

CHAPTER - III

CHAPTER III

FUZZY SOFT MAPPINGS

Definition 3.1

Let (U, E) and (U^*, E^*) be classes of fuzzy soft sets over U and U^* with attributes from E and E^* respectively. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings. Then a **fuzzy soft mapping** $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ would be defined as follows.

For a fuzzy soft set F_A in (U, E) , $f(F_A)$ is a fuzzy soft set in (U^*, E^*) obtained as follows : for $\beta \in \psi(E) \subseteq E^*$ and $y \in E^*$,

$$f(F_A)(\beta)(y) = \begin{cases} \bigvee_{x \in \rho^{-1}(y)} \left(\bigvee_{\alpha \in \psi^{-1}(\beta)} F_A(\alpha) \right) (x), & \text{if } \rho^{-1}(y) \neq \phi, \psi^{-1}(\beta) \neq \phi. \\ 0, & \text{otherwise} \end{cases}$$

$f(F_A)$ is called fuzzy soft set image of the fuzzy soft set F_A .

Example 3.2

Let $U = \{a, b, c\}$ and $U^* = \{x, y, z\}$, $E = \{e_1, e_2, e_3, e_4\}$, $E^* = \{e'_1, e'_2, e'_3\}$ and (U, E) , (U^*, E^*) classes of fuzzy soft sets.

Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings defined as $\rho(a) = z$, $\rho(b) = y$, $\rho(c) = y$ and $\psi(e_1) = e'_1$, $\psi(e_2) = e'_1$, $\psi(e_3) = e'_3$, $\psi(e_4) = e'_2$.

Let $A = \{e_1, e_2, e_4\}$. Let us consider a fuzzy soft set F_A in (U, E) as

$$F_A = \{F(e_1) = \{(a, 0.5), (b, 0), (c, 0.8)\}$$

$$F(e_2) = \{(a, 0.1), (b, 0.9), (c, 0.5)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$F(e_4) = \{(a, 0.4), (b, 0.3), (c, 0.6)\}$$

Now,

$$\begin{aligned} f(F_A)(e'_1)(x) &= V_{s \in \rho^{-1}(x)} \left(V_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\ &= 0, \quad \text{Since } \rho^{-1}(x) = \varnothing. \end{aligned}$$

$$\begin{aligned} f(F_A)(e'_1)(y) &= V_{s \in \rho^{-1}(y)} \left(V_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\ &= V_{s \in \{b,c\}} \left(V_{\alpha \in \{e_1, e_2\}} F_A(\alpha) \right) (s) \\ &= V_{s \in \{b,c\}} \left((F_A(e_1) \vee F_A(e_2)) \right) (s) \\ &= V_{s \in \{b,c\}} \left(\{(a, 0.5), (b, 0.9), (c, 0.8)\} \right) (s) \\ &= V(0.9, 0.8) \\ &= 0.9 \end{aligned}$$

$$\begin{aligned} f(F_A)(e'_1)(z) &= V_{s \in \rho^{-1}(z)} \left(V_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\ &= V_{s \in \{a\}} \left(V_{\alpha \in \{e_1, e_2\}} F_A(\alpha) \right) (s) \\ &= V_{s \in \{a\}} \left((F_A(e_1) \vee F_A(e_2)) \right) (s) \\ &= V_{s \in \{a\}} \left(\{(a, 0.5), (b, 0.9), (c, 0.8)\} \right) (s) \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} f(F_A)(e'_2)(x) &= V_{s \in \rho^{-1}(x)} \left(V_{\alpha \in \psi^{-1}(e'_2)} F_A(\alpha) \right) (s) \\ &= 0, \text{ since } \rho^{-1}(x) = \varnothing \end{aligned}$$

$$\begin{aligned} f(F_A)(e'_1)(y) &= V_{s \in \rho^{-1}(y)} \left(V_{\alpha \in \psi^{-1}(e'_2)} F_A(\alpha) \right) (s) \\ &= V_{s \in \{b,c\}} \left(V_{\alpha \in \{e_4\}} F_A(\alpha) \right) (s) \\ &= V_{s \in \{b,c\}} \left(F_A(e_4) \right) (s) \\ &= V_{s \in \{b,c\}} \left(\{(a, 0.4), (b, 0.3), (c, 0.6)\} \right) (s) \\ &= V(0.3, 0.6) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_1)(z) &= \bigvee_{s \in \rho^{-1}(z)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} \left(\bigvee_{\alpha \in \{e_4\}} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} (F_A(e_1)) (s) \\
&= \bigvee_{s \in \{a\}} (\{(a, 0.4), (b, 0.3), (c, 0.6)\}) (s) \\
&= 0.4
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_3)(x) &= \bigvee_{s \in \rho^{-1}(x)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_3)} F_A(\alpha) \right) (s) \\
&= 0, \quad \text{Since } \rho^{-1}(x) = \emptyset.
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_3)(y) &= \bigvee_{s \in \rho^{-1}(y)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_3)} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{b,c\}} \left(\bigvee_{\alpha \in \{e_3\}} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{b,c\}} (F_A(e_3)) (s) \\
&= \bigvee_{s \in \{b,c\}} (\{(a, 0), (b, 0), (c, 0)\}) (s) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_3)(z) &= \bigvee_{s \in \rho^{-1}(z)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_3)} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} \left(\bigvee_{\alpha \in \{e_3\}} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} (F_A(e_3)) (s) \\
&= \bigvee_{s \in \{a\}} (\{(a, 0), (b, 0), (c, 0)\}) (s) \\
&= 0
\end{aligned}$$

