CHAPTER - II

CHAPTER -II

ON GENERALIZED b-CLOSED SETS AND THEIR RELATIONSHIP

In this chapter the relationship between gb-closed sets with the other forms of closed sets furnished by Adea Khaliefa Hussein [2] was discussed. Several equivalence relationships between gb-closed sets and other forms of closed sets were analyzed. The chapter was concluded with the equivalence relationships between the already existing closed sets, extremely disconnected spaces, T_{gs}-spaces and sg-submaximal spaces.

Section 2.1

Preliminaries

Definition 2.1.1

A subset A of a topological space (X, τ) is called an **generalized α-closed set** (briefly, gα-closed) [34] if αcl (A) ⊂ U Whenever A ⊂ U and U is open.

The complement of gα-closed set is called an **gα-open set**.

Definition 2.1.2

A point p of a subset A in a topological space X is said to be a **condensation point** if every open neighborhood of p contains uncountably many points of A.

If every point in A is a condensation point then A is called as the **condense set**.

Definition 2.1.3

A topological space (X, τ) is said to be **sg-submaximal** [29] if every condense subset of X is a sg-closed set.

Definition 2.1.4

A topological space (X, τ) is said to be **resolvable** [29] if X is the union of two disjoint dense subsets.

Definition 2.1.5

A closed set A of a topological space (X, τ) is said to be **nowhere dense** if it has no interior points or equivalently if the exterior of A is dense in X.

Theorem 2.1.6

For a space X the following are equivalent

(1) Every semi-closed subset of X is α-closed

(2) Every semi-closed subset of X is gα-closed

Section 2.2

gb-closed sets and their relationship

In this section several equivalence relationships between gb-closed sets with other types of closed sets, extremely disconnected spaces , T_{gs}-spaces and sg-submaximal spaces were discussed.

Theorem 2.2.1

Let (X, τ) be any topological space then the following are equivalent:

(1) Every gb-closed set is gp-closed.

(2) Every b-closed set is gp-closed.

Proof

(1) $\Rightarrow$ (2)

Let A be a b-closed set such that, A $\subset$ U where U is open in X.

Since A is b-closed, bcl(A) = A $\subset$ U $\Rightarrow$ A is gb-closed

$\Rightarrow$ A is gp-closed (by our assumption).

(2) $\Rightarrow$ (1)

Let A be a gb-closed set such that, A $\subset$ U where U is open in X.

Since A is gb-closed, bcl(A) $\subset$ U. bcl(A) is b-closed then by our assumption bcl(A) is gp-closed.

Therefore pcl(A) $\subset$ pcl(bcl(A)) $\subset$ U

$\Rightarrow$ A is gp-closed (by our assumption).

Theorem 2.2.2

Let (X, τ) be any topological space then the following are equivalent:

(1) Every gsp-closed set is gp-closed.

(2) Every sp-closed set is gp-closed.

(3) Every gs-closed set is gp-closed.

(4) Every sg-closed set is gp-closed.

(5) X is extremely disconnected.

Proof

(1)$\Rightarrow$ (2)

Suppose every gsp-closed set is gp-closed.

Let $A \subset (X, \tau)$ be sp-closed. Since every sp-closed set is gp-closed, A is gp-closed.

(2)$\Longrightarrow$ (1)

Suppose every sp-closed set is gp-closed.

Let $A \subset (X, \tau)$ be gsp-closed and $A \subset U$ where U is open in X.

Then since A is gsp-closed, $spcl(A) \subset U$.

spcl(A) is sp-closed and hence is gp-closed thus, $pcl(spcl(A)) \subset U$

$\Longrightarrow pcl(A) \subset pcl(spcl(A)) \subset U \Longrightarrow$ A is gp-closed.

(2)$\Longrightarrow$ (3)

Suppose every sp-closed set is gp-closed.

Let $A \subset (X, \tau)$ be gs-closed and $A \subset U$ where U is open in X.

Then since A is gs-closed, $scl(A) \subset U$.

scl(A) is semi-closed and hence sp- closed therefore by our hypothesis scl(A) is gp-closed.

