that $x * y \in U(\mu, s)$.

This shows that $\mu(x * y) \geq \{\mu(x) \wedge \mu(y)\} \geq s$.

Thus we have, $\mu(x * y) \geq s$ as well as $\mu(x * y) \geq s$ as well as $\mu(x * y) \geq t$.

Whenever $x, y \in U(\mu, t) = U(\mu, s)$.

Similarly, we can prove that, $\mu(x * y) \geq s$ as well as $\mu(x * y) \geq t$

Whenever $x, y \in U(\mu, s) = U(\mu, t)$.

This proves that s = t.

Lemma 5.1.13: Let μ and λ be two L-fuzzy BP-algebras of X with identical family of level BP-Subalgebras. If Im(μ) = $\{t_1, t_2, t_3,,,, t_n\}$ and Im(λ) = $\{s_1, s_2, s_3,,,, s_n\}$ where $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_n$ and $s_1 \geq s_2 \geq s_3 \geq \ldots\ldots \geq s_m$ then

(i) m = n

(ii) U(μ, t_i) = U(λ, s_i) for I = 1,2,,,,,,n.

(iii) If μ (x) = s_i, then λ (x) = s_i, $\forall$ x $\in$X and i = 1,2,,,,,,n.

Proof: Let μ and λ be two L-fuzzy BP-Subalgebras of X with identical family of level BP-Subalgebras F(μ) = F(λ).

Let Im(μ) = $\{t_1, t_2, t_3,,,, t_n\}$ where $t_1 \geq t_2 \geq t_3 \geq \ldots\ldots \geq t_n$ ………………….(1) and

Im(λ) = $\{s_1, s_2, s_3,,,, s_n\}$ where $s_1 \geq s_2 \geq s_3 \geq \ldots.. \geq s_m$ ………………….(2)

Therefore, U(μ, t_1)$\subseteq$U(μ, t_2)$\subseteq$U(μ, t_3)$\subseteq$………$\subseteq$U(μ, t_n) = X…………..(3)

And U(λ, s_1)$\subseteq$U(λ, s_2)$\subseteq$U(λ, s_3)$\subseteq$……...$\subseteq$U(λ, s_m) = X …………(4)

Also F(μ) = {U(μ, t_i): 1$\leq$ i $\leq$n},

F(λ) = {U(λ, s_i): 1$\leq$ j$\leq$ m}

Assume m$\neq$ n.

Then, m$\geq$ n of n$\geq$ m.

Let m$\geq$ n.

Then U(μ, t_i) = U(λ, s_i), i =1, 2,……n.

This shows that both t_i and s_i $\in$ Im(μ).

For i > n we observe that $t_i \notin$ Im(μ) and hence, U(μ, t_i) ≠ U(λ, s_i) where i = n+1,n+2,…m

Let n ≥ m. Then U(μ, t_i)=U(λ, s_i) where i = 1,2,……m.

This shows that both t_i and $s_i \in$ Im(λ).

For j > m we observe that $s_j \notin$ Im(μ) and hence, U(μ, t_i) ≠ U(λ, s_i) where i = m+1,m+2,…..n.

From (3) and (4) implies $t_i \neq s_i$, $\forall$ i =1, 2,……,n.

Hence we can find some i such that U(μ, t_i) ≠ U(λ, s_i).

This contradicts that F(μ) = F(λ).

Hence we conclude that m = n.

Since μ and λ have identical family of level subalgebras, we have U(μ, t_i) = U(λ, s_i), i =1, 2,……n.

Hence (ii) is obtained.

(iii) Follows from (i) and (ii)

Let μ(x) = t_i, implies λ(x) = s_i, for i = 1,2,….n.

Theorem 5.1.14: Let μ and λ be two L-fuzzy subalgebras of X with identical family of level subalgebras. Then Im(μ) = Im(λ) implies $\mu = \lambda$.

Proof: Let μ and λ be two L-fuzzy subalgebras of X with identical family of level subalgebras.

Let Im(μ) = Im(λ) = $\{s_1, s_2, s_3, ,,, s_n\}$, where $s_1 \geq s_2 \geq s_3 \geq \ldots.. \geq s_n$

By lemma [5.1.13] for any

x ∈ X, there exists s_i such that $\mu(x) = s_i = \lambda(x)$

Thus $\mu(x) = \lambda(x) \quad \forall x \in X$, proving that $\mu = \lambda$.

Theorem 5.1.15: Two level BP-Subalgebras $U(\mu, s)$ and $U(\mu, t)$, $(s < t)$ of a L-fuzzy BP-Subalgebra μ are equal if and only if there is no $x \in X$ such that $s \le \mu(x) < t$ where $s, t \in L$.

Proof: Let μ be a L-fuzzy BP-Subalgebra and $s, t \in L$.

Let $U(\mu, s)$ and $U(\mu, t)$ be two level BP-Subalgebras of L-fuzzy BP-Subalgebra μ of X.

Suppose that $U(\mu, s) = U(\mu, t)$ for some $s < t$.

Suppose there is one $x \in X$ such that $s \le \mu(x) < t$.

Then, $\mu(x) \ge s$ and $\mu(x) < t$.

That is, $x \in U(\mu, s)$ and $x \notin U(\mu, t)$.

This contradicts to $U(\mu, s) = U(\mu, t)$.

Conversely, assume that there is no $x \in X$ such that $s \le \mu(x) < t$.

Such that $s \le \mu(x) < t$.

Suppose, $U(\mu, s) \ne U(\mu, t)$.

For, , $x \in U(\mu, t) \Rightarrow \mu(x) \ge t > s$.

$\Rightarrow \mu(x) > s \Rightarrow$, $x \in U(\mu, s)$.

Since $U(\mu, s) \ne U(\mu, t)$.

Choose, $U(\mu, s) \not\subset U(\mu, t)$.

Hence there is an , $x \in U(\mu, s)$ and $x \notin U(\mu, t)$

$\Rightarrow \mu(x) \ge s$ and $\mu(x) < t$.

Thus there exists an element x $\in$X such that s $\leq \mu$(x) $<$ t, thus contradicting our hypothesis.

Hence U(μ,s) = U(μ,t).