Thus

$$\begin{aligned}
&f(F_A) \\
&= \{(e'_1, \{(x, 0), (y, 0.9), (z, 0.5)\}), (e'_2, \{(x, 0), (y, 0.6), (z, 0.4)\}), (e'_3, \{(x, 0), (y, 0), (z, 0)\})\}
\end{aligned}$$

Definition 3.3

Let (U, E) and (U^*, E^*) be classes of fuzzy soft sets over U and U^* with attributes from E and E^* respectively. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings and

$f: (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be a fuzzy soft mapping. Then $f = (\rho, \psi)$ is said to be **one-one or injective** (respectively, onto or surjective) if both $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ are one-one or injective (respectively, onto or surjective) mappings. If $f = (\rho, \psi)$ is both injective as well as surjective, then $f = (\rho, \psi)$ is said to be a fuzzy soft bijective mapping.

Example 3.4

Let $U = \{a, b, c\}$ and $U^* = \{x, y, z\}$, $E = \{e_1, e_2, e_3\}$, $E^* = \{e'_1, e'_2, e'_3\}$ and (U, E) , (U^*, E^*) classes of fuzzy soft sets.

Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings defined as $\rho(a) = z$, $\rho(b) = x$, $\rho(c) = y$ and $\psi(e_1) = e'_2$, $\psi(e_2) = e'_3$, $\psi(e_3) = e'_1$.

Let $A = \{e_1, e_2\}$. Let us consider a fuzzy soft set F_A in (U, E) as

$$F_A = \{F(e_1) = \{(a, 0.5), (b, 0.4), (c, 0.8)\}$$

$$F(e_2) = \{(a, 0.1), (b, 0.9), (c, 0.5)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$\begin{aligned} f(F_A)(e'_1)(x) &= \bigvee_{s \in \rho^{-1}(x)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\ &= \bigvee_{s \in \{b\}} \left(\bigvee_{\alpha \in \{e_3\}} F_A(\alpha) \right) (s) \\ &= \bigvee_{s \in \{b\}} (F_A(e_3))(s) \\ &= \bigvee_{s \in \{b, c\}} (\{(a, 0), (b, 0), (c, 0)\}) (s) \\ &= 0 \end{aligned}$$

$$\begin{aligned} f(F_A)(e'_1)(y) &= \bigvee_{s \in \rho^{-1}(y)} \left(\bigvee_{\alpha \in \psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\ &= \bigvee_{s \in \{c\}} \left(\bigvee_{\alpha \in \{e_3\}} F_A(\alpha) \right) (s) \\ &= \bigvee_{s \in \{c\}} (F_A(e_3))(s) \\ &= \bigvee_{s \in \{c\}} (\{(a, 0), (b, 0), (c, 0)\}) (s) \\ &= 0 \end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_1)(z) &= V_{s \in \rho^{-1}(z)} \left(V_{\alpha \in \Psi^{-1}(e'_1)} F_A(\alpha) \right) (s) \\
&= V_{s \in \{a\}} \left(V_{\alpha \in \{e_3\}} F_A(\alpha) \right) (s) \\
&= V_{s \in \{a\}} (F_A(e_3)) (s) \\
&= V_{s \in \{c\}} (\{(a, 0), (b, 0), (c, 0)\}) (s) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_2)(x) &= V_{s \in \rho^{-1}(x)} \left(V_{\alpha \in \Psi^{-1}(e'_2)} F_A(\alpha) \right) (s) \\
&= V_{s \in \{b\}} \left(V_{\alpha \in \{e_1\}} F_A(\alpha) \right) (s) \\
&= V_{s \in \{b\}} (F_A(e_1)) (s) \\
&= V_{s \in \{c\}} (\{(a, 0.5), (b, 0.4), (c, 0.8)\}) (s) \\
&= 0.4
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_2)(y) &= V_{s \in \rho^{-1}(y)} \left(V_{\alpha \in \Psi^{-1}(e'_2)} F_A(\alpha) \right) (s) \\
&= V_{s \in \{c\}} \left(V_{\alpha \in \{e_1\}} F_A(\alpha) \right) (s) \\
&= V_{s \in \{c\}} (F_A(e_1)) (s) \\
&= V_{s \in \{c\}} (\{(a, 0.5), (b, 0.4), (c, 0.8)\}) (s) \\
&= 0.8
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_2)(z) &= V_{s \in \rho^{-1}(z)} \left(V_{\alpha \in \Psi^{-1}(e'_2)} F_A(\alpha) \right) (s) \\
&= V_{s \in \{a\}} \left(V_{\alpha \in \{e_1\}} F_A(\alpha) \right) (s) \\
&= V_{s \in \{a\}} (F_A(e_1)) (s) \\
&= V_{s \in \{a\}} (\{(a, 0.5), (b, 0.4), (c, 0.8)\}) (s) \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
f(F_A)(e'_3)(x) &= V_{s \in \rho^{-1}(x)} \left(V_{\alpha \in \Psi^{-1}(e'_3)} F_A(\alpha) \right) (s) \\
&= V_{s \in \{b\}} \left(V_{\alpha \in \{e_2\}} F_A(\alpha) \right) (s)
\end{aligned}$$