$\Longrightarrow pcl(A) \subset pcl(scl(A)) \subset U \Longrightarrow$ A is gp-closed

(3)$\Longrightarrow$ (4)

Suppose every gs-closed set is gp-closed.

Let $A \subset (X, \tau)$ be sg-closed and every sg-closed set is gs-closed, therefore is gp-closed

$\Longrightarrow$ A is gp-closed

(4)$\Longrightarrow$ (3)

Suppose every sg-closed set is gp-closed.

Let $A \subset (X, \tau)$ be gs-closed and $A \subset U$ where U is open in X.

Then since A is gs-closed, $scl(A) \subset U$.

Every open set is semi-open

Thus $scl(A) \subset U$ whenever $A \subset U$ and U is semi-open .

$\Longrightarrow$ A is sg-closed $\Longrightarrow$ A is gp-closed

(2)$\Longrightarrow$ (5)

Suppose every sp-closed set is gp-closed.

We need to show that X is extremely disconnected. It is enough to show that every gb-closed set is gp-closed.

Let $A \subset (X, \tau)$ be gb-closed and $A \subset U$ where U is open in X.

Then since A is gb-closed,bcl(A) $\subset$ U.

bcl(A) is b-closed and hence sp- closed therefore by our hypothesis bcl(A) is gp-closed.

$\Rightarrow$ pcl(A) $\subset$ pcl(scl(A)) $\subset$ U $\Rightarrow$ A is gp-closed

Theorem 2.2.3

Let (X, τ) be any topological space then the following are equivalent:

(1) Every sp-closed set is pre-closed.

(2) Every gb-closed set is gp-closed.

(3) Every b-closed set is gp-closed.

(4) X is extremely disconnected.

Proof

(1)$\Rightarrow$ (2)

Suppose every sp-closed set is pre-closed.

Let A $\subset$ (X, τ) be gb-closed and A $\subset$ U where U is open in X.

Then since A is gb-closed, bcl(A) $\subset$ U.

Every b-closed set is sp-closed, therefore by our hypothesis bcl(A) is pre-closed.

$\Rightarrow$ pcl(A) $\subset$ pcl(bcl(A)) = bcl(A) $\subset$ U $\Rightarrow$ A is gp-closed

(2)$\Rightarrow$ (3)

Suppose every gb-closed set is gp-closed.

Let A be a b-closed set in X then A is gb-closed and hence is gp-closed.

(3)$\Rightarrow$ (2)

Suppose every b-closed set is gp-closed.

Let A $\subset$ (X, τ) be gb-closed and A $\subset$ U where U is open in X.

Then since A is gb-closed, bcl(A) $\subset$ U.

bcl(A) is b-closed and hence gp- closed therefore by our hypothesis bcl(A) is gp-closed.

$\Rightarrow$ pcl(A) $\subset$ pcl(bcl(A)) $\subset$ U $\Rightarrow$ A is gp-closed.

(3)$\Rightarrow$ (4)

Suppose every b-closed set is gp-closed we need to show that X is extremely disconnected.

Let A be regular open.

bcl(A) = A $\cup$ (int(cl(A)) $\cap$ cl(int(A)))

= A $\cup$ (A $\cap$ cl(int(A))) $\subset$ A

$\Rightarrow$ bcl(A) = A $\Rightarrow$ A is b-closed $\Rightarrow$ A is gp-closed.

Since every regular open set is semi-open, A is semi-open.

Thus cl(A) = pcl(A) $\subset$ A.

Hence cl(A) $\subset$ A $\Rightarrow$ A is closed.

Therefore X is extremely disconnected.

(4)$\Rightarrow$ (3)

Suppose (X, τ) is extremely disconnected.

Let A $\subset$ (X, τ) be b-closed and A $\subset$ U where U is open in X.

Then since A is b-closed, bcl(A) = A $\subset$ U.

Now consider bcl(A) = A $\cup$ (int(cl(A)) $\cap$ cl(int(A))) $\subset$ U

Since int(cl(A)) is closed,

cl(int(A)) $\subset$ cl(int(cl(A)) $\cap$ cl(int(A))) $\subset$ U

Thus cl(int(A)) $\subset$ U $\Rightarrow$ pcl(A) $\subset$ A $\cup$ (cl(int (A))) $\subset$ U $\Rightarrow$ A is gp-closed in X.