Lemma 5.1.14: If μ_1 and μ_2 are fuzzy BP-Subalgebras of X, then $\mu = \mu_1 \times \mu_2$ is a fuzzy BP-Subalgebra of X $\times$X.

Proof: For any (x_1, x_2) and $(y_1, y_2) \in$ X$\times$X we have,

$$\mu[(x_1, x_2)*(y_1, y_2)] = \mu\,(x_1 * y_1,, x_2 * y_2)$$
$$= (\mu_1 \times \mu_2)[x_1 * y_1,\ x_2 * y_2]$$
$$= \{\mu_1(x_1 * y_1) \wedge \mu_2(x_2 * y_2)\}$$
$$\geq \{(\mu_1(x_1) \wedge \mu_1(y_1)) \wedge (\mu_2(x_2) \wedge \mu_2(y_2))\}$$
$$= \{(\mu_1(x_1) \wedge (\mu_2(x_2)) \wedge (\mu_1(y_1) \wedge \mu_2(y_2))\}$$
$$= \{(\mu_1 \times \mu_2)(x_1, x_2) \wedge (\mu_1 \times \mu_2)(y_1, y_2)\}$$
$$= \{\mu(x_1, x_2) \wedge \mu(y_1, y_2)\}$$

Hence $\mu = \mu_1 \times \mu_2$ is a fuzzy BP-Subalgebra of X $\times$X.

Lemma 5.1.15: Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two BP-algebras. Let f : $X_1 \to X_2$ be an epimorphism. If σ is fuzzy BP-Subalgebra of X_2 then $f^{-1}(\sigma)$ is a fuzzy BP-Subalgebra of X_1.

Alternatively, we have epimorphic pre image of a fuzzy BP-Subalgebra is a fuzzy BP-Subalgebra.

Proof: $(f^{-1}(\sigma))$(x $*_1$ y) = σ(f (x $*_1$ y))

$= \sigma$(f(x) $*_2$ f(y)) (since f is an epimorphism)

$\geq \{\sigma$(f(x) $\wedge\ \sigma$(f(y)} (since σ is a fuzzy BP-Subalgebra)

$= [(f^{-1}(\sigma))$(x) $\wedge$ $(f^{-1}(\sigma))$(x))] $\forall$ x, y $\in$X

Thus $f^{-1}(\sigma)$ is a fuzzy BP-Subalgebra of X_1.

Section 5.2:

L-Fuzzy BP-ideals in BP-Algebras

Definition 5.2.1: Let X be a BP-algebra. A L-fuzzy set μ of X is said to be a L-fuzzy BP-ideal of X if it satisfies the following conditions: $\forall$ x, y $\in$ X.

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x) \geq \{\mu(x * y) \wedge \mu(y)\}$.

Example 5.2.2: Let (X = {0,1,2,3}, $*$,0) be a BP-algebra with the following cayley table:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define μ ; X$\rightarrow$[0,1] by

$$\mu(x) = \begin{cases} 1 & if\ x = 0 \\ t_1 & if\ x = 2 \\ t_2 & if\ x = 1 \\ 0 & if\ x = 3 \end{cases}$$

t_1 , t_2 $\in$ L and inf L $\leq t_1 \leq t_2 \leq$ sup L.

Therefore μ is a L-fuzzy BP-ideal of the BP-algebra X.

Proposition 5.2.3: Intersection of two L-fuzzy BP-ideals of X is again a fuzzy BP-ideal of X.

Proof: Let μ and ψ be any two fuzzy BP-ideal of X.

$(\mu \cap \psi)(0) = (\mu \cap \psi)(x * x)$

$\geq \{ \mu (x* x) \wedge \psi (x* x)\}$

$\geq \{\{ \mu (x) \wedge \mu (x)\} \wedge \{ \psi (x) \wedge \psi (x)\}\}$

$= \{\{ \mu (x) \wedge \psi (x)\} \wedge \{ \mu (x) \wedge \psi (x)\}\}$

$= \{(\mu \cap \psi)(x) \wedge (\mu \cap \psi)(x)\}$

$= \{(\mu \cap \psi)(x)\}$

Therefore $(\mu \cap \psi)(0) \geq (\mu \cap \psi)(x)$

$(\mu \cap \psi)(x) = \{ \mu (x) \wedge \psi (x)\}$

$= \{(\mu (x*y) \wedge \mu (y)) \wedge (\psi (x* y) \wedge \psi (y))\}$

$= \{(\mu (x*y) \wedge \psi (x* y)) \wedge (\mu (y) \wedge \psi (y))\}$

$= \{(\mu \cap \psi)(x* y) \wedge (\mu \cap \psi)(y)\}$, for all $x, y \in X$.

Hence $\mu \cap \psi$ is a L-fuzzy BP-ideal of X.

Proposition 5.2.4: If μ is a L-fuzzy BP-ideal of a BP-algebra $(X, *, 0)$, then $\forall\ x, y \in X$.

(iii) μ is order reversing. ie, $x \leq y$ implies $\mu (x) \geq \mu (y)$

(iv) $\mu (x*(x* y)) \geq \mu (y)$.

Proof: Since μ is a L-fuzzy BP-ideal of X.

Let $x \leq y \Rightarrow x* y = 0$

$\Rightarrow \mu (x*y) = \mu (0)$

Therefore $\mu (x*y) = \mu (0) \geq \mu (x)$

$\mu (x) \geq \{ \mu (x*y) \wedge \mu (y)\}$

$\geq \{ \mu (0) \wedge \mu (y)\}$

$\mu (x) \geq \mu (y)$.

By definition of BP-algebra $x*(x*y) = y$.

$(x*(x*y)) * y = y* y$.

$\Rightarrow (x* (x*y)) * y = 0$

$x*(x*y) \leq y$.

By (i) μ is order reversing.

Therefore $\mu(x*(x*y)) \geq \mu(y)\ \forall\ x, y \in X$.

Proposition 5.2.5: If μ is a L-fuzzy ideal of a BP-algebra $(X, *, 0)$ and

$$\mu_\alpha(x) = \{\alpha \wedge \mu(x)\}$$

$\forall x \in X$ and $\alpha \in [0,1]$, then $\mu_\alpha(x)$ is a L-fuzzy BP-ideal of X.

