CHAPTER – 3

SECOND ORDER BIPOLAR FUZZY CONTINUITY AND SECOND ORDER BIPOLAR FUZZY PRODUCT TOPOLOGY

In the first section, second order bipolar fuzzy continuity is defined. It is proved that the associations R_1, R_3 and R_5 preserve continuity. It is also proved that the associations $\hat{\tau}_{\mathfrak{B}} \to i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}})$, $\hat{\tau}_{\mathfrak{B}} \to i^{*}(\hat{\tau}_{\mathfrak{B}})$, $\hat{\tau}_{\mathfrak{B}} \to i(\hat{\tau}_{\mathfrak{B}})$, $\tau \to \widehat{\omega_{\varepsilon}(\tau)}$, $\tau \to \widehat{\omega_{*}(\tau)}$ and $\tau \to \widehat{\omega(\tau)}$ are functorial.

In the second section, first order bipolar fuzzy product topology is introduced and its properties are analysed.

In the third section, second order bipolar fuzzy product topology is defined. It is proved that the associations R_1, R_3 and R_5 preserve product. Further the following results are obtained.

(i) $i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_1}) \times i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i_{\varepsilon}(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

(ii) $i^{*}(\hat{\tau}_{\mathfrak{B}_1}) \times i^{*}(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i^{*}(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

(iii) $\widehat{\omega_{\varepsilon}(\tau)} \times \widehat{\omega_{\varepsilon}(\tau')} \subseteq \omega_{\varepsilon}\widehat{(\tau \times \tau')}$.

(iv) $\widehat{\omega_{*}(\tau)} \times \widehat{\omega_{*}(\tau')} \subseteq \omega_{*}\widehat{(\tau \times \tau')}$.

(v) $S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right) \times S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right) \subseteq S_2\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)$.

SECTION 3.1

SECOND ORDER BIPOLAR FUZZY CONTINUITY

Definition:3.1.1

Let $\widehat{A}_{bp} \in \mathrm{SBPF}(X)$ and $\widehat{B}_{bp} \in \mathrm{SBPF}(Y)$ be two second order bipolar fuzzy sets and let $\theta: X \to Y$ be a mapping. Then

(i) The **image** of $\widehat{A}_{bp}$ under θ, denoted by $\theta(\widehat{A}_{bp}) = \left(\theta\left({\widehat{A}_{bp}}^{+}\right), \theta({\widehat{A}_{bp}}^{-})\right)$, is a second order bipolar fuzzy set in Y defined as follows: for every $y \in Y$

$$\left(\theta\left({\widehat{A}_{bp}}^{+}\right)\right)(y) = \begin{cases} \bigvee_{x\in\theta^{-1}(y)} {\widehat{A}_{bp}}^{+}(x), & \text{if } \theta^{-1}(y) \neq \emptyset \\ \mathbf{0} & \text{otherwise} \end{cases}$$

and

$$\left(\theta({\widehat{A}_{bp}}^{-})\right)(y) = \begin{cases} \bigwedge_{x\in\theta^{-1}(y)} {\widehat{A}_{bp}}^{-}(x), & \text{if } \theta^{-1}(y) \neq \emptyset \\ \mathbf{0} & \text{otherwise} \end{cases}$$

(ii) The **pre-image** of $\widehat{B}_{bp}$ under θ, denoted by $\theta^{-1}(\widehat{B}_{bp}) = \left(\theta^{-1}\left({\widehat{B}_{bp}}^{+}\right), \theta^{-1}({\widehat{B}_{bp}}^{-})\right)$, is a second order bipolar fuzzy set in X defined as follows: for every $x \in X$

$$\left(\theta^{-1}\left({\widehat{B}_{bp}}^{+}\right)\right)(x) = {\widehat{B}_{bp}}^{+}(\theta(x)) \text{ and } \left(\theta^{-1}({\widehat{B}_{bp}}^{-})\right)(x) = {\widehat{B}_{bp}}^{-}(\theta(x)).$$

Definition:3.1.2

Let $(X, \hat{\tau}_{\mathfrak{B}_1})$, $(Y, \hat{\tau}_{\mathfrak{B}_2})$ be two second order bipolar fuzzy topological spaces. Then a function $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \to (Y, \hat{\tau}_{\mathfrak{B}_2})$ is said to be **second order bipolar fuzzy continuous**, if the following condition is satisfied:

$$\theta^{-1}(\widehat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}, \text{ if } \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$$

Theorem:3.1.3

A function $\theta: (X, \tau_{\mathfrak{B}_1}) \to (Y, \tau_{\mathfrak{B}_2})$ is first order bipolar fuzzy continuous if and only if $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \to (Y, \hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy continuous where $\hat{\tau}_{\mathfrak{B}_1}$ and $\hat{\tau}_{\mathfrak{B}_2}$ are from $\tau_{\mathfrak{B}_1}$ and $\tau_{\mathfrak{B}_2}$, respectively, through the relation R_1.

Proof:

Given $\tau_{\mathfrak{B}}$, $\hat{\tau}_{\mathfrak{B}} = \{\widehat{A}_{bp} \,/\, A_{bp} \in \tau_{\mathfrak{B}}\}$, where $\widehat{A}_{bp}{}^{+}(x)(\alpha) = A_{bp}{}^{+}(x)$ and $\widehat{A}_{bp}{}^{-}(x)(\alpha) = A_{bp}{}^{-}(x)$, for every $x \in X$ and for every $\alpha \in I$.

Assume $\theta: (X, \tau_{\mathfrak{B}_1}) \to (Y, \tau_{\mathfrak{B}_2})$ is bipolar fuzzy continuous.

Consider $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$.
Then $A_{bp} \in \tau_{\mathfrak{B}_2}$ (through the relation R_1).

Since θ is bipolar fuzzy continuous, $\theta^{-1}(A_{bp}) \in \tau_{\mathfrak{B}_1}$.
Let $B_{bp} = \theta^{-1}(A_{bp}) = \left(\theta^{-1}(A_{bp}{}^{+}), \theta^{-1}(A_{bp}{}^{-})\right) \in \tau_{\mathfrak{B}_1}$, then $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$.

For $x \in X$ and $\alpha \in I$, consider

$$\begin{aligned}
\widehat{B}_{bp}{}^{+}(x)(\alpha) &= B_{bp}{}^{+}(x) \\
&= \left(\theta^{-1}(A_{bp}{}^{+})\right)(x) \\
&= A_{bp}{}^{+}(\theta(x)) \\
&= \widehat{A}_{bp}{}^{+}\left(\theta(x)\right)(\alpha) \\
&= \theta^{-1}(\widehat{A}_{bp}{}^{+})(x)(\alpha).
\end{aligned}$$

Similarly, $\widehat{B}_{bp}{}^{-}(x)(\alpha) = \theta^{-1}(\widehat{A}_{bp}{}^{-})(x)(\alpha)$, then $\theta^{-1}(\widehat{A}_{bp}) = \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$.

Therefore θ is second order bipolar fuzzy continuous.

Conversely, assume $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \to (Y, \hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy continuous.

Consider, $A_{bp} \in \tau_{\mathfrak{B}_2}$, then $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ (through the relation R_1).