$$\begin{aligned}
&= V_{s \in \{b\}}(F_A(e_2))(s) \\
&= V_{s \in \{b\}}(\{(a, 0.1), (b, 0.9), (c, 0.5)\})(s) \\
&= 0.9 \\
f(F_A)(e'_3)(y) &= V_{s \in \rho^{-1}(y)}(V_{\alpha \in \psi^{-1}(e'_3)} F_A(\alpha))(s) \\
&= V_{s \in \{c\}}(V_{\alpha \in \{e_3\}} F_A(\alpha))(s) \\
&= V_{s \in \{c\}}(F_A(e_2))(s) \\
&= V_{s \in \{c\}}(\{(a, 0.1), (b, 0.9), (c, 0.5)\})(s) \\
&= 0.5 \\
f(F_A)(e'_3)(z) &= V_{s \in \rho^{-1}(z)}(V_{\alpha \in \psi^{-1}(e'_3)} F_A(\alpha))(s) \\
&= V_{s \in \{a\}}(V_{\alpha \in \{e_2\}} F_A(\alpha))(s) \\
&= V_{s \in \{a\}}(F_A(e_2))(s) \\
&= V_{s \in \{a\}}(\{(a, 0.1), (b, 0.9), (c, 0.5)\})(s) \\
&= 0.1
\end{aligned}$$

Thus

$$\begin{aligned}
&f(F_A) \\
&= \{(e'_1, \{(x, 0), (y, 0), (z, 0)\}), (e'_2, \{(x, 0.4), (y, 0.8), (z, 0.5)\}), (e'_3, \{(x, 0.9), (y, 0.5), (z, 0.1)\})\}
\end{aligned}$$

Definition 3.5

Let (U, E) and (U^*, E^*) be classes of fuzzy soft sets over U and U^* with attributes from E and E^* respectively. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings and $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be a fuzzy soft mapping. Then for a fuzzy soft set G_B in (U^*, E^*) , $f^{-1}(G_B)$ is a fuzzy soft set in (U, E) obtained as follows. For $\alpha \in \psi^{-1}(E^*) \subseteq E$ and $x \in U$, $f^{-1}(G_B)(\alpha)(x) = G_B(\psi(\alpha))(\rho(x))$. $f^{-1}(G_B)$ is called a **fuzzy soft inverse image of the fuzzy soft set G_B** .

Example 3.6

Let $U = \{a, b, c\}$ and $U^* = \{x, y, z\}$, $E = \{e_1, e_2, e_3, e_4\}$, $E^* = \{e'_1, e'_2, e'_3\}$ and (U, E) , (U^*, E^*) classes of fuzzy soft sets.

Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings defined as $\rho(a) = z$, $\rho(b) = y$, $\rho(c) = y$ and $\psi(e_1) = e'_1$, $\psi(e_2) = e'_1$, $\psi(e_3) = e'_3$, $\psi(e_4) = e'_2$.

Let $B = \{e'_1, e'_2\}$. Let us consider a fuzzy soft set G_B in (U, E) as

$$G_B = \{G(e'_1) = \{(a, 0.2), (b, 0.6), (c, 0.4)\}\}$$

$$G(e'_2) = \{(a, 0.5), (b, 0.9), (c, 0.3)\}$$

$$G(e'_3) = \{(a, 0), (b, 0), (c, 0)\}$$

Now

$$f^{-1}(G_B)(e_1)(a) = G_B(\psi(e_1))(\rho(a)) = G_B(e'_1)(z) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(z) = 0.4$$

$$f^{-1}(G_B)(e_1)(b) = G_B(\psi(e_1))(\rho(b)) = G_B(e'_1)(y) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(y) = 0.6$$

$$f^{-1}(G_B)(e_1)(c) = G_B(\psi(e_1))(\rho(c)) = G_B(e'_1)(y) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(y) = 0.6$$

$$f^{-1}(G_B)(e_2)(a) = G_B(\psi(e_2))(\rho(a)) = G_B(e'_1)(z) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(z) = 0.4$$

$$f^{-1}(G_B)(e_2)(b) = G_B(\psi(e_2))(\rho(b)) = G_B(e'_1)(y) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(y) = 0.6$$

$$f^{-1}(G_B)(e_2)(c) = G_B(\psi(e_2))(\rho(c)) = G_B(e'_1)(y) = (\{(x, 0.2), (y, 0.6), (z, 0.4)\})(y) = 0.6$$

$$f^{-1}(G_B)(e_3)(a) = G_B(\psi(e_3))(\rho(a)) = G_B(e'_3)(z) = (\{(x, 0), (y, 0), (z, 0)\})(z) = 0$$