Theorem 2.2.4

Let (X, τ) be any topological space then the following are equivalent:

(1) Every gb-closed set is b-closed.
(2) Every gs-closed set is b-closed.
(3) Every sg-closed set is b-closed.
(4) X is T_{gs}

Proof

(1)$\Rightarrow$ (2)

Suppose every gb-closed set is b-closed.

Let A be gs-closed set $\Rightarrow$ A is gb-closed $\Rightarrow$ A is b-closed.

(2)$\Rightarrow$ (3)

Suppose every gs-closed set is b-closed.

Let A be sg-closed set $\Rightarrow$ A is gs-closed $\Rightarrow$ A is b-closed.

(3)$\Rightarrow$ (4)

Suppose every sg-closed set is b-closed. We need to show that X is a T_{gs} –space for that it is

enough to show that every singleton is either pre-open or closed.

Since (X, τ) being a topological space every singleton is either pre-open or nowhere dense.

Let x $\in$ X, if {x} is pre-open, then we are done. Suppose {x} is nowhere dense thus int(cl({x})) = φ and not closed. Also X - {x} is not open.

Hence X is the only open set containing X – {x} $\Longrightarrow$ X - {x} is sg-closed.

By our assumption X - {x} is b-closed $\Longrightarrow$ {x} is b-open

$\Longrightarrow$ {x} $\subset$ int(cl({x})) $\cup$ cl(int({x})) = φ, since {x} is nowhere dense.

Thus {x} is closed and X is T_{gs}.

(4)$\Longrightarrow$ (1)

Suppose X is T_{gs}

Let A be a gb-closed in X we need to show that A is b-closed it is enough to show that bcl(A) = A

Let x $\in$ bcl(A) , assume x $\notin$ A $\Longrightarrow$ A $\subset$ X – {x}. Since X being a T_{gs}-space every singleton is

either pre-open or closed.

Suppose {x} is closed then X – {x} is open and A is gb-closed

Hence bcl(A) $\subset$ X – {x}, not possible since x $\in$ bcl(A).

For if {x} is pre-open then X - {x} is pre-closed and every pre-closed set is b-closed $\Longrightarrow$ X - {x}

is b-closed . Thus bcl(A) $\subset$ X - {x}, which is not possible.

Therefore bcl(A) = A $\Longrightarrow$ A is b-closed.

Theorem 2.2.5

Let (X, τ) be any topological space then the following are equivalent:

(1) Every gp-closed set is pre-closed.

(2) Every gsp-closed set is sp-closed.

(3) Every gp-closed set is sp-closed.

(4) X is T_{gs}

(5) Every gb-closed set is sp-closed.

Proof

(4)$\Longrightarrow$ (2)

Suppose X is T_{gs} ,

Let A be a gsp-closed set in we need to show that A is sp-closed it is enough to show that spcl(A) = A.

Let x $\in$ spcl(A) and assume x $\notin$ A, then A $\subset$ X – {x}.

Since X is T_{gs} , {x} is either pre-open or closed.

If {x} is pre-open then X – {x} is pre-closed and every pre-closed set is semipre-closed, thus

spcl(A) $\subset$ spcl (X – {x}) = X – {x}, a contradiction.

Therefore, {x} $\in$ A$\Longrightarrow$ A is sp-closed.

(2)$\Longrightarrow$ (3)

Suppose every gsp-closed set is sp-closed.

Let A be gp-closed $\Longrightarrow$ A is gsp-closed $\Longrightarrow$ A is sp-closed.

(3)$\Longrightarrow$ (4)

Suppose every gp-closed set is sp-closed. We need to show that X is a T_{gs} –space for that it is enough to show that every singleton is either pre-open or closed.

Since (X, τ) being a topological space every singleton is either pre-open or nowhere dense.

Let x $\in$ X, if {x} is pre-open, then we are done. Suppose {x} is nowhere dense

thus int(cl({x})) = φ and not closed. Also X - {x} is not open.

Hence X is the only open set containing X – {x} $\Longrightarrow$ X - {x} is gp-closed.