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}(\widehat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$.

Let $\widehat{B}_{bp} = \theta^{-1}(\widehat{A}_{bp}) = \left(\theta^{-1}\left(\widehat{A}_{bp}{}^{+}\right),\ \theta^{-1}(\widehat{A}_{bp}{}^{-})\right) \in \hat{\tau}_{\mathfrak{B}_1}$, then $B_{bp} \in \tau_{\mathfrak{B}_1}$.

For x ∈ X and α ∈ I, consider

$$\begin{aligned} \mathrm{B_{bp}}^{+}(\mathrm{x}) &= \widehat{\mathrm{B}}_{\mathrm{bp}}{}^{+}(\mathrm{x})(\alpha) \\ &= \left(\theta^{-1}(\widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+})\right)(\mathrm{x})(\alpha) \\ &= \widehat{\mathrm{A}}_{\mathrm{bp}}{}^{+}\big(\theta(\mathrm{x})\big)(\alpha) \\ &= \mathrm{A_{bp}}^{+}\big(\theta(\mathrm{x})\big) \\ &= \theta^{-1}(\mathrm{A_{bp}}^{+})(\mathrm{x}). \end{aligned}$$

Similarly, $\mathrm{B_{bp}}^{-}(\mathrm{x}) = \theta^{-1}(\mathrm{A_{bp}}^{-})(\mathrm{x})$.

Therefore $\theta^{-1}\big(\mathrm{A_{bp}}\big) = \mathrm{B_{bp}} \in \tau_{\mathfrak{B}_1}$.

Thus θ is bipolar fuzzy continuous.

Example:3.1.4

Define $\theta\colon \mathrm{X} \to \mathrm{Y}$ as $\theta(\mathrm{x}) = \mathrm{y}$. Consider two first order bipolar fuzzy topologies $\tau_{\mathfrak{B}_1} = \{0_{\mathrm{bp}}, 1_{\mathrm{bp}}, \mathrm{A_{bp}}\}$ and $\tau_{\mathfrak{B}_2} = \{0_{\mathrm{bp}}, 1_{\mathrm{bp}}, \mathrm{B_{bp}}\}$ as the collection of first order bipolar fuzzy sets on X and Y respectively, where

$\mathrm{A_{bp}}^{+}(x) = 0.5$, $\mathrm{A_{bp}}^{-}(x) = -0.3$ in X and $\mathrm{B_{bp}}^{+}(y) = 0.3$, $\mathrm{B_{bp}}^{-}(y) = -0.2$ in Y.

Let us assume that θ is first order bipolar fuzzy continuous.

Consider $\widehat{\mathrm{B}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}_2}$ implies $\mathrm{B_{bp}} \in \tau_{\mathfrak{B}_2}$ (through R_1).

implies $\theta^{-1}\big(\mathrm{B_{bp}}\big) \in \tau_{\mathfrak{B}_1}$

So, $\theta^{-1}\left(\mathrm{B_{bp}}^{+}(x)\right) = 0.5$, $\theta^{-1}\left(\mathrm{B_{bp}}^{-}(x)\right) = -0.3$

implies $\left(\theta^{-1}\left(\widehat{\mathrm{B}}_{\mathrm{bp}}{}^{+}\right)\right)(x)(\alpha) = 0.5$, $\left(\theta^{-1}\big(\widehat{\mathrm{B}}_{\mathrm{bp}}{}^{-}\big)\right)(x)(\alpha) = -0.3$

implies $\theta^{-1}\big(\widehat{\mathrm{B}}_{\mathrm{bp}}\big) \in \hat{\tau}_{\mathfrak{B}_1}$

Hence θ is second order bipolar fuzzy continuous.

Consider two second order bipolar fuzzy topologies $\hat{\tau}_{\mathfrak{B}_1} = \{\hat{0}_{bp}, \hat{1}_{bp}, \hat{A}_{bp}\}$ and $\hat{\tau}_{\mathfrak{B}_2} = \{\hat{0}_{bp}, \hat{1}_{bp}, \hat{B}_{bp}\}$ as the collection of second order bipolar fuzzy sets on X and Y respectively, where $\hat{A}_{bp}{}^{+}(x)(\alpha) = 0.5$, $\hat{A}_{bp}{}^{-}(x)(\alpha) = -0.8$, for every $\alpha \in I$ and

$\hat{B}_{bp}{}^{+}(y)(\alpha) = 0.5$, $\hat{B}_{bp}{}^{-}(y)(\alpha) = -0.8$, for every $\alpha \in I$.

Conversely, assume θ is second order bipolar fuzzy continuous.

Consider $B_{bp} \in \tau_{\mathfrak{B}_2}$ implies $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ (through R_1).

implies $\theta^{-1}(\hat{B}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$

implies $\left(\theta^{-1}\left(\hat{B}_{bp}{}^{+}\right)\right)(x)(\alpha) = 0.5$, $\left(\theta^{-1}(\hat{B}_{bp}{}^{-})\right)(x)(\alpha) = -0.8$

So, $\theta^{-1}\left(B_{bp}{}^{+}(x)\right) = 0.5$, $\theta^{-1}\left(B_{bp}{}^{-}(x)\right) = -0.8$ implies $\theta^{-1}(B_{bp}) \in \tau_{\mathfrak{B}_1}$

Hence θ is first order bipolar fuzzy continuous.

Theorem:3.1.5

If $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \to (Y, \hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy continuous then for $\alpha \in I$, $\theta: \left(X, \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha}\right) \to \left(Y, \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}\right)$ is first order bipolar fuzzy continuous where $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha}$ and $\left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$ are from $\hat{\tau}_{\mathfrak{B}_1}$ and $\hat{\tau}_{\mathfrak{B}_2}$, respectively, through the relation R_3.

Proof:

Any element of $\left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$ is of the form $(\hat{A}_{bp})_{\alpha}$ where $\left(\hat{A}_{bp}{}^{+}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{+}(x)(\alpha)$ and $\left(\hat{A}_{bp}{}^{-}\right)_{\alpha}(x) = \hat{A}_{bp}{}^{-}(x)(\alpha)$, for some $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ and for every $x \in X$.

Consider $(\hat{A}_{bp})_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$, then there exists $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ (through the relation R_3)

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}(\hat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$.

Let $\hat{B}_{bp} = \theta^{-1}(\hat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$, then $(\hat{B}_{bp})_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha}$.

For $x \in X$ and $\alpha \in I$, consider

$$\begin{aligned}\left(\widehat{B}_{bp}{}^{+}\right)_{\alpha}(x) &= \widehat{B}_{bp}{}^{+}(x)(\alpha)\\ &= \left(\theta^{-1}(\widehat{A}_{bp}{}^{+})\right)(x)(\alpha)\\ &= \widehat{A}_{bp}{}^{+}\big(\theta(x)\big)(\alpha)\\ &= \left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}\big(\theta(x)\big)\\ &= \theta^{-1}\left(\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha}\right)(x).\end{aligned}$$

Similarly, $\left(\widehat{B}_{bp}{}^{-}\right)_{\alpha}(x) = \theta^{-1}\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}\right)(x)$ implies $\theta^{-1}\left(\widehat{A}_{bp}\right)_{\alpha} = \left(\widehat{B}_{bp}\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha}$.