$$f^{-1}(G_B)(e_3)(b) = G_B(\psi(e_3))(\rho(b)) = G_B(e'_3)(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(G_B)(e_3)(c) = G_B(\psi(e_3))(\rho(c)) = G_B(e'_3)(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(G_B)(e_4)(a) = G_B(\psi(e_4))(\rho(a)) = G_B(e'_2)(z) = (\{(x, 0.5), (y, 0.9), (z, 0.3)\})(z) = 0.3$$

$$f^{-1}(G_B)(e_4)(b) = G_B(\psi(e_4))(\rho(b)) = G_B(e'_2)(y) = (\{(x, 0.5), (y, 0.9), (z, 0.3)\})(y) = 0.9$$

$$f^{-1}(G_B)(e_4)(c) = G_B(\psi(e_4))(\rho(c)) = G_B(e'_2)(y) = (\{(x, 0.5), (y, 0.9), (z, 0.3)\})(y) = 0.9$$

Thus

$$\begin{aligned}
& f^{-1}(G_B) \\
&= \{(e_1, \{(x, 0.4), (y, 0.6), (z, 0.6)\}), (e_2, \{(x, 0.4), (y, 0.6), (z, 0)\}), (e_3, \{(x, 0), (y, 0), (z, 0)\}), \\
&\quad (e_4, \{(x, 0.3), (y, 0.9), (z, 0.9)\})\}
\end{aligned}$$

Definition 3.7

Let (U, E) be classes of fuzzy soft sets over U with attributes from E . Let $\rho: U \rightarrow U$ and $\psi: E \rightarrow E$ be mappings and $f: (\rho, \psi): (U, E) \rightarrow (U, E)$ be a fuzzy soft mapping. Then $f = (\rho, \psi)$ is said to be **fuzzy soft identity mapping** if both $\rho: U \rightarrow U$ and $\psi: E \rightarrow E$ are identity mappings.

Example 3.8

Let $U = \{a, b, c\}$ and $E = \{e_1, e_2, e_3\}$ and (U, E) be a class of fuzzy soft sets. Let $\rho: U \rightarrow U$ and $\psi: E \rightarrow E$ be mappings defined as $\rho(a) = a, \rho(b) = b, \rho(c) = c$ and $\psi(e_1) = e_1, \psi(e_2) = e_2, \psi(e_3) = e_3$. Here both ρ and ψ are identity mappings and hence by our definition $f: (\rho, \psi): (U, E) \rightarrow (U, E)$ is a fuzzy soft identity mapping denoted by 'i'.

Let $A = \{e_1, e_2\}$. Let us consider a fuzzy soft set F_A in (U, E) as

$$F_A = \{F(e_1) = \{(a, 0.5), (b, 0.4), (c, 0.8)\}\}$$

$$F(e_2) = \{(a, 0.1), (b, 0.9), (c, 0.5)\}$$

$$F(e_3) = \{(a, 0), (b, 0), (c, 0)\}$$

$$\begin{aligned}
i(F_A)(e_1)(a) &= \bigvee_{s \in \rho^{-1}(a)} \left(\bigvee_{\alpha \in \psi^{-1}(e_1)} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} \left(\bigvee_{\alpha \in \{e_1\}} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{a\}} (F_A(e_1))(s) \\
&= \bigvee_{s \in \{a\}} \{(a, 0.5), (b, 0.4), (c, 0.8)\} (s) \\
&= 0.5
\end{aligned}$$

$$\begin{aligned}
i(F_A)(e_1)(b) &= \bigvee_{s \in \rho^{-1}(b)} \left(\bigvee_{\alpha \in \psi^{-1}(e_1)} F_A(\alpha) \right) (s) \\
&= \bigvee_{s \in \{b\}} \left(\bigvee_{\alpha \in \{e_1\}} F_A(\alpha) \right) (s)
\end{aligned}$$

$$\begin{aligned}
&= V_{s \in \{b\}}(F_A(e_1))(s) \\
&= V_{s \in \{b\}}(\{(a, 0.5), (b, 0.4), (c, 0.8)\})(s) \\
&= 0.4 \\
i(F_A)(e_1)(c) &= V_{s \in \rho^{-1}(c)}(V_{\alpha \in \psi^{-1}(e_1)} F_A(\alpha))(s) \\
&= V_{s \in \{c\}}(V_{\alpha \in \{e_1\}} F_A(\alpha))(s) \\
&= V_{s \in \{c\}}(F_A(e_1))(s) \\
&= V_{s \in \{c\}}(\{(a, 0.5), (b, 0.4), (c, 0.8)\})(s) \\
&= 0.8
\end{aligned}$$

In a similar way

$$i(F_A)(e_2)(a) = 0.1, i(F_A)(e_2)(b) = 0.9, i(F_A)(e_2)(c) = 0.5$$

$$i(F_A)(e_3)(a) = 0, i(F_A)(e_3)(b) = 0, i(F_A)(e_3)(c) = 0$$

Thus

$i(F_A)$

$$= \{(e_1, \{(x, 0.5), (y, 0.4), (z, 0.8)\}), (e_2, \{(x, 0.1), (y, 0.9), (z, 0.5)\}), (e_3, \{(x, 0), (y, 0), (z, 0)\})\}$$