By our assumption X - {x} is sp-closed $\Longrightarrow$ {x} is sp-open

$\Longrightarrow$ {x} $\subset$ cl(int(cl({x}))) , a contradiction.

Thus {x} is closed and X is T_{gs}.

Theorem 2.2.6

Let (X, τ) be any topological space then the following are equivalent:

(1) Every b-closed set is sg-closed.

(2) Every b-closed set is gs-closed.

(3) Every gb-closed set is gs-closed.

(4) Every gp-closed set is gs-closed.

(5) Every pre-closed set is gs-closed.

(6) Every pre-closed set is sg-closed.

(7) X is sg-submaximal.

Proof

(1)$\Longrightarrow$ (2)

Suppose every b-closed set is sg-closed.

Let A be b-closed $\Longrightarrow$ A is sg-closed $\Longrightarrow$ A is gs-closed.

(2)$\Longrightarrow$ (3)

Let A be gb-closed such that $A \subset U$, U is open.

Then $bcl(A) \subset U$, bcl(A) is b-closed and hence is gs-closed. Thus,

$scl(A) \subset scl(bcl(A)) \subset U \Longrightarrow$ A is gs-closed.

(3)$\Longrightarrow$ (4)

Suppose every gb-closed set is gs-closed.

Let A be gp-closed $\Longrightarrow$ A is gb-closed $\Longrightarrow$ A is gs-closed.

Similarly we can prove the other implications.

Theorem 2.2.7

If every gp-closed set of a topological space (X, τ) is sg-closed then X is T_{gs}.

Theorem 2.2.8

For a space (X, τ) the following are equivalent.

(1) Every gp-closed set is sg-closed.

(2) X is T_{gs} and sg-submaximal.

Proof

(1) $\Longrightarrow$(2)

Let A be gp-closed subset of X then by (1)

A is sg-closed $\Longrightarrow$ A is gs-closed.

Thus every gp-closed set is sg-closed $\Longrightarrow$ X is T_{gs}.

Also every gp-closed set is gs-closed $\Longrightarrow$ X is sg-submaximal.

(2) $\Longrightarrow$(1)

Since X is T_{gs}, every gp-closed set is pre-closed and since X is sg-submaximal, every pre-closed set is sg-closed $\Longrightarrow$ every gp-closed set is sg-closed.

Theorem 2.2.9

For a space (X, τ) the following are equivalent.

(1) Every gsp-closed set is pre-closed.

(2) Every gb-closed set is pre-closed

(3) X is T_{gs} and extremely disconnected.

Proof

(1) ⟹(2)

Let A be gb-closed set then A is gsp-closed again by (1) A is pre-closed.

(2) ⟹(3)

Let A be gb-closed set then A is gsp-closed again by (1) A is pre-closed.

Thus every gb-closed set is gp-closed ⟹ X is extremely disconnected.

Also every gb-closed set is pre-closed ⟹ every gb-closed set is semipre-closed ⟹ X is T_{gs}.

(3) ⟹(1)

Let A be gsp-closed then X is T_{gs} every gsp-closed set is sp-closed.

Again since X is extremely disconnected every sp-closed set pre-closed.

Theorem 2.2.10

Every sp-closed set is b-closed if and only if cl(W) is open for every open solvable subspace W of X.

Theorem 2.2.11

Every sg-closed set in a space (X, τ) is b-closed.

Theorem 2.2.12

For any space (X, τ) the following are equivalent.

(1) Every sp-closed set sg-closed.

(2) X is sg-submaximal and cl(W) is open for every open solvable subspace W of X.

Proof

(1) ⟹(2)

Let A sp-closed ⟹ A is sg-closed.

Then by theorem 2.2.11 A is b-closed.

Therefore by theorem 2.2.10 cl(W) is open for every open solvable subspace W of X

Now let A be a b-closed set. Hence A is sp-closed so by (1) A is sg-closed.

Thus every b-closed set is sg-closed ⟹ X is sg-submaximal.

(2) ⟹(1)

Let A be sp-closed.

Then by (2) cl(W) is open for every open solvable subspace W of X $\Longrightarrow$ A is b-closed. Again since X is sg-submaximal every b-closed set is sg-closed $\Longrightarrow$ Every sp-closed set is sg-closed.