Thus θ is bipolar fuzzy continuous.

Example:3.1.6

Define $\theta: X \to Y$ as $\theta(x) = y$. Consider two second order bipolar fuzzy topologies $\hat{\tau}_{\mathfrak{B}_1} = \left\{\hat{0}_{bp},\ \hat{1}_{bp}, \widehat{A}_{bp}\right\}$ and $\hat{\tau}_{\mathfrak{B}_2} = \left\{\hat{0}_{bp},\ \hat{1}_{bp}, \widehat{B}_{bp}\right\}$ as a collection of second order bipolar fuzzy sets in X and Y respectively. For $\alpha_1 \in I$,

$\widehat{A}_{bp}{}^{+}(x)(\alpha_1) = 0.5, \widehat{A}_{bp}{}^{-}(x)(\alpha_1) = -0.8$, for every $x \in X$ and

$\widehat{B}_{bp}{}^{+}(y)(\alpha_1) = 0.3, \widehat{B}_{bp}{}^{-}(y)(\alpha_1) = -0.5$, for every $y \in Y$.

Let us assume that θ is second order bipolar fuzzy continuous.

Consider $\left(\widehat{B}_{bp}\right)_{\alpha_1} \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha_1}$, then $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ implies $\theta^{-1}\left(\widehat{B}_{bp}\right) \in \hat{\tau}_{\mathfrak{B}_1}$

implies $\left(\theta^{-1}\left(\widehat{B}_{bp}{}^{+}\right)\right)(x)(\alpha) = 0.5, \left(\theta^{-1}\left(\widehat{B}_{bp}{}^{-}\right)\right)(x)(\alpha) = -0.8$.

Therefore, $\theta^{-1}\left(\left(\widehat{B}_{bp}{}^{+}\right)_{\alpha_1}\right)(x) = 0.5, \theta^{-1}\left(\left(\widehat{B}_{bp}{}^{-}\right)_{\alpha_1}\right)(x) = -0.8$.

Hence $\theta^{-1}\left(\left(\widehat{B}_{bp}\right)_{\alpha_1}\right) \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha_1}$.

Therefore θ is first order bipolar fuzzy continuous.

Theorem:3.1.7

A function $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \rightarrow (Y, \hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy continuous if and only if $\theta: \left(X, \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c\right) \rightarrow \left(Y, \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c\right)$ is second order bipolar fuzzy continuous where $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_c$ and $\left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$ are from $\hat{\tau}_{\mathfrak{B}_1}$ and $\hat{\tau}_{\mathfrak{B}_2}$, respectively, through the relation R_5.

Proof:

Suppose $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \rightarrow (Y, \hat{\tau}_{\mathfrak{B}_2})$ is a second order bipolar fuzzy continuous function.

Any element $\left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$ is of the form $\left(\hat{A}_{bp}\right)_c = \left(\left(\hat{A}_{bp}{}^{+}\right)_c, \left(\hat{A}_{bp}{}^{-}\right)_c\right)$ where

$\left(\hat{A}_{bp}{}^{+}\right)_c(x)(\alpha) = \hat{A}_{bp}{}^{+}(x)(1-\alpha)$ and $\left(\hat{A}_{bp}{}^{-}\right)_c(x)(\alpha) = \hat{A}_{bp}{}^{-}(x)(1-\alpha)$, for every $x \in X$, for every $\alpha \in I$.

Consider $\left(\hat{A}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$, then $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ (through the relation R_5).
Since θ is a second order bipolar fuzzy continuous function, $\theta^{-1}(\hat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$.
Let $\hat{B}_{bp} = \theta^{-1}(\hat{A}_{bp}) = \left(\theta^{-1}\left(\hat{A}_{bp}{}^{+}\right), \theta^{-1}(\hat{A}_{bp}{}^{-})\right) \in \hat{\tau}_{\mathfrak{B}_1}$, then $\left(\hat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c$.

For $x \in X$ and $\alpha \in I$, consider

$$\begin{aligned}
\left(\hat{B}_{bp}{}^{+}\right)_c(x)(\alpha) &= \hat{B}_{bp}{}^{+}(x)(1-\alpha) \\
&= \left(\theta^{-1}\left(\hat{A}_{bp}{}^{+}\right)\right)(x)(1-\alpha) \\
&= \hat{A}_{bp}{}^{+}(\theta(x))(1-\alpha) \\
&= \left(\hat{A}_{bp}{}^{+}\right)_c(\theta(x))(\alpha) \\
&= \left(\theta^{-1}\left(\hat{A}_{bp}{}^{+}\right)_c\right)(x)(\alpha).
\end{aligned}$$

Similarly, $\left(\hat{B}_{bp}{}^{-}\right)_c(x)(\alpha) = \left(\theta^{-1}\left(\hat{A}_{bp}{}^{-}\right)_c\right)(x)(\alpha)$.

Thus $\theta^{-1}\left(\hat{A}_{bp}\right)_c = \left(\hat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c$.
Hence $\theta: \left(X, \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c\right) \rightarrow \left(Y, \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c\right)$ is second order bipolar fuzzy continuous.

Conversely, assume $\theta: \left(X, \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c\right) \to \left(Y, \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c\right)$ is second order bipolar fuzzy continuous Consider, $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$, then $\left(\hat{A}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$ (through the relation R5).

Since θ is second order bipolar fuzzy continuous, $\theta^{-1}\left(\hat{A}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c$.

Let $\hat{B}_{bp} = \theta^{-1}\left(\hat{A}_{bp}\right)_c = \left(\theta^{-1}\left(A_{bp}{}^{+}\right)_c, \theta^{-1}\left(A_{bp}{}^{-}\right)_c\right)$, then $\left(\hat{B}_{bp}\right)_c \in \hat{\tau}_{\mathfrak{B}_1}$.

For x ∈ X and α ∈ I, consider

$$\begin{aligned}\left(\hat{B}_{bp}{}^{+}\right)_c(x)(\alpha) &= \hat{B}_{bp}{}^{+}(x)(1-\alpha)\\ &= \left(\theta^{-1}(\hat{A}_{bp}{}^{+})_c\right)(x)(1-\alpha)\\ &= \left(\hat{A}_{bp}{}^{+}\right)_c \theta(x)(1-\alpha)\\ &= \hat{A}_{bp}{}^{+}\theta(x)(\alpha)\\ &= \theta^{-1}\left(\hat{A}_{bp}{}^{+}\right)(x)(\alpha)\end{aligned}$$

implies $\left(\hat{B}_{bp}{}^{+}\right)_c = \theta^{-1}\left(\hat{A}_{bp}{}^{+}\right)$.

Similarly, $\left(\hat{B}_{bp}{}^{-}\right)_c = \theta^{-1}\left(\hat{A}_{bp}{}^{-}\right)$.

Thus $\theta^{-1}\left(\hat{A}_{bp}\right) = \left(\hat{B}_{bp}\right)_c \in \hat{\tau}_{\mathfrak{B}_1}$.