Theorem 3.9

Let (U, E) and (U^*, E^*) be classes of fuzzy soft sets over U and U^* with attributes from E and E^* respectively. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings and $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be a fuzzy soft mapping. Then

- i. $f(\tilde{\Phi}) = \tilde{\Phi}^*$, $f(\tilde{E}) \subseteq \tilde{E}^*$
- ii. $f^{-1}(\tilde{\Phi}^*) = \tilde{\Phi}$, $f^{-1}(\tilde{E}^*) = \tilde{E}$
- iii. $F_{A_1}^1 \subseteq F_{A_2}^2 \Rightarrow f(F_{A_1}^1) \subseteq f(F_{A_2}^2) \forall F_{A_1}^1, F_{A_2}^2 \in (U, E)$
- iv. $G_{B_1}^1 \subseteq G_{B_2}^2 \Rightarrow f^{-1}(G_{B_1}^1) \subseteq f^{-1}(G_{B_2}^2)$
- v. $f(f^{-1}(G_B)) \subseteq G_B, \forall G_B \in (U^*, E^*)$, the equality holds if f is surjective.
- vi. $F_A \subseteq f^{-1}(f(F_A)), \forall F_A \in (U, E)$, the equality holds if f is injective.
- vii. $f(\tilde{U}_{\alpha \in \Delta} F_{A_\alpha}^\alpha) = \tilde{U}_{\alpha \in \Delta} (f(F_{A_\alpha}^\alpha)) \forall F_{A_\alpha}^\alpha \in (U, E), \alpha \in \Delta$

$$f^{-1}(\tilde{U}_{\alpha \in \Delta} G_{B_\alpha}^\alpha) = \tilde{U}_{\alpha \in \Delta} (f^{-1}(G_{B_\alpha}^\alpha)) \quad \forall G_{B_\alpha}^\alpha \in (U^*, E^*), \alpha \in \Delta.$$

$$\text{viii. } f(\tilde{\cap}_{\alpha \in \Delta} F_{A_\alpha}^\alpha) = \tilde{\cap}_{\alpha \in \Delta} (f(F_{A_\alpha}^\alpha)) \quad \forall F_{A_\alpha}^\alpha \in (U, E), \alpha \in \Delta$$

$$f^{-1}(\tilde{\cap}_{\alpha \in \Delta} G_{B_\alpha}^\alpha) = \tilde{\cap}_{\alpha \in \Delta} (f^{-1}(G_{B_\alpha}^\alpha)) \quad \forall G_{B_\alpha}^\alpha \in (U^*, E^*), \alpha \in \Delta. .$$

$$\text{ix. } f^{-1}(G_B^C) = (f^{-1}(G_B))^C$$

$$\text{x. } (f(F_A))^C \cong (f(F_A^C)) \quad \forall F_A \in (U, E).$$

Definition 3.10

Let (U, E, τ) and (U^*, E^*, τ^*) be fuzzy soft topological spaces. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings and $f: (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be a fuzzy soft mapping. Then $f: (\rho, \psi)$ is a **fuzzy soft continuous** if the inverse image under $f: (\rho, \psi)$ of any $G_B \in \tau^*$ is a fuzzy soft set $F_A \in \tau$ i.e., $f^{-1}(G_B) \in \tau$ whenever $G_B \in \tau^*$.

Example 3.11

Let (U, E, τ) and (U^*, E^*, τ^*) be fuzzy soft topological spaces. Let $U = \{a, b, c\}$ and $U^* = \{x, y, z\}$, $E = \{e_1, e_2, e_3, e_4\}$, $E^* = \{e'_1, e'_2, e'_3\}$ and (U, E) , (U^*, E^*) classes of fuzzy soft sets. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings defined as $\rho(a) = z$, $\rho(b) = y$, $\rho(c) = y$ and $\psi(e_1) = e'_1$, $\psi(e_2) = e'_1$, $\psi(e_3) = e'_3$, $\psi(e_4) = e'_2$. Let $A_1 = \{e_1, e_2\}$, $A_2 = \{e_1, e_2, e_3\}$, $B_1 = \{e'_1, e'_2\}$, $B_2 = \{e'_1, e'_2, e'_3\}$. Let us consider a fuzzy soft sets $F_{A_1}^1$ and $F_{A_2}^2$ in (U, E) and $G_{B_1}^1$ and $G_{B_2}^2$ in (U^*, E^*) as

$$F_{A_1}^1$$

$$= \{(e_1, \{(a, 0.1), (b, 0.2), (c, 0.2)\}), (e_2, \{(a, 0.1), (b, 0.2), (c, 0.2)\}), (e_3, \{(a, 0), (b, 0), (c, 0)\})\}$$

$$F_{A_2}^2$$

$$= \{(e_1, \{(a, 0.6), (b, 0.7), (c, 0.7)\}), (e_2, \{(a, 0.6), (b, 0.7), (c, 0.7)\}), (e_3, \{(a, 0.6), (b, 0.4), (c, 0.4)\})\}$$

$$G_{B_1}^1$$

$$= \{(e_1, \{(a, 0.3), (b, 0.2), (c, 0.1)\}), (e_2, \{(a, 0.5), (b, 0.9), (c, 0.6)\}), (e_3, \{(a, 0), (b, 0), (c, 0)\})\}$$

$$G_{B_2}^2$$

$$= \{(e_1, \{(a, 0.3), (b, 0.7), (c, 0.6)\}), (e_2, \{(a, 0.8), (b, 0.9), (c, 0.7)\}), (e_3, \{(a, 0.2), (b, 0.4), (c, 0.6)\})\}$$