Proof: Let μ be a L-fuzzy ideal of the BP-algebra $(X, *, 0)$ and $\alpha \in [0,1]$.

Therefore $\mu(0) \geq \mu(x)\ \forall\ x \in X$.

Now, $\mu_\alpha(x)(0) = \{\alpha \wedge \mu(0)\}$

$\geq \{\alpha \wedge \mu(x)\}$

$= \mu_\alpha(x)\quad \forall x \in X$.

Also, μ is a L-fuzzy ideal of X shows that

$\mu(x) \geq \{\mu(x*y) \wedge \mu(y)\}\quad \forall\ x, y \in X$.

$\mu_\alpha(x) = \{\alpha \wedge \mu(x)\}$

$\geq \{\alpha \wedge (\mu(x*y) \wedge \mu(y))\}$

$= \{(\alpha \wedge \mu(x*y)) \wedge (\alpha \wedge \mu(y))\}$

$= \{\mu_\alpha(x)(x*y) \wedge \mu_\alpha(x)(y)\}$.

$\Rightarrow \mu_\alpha(x)$ is a L-fuzzy ideal of X.

Since this is true for all $\alpha \in L$, $\mu_\alpha(x)$ is a L-fuzzy BP-ideal of X for all $\alpha \in [0, 1]$.

Corollary 5.2.6: If μ is a L-fuzzy BP-ideal of a BP-algebra X and

$\mu_{\mu(\alpha)}(x) = \{(\mu(\alpha) \wedge \mu(x)\}\ \forall\ \alpha, x \in X$.

Then $\mu_{\mu(\alpha)}$ is a L-fuzzy BP-ideal of X.

Theorem 5.2.7: A L-fuzzy set of a BP-algebra (X, $*$,0) is a L-fuzzy BP-ideal if and only if for any $\lambda \in [0, 1]$, U(μ, λ) = {x : x $\in$ X, μ (x) $\geq \lambda$ } is an ideal of X where U(μ, λ) $\neq \phi$.

Proof: Suppose μ is a L-fuzzy ideal of X and U(μ, λ) $\neq \phi$ for $\lambda \in [0,1]$.

Let x $\in$ U(μ, λ), μ(x) $\geq \lambda$.

By definition of L-fuzzy BP-ideal.

We have $\mu(0) \geq \mu(x) \geq \lambda$.

Thus 0 $\in$U(μ, λ).

Suppose x$*$y $\in$ U(μ, λ) and y $\in$ U(μ, λ).

Therefore, $\mu(x*y) \geq \lambda$ and $\mu(y) \geq \lambda$.

By definition, we have $\mu(x) \geq \{ \mu(x*y) \wedge \mu(y)\} \geq \lambda$.

x $\in$U(μ, λ).

Hence (μ, λ) is an BP-ideal of X.

Conversely, suppose that for each $\lambda \in [0,1]$, U(μ, λ) is either empty or an ideal of X.

For any x $\in$X, let $\mu(x) = \lambda$.

Then x $\in$U(μ, λ).

Since U(μ, λ) $\neq \phi$ is an ideal of X.

We have 0 $\in$U(μ, λ) and hence $\mu(0) \geq \lambda = \mu(x)$.

Thus $\mu(0) \geq \mu(x)$ $\forall$ x$\in$X.

Assume $\mu(x) \geq \{ \mu(x*y) \wedge \mu(y)\}$ $\forall$ x, y $\in$ X is not true.

Then there exists $x_0, y_0 \in X$ such that

$\mu(x_0) \leq \{\mu(x_0 * y_0) \wedge \mu(y_0)\}$

$\Rightarrow \mu(x_0) < \lambda_0 < \{\mu(x_0 * y_0) \wedge \mu(y_0)\}$

We have $x_0 * y_0$, $y_0 \in U(\mu, \lambda_0)$ and $U(\mu, \lambda_0) \neq \phi$.

But $U(\mu, \lambda_0)$ is an ideal of X.

So $x_0 \in U(\mu, \lambda_0)$ by the definition of BP-ideal, $\mu(x_0) \geq \lambda_0$,

contradicting ($\mu(0) \geq \mu(x)$, $\forall x \in X$).

Therefore $\mu(x) \geq \{\mu(x * y) \wedge \mu(y)\}$.

Theorem 5.2.8: A fuzzy set μ of a BP-algebra (X, $*$,0) is a L-fuzzy BP-ideal if and only if every nonempty level subset $U(\mu, s)$ of μ, $s \in Im(\mu)$ is a BP-ideal.

Proof: Let μ be a L-fuzzy BP-ideal.

Claim: $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal.

Since $U(\mu, s) \neq \phi$ there exist $x \in U(\mu, s)$.

Such that $\mu(x) \geq s$.

Since μ is a fuzzy BP-ideal, $\mu(0) \geq \mu(x)$ $\forall x \in X$.

Hence for this $x \in U(\mu, s)$, $\mu(0) \geq s$ which shows that $0 \in U(\mu, s)$.

Now, for any $x, y \in X$.

Assume that $x * y \in U(\mu, s)$ and $y \in U(\mu, s)$.

$x * y \in U(\mu, s) \Rightarrow \mu(x * y) \geq s$

Also $y \in U(\mu, s) \Rightarrow \mu(y) \geq s$

Therefore $\mu(x*y) \geq s$ and $\mu(y) \geq s$.

$\Rightarrow \{\mu(x*y) \wedge \mu(y)\} \geq s$.

Since μ is a L-fuzzy BP-ideal,

$\mu(x) \geq \{\mu(x*y) \wedge \mu(y)\} \geq s$.

This implies $x \in U(\mu, s)$.

This proves that $U(\mu, s)$ is a BP-ideal of X.

Conversely, let $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal of X.

Claim: μ is a L-fuzzy BP-ideal.

Let $x, y \in X$.

For any $s \in Im(\mu)$, let $s = \{\mu(x*y) \wedge \mu(y)\}$.

Therefore, $\mu(x*y) \geq s$ and $\mu(y) \geq s$.

This shows that $x*y, y \in U(\mu, s)$.