Therefore, $\theta: \left(X, \hat{\tau}_{\mathfrak{B}_1}\right) \to \left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy continuous.

Example:3.1.8

Define $\theta: X \to Y$ as $\theta(x) = y$. Consider two second order bipolar fuzzy topologies $\hat{\tau}_{\mathfrak{B}_1} = \{\hat{0}_{bp}, \hat{1}_{bp}, \hat{A}_{bp}\}$ and $\hat{\tau}_{\mathfrak{B}_2} = \{\hat{0}_{bp}, \hat{1}_{bp}, \hat{B}_{bp}\}$ as the collection of second order bipolar fuzzy sets on X and Y respectively. For α ∈ I,

$\hat{A}_{bp}{}^{+}(x)(\alpha) = 0.5$, $\hat{A}_{bp}{}^{-}(x)(\alpha) = -0.3$, for every x ∈ X and

$\hat{B}_{bp}{}^{+}(y)(\alpha) = 0.5$, $\hat{B}_{bp}{}^{-}(y)(\alpha) = -0.8$, for every y ∈ Y.

As $(1-\alpha) \in I$, $\hat{A}_{bp}{}^{+}(x)(1-\alpha) = 0.5$, $\hat{A}_{bp}{}^{-}(x)(1-\alpha) = -0.3$, for every x ∈ X and

$\hat{B}_{bp}{}^{+}(y)(1-\alpha) = 0.5$, $\hat{B}_{bp}{}^{-}(y)(1-\alpha) = -0.8$, for every y ∈ Y.

Let us assume that θ is second order bipolar fuzzy continuous.

Consider $(\widehat{B}_{bp})_c \in (\hat{\tau}_{\mathfrak{B}_2})_c$ implies $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ (through R_5) then $\theta^{-1}(\widehat{B}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$

implies $\left(\theta^{-1}\left(\widehat{B}_{bp}{}^{+}\right)\right)(x)(\alpha) = 0.5, \left(\theta^{-1}(\widehat{B}_{bp}{}^{-})\right)(x)(\alpha) = -0.3$

implies $\left(\theta^{-1}\left(\widehat{B}_{bp}{}^{+}\right)\right)(x)(1-\alpha) = 0.5, \left(\theta^{-1}(\widehat{B}_{bp}{}^{-})\right)(x)(1-\alpha) = -0.3.$

Therefore $\left(\theta^{-1}(\widehat{B}_{bp})\right)_c \in (\hat{\tau}_{\mathfrak{B}_2})_c$.

Theorem:3.1.9

If $\theta: (X, \hat{\tau}_{\mathfrak{B}_1}) \to (Y, \hat{\tau}_{\mathfrak{B}_2})$ is second order bipolar fuzzy continuous, then

(i) For $\varepsilon \in (0,1)$, $\theta: \left(X, i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1})\right) \to \left(Y, i_\varepsilon(\hat{\tau}_{\mathfrak{B}_2})\right)$ is continuous.

(ii) $\theta: \left(X, i^*(\hat{\tau}_{\mathfrak{B}_1})\right) \to \left(Y, i^*(\hat{\tau}_{\mathfrak{B}_2})\right)$ is continuous.

(iii) $\theta: \left(X, i(\hat{\tau}_{\mathfrak{B}_1})\right) \to \left(Y, i(\hat{\tau}_{\mathfrak{B}_2})\right)$ is continuous.

Proof:

(i) Consider a basis element $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon$ of $i_\varepsilon(\hat{\tau}_{\mathfrak{B}_2})$ where $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$. So, $\theta^{-1}(\widehat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$.

Let $\widehat{B}_{bp} = \theta^{-1}(\widehat{A}_{bp}) = \left(\theta^{-1}\left(\widehat{A}_{bp}{}^{+}\right), \theta^{-1}(\widehat{A}_{bp}{}^{-})\right) \in \hat{\tau}_{\mathfrak{B}_1}$.

Therefore $\left(L_{\widehat{B}_{bp}}\right)_\varepsilon$ is a basis element of $i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1})$.

Consider $x \in \left(L_{\widehat{B}_{bp}}\right)_\varepsilon$

$$\Leftrightarrow \left(\widehat{B}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon, 1] = I, \left(\widehat{B}_{bp}{}^{-}(x)\right)^{-1}[-1, -\varepsilon) = I$$

$$\Leftrightarrow \left(\theta^{-1}\left(\widehat{A}_{bp}{}^{+}\right)(x)\right)^{-1}(\varepsilon, 1] = I, \left(\theta^{-1}(\widehat{A}_{bp}{}^{-})(x)\right)^{-1}[-1, -\varepsilon) = I$$

$$\Leftrightarrow \left(\left(\widehat{A}_{bp}{}^{+}\right)\theta(x)\right)^{-1}(\varepsilon, 1] = I, \left((\widehat{A}_{bp}{}^{-})\theta(x)\right)^{-1}[-1, -\varepsilon) = I$$

$$\Leftrightarrow \theta(x) \in \left(L_{\widehat{A}_{bp}}\right)_\varepsilon$$

$\Leftrightarrow x \in \theta^{-1}\left(\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}\right)$.

Therefore $\left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} = \theta^{-1}\left(\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon}\right) \in i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_1}\right)$.

Hence $\theta: \left(X, i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_1}\right)\right) \to \left(Y, i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_2}\right)\right)$ is continuous.

The proof is similar for (ii) and (iii).

Example:3.1.10

Define $\theta: X \to Y$ as $\theta(x) = y$. Consider two second order bipolar fuzzy topologies $\hat{\tau}_{\mathfrak{B}_1} = \{\hat{0}_{bp}, \hat{1}_{bp}, \widehat{A}_{bp}\}$ and $\hat{\tau}_{\mathfrak{B}_2} = \{\hat{0}_{bp}, \hat{1}_{bp}, \widehat{B}_{bp}\}$ as the collection of second order bipolar fuzzy sets on X and Y respectively.

Let $\theta: \left(X, i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_1}\right)\right) \to \left(Y, i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_2}\right)\right)$ where $i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_1}\right)$ is the topology generated by the collection $\left\{\left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}\right\}$ and $i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_2}\right)$ is the topology generated by the collection $\left\{\left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} / \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}\right\}$.

Let $\left(L_{\widehat{B}_{bp}}\right)_{\varepsilon} \in i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_2}\right)$, then $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$.

Since θ is second order bipolar fuzzy continuous, $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ implies $\theta^{-1}\left(\widehat{B}_{bp}\right) = \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$.

Hence $\left(L_{\theta^{-1}(\widehat{B}_{bp})}\right)_{\varepsilon} = \left(L_{\widehat{A}_{bp}}\right)_{\varepsilon} \in i_{\varepsilon}\left(\hat{\tau}_{\mathfrak{B}_1}\right)$.

Theorem:3.1.11

If $\theta: (X, \tau) \to (Y, \tau')$ is continuous, then

(i) For $\varepsilon \in (0,1)$, $\theta: \left(X, \widehat{\omega_{\varepsilon}(\tau)}\right) \to \left(Y, \widehat{\omega_{\varepsilon}(\tau')}\right)$ is second order bipolar fuzzy continuous.