Let $\tau = \{\tilde{\Phi}, \tilde{E}, F_{A_1}^1, F_{A_2}^2\}$ and $\tau^* = \{\tilde{\Phi}^*, \tilde{E}^*, G_{B_1}^1, G_{B_2}^2\}$, where

$$\tilde{\Phi} = \{(e_1, \{(a, 0), (b, 0), (c, 0)\}), (e_2, \{(a, 0), (b, 0), (c, 0)\}), (e_3, \{(a, 0), (b, 0), (c, 0)\})\}$$

$$\tilde{E} = \{(e_1, \{(a, 1), (b, 1), (c, 1)\}), (e_2, \{(a, 1), (b, 1), (c, 1)\}), (e_3, \{(a, 1), (b, 1), (c, 1)\})\}$$

$$\tilde{\Phi}^* = \{(e_1, \{(a, 0), (b, 0), (c, 0)\}), (e_2, \{(a, 0), (b, 0), (c, 0)\}), (e_3, \{(a, 0), (b, 0), (c, 0)\})\}$$

$$\tilde{E}^* = \{(e_1, \{(a, 1), (b, 1), (c, 1)\}), (e_2, \{(a, 1), (b, 1), (c, 1)\}), (e_3, \{(a, 1), (b, 1), (c, 1)\})\}$$

(i)

$$f^{-1}(\tilde{\Phi}^*)(e_1)(a) = \tilde{\Phi}^*(\psi(e_1))(\rho(a)) = \tilde{\Phi}^*(e_1')(z) = (\{(x, 0), (y, 0), (z, 0)\})(z) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_1)(b) = \tilde{\Phi}^*(\psi(e_1))(\rho(b)) = \tilde{\Phi}^*(e_1')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_1)(c) = \tilde{\Phi}^*(\psi(e_1))(\rho(c)) = \tilde{\Phi}^*(e_1')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_2)(a) = \tilde{\Phi}^*(\psi(e_2))(\rho(a)) = \tilde{\Phi}^*(e_1')(z) = (\{(x, 0), (y, 0), (z, 0)\})(z) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_2)(b) = \tilde{\Phi}^*(\psi(e_2))(\rho(b)) = \tilde{\Phi}^*(e_1')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_2)(c) = \tilde{\Phi}^*(\psi(e_2))(\rho(c)) = \tilde{\Phi}^*(e_1')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_3)(a) = \tilde{\Phi}^*(\psi(e_3))(\rho(a)) = \tilde{\Phi}^*(e_3')(z) = (\{(x, 0), (y, 0), (z, 0)\})(z) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_3)(b) = \tilde{\Phi}^*(\psi(e_3))(\rho(b)) = \tilde{\Phi}^*(e_3')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(\tilde{\Phi}^*)(e_3)(c) = \tilde{\Phi}^*(\psi(e_3))(\rho(c)) = \tilde{\Phi}^*(e_3')(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

Thus $f^{-1}(\tilde{\Phi}^*) = \tilde{\Phi}$

(ii)

$$f^{-1}(\tilde{E}^*)(e_1)(a) = \tilde{E}^*(\psi(e_1))(\rho(a)) = \tilde{E}^*(e_1')(z) = (\{(x, 1), (y, 1), (z, 1)\})(z) = 1$$

$$f^{-1}(\tilde{E}^*)(e_1)(b) = \tilde{E}^*(\psi(e_1))(\rho(b)) = \tilde{E}^*(e_1')(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*)(e_1)(c) = \tilde{E}^*(\psi(e_1))(\rho(c)) = \tilde{E}^*(e_1')(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*)(e_2)(a) = \tilde{E}^*(\psi(e_2))(\rho(a)) = \tilde{E}^*(e_1')(z) = (\{(x, 1), (y, 1), (z, 1)\})(z) = 1$$

$$f^{-1}(\tilde{E}^*)(e_2)(b) = \tilde{E}^*(\psi(e_2))(\rho(b)) = \tilde{E}^*(e_1')(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*)(e_2)(c) = \tilde{E}^*(\psi(e_2))(\rho(c)) = \tilde{E}^*(e'_1)(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*)(e_3)(a) = \tilde{E}^*(\psi(e_3))(\rho(a)) = \tilde{E}^*(e'_3)(z) = (\{(x, 1), (y, 1), (z, 1)\})(z) = 1$$