Since $U(\mu, s)$ is a BP-ideal we have $x \in U(\mu, s)$.

This proves that $\mu(x) \geq s = \{\mu(x*y) \wedge \mu(y)\}$

This shows that μ is a L-fuzzy BP-ideal of X.

Theorem 5.2.9: Let μ be a L-fuzzy BP-ideal of BP-algebra X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in U(\mu, t)$ but $x \notin U(\mu, s)$, $\forall\ s > t$.

Proof: Obivious.

Theorem 5.2.10: Let X be a BP-algebra. Let λ and μ be the L-fuzzy BP-ideals of X. Then $\lambda \times \mu$ is a L-fuzzy BP-ideal of $X \times X$.

Proof: Let X be a BP-algebra and let λ and μ be the L-fuzzy BP-ideals of X. For any (x, y) $\in$ X$\times$X.

$$(\lambda \times \mu)(0, 0) = \{\lambda(0) \wedge \mu(0)\}$$
$$\geq \{\lambda(x) \wedge \mu(x)\}$$
$$= (\lambda \times \mu)(x).$$

Let (x_1, x_2) and $(y_1, y_2) \in$ X $\times$ X, x = (x_1, x_2) and y = (y_1, y_2).

$$(\lambda \times \mu)(x) = (\lambda \times \mu)(x_1, x_2)$$
$$= \{\lambda(x_1) \wedge \mu(x_2)\}$$
$$\geq \{(\lambda(x_1 * y_1) \wedge \lambda(y_1)) \wedge (\mu(x_2 * y_2) \wedge \mu(y_2))\}$$
$$= \{(\lambda(x_1 * y_1) \wedge \mu(x_2 * y_2)) \wedge \lambda((y_1) \wedge \mu(y_2)\}$$
$$= \{(\lambda \times \mu)(x_1 * y_1 \wedge x_2 * y_2) \wedge (\lambda \times \mu)(y_1, y_2)\}$$
$$= \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \wedge (\lambda \times \mu)(y_1, y_2)\}$$
$$= \{(\lambda \times \mu)(x, y) \wedge (\lambda \times \mu)(y)\}$$

Thus $\lambda \times \mu$ is a L-fuzzy BP-ideal of X $\times$ X.

Theorem 5.2.11: For any two L-fuzzy sets λ and μ of X, if $\lambda \times \mu$ is a L-fuzzy BP-ideal of X, then either λ or μ is a L-fuzzy BP-ideal of X.

Proof: Let λ and μ be L-fuzzy sets of X such that $\lambda \times \mu$ is a L-fuzzy BP-ideal of X.

$(\lambda \times \mu)(0, 0) \geq (\lambda \times \mu)(x, y)$ $\forall$ x, y$\in$X$\times$X.

Assume $\lambda(x) > \lambda(0)$ and $\mu(y) > \mu(0)$ for some x, y$\in$X.

Then $(\lambda \times \mu)(x, y) = \{\lambda(x) \wedge \mu(y)\}$
$$> \{\lambda(0) \wedge \mu(0)\}$$
$$= (\lambda \times \mu)(0) \quad \forall\ x, y \in X \times X.$$

which is contradiction.

Thus $\lambda(x) \geq \lambda(0)$ or $\mu(0) > \mu(x)$, $\forall$ x, y$\in$X.

Let x = (x_1, x_2) and y = $(y_1, y_2) \in$ X$\times$X.

$(\lambda \times \mu)(x) \geq \{(\lambda \times \mu)(x * y) \wedge (\lambda \times \mu)(y)\}$

$= \{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2) \wedge (\lambda \times \mu)(y_1, y_2)\}$

$= \{(\lambda(x_1 * y_1) \wedge \mu(x_2 * y_2)) \wedge (\lambda(y_1) \wedge \mu(y_2))\}$

$\Rightarrow \{\lambda(x_1) \wedge \mu(x_2)\} \geq \{(\lambda(x_1 * y_1) \wedge \lambda(y_1)) \wedge (\mu(x_2 * y_2) \wedge \mu(y_2))\}$.

$\Rightarrow$ either $\lambda(x_1) \geq \{\lambda(x_1 * y_1) \wedge \lambda(y_1)\}$

Or $\mu(x_2) \geq \{\mu(x_1, y_2) \wedge \mu(y_2)\}$

$\Rightarrow$ λ or μ is a L-fuzzy ideal of X.

Theorem 5.2.12: Let λ and μ be L-fuzzy BP-ideals of $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ respectively. Then $\lambda \times \mu$ is a L-fuzzy BP-ideal of $(X_1 \times X_2, *, 0)$.

Proof: Let λ be a L-fuzzy BP-ideal of X_1.

Let μ be a L-fuzzy BP-ideal of X_2.

Claim: $\lambda \times \mu$ is L-fuzzy BP-ideals of $X_1 \times X_2$.

For any (x, y) $\in X_1 \times X_2$.

$(\lambda \times \mu)(0, 0) = \{\lambda(0) \wedge \mu(0)\}$

$\geq \{\lambda(x) \wedge \mu(y)\}$

$= (\lambda \times \mu)(x, y)$.

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$.

$(\lambda \times \mu)(x_1, x_2) = \{\lambda(x_1) \wedge \mu(x_2)\}$

$\geq \{(\lambda(x_1 *_1 y_1) \wedge \lambda(y_1)) \wedge (\mu(x_2 *_2 y_2) \wedge \mu(y_2))\}$

$= \{(\lambda(x_1 *_1 y_1) \wedge \mu(x_2 *_2 y_2)) \wedge (\lambda(y_1) \wedge \mu(y_2))\}$

$= \{(\lambda \times \mu)(x_1 *_1 y_1 \wedge x_2 *_2 y_2) \wedge (\lambda \times \mu)(y_1, y_2)\}$

$= \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \wedge (\lambda \times \mu)(y_1 y_2)\}$

Thus $\lambda \times \mu$ is a L-fuzzy BP-ideal of $X_1 \times X_2$.

Theorem 5.2.13: Inverse image of L-fuzzy BP-ideal is again a L-fuzzy BP-ideal.

Proof: Let f : $(X_1,*_1,0_1) \to (X_2,*_2,0_2)$ be an epimorphism.