(ii) $\theta: \left(X, \widehat{\omega(\tau)}\right) \to \left(Y, \widehat{\omega(\tau')}\right)$ is second order bipolar fuzzy continuous.

(iii) $\theta: \left(X, \widehat{\omega_{*}(\tau)}\right) \to \left(Y, \widehat{\omega_{*}(\tau')}\right)$ is second order bipolar fuzzy continuous.

Proof:

(i) By the definition, a base for $\widehat{\omega_\varepsilon(\tau')}$ is $\left(\left(\widehat{K}_{bp}\right)_\varepsilon\right)' = \{\widehat{A}_{bp} \in \mathrm{SBPF}(Y) / \left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau'\}$.

Let $\widehat{A}_{bp} \in \left(\left(\widehat{K}_{bp}\right)_\varepsilon\right)'$, then $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau'$.

Since θ is continuous, $\theta^{-1}\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau$ implies $\left(L_{\theta^{-1}(\widehat{A}_{bp})}\right)_\varepsilon \in \tau$.

Therefore $\theta^{-1}\left(\widehat{A}_{bp}\right) \in \left(\widehat{K}_{bp}\right)_\varepsilon$ which is a base for $\widehat{\omega_\varepsilon(\tau)}$.

Hence $\theta: \left(X, \widehat{\omega_\varepsilon(\tau)}\right) \rightarrow \left(Y, \widehat{\omega_\varepsilon(\tau')}\right)$ is second order bipolar fuzzy continuous.

Proofs are similar for (ii) and (iii).

Example:3.1.12

Let $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2\}$. Define $\theta: X \rightarrow Y$ as $\theta(x_1) = y_1$ and $\theta(x_2) = y_2$.

For $\varepsilon = 0.2$, then

$$\left(L_{\widehat{A}_{bp}}\right)_{0.2} = \left\{\begin{array}{r} x_1, x_2 \in X : \widehat{A}_{bp}{}^{+}(x_1)(\alpha) = 0.5 > 0.2, \widehat{A}_{bp}{}^{-}(x_1)(\alpha) = -0.8 < -0.2 \\ \widehat{A}_{bp}{}^{+}(x_2)(\alpha) = 0.3 > 0.2, \widehat{A}_{bp}{}^{-}(x_2)(\alpha) = -0.3 < -0.2 \end{array}\right\}$$

$$\left(L_{\widehat{B}_{bp}}\right)_{0.2} = \left\{\begin{array}{r} y_1, y_2 \in Y : \widehat{B}_{bp}{}^{+}(y_1)(\alpha) = 0.5 > 0.2, \widehat{B}_{bp}{}^{-}(y_1)(\alpha) = -0.8 < -0.2 \\ \widehat{B}_{bp}{}^{+}(y_2)(\alpha) = 0.3 > 0.2, \widehat{B}_{bp}{}^{-}(y_2)(\alpha) = -0.3 < -0.2 \end{array}\right\}$$

Let $\left(\widehat{K}_{bp}\right)_\varepsilon = \{\widehat{A}_{bp} \in \mathrm{SBPF}(X) / \left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau\}$ is a base for $\widehat{\omega_\varepsilon(\tau)}$ and

$\left(\left(\widehat{K}_{bp}\right)_\varepsilon\right)' = \{\widehat{B}_{bp} \in \mathrm{SBPF}(Y) / \left(L_{\widehat{B}_{bp}}\right)_\varepsilon \in \tau'\}$ is a base for $\widehat{\omega_\varepsilon(\tau')}$.

Let $\widehat{B}_{bp} \in \widehat{\omega_\varepsilon(\tau')}$, then $\left(L_{\widehat{B}_{bp}}\right)_\varepsilon \in \tau'$.

Since θ is continuous, $\left(L_{\theta^{-1}(\widehat{B}_{bp})}\right)_\varepsilon \in \tau$.

Therefore $\theta^{-1}\left(\widehat{B}_{bp}\right) \in \widehat{\omega_\varepsilon(\tau)}$.

Hence $\theta: \left(X, \widehat{\omega_\varepsilon(\tau)}\right) \rightarrow \left(Y, \widehat{\omega_\varepsilon(\tau')}\right)$ is second order bipolar fuzzy continuous.

Theorem:3.1.13

If $\theta:\left(X,\hat{\tau}_{\mathfrak{B}_1}\right) \rightarrow \left(Y,\hat{\tau}_{\mathfrak{B}_2}\right)$ is second order bipolar fuzzy continuous, then $\theta:\left(X,S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)\right) \rightarrow \left(Y,S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)\right)$ is continuous.

Proof:

A base for $S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)$ is $\{ S_2\left(\widehat{A}_{bp}\right) / \widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}\}$.

Consider a basis element $S_2\left(\widehat{A}_{bp}\right)$ of $S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)$, where $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_2} \Rightarrow \theta^{-1}(\widehat{A}_{bp}) \in \hat{\tau}_{\mathfrak{B}_1}$.

Let $\widehat{B}_{bp} = \theta^{-1}\left(\widehat{A}_{bp}\right) = \left(\theta^{-1}\left({\widehat{A}_{bp}}^{+}\right), \theta^{-1}\left({\widehat{A}_{bp}}^{-}\right)\right)$.

Therefore $S_2\left(\widehat{B}_{bp}\right)$ is a basis element of $S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)$.

Consider $x \in S_2\left(\widehat{B}_{bp}\right)$

$\Leftrightarrow {\widehat{B}_{bp}}^{+}(x)(\alpha) > 0, {\widehat{B}_{bp}}^{-}(x)(\alpha) < 0$, for every $\alpha \in I$

$\Leftrightarrow \left(\theta^{-1}\left({\widehat{A}_{bp}}^{+}\right)\right)(x)(\alpha) > 0, \left(\theta^{-1}\left({\widehat{A}_{bp}}^{-}\right)\right)(x)(\alpha) < 0$, for every $\alpha \in I$

$\Leftrightarrow {\widehat{A}_{bp}}^{+}\left(\theta(x)\right)(\alpha) > 0, {\widehat{A}_{bp}}^{-}\left(\theta(x)\right)(\alpha) < 0$, for every $\alpha \in I$

$\Leftrightarrow \theta(x) \in S_2\left(\widehat{A}_{bp}\right)$
$\Leftrightarrow x \in \theta^{-1}\left(S_2\left(\widehat{A}_{bp}\right)\right)$.

Therefore $\theta^{-1}\left(S_2\left(\widehat{A}_{bp}\right)\right)$ is a basis element of $S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)$.

Hence $\theta:\left(X,S_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)\right) \rightarrow \left(Y,S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)\right)$ is continuous.

Example:3.1.14

Define $\theta: X \rightarrow Y$ as $\theta(x) = y$. Consider two second order bipolar fuzzy topologies $\hat{\tau}_{\mathfrak{B}_1} = \left\{\hat{0}_{bp}, \hat{1}_{bp}, \widehat{A}_{bp}\right\}$ and $\hat{\tau}_{\mathfrak{B}_2} = \left\{\hat{0}_{bp}, \hat{1}_{bp}, \widehat{B}_{bp}\right\}$ as the collection of second order bipolar fuzzy sets on X and Y respectively.