$$f^{-1}(\tilde{E}^*)(e_3)(b) = \tilde{E}^*(\psi(e_3))(\rho(b)) = \tilde{E}^*(e'_3)(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*)(e_3)(c) = \tilde{E}^*(\psi(e_3))(\rho(c)) = \tilde{E}^*(e'_3)(y) = (\{(x, 1), (y, 1), (z, 1)\})(y) = 1$$

$$f^{-1}(\tilde{E}^*) = \tilde{E}$$

(iii)

$$f^{-1}(G_{B_1}^1)(e_1)(a) = G_{B_1}^1(\psi(e_1))(\rho(a)) = G_{B_1}^1(e'_1)(z) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(z) = 0.1$$

$$f^{-1}(G_{B_1}^1)(e_1)(b) = G_{B_1}^1(\psi(e_1))(\rho(b)) = G_{B_1}^1(e'_1)(y) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(y) = 0.2$$

$$f^{-1}(G_{B_1}^1)(e_1)(c) = G_{B_1}^1(\psi(e_1))(\rho(c)) = G_{B_1}^1(e'_1)(y) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(y) = 0.2$$

$$f^{-1}(G_{B_1}^1)(e_2)(a) = G_{B_1}^1(\psi(e_2))(\rho(a)) = G_{B_1}^1(e'_1)(z) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(z) = 0.1$$

$$f^{-1}(G_{B_1}^1)(e_2)(b) = G_{B_1}^1(\psi(e_2))(\rho(b)) = G_{B_1}^1(e'_1)(y) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(y) = 0.2$$

$$f^{-1}(G_{B_1}^1)(e_2)(c) = G_{B_1}^1(\psi(e_2))(\rho(c)) = G_{B_1}^1(e'_1)(y) = (\{(x, 0.3), (y, 0.2), (z, 0.1)\})(y) = 0.2$$

$$f^{-1}(G_{B_1}^1)(e_3)(a) = G_{B_1}^1(\psi(e_3))(\rho(a)) = G_{B_1}^1(e'_3)(z) = (\{(x, 0), (y, 0), (z, 0)\})(z) = 0$$

$$f^{-1}(G_{B_1}^1)(e_3)(b) = G_{B_1}^1(\psi(e_3))(\rho(b)) = G_{B_1}^1(e'_3)(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(G_{B_1}^1)(e_3)(c) = G_{B_1}^1(\psi(e_3))(\rho(c)) = G_{B_1}^1(e'_3)(y) = (\{(x, 0), (y, 0), (z, 0)\})(y) = 0$$

$$f^{-1}(G_{B_1}^1)$$

$$= \{(e_1, \{(a, 0.1), (b, 0.2), (c, 0.2)\}), (e_2, \{(a, 0.1), (b, 0.2), (c, 0.2)\}), (e_3, \{(a, 0), (b, 0), (c, 0)\})\}$$

$$= F_{A_1}^1$$

(iv)

$$f^{-1}(G_{B_2}^2)(e_1)(a) = G_{B_2}^2(\psi(e_1))(\rho(a)) = G_{B_2}^2(e'_1)(z) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(z) = 0.6$$

$$f^{-1}(G_{B_2}^2)(e_1)(b) = G_{B_2}^2(\psi(e_1))(\rho(b)) = G_{B_2}^2(e'_1)(y) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(y) = 0.7$$

$$f^{-1}(G_{B_2}^2)(e_1)(c) = G_{B_2}^2(\psi(e_1))(\rho(c)) = G_{B_2}^2(e'_1)(y) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(y) = 0.7$$

$$f^{-1}(G_{B_2}^2)(e_2)(a) = G_{B_2}^2(\psi(e_2))(\rho(a)) = G_{B_2}^2(e'_1)(z) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(z) = 0.6$$

$$f^{-1}(G_{B_2}^2)(e_2)(b) = G_{B_2}^2(\psi(e_2))(\rho(b)) = G_{B_2}^2(e'_1)(y) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(y) = 0.7$$

$$f^{-1}(G_{B_2}^2)(e_2)(c) = G_{B_2}^2(\psi(e_2))(\rho(c)) = G_{B_2}^2(e'_1)(y) = (\{(x, 0.3), (y, 0.7), (z, 0.6)\})(y) = 0.7$$

$$f^{-1}(G_{B_2}^2)(e_3)(a) = G_{B_2}^2(\psi(e_3))(\rho(a)) = G_{B_2}^2(e'_3)(z) = (\{(x, 0.2), (y, 0.4), (z, 0.6)\})(z) = 0.6$$

$$f^{-1}(G_{B_2}^2)(e_3)(b) = G_{B_2}^2(\psi(e_3))(\rho(b)) = G_{B_2}^2(e'_3)(y) = (\{(x, 0.2), (y, 0.4), (z, 0.6)\})(y) = 0.4$$

$$f^{-1}(G_{B_2}^2)(e_3)(c) = G_{B_2}^2(\psi(e_3))(\rho(c)) = G_{B_2}^2(e'_3)(y) = (\{(x, 0.2), (y, 0.4), (z, 0.6)\})(y) = 0.4$$

Thus

$$f^{-1}(G_{B_2}^2)$$

$$= \{(e_1, \{(a, 0.6), (b, 0.7), (c, 0.7)\}), (e_2, \{(a, 0.6), (b, 0.7), (c, 0.7)\}), (e_3, \{(a, 0.6), (b, 0.4), (c, 0.4)\})\}$$

$$= F_{A_2}^2$$

It follows that $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ is fuzzy soft continuous.