Let σ be L-fuzzy BP-ideal of X_2.

To prove: $f^{-1}(\sigma)$ is a L-fuzzy BP-ideal of X_1.

Let x, y $\in X_1$.

$$(f^{-1}(\sigma))(x) = \sigma(f(x))$$
$$\geq \{\sigma(f(x) *_2 f(y)) \wedge \sigma(f(y))\}$$
$$= \{\sigma(f(x *_1 y) \wedge \sigma(f(y))\} \text{ (since f is epimorphism)}$$
$$= \{(f^{-1}(\sigma))(x *_1 y) \wedge (f^{-1}(\sigma))(y)\} \ \forall \ x, y \in X.$$

Thus $f^{-1}(\sigma)$ is a L-fuzzy BP-ideal of X_1.

Theorem 5.2.14: Let f : $\mathbf{(}X_1,*_1,0_1\mathbf{)} \to \mathbf{(}X_2,*_2,0_2\mathbf{)}$ be an epimorphism of BP-algebras. Let μ be a L-fuzzy set of X_2. If $f^{-1}(\mu)$ is a L-fuzzy BP-ideal of X_1, then μ is a L-fuzzy BP-ideal of X_2.

Proof: Let f : $X_1 \to X_2$ be an epimorphism of BP-algebras.

Let μ be a L-fuzzy set of X_2.

Let $f^{-1}(\mu)$ is a L-fuzzy BP-ideal of X_1.

Claim: μ is a L-fuzzy BP-ideal of X_2.

$$\mu(0_2) = \mu(f(0_1)) \geq f^{-1}(\mu(x_1))$$
$$= \mu(f(x_1)) = \mu(x_2).$$

Let x_2, $y_2 \in X_2$. Since f is an epimorphism, x_1, $y_1 \in X_1$ such that f $(x_1) = x_2$ and f$(y_1) = y_2$.

i.e, $x_1 = f^{-1}(x_2)$ and $y_1 = f^{-1}(y_2)$

$\mu(x_2) = \mu(f(x_1))$

$= f^{-1}(\mu(x_1))$

$\geq \{ f^{-1}(\mu(x_1 *_1 y_1)) \wedge f^{-1}(\mu(y_1))\}$

$= \{ \mu(f(x_1 *_1 y_1) \wedge \mu(f(y_1))\}$

$= \{ \mu((f(x_1) *_2 f(y_1)) \wedge \mu(f(y_1))\}$

$= \{ \mu(x_2 *_2 y_2) \wedge \mu(y_2)\}$

Therefore μ is a L-fuzzy BP-ideal of X_2.

L-Fuzzy T-Ideals in BP-algebras

Definition 5.2.15: A L-fuzzy set μ in a BP-algebra X is called a L-fuzzy T-ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x)$

(ii) $\mu(x * z) \geq \{ \mu((x * y) * z) \wedge \mu(y)\}, \ \forall\ x, y, z \in X.$

Theorem 5.2.16: Every L-fuzzy T-ideal μ of a BP-algebra X is order reversing if $x \leq y$ then $\mu(x) \geq \mu(y)\ \forall\ x, y \in X$.

Proof: Let $x, y \in X$ such that $x \leq y$.

Therefore $x * y = 0$

Now, $\mu(x) = \mu(x * 0)$

$\geq \{ \mu((x * y) * 0) \wedge \mu(y)\}$

$= \{ \mu(0 * 0) \wedge \mu(y)\}$

$= \{ \mu(0) \wedge \mu(y)\}$

$= \mu(y)$

Theorem 5.2.17: A L-fuzzy set μ in a BP-algebra X is a L-fuzzy T-ideal if and only if it is a L-fuzzy BP-ideal μ of X.

Proof: Let μ be a L-fuzzy T-ideal of X

$\mu(0) \geq \mu(x)$.

$\mu(x*z) \geq \{\mu((x*y)*z) \wedge \mu(y)\}, \quad \forall \ x, y, z \in X$.

Put z = 0, then

We get $\mu(x) \geq \{\mu(x*y) \wedge \mu(y)\}$

Hence μ is a L-fuzzy BP-ideal of X.

Conversely, μ is a L-fuzzy BP-ideal of X.

Then, $\mu(x*z) \geq \{\mu((x*z)*y) \wedge \mu(y)\}, \quad \forall \ x, y, z \in X$.

$$= \{\mu((x*y)*z) \wedge \mu(y)\}, \quad \forall \ x, y, z \in X,$$

Which proves the result.

Theorem 5.2.18: Let μ be a L-fuzzy set in a BP-algebra X and let $t \in Im(\mu)$. Then μ is a L-fuzzy T-ideal of X if and only if the level subset $\mu t = \{x \in X / \mu(x) \geq t\}$ is a T-ideal of X, which is called a level T-ideal of μ.

Proof: Assume that μ is a L-fuzzy T-ideal of X.

Clearly, $0 \in \mu_t$

Let $(x*y)*z \in \mu_t$ and $y \in \mu_t$

Then $\mu((x*y)*z) \geq t$ and $\mu(y) \geq t$.

Now, $\mu(x*z) \geq \{\mu((x*y)*z) \wedge \mu(y)\}$

$$\geq \{t, t\} = t.$$

Hence μ_t is T-ideal of X.

Conversely, let μ_t be T-ideal of X for any $t \in [0,1]$.

Suppose, assume that there exist some $x_0 \in$ X.

Such that $\mu(0) < \mu(x_0)$.

Take S = ½ [$\mu(0) + \mu(x_0)$]

$\Rightarrow$ S $< \mu(x_0)$ and $0 \le \mu(0) < s < 1$.

$\Rightarrow x_0 \in \mu_s$ and $0 \notin \mu_s$, which is a contradiction, since μ_s is a T-ideal of X.

Therefore, $\mu(0) \ge \mu(x), \forall x \in X$.

Assume that $x_0, y_0, z_0 \in$ X.