Let $\mathbb{S}_2\left(\widehat{B}_{bp}\right) \in S_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)$, then $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$.

Since θ is second order bipolar fuzzy continuous, $\widehat{\mathrm{B}}_{\mathrm{bp}} \in \hat{\tau}_{\mathfrak{B}_2}$, then $\theta^{-1}\left(\widehat{\mathrm{B}}_{\mathrm{bp}}\right) \in \hat{\tau}_{\mathfrak{B}_1}$.

Therefore $\mathbb{S}_2\left(\theta^{-1}\left(\widehat{\mathrm{B}}_{\mathrm{bp}}\right)\right) = \mathrm{S}_2\left(\widehat{\mathrm{A}}_{\mathrm{bp}}\right) \in \mathrm{S}_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)$.

Hence $\theta: \left(\mathrm{X}, \mathrm{S}_2\left(\hat{\tau}_{\mathfrak{B}_1}\right)\right) \to \left(\mathrm{Y}, \mathrm{S}_2\left(\hat{\tau}_{\mathfrak{B}_2}\right)\right)$ is second order bipolar fuzzy continuous.

SECTION – 3.2

FIRST ORDER BIPOLAR FUZZY PRODUCT TOPOLOGY

Definition:3.2.1

Let $(X, \tau_{\mathfrak{B}_1})$ and $(Y, \tau_{\mathfrak{B}_2})$ be two bipolar fuzzy topological spaces. If $A_{bp} \in \tau_{\mathfrak{B}_1}$ and $B_{bp} \in \tau_{\mathfrak{B}_2}$ then the **product** of $A_{bp} \times B_{bp}$ on $X \times Y$ is defined as follows:

$A_{bp} \times B_{bp} = ({A_{bp}}^{+} \times {B_{bp}}^{+}, {A_{bp}}^{-} \times {B_{bp}}^{-})$ where

$({A_{bp}}^{+} \times {B_{bp}}^{+})(x, y) = \min\{{A_{bp}}^{+}(x), {B_{bp}}^{+}(y)\}$ and

$({A_{bp}}^{-} \times {B_{bp}}^{-})(x, y) = \max\{{A_{bp}}^{-}(x), {B_{bp}}^{-}(y)\}$, for every $(x, y) \in X \times Y$.

Definition:3.2.2

Let $(X, \tau_{\mathfrak{B}_1})$ and $(Y, \tau_{\mathfrak{B}_2})$ be two bipolar fuzzy topological spaces. The **product topology** $\tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}$ on $X \times Y$ is the bipolar fuzzy topology having the collection $\{A_{bp} \times B_{bp} \,/\, A_{bp} \in \tau_{\mathfrak{B}_1}, B_{bp} \in \tau_{\mathfrak{B}_2}\}$ as a basis.

Definition:3.2.3

Let $\{(X_\lambda, (\tau_{\mathfrak{B}})_\lambda) \,/\, \lambda \in \Lambda\}$ be a family of bipolar fuzzy topological spaces and $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\left\{(A_{bp})_\lambda = \left(({A_{bp}}^{+})_\lambda, ({A_{bp}}^{-})_\lambda\right) / \lambda \in \Lambda\right\}$ be a collection of bipolar fuzzy sets in X_λ. The **product** $\prod_{\lambda \in \Lambda}(A_{bp})_\lambda$ is a bipolar fuzzy set in $\prod_{\lambda \in \Lambda} X_\lambda$ is defined as

$\prod_{\lambda \in \Lambda}(A_{bp})_\lambda = \left(\bigwedge_{\lambda \in \Lambda}({A_{bp}}^{+})_\lambda, \bigvee_{\lambda \in \Lambda}({A_{bp}}^{-})_\lambda\right)$ where

$\bigwedge_{\lambda \in \Lambda}({A_{bp}}^{+})_\lambda(x) = \min\left\{({A_{bp}}^{+})_\lambda(x)\right\}$, for every $(x_\lambda)_{\lambda \in \Lambda} \in \prod x_\lambda$ and

$\bigvee_{\lambda \in \Lambda}({A_{bp}}^{-})_\lambda(x) = \max\left\{({A_{bp}}^{-})_\lambda(x)\right\}$, for every $(x_\lambda)_{\lambda \in \Lambda} \in \prod x_\lambda$.

The bipolar fuzzy product topology on X is the one with basic bipolar fuzzy open sets of the form $\prod_{\lambda \in \Lambda}(A_{bp})_\lambda$ where $(A_{bp})_\lambda \in \tau_{\mathfrak{B}_\lambda}$ and $(A_{bp})_\lambda = 1_{bp}$ except for finitely many λ's.

Proposition:3.2.4

Let $\left(X, \tau_{\mathfrak{B}_1}\right)$ and $\left(Y, \tau_{\mathfrak{B}_2}\right)$ be two bipolar fuzzy topological spaces. If $A_{bp} \in \tau_{\mathfrak{B}_1}$ and $B_{bp} \in \tau_{\mathfrak{B}_2}$, then

(i) $A_{bp} \times 0_{bp} = 0_{bp} \times B_{bp} = 0_{bp} = 0_{bp} \times 0_{bp}$.

(ii) $A_{bp} \times 1_{bp} = \left({A_{bp}}^{+} \times {1_{bp}}^{+}, {A_{bp}}^{-} \times {1_{bp}}^{-}\right)$ where

$\left({A_{bp}}^{+} \times {1_{bp}}^{+}\right)(x, y) = {A_{bp}}^{+}(x)$, $\left({A_{bp}}^{-} \times {1_{bp}}^{-}\right)(x, y) = {A_{bp}}^{-}(x)$, for every $x \in X$ and for every $y \in Y$

and $1_{bp} \times B_{bp} = \left({1_{bp}}^{+} \times {B_{bp}}^{+}, {1_{bp}}^{-} \times {B_{bp}}^{-}\right)$ where

$\left({1_{bp}}^{+} \times {B_{bp}}^{+}\right)(x, y) = {B_{bp}}^{+}(x)$,$\left({1_{bp}}^{-} \times {B_{bp}}^{-}\right)(x, y) = {B_{bp}}^{-}(x)$, for every $x \in X$ and for every $y \in Y$.

(iii) $\left((A_{bp})_1 \cap (A_{bp})_2\right) \times \left((B_{bp})_1 \cap (B_{bp})_2\right) = \left((A_{bp})_1 \times (B_{bp})_1\right) \cap \left((A_{bp})_2 \times (B_{bp})_2\right)$.

Proof:

(i) Let $A_{bp} \times 0_{bp} = \left({A_{bp}}^{+} \times {0_{bp}}^{+}, {A_{bp}}^{-} \times {0_{bp}}^{-}\right)$.