Theorem 3.12

The fuzzy soft identity mapping $i: (U, E) \rightarrow (U, E)$ on a fuzzy soft topological space (U, E, τ) is fuzzy soft continuous.

Proof

Let $i = (\rho, \psi): (U, E) \rightarrow (U, E)$ be the fuzzy soft identity mapping where $\rho: U \rightarrow U$ and $\psi: E \rightarrow E$ are identity mappings. Then $\rho(x) = x, \forall x \in U$ and $\psi(\alpha) = \alpha, \forall \alpha \in E$. Let $G_B \in \tau$.

$$i^{-1}(G_B)(\alpha)(x) = G_B(\psi(\alpha))(\rho(x))$$

$$= G_B(\alpha)(x), \forall \alpha \in E, x \in U$$

It follows that $i^{-1}(G_B) = G_B \in \tau$ and hence $i = (\rho, \psi): (U, E) \rightarrow (U, E)$ on a fuzzy soft topological space (U, E, τ) is fuzzy soft continuous.

Theorem 3.13

Let (U, E, τ) and (U^*, E^*, τ^*) be fuzzy soft topological spaces. Let $\rho: U \rightarrow U^*$ and $\psi: E \rightarrow E^*$ be mappings and $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be a fuzzy soft mapping. Then the following statements are equivalent:

- i. The fuzzy soft function $f: (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ is continuous.

- ii. The inverse of every fuzzy soft closed set is closed.
- iii. For each fuzzy soft point $e(F_A)$ in (U, E) , the fuzzy soft inverse of every neighborhood of $f(e(F_A))$ under f is a neighborhood of $e(F_A)$.
- iv. For each fuzzy soft point $e(F_A)$ in (U, E) and each neighborhood G_B of $f(e(F_A))$, there is a neighborhood H_C of $e(F_A)$ such that $f(H_C) \cong G_B$.
- v. $f(\overline{F_A}) \cong \overline{f(F_A)}$

Proof

(i) \Rightarrow (ii) Let $f = (\rho, \psi): (U, E) \rightarrow (U^*, E^*)$ be continuous and $G_B \in \tau^*$, then $f^{-1}(G_B) \in \tau$. $G_B \in \tau^* \Rightarrow G_B^C$ is closed in (U^*, E^*, τ^*) .

Again $f^{-1}(G_B^C) = (f^{-1}(G_B))^C$ and f being continuous, $f^{-1}(G_B)$ is open in (U, E, τ) .

It follows that $(f^{-1}(G_B))^C$ is closed in (U, E, τ) .

$\Rightarrow f^{-1}(G_B^C)$ is closed in (U, E, τ) .

(ii) \Rightarrow (iii) Let H_C be a neighborhood of $f(e(F_A))$. Then there is a $G_B \in \tau^*$ such that $f(e(F_A)) \cong G_B \cong H_C$. Now $e(F_A) \cong f^{-1}(f(e(F_A))) \cong f^{-1}(G_B) \cong f^{-1}(H_C)$

i.e. $e(F_A) \cong f^{-1}(G_B) \cong f^{-1}(H_C)$, $f^{-1}(G_B)$ being open in (U, E, τ) , the result follows.

(iii) \Rightarrow (iv) Let $e(F_A) \cong (U, E)$ and G_B be a neighborhood of $f(e(F_A))$. Then $f^{-1}(G_B)$ is a neighborhood of $e(F_A)$. Thus there exists a fuzzy soft open set H_C in (U, E) such that

$$e(F_A) \cong H_C \cong f^{-1}(G_B)$$

$$\Rightarrow H_C \cong f^{-1}(G_B)$$

$$\Rightarrow f(H_C) \cong f(f^{-1}(G_B)) \cong G_B$$

i.e. $f(H_C) \cong G_B$ and the results follows.

(iv) \Rightarrow (v) Since $\overline{f(F_A)}$ is fuzzy soft closed in (U^*, E^*) , $f^{-1}(\overline{f(F_A)})$ is a fuzzy soft closed in (U, E) .

$$\text{Thus } f^{-1}(\overline{f(F_A)}) = \overline{f^{-1}(f(F_A))}$$

$$\text{Now, } F_A \cong f^{-1}(F_A) \cong f^{-1}(\overline{f(F_A)}), \text{ as } f(F_A) \cong \overline{f(F_A)}$$

$$\Rightarrow F_A \cong f^{-1}(\overline{f(F_A)}) = \overline{f^{-1}(f(F_A))}$$

$$\Rightarrow F_A \cong \overline{f^{-1}(f(F_A))}$$

$$\Rightarrow \overline{F_A} \cong f^{-1}(\overline{f(F_A)})$$

$$\Rightarrow \overline{F_A} \cong f(f^{-1}(\overline{f(F_A)})) \cong \overline{f(F_A)}$$

$$\text{i.e. } f(\overline{F_A}) \cong \overline{f(F_A)}$$