Such that,

$\mu(x_0 * z_0) \ge \{\mu((x_0 * y_0) * z_0) \wedge \mu(y_0)\}$

Let s = ½ [$\mu(x_0 * z_0) + \{\mu((x_0 * y_0) * z_0) \wedge \mu(y_0)\}$]

$\Rightarrow$ S $> \mu(x_0 * z_0)$

And S $< \{\mu((x_0 * y_0) * z_0) \wedge \mu(y_0)\}$

$\Rightarrow$ S $> \mu(x_0 * z_0)$, S $< \{\mu((x_0 * y_0) * z_0)\}$ and S $< \mu(y_0)$.

$\Rightarrow x_0 * z_0 \notin \mu_s$, which is a contrsdiction, since μ_s is a T-ideal of X.

Therefore, $\mu(x * z) \ge \{\mu((x * y) * z) \wedge \mu(y)\} \quad \forall$ x, y, z$\in$ X.

Theorem 5.2.19: If μ and λ are L-fuzzy T-ideal in a BP-algebra of X, then $\mu \times \lambda$ is a L-fuzzy T-ideal in X $\times$X.

Proof: For any (x, y) $\in$ X $\times$ X we have,

$(\mu \times \lambda)(0, 0) = \{\mu(0) \wedge \lambda(0)\}$

$\ge \{\mu(x) \wedge \lambda(y)\}$

$= (\mu \times \lambda) (x, y)$

Let (x_1, x_2), (y_1, y_2) and $(z_1, z_2) \in X \times X$.

$(\mu \times \lambda)[(x_1, x_2) * (z_1, z_2)] = (\mu \times \lambda)[(x_1 * z_1, x_2 * z_2)]$

$= \{ \mu(x_1 * z_1) \wedge \lambda(x_2 * z_2)]$

$\geq \{\{ \mu((x_1 * y_1) * z_1) \wedge \mu(y_1)\} \wedge \{ \lambda((x_2 * y_2) * z_2) \wedge \lambda(y_2)\}\}$s

$= \{\{ \mu((x_1 * y_1) * z_1) \wedge \lambda((x_2 * y_2) * z_2)\} \wedge \{ \mu(y_1) \wedge \lambda(y_2)\}\}$

$= \{(\mu \times \lambda)[((x_1 * y_1) * z_1) \wedge ((x_2 * y_2) * z_2)] \wedge (\mu \times \lambda)(y_1, y_2)\}$

$= \{(\mu \times \lambda)[(x_1 * y_1, x_2 * y_2) * (z_1, z_2)] \wedge (\mu \times \lambda)(y_1, y_2)\}$

$= \{(\mu \times \lambda)[((x_1, x_2) * (y_1, y_2)) * (z_1, z_2)] \wedge (\mu \times \lambda)(y_1, y_2)\}$

Hence $\mu \times \lambda$ is a L-fuzzy T-ideal in $X \times X$.

Section 5.3:

Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras

Definition 5.3.1: Let (L, $\leq$) be a complete lattice with least element 0 and greatest element 1 and an involute order reversing operation N : L$\rightarrow$L. Then an intuitionistic L-fuzzy set (ILFS) A in a nonempty set X is defined as an object of the form A = {< x, μ_A(x), ν_A(x) > / x$\in$X} where μ_A:X$\rightarrow$L is the degree of membership an ν_A:X $\rightarrow$ L is the degree of non membership of the element x X satisfying $\mu_A(x) \leq N(\nu_A(x))$.

Definition 5.3.2: An Intuitionistic L-fuzzy set A in a BP-algebra X is said to be an Intuitionistic L-fuzzy BP-Subalgebra of X if

(i) $\mu_A(x * y) \geq \mu_A(x) \wedge \mu_A(y)$

(ii) $\nu_A(x * y) \leq \nu_A(x) \vee \nu_A(y)$ $\forall$ x, y $\in$X.

Example 5.3.3: Consider the BP-algebra X = {0,1, 2, 3} with the following Cayley's table:

*	0	1	2	3
0	0	1	2	3

1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

The Intuitionistic L-fuzzy set A = {< x, μ_A(x), ν_A(x) > / x$\in$X} of X given by

$$\mu_A(\mathrm{x}) = \begin{cases} 0.6 \; if \;\; x \neq 2 \\ 0.1 \; if \;\; x = 2 \end{cases} \quad \text{and}$$

$$\nu_A(\mathrm{x}) = \begin{cases} 0.2 \; if \;\; x \neq 2 \\ 0.8 \; if \;\; x = 2 \end{cases}$$

Is an Intuitionistic L-fuzzy BP-Subalgebra of X.

Lemma: 5.3.4: In an Intuitionistic L-fuzzy BP-subalgebra A of X we have

(i) $\mu_A(0) \geq \mu_A(\mathrm{x})$

(ii) $\nu_A(0) \leq \nu_A(\mathrm{x}) \;\; \forall \, \mathrm{x} \in \mathrm{X}$.

Proof: $\mu_A(0) = \mu_A(\mathrm{x} * \mathrm{x}) \geq \mu_A(\mathrm{x}) \wedge \mu_A(\mathrm{x}) = \mu_A(\mathrm{x})$.

Similarly,

$\nu_A(0) = \nu_A(\mathrm{x} * \mathrm{x}) \leq \nu_A(\mathrm{x}) \vee \nu_A(\mathrm{x}) = \nu_A(\mathrm{x})$.

Theorem 5.3.5: Intersection of any two Intuitionistic L-fuzzy BP-subalgebras of X is again an Intuitionistic L-fuzzy BP-subalgebra of X.

Proof: Let A = {< x, μ_A(x), ν_A(x) > / x$\in$X} and B = {< x, μ_B(x), ν_B(x) > / x$\in$X } be Intuitionistic L-fuzzy BP subalgebras of X and let C = A$\cap$B. Now, for every x, y$\in$X.

$\mu_C(\mathrm{x} * \mathrm{y}) = \mu_{A \cap B}(\mathrm{x} * \mathrm{y})$

$= \mu_A(\mathrm{x} * \mathrm{y}) \wedge \mu_B(\mathrm{x} * \mathrm{y})$

$\geq (\mu_A(\mathrm{x}) \wedge \mu_A(\mathrm{y})) \wedge (\mu_B(\mathrm{x}) \wedge \mu_B(\mathrm{y}))$

$= (\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y))$

$= \mu_C(x) \wedge \mu_C(y)$

$\nu_C(x * y) = \nu_{A \cap B}(x * y)$

$= \nu_A(x * y) \vee \nu_B(x * y)$

$\leq (\nu_A(x) \vee \nu_A(y)) \vee (\nu_B(x) \vee \nu_B(y))$

$= (\nu_A(x) \vee \nu_B(x)) \vee (\nu_A(y) \vee \nu_B(y))$

$= \nu_C(x) \vee \nu_C(y)$

Hence, C is an Intuitionistic L-fuzzy BP-algebra of X.