Consider for $(x, y) \in X \times Y$

$$\begin{aligned}\left({A_{bp}}^{+} \times {0_{bp}}^{+}\right)(x, y) &= \min\{{A_{bp}}^{+}(x), {0_{bp}}^{+}(y)\} \\ &= \min\{{A_{bp}}^{+}(x), \mathbf{0}\} \\ &= \mathbf{0}\end{aligned}$$

$$\begin{aligned}\left({A_{bp}}^{-} \times {0_{bp}}^{-}\right)(x, y) &= \max\{{A_{bp}}^{-}(x), {0_{bp}}^{-}(y)\} \\ &= \max\{{A_{bp}}^{-}(x), \mathbf{0}\} \\ &= \mathbf{0}.\end{aligned}$$

Therefore $A_{bp} \times 0_{bp} = (\mathbf{0}, \mathbf{0}) = 0_{bp}$.

Similarly, $0_{bp} \times B_{bp} = (\mathbf{0}, \mathbf{0}) = 0_{bp}$ and $0_{bp} \times 0_{bp} = (\mathbf{0}, \mathbf{0}) = 0_{bp}$.

(ii) Let $A_{bp} \times 1_{bp} = \left({A_{bp}}^{+} \times {1_{bp}}^{+}, {A_{bp}}^{-} \times {1_{bp}}^{-}\right)$

$$\left({A_{bp}}^{+} \times {1_{bp}}^{+}\right)(x,y) = \min\{{A_{bp}}^{+}(x), {1_{bp}}^{+}(y)\}, \text{ for every } x \in X \text{ and for every } y \in Y$$

$$= \min\{{A_{bp}}^{+}(x), \mathbf{1}\}, \text{ for every } x \in X$$

$$= {A_{bp}}^{+}(x), \text{ for every } x \in X.$$

$$\left({A_{bp}}^{-} \times {1_{bp}}^{-}\right)(x,y) = \max\{{A_{bp}}^{-}(x), {1_{bp}}^{-}(y)\}, \text{ for every } x \in X \text{ and for every } y \in Y$$

$$= \max\{{A_{bp}}^{-}(x), -\mathbf{1}\}, \text{ for every } x \in X$$

$$= {A_{bp}}^{-}(x), \text{ for every } x \in X.$$

Similarly, $1_{bp} \times B_{bp} = \left({1_{bp}}^{+} \times {B_{bp}}^{+}, {1_{bp}}^{-} \times {B_{bp}}^{-}\right)$,

where $\left({1_{bp}}^{+} \times {B_{bp}}^{+}\right)(x,y) = {B_{bp}}^{+}(y)$ & $\left({1_{bp}}^{-} \times {B_{bp}}^{-}\right)(x,y) = {B_{bp}}^{-}(y)$.

(iii) Let $\left(A_{bp}\right)_1 \times \left(B_{bp}\right)_1 = \left(\left({A_{bp}}^{+}\right)_1 \times \left({B_{bp}}^{+}\right)_1, \left({A_{bp}}^{-}\right)_1 \times \left({B_{bp}}^{-}\right)_1\right)$

$$\left(A_{bp}\right)_2 \times \left(B_{bp}\right)_2 = \left(\left({A_{bp}}^{+}\right)_2 \times \left({B_{bp}}^{+}\right)_2, \left({A_{bp}}^{-}\right)_2 \times \left({B_{bp}}^{-}\right)_2\right)$$

$$\left(\left(A_{bp}\right)_1 \times \left(B_{bp}\right)_1\right) \cap \left(\left(A_{bp}\right)_2 \times \left(B_{bp}\right)_2\right)$$

$$= \left(\left(\left({A_{bp}}^{+}\right)_1 \times \left({B_{bp}}^{+}\right)_1\right) \wedge \left(\left({A_{bp}}^{+}\right)_2 \times \left({B_{bp}}^{+}\right)_2\right), \left(\left({A_{bp}}^{-}\right)_1 \times \left({B_{bp}}^{-}\right)_1\right) \vee \left(\left({A_{bp}}^{-}\right)_2 \times \left({B_{bp}}^{-}\right)_2\right)\right).$$

For $(x,y) \in X \times Y$,

Consider

$$\left(\left(\left({A_{bp}}^{+}\right)_1 \times \left({B_{bp}}^{+}\right)_1\right) \wedge \left(\left({A_{bp}}^{+}\right)_2 \times \left({B_{bp}}^{+}\right)_2\right)\right)(x,y)$$

$$= \left(\left({A_{bp}}^{+}\right)_1 \times \left({B_{bp}}^{+}\right)_1\right)(x,y) \wedge \left(\left({A_{bp}}^{+}\right)_2 \times \left({B_{bp}}^{+}\right)_2\right)(x,y)$$

$$= \min[\,\min\{\left({A_{bp}}^{+}\right)_1(x), \left({B_{bp}}^{+}\right)_1(y)\}, \min\{\left({A_{bp}}^{+}\right)_2(x), \left({B_{bp}}^{+}\right)_2(y)\}]$$

$$= \min[\,\min\{\left({A_{bp}}^{+}\right)_1(x), \left({A_{bp}}^{+}\right)_2(x)\}, \min\{\left({B_{bp}}^{+}\right)_1(y), \left({B_{bp}}^{+}\right)_2(y)\}]$$

$$= \min\{\left(\left({A_{bp}}^{+}\right)_1 \wedge \left({A_{bp}}^{+}\right)_2\right)(x), \left(\left({B_{bp}}^{+}\right)_1 \wedge \left({B_{bp}}^{+}\right)_2\right)(y)\}$$

$$= \left(\left(\left({A_{bp}}^{+}\right)_1 \wedge \left({A_{bp}}^{+}\right)_2\right) \times \left(\left({B_{bp}}^{+}\right)_1 \wedge \left({B_{bp}}^{+}\right)_2\right)\right)(x,y).$$

Also,

$$\left(\left(\left({A_{bp}}^{-}\right)_1 \times \left({B_{bp}}^{-}\right)_1\right) \vee \left(\left({A_{bp}}^{-}\right)_2 \times \left({B_{bp}}^{-}\right)_2\right)\right)(x, y)$$

$$= \left(\left({A_{bp}}^{-}\right)_1 \times \left({B_{bp}}^{-}\right)_1\right)(x, y) \vee \left(\left({A_{bp}}^{-}\right)_2 \times \left({B_{bp}}^{-}\right)_2\right)(x, y)$$

$$= \max\left[\, \max\{\left({A_{bp}}^{-}\right)_1(x), \left({B_{bp}}^{-}\right)_1(y)\}, \max\{\left({A_{bp}}^{-}\right)_2(x), \left({B_{bp}}^{-}\right)_2(y)\}\right]$$

$$= \max\left[\, \max\{\left({A_{bp}}^{-}\right)_1(x), \left({A_{bp}}^{-}\right)_2(x)\}, \max\{\left({B_{bp}}^{-}\right)_1(y), \left({B_{bp}}^{-}\right)_2(y)\}\right]$$

$$= \max\left\{\left(\left({A_{bp}}^{-}\right)_1 \vee \left({A_{bp}}^{-}\right)_2\right)(x), \left(\left({B_{bp}}^{-}\right)_1 \vee \left({B_{bp}}^{-}\right)_2\right)(y)\right\}$$

$$= \left(\left(\left({A_{bp}}^{-}\right)_1 \vee \left({A_{bp}}^{-}\right)_2\right) \times \left(\left({B_{bp}}^{-}\right)_1 \vee \left({B_{bp}}^{-}\right)_2\right)\right)(x, y).$$

Hence $\left(\left(A_{bp}\right)_1 \cap \left(A_{bp}\right)_2\right) \times \left(\left(B_{bp}\right)_1 \cap \left(B_{bp}\right)_2\right) = \left(\left(A_{bp}\right)_1 \times \left(B_{bp}\right)_1\right) \cap \left(\left(A_{bp}\right)_2 \times \left(B_{bp}\right)_2\right)$.