We can generalize the above result as follows:

Corollary 5.3.6: Intersection of any family Intuitionistic L-fuzzy BP-subalgebras of X is again an Intuitionistic L-fuzzy BP-subalgebra of X.

Theorem 5.3.7: If A is an Intuitionistic L-fuzzy BP-subalgebra of X, then $\overline{A}$ is an Intuitionistic L-fuzzy BP-subalgebra of X.

Proof: Obvious.

Theorem 5.3.8: An Intuitionistic L-fuzzy set A of X is an Intuitionistic L-fuzzy BP-Subalgebra of X if and only if the L-fuzzzy sets μ_A and $\overline{\nu}_A$ = 1- ν_A are L-fuzzy BP-subalgebras of X.

Proof: Let A ={< x, $\mu_A(x)$, $\nu_A(x)$ > / x $\in$ X } be an Intuitionistic L-fuzzy BP-subalgebra of X. Clearly, μ_A is a L-fuzzzy BP-subalgebra of X. For all x, y $\in$ X,

$\overline{\nu}_A(x * y) = 1 - \nu_A(x * y)$

$\geq 1 - [\nu_A(x) \vee \nu_A(y)]$

$= \{(1 - \nu_A(x)) \wedge (1 - \nu_A(y))\}$

$= \overline{\nu}_A(x) \wedge \overline{\nu}_A(y)$

This proves that $\overline{\nu}_A$ is a L-fuzzy BP subalgebra of X.

Conversely, assume μ_A and $\overline{\nu}_A$ are L-fuzzy BP-subalgebras of X.

Then we have, $\mu_A(x * y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\overline{\nu}_A(x * y) \geq \overline{\nu}_A(x) \wedge \overline{\nu}_A(y)$ $\forall$ x, y $\in$ X.

Hence, to prove that A = {< x, μ_A(x), ν_A(x) > / x$\in$X} is an Intuitionistic L-fuzzy BP-subalgebra of X, it is enough to prove that $\nu_A(x*y) \leq \nu_A(x) \vee \nu_A(y)$ $\forall$ x, y$\in$X.

Since $\overline{\nu}_A$ is L-fuzzy subalgebra of X,

$\overline{\nu}_A(x*y) \geq \overline{\nu}(x) \vee \overline{\nu}(y)$

$1-\nu_A(x*y) \geq (1-\nu_A(x)) \wedge (1-\nu_A(y))$

$= 1-[\nu_A(x) \vee \nu_A(y)]$

That is, $\nu_A(x*y) \leq \nu_A(x) \vee \nu_A(y)$ $\forall$ x, y$\in$X. This completes the proof.

Intuitionistic L-fuzzy BP-ideals

Definition 5.3.9: An Intuitionistic L-fuzzy set A of a BP-algebra X is said to be an Intuitionistic L-fuzzy BP-ideal of X if,

(i) $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$

(ii) $\mu_A(x) \geq \mu_A(x*y) \wedge \mu_A(y)$

(iii) $\nu_A(x) \leq \nu_A(x*y) \vee \nu_A(y)$ for all x, y $\in$ X.

Example 5.3.10: Consider the BP-Algebra X = {0, 1, 2, 3} in the example of fuzzy BP-Subalgebra. Then A = (μ_A, ν_A) defined as follows is an Intuitionistic L-Fuzzy BP-Ideal of X. $\mu_A(0) = \mu_A(2) = \mu_A(1) = \mu_A(3)$ = t; and $\nu_A(0) = \nu_A(2) = 0$; $\nu_A(1) = \nu_A(3) = s$ where t, s$\in$[0, 1] and t + s $\leq$ 1.

Lemma 5.3.11: Let A = (μ_A, ν_A) in X be an Intuitionistic L-fuzzy BP-Ideal of X.

If $x*y \leq z$ then,

(i) $\mu_A(x) \geq \{\mu_A(x*y) \wedge \mu_A(y)\}$

(ii) $\nu_A(x) \leq \{\nu_A(y) \vee \nu_A(z)\}$

Proof: Let x, y z$\in$X such that $x*y \leq z$.

Then $(x*y)*z = 0$ and thus

$\mu_A(x) \geq \{\mu_A(x*y) \wedge \mu_A(y)\}$

$\geq \{\{\mu_A((x*y)*z) \wedge \mu_A(z)\}, \mu_A(y)\}$

$=\{\{\mu_A(0) \wedge \mu_A(z)\} \wedge \mu_A(y)\}$

$=\{\mu_A(y) \wedge \mu_A(z)\}$

Similarly, $\nu_A(x) \leq \{\nu_A(y) \vee \nu_A(z)\}$.

Lemma 5.3.12: Let A = (μ_A, ν_A) in X be an Intuitionistic L-fuzzy BP-ideal of X.

If x $\leq$ y then:

$\mu_A(x) \geq \mu_A(y)$, $\nu_A(x) \leq \nu_A(y)$ that is, μ_A is order reversing and ν_A is order preserving.

Proof: Let x, y, z $\in$ X be such that $x*y = 0$.

$\mu_A(x) \geq \{\mu_A(x*y) \wedge \mu_A(y)\}$

$=\{\mu_A(0) \wedge \mu_A(y)\}$

$=\mu_A(y)$

$\nu_A(x) \leq \{\nu_A(x*y) \vee \nu_A(y)\}$

$=\{\nu_A(0) \vee \nu_A(y)\}$

$=\nu_A(y)$

Definition 5.3.13: Let f : X$\rightarrow$Y be a homomorphism of BP-algebras. For any Intuitionistic L-Fuzzy Set A=(μ_A, ν_A) in Y, we define a new Intuitionistic L-Fuzzy Set by $A^f = (\mu_A{}^f, \nu_A{}^f)$ in X by $\mu_A{}^f(x) = \mu_A(f(x))$, $\nu_A{}^f = \nu_A(f(x))$ for all x, y$\in$X.

Theorem 5.3.14: Let f : X$\rightarrow$Y be a homomorphism of BP-algebras. If an Intuitionistic L-Fuzzy set A = (μ_A, ν_A) is an Intuitionistic L-fuzzy BP-ideal of Y, then an Intuitionistic L-Fuzzy Set $A^f = (\mu_A^f, \nu_A^f)$ in X is an Intuitionistic L-fuzzy BP-ideal of X.

Proof: $\mu_A{}^f(x) = \mu_A(f(x)) \leq \mu_A(0') = \mu_A(f(0)) = \mu_A{}^f(0)$

$\nu_A{}^f(x) = \nu_A(f(x)) \geq \nu_A(0') = \nu_A(f(0)) = \nu_A{}^f(0)\ \forall x \in X.$

Let x, y$\in$X. Then

$\{\mu_A^f(x*y) \wedge \mu_A^f(y)\} = \{\mu_A(f(x*y)) \wedge \mu_A(f(y))\}$

$= \{\mu_A(f(x)*' f(y)) \wedge \mu_A(f(y))\}$

$\leq \mu_A(f(x))$

$= \mu_A^f(x)$

$\{\nu_A^f(x*y) \vee \nu_A^f(y)\} = \{\nu_A(f(x*y)) \vee \nu_A(f(y))\}$

$= \{\nu_A(f(x)*' f(y)) \vee \nu_A(f(y))\}$

$\geq \nu_A(f(x))$

$= \nu_A^f(x)$

Hence $A^f = (\mu_A^f, \nu_A^f)$ is an Intuitionistic L-fuzzy BP-ideal of X.

Theorem 5.3.15: Let f : X→Y be an epimorphism of BP-algebra and let A = (μ_A, ν_A) be an Intuitionistic L-Fuzzy Set in Y. If $A^f = (\mu_A^f, \nu_A^f)$ is an Intuitionistic L-fuzzy BP-ideal of X. Then A = (μ_A, ν_A) is an Intuitionistic L-Fuzzy BP-Ideal of Y.

Proof: For any $x \in Y$ there exist $a \in X$ such that f(a) = x.

Then,

$\mu_A(x) = \mu_A(f(a)) = \mu_A^f(a) \leq \mu_A^f(0) = \mu_A(f(0)) = \mu_A(0')$

$\nu_A(x) = \nu_A(f(a)) = \nu_A^f(a) \geq \nu_A^f(0) = \nu_A(f(0)) = \nu_A(0')$

Let x, $y \in Y$. Then f(a) = x and f(b) = y for some a, $b \in X$.

Thus

$\mu_A(x) = \mu_A(f(a)) = \mu_A^f(a)$

$\geq \{\mu_A^f(a*b) \wedge \mu_A^f(b)\}$

$= \{\mu_A(f(a*b) \wedge \mu_A(f(b))\}$

$= \{\mu_A(f(a) *' f(b)) \wedge \mu_A(f(b))\}$

$= \{\mu_A(x*' y) \wedge \mu_A(y)\}$

$\nu_A(x) = \nu_A(f(a)) = \nu_A^f(a)$

$\leq \{\nu_A^f(a*b) \vee \nu_A^f(b)\}$

$= \{\nu_A(f(a * b) \vee \nu_A(f(b))\}$

$= \{\nu_A(f(a) *' f(b)) \vee \nu_A(f(b))\}$

$= \{\nu_A(x *' y) \vee \nu_A(y)\}$

Hence A = (μ_A, ν_A) is an Intuitionistic L-Fuzzy BP-Ideal of Y.

SUMMARY AND CONCLUSION

In 1966, BCK and BCI algebras are two classes of algebras based on propositional calculi or logic introduced by Imai and Iseki [11]. In 1978, Iseki and Tanaka [12] introduced the theory of BCK-Algebras. In 2013, Ahn and Han [1] introduced the notion of BP-Algebras.

In 1965, Zadeh [35] introduced the notion of fuzzy sets. In 1967, Goguen [9] extended the notion of fuzzy sets into L-Fuzzy sets where L is a complete lattice. In 1971, Rosenfeld [29] initiated the study of fuzzy algebraic structures.

Xi [34] applied the concept of fuzzy sets to BCK-Algebras and got some results in 1991. In 1986, Atanassov [2] generalized the concept of fuzzy sets into Intuitionistic fuzzy sets.

In this thesis, we have made an attempt to study the properties of fuzzy BP-Subalgebras and Fuzzy BP-Ideals and Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras.

In Chapter 1, the preliminaries of BP-Algebras and quadratic BP-Algebras are presented. Also the relations between BP-Algebra and several other algebras are discussed due to Ahn and Han [1].

In Chapter 2, f-derivations, regular f-derivations, Composition of f-derivations on BP-Algebras and their properties are discussed due to Kandaraj and Arul Devi [16].

In Chapter 3, properties of Fuzzy BP-Subalgebras are discussed and obtained some results due to Christopher Jefferson and Chandramouleeswaran [3].

In Chapter 4, The properties of Fuzzy BP-Ideals, Fuzzy T-Ideals, Cartesian product of Fuzzy T-Ideals in BP-Algebras are discussed due to Christopher Jefferson and Chandramouleeswaran [4, 5]. Also the notion of Intuitionistic Fuzzy BP-Subalgebras and Intuitionistic fuzzy BP-Ideals in BP-Algebras are introduced and studied their properties due to Shalini and Jeyalakshmi [30].

In Chapter 5, the properties of L-Fuzzy BP-Subalgebras, L-Fuzzy BP-Ideals, L-Fuzzy T-ideals and Intuitionistic L-Fuzzy BP-Ideals in BP-Algebras are established due to Christopher Jefferson and Chandramouleeswaran [6, 7, 8].

A deep study of Fuzzy BP-Subalgebras and Fuzzy BP-Ideals in BP-Algebras can be extended to different types of algebras. So it provides a lot of scope for further research.

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