Theorem : 3.2.5

Let $\left(X, \tau_{\mathfrak{B}_1}\right)$ and $\left(Y, \tau_{\mathfrak{B}_2}\right)$ be two bipolar fuzzy topological spaces. If A_{bp} and B_{bp} be two bipolar fuzzy closed subsets in X and Y respectively, then $A_{bp} \times B_{bp}$ is a bipolar fuzzy closed subset in $X \times Y$.

Proof:

If A_{bp} be a bipolar fuzzy closed set in X then ${A_{bp}}^{c}$ is a bipolar fuzzy open set in X. Therefore $\left({A_{bp}}^{c} \times 1_{bp} \text{ in } Y\right)$ is a bipolar fuzzy open set in $X \times Y$.

Also, if B_{bp} is a bipolar fuzzy closed set in Y then ${B_{bp}}^{c}$ is a bipolar fuzzy open set in Y. Therefore $\left(1_{bp} \text{ in } X \times {B_{bp}}^{c}\right)$ is bipolar fuzzy open in $X \times Y$.

$$\left(A_{bp} \times B_{bp}\right)^{c} = \left(\left({A_{bp}}^{+} \times {B_{bp}}^{+}\right)^{c}, \left({A_{bp}}^{-} \times {B_{bp}}^{-}\right)^{c}\right).$$

For $(x, y) \in X \times Y$, we have

$$\left({A_{bp}}^{+} \times {B_{bp}}^{+}\right)^{c}(x, y) = 1 - \left({A_{bp}}^{+} \times {B_{bp}}^{+}\right)(x, y)$$

$$= 1 - \min\{{A_{bp}}^{+}(x), {B_{bp}}^{+}(y)\}$$

$$= \max\{1 - {A_{bp}}^{+}(x), 1 - {B_{bp}}^{+}(y)\}$$

$$= \max\{(A_{bp}{}^{+})^{c}(x), (B_{bp}{}^{+})^{c}(y)\}$$

$$= \max\{(A_{bp}{}^{+})^{c}(x) \wedge 1_{bp}{}^{+}(y), 1_{bp}{}^{+}(x) \wedge (B_{bp}{}^{+})^{c}(y)\}$$

$$= \max\{((A_{bp}{}^{+})^{c} \times 1_{bp}{}^{+})(x,y), (1_{bp}{}^{+} \times (B_{bp}{}^{+})^{c})(x,y)\}$$

$$= ((A_{bp}{}^{+})^{c} \times 1_{bp}{}^{+}) \vee (1_{bp}{}^{+} \times (B_{bp}{}^{+})^{c})(x,y).$$

Therefore $(A_{bp}{}^{+} \times B_{bp}{}^{+})^{c} = ((A_{bp}{}^{+})^{c} \times 1_{bp}{}^{+}) \vee (1_{bp}{}^{+} \times (B_{bp}{}^{+})^{c})$ is open in $X \times Y$.

Similarly $(A_{bp}{}^{-} \times B_{bp}{}^{-})^{c} = ((A_{bp}{}^{-})^{c} \times 1_{bp}{}^{-}) \wedge (1_{bp}{}^{-} \times (B_{bp}{}^{-})^{c})$ is open in $X \times Y$.

Thus $(A_{bp} \times B_{bp})^{c}$ is a bipolar fuzzy open set in $X \times Y$.

Hence $(A_{bp} \times B_{bp})$ is bipolar fuzzy closed in $X \times Y$.

SECTION – 3.3

SECOND ORDER BIPOLAR FUZZY PROUCT TOPOLOGY

Definition : 3.3.1

Let $(X, \hat{\tau}_{\mathfrak{B}_1})$, $(Y, \hat{\tau}_{\mathfrak{B}_2})$ be two second order bipolar fuzzy topological spaces. If $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ and $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$ then the **product** $\hat{A}_{bp} \times \hat{B}_{bp}$ on $X \times Y$ is defined as follows:

$\hat{A}_{bp} \times \hat{B}_{bp} = \left(\left(\hat{A}_{bp}{}^{+} \times \hat{B}_{bp}{}^{+}\right), \left(\hat{A}_{bp}{}^{-} \times \hat{B}_{bp}{}^{-}\right)\right)$ such that

$\left(\hat{A}_{bp}{}^{+} \times \hat{B}_{bp}{}^{+}\right)(x, y)(\alpha) = \hat{A}_{bp}{}^{+}(x)(\alpha) \wedge \hat{B}_{bp}{}^{+}(y)(\alpha)$ and

$\left(\hat{A}_{bp}{}^{-} \times \hat{B}_{bp}{}^{-}\right)(x, y)(\alpha) = \hat{A}_{bp}{}^{-}(x)(\alpha) \vee \hat{B}_{bp}{}^{-}(y)(\alpha)$, for every $(x, y) \in X \times Y$ and

for every $\alpha \in I$.

The **product topology** $\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$ on $X \times Y$ is the second order bipolar fuzzy topology having the collection $\{\hat{A}_{bp} \times \hat{B}_{bp} \,/\, \hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}, \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}\}$ as a basis.

Definition: 3.3.2

Let $\{(X_\lambda, \hat{\tau}_{\mathfrak{B}_\lambda}) \,/\, \lambda \in \Lambda\}$ be a family of second order bipolar fuzzy topological spaces and $X = \prod_{\lambda \in \Lambda} X_\lambda$. Let $\left\{(\hat{A}_{bp})_\lambda = \left(\left(\hat{A}_{bp}{}^{+}\right)_\lambda, \left(\hat{A}_{bp}{}^{-}\right)_\lambda\right) / \lambda \in \Lambda\right.$, where $(\hat{A}_{bp})_\lambda$ is a second order bipolar fuzzy set in $\left.X_\lambda\right\}$. The **product** $\prod_{\lambda \in \Lambda}(\hat{A}_{bp})_\lambda$ is a second order bipolar fuzzy set in $\prod_{\lambda \in \Lambda} X_\lambda$ defined as $\prod_{\lambda \in \Lambda}(\hat{A}_{bp})_\lambda = \left(\bigwedge_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{+}\right)_\lambda, \ \bigvee_{\lambda \in \Lambda}\left(\hat{A}_{bp}{}^{-}\right)_\lambda\right)$,

for every $(x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_\lambda$ and for every $\alpha \in I$.

The second order bipolar fuzzy product topology on X is the one with basic second order bipolar fuzzy open sets of the form $\prod_{\lambda \in \Lambda}(\hat{A}_{bp})_\lambda$, where $(\hat{A}_{bp})_\lambda \in \hat{\tau}_{\mathfrak{B}_\lambda}$ and $(\hat{A}_{bp})_\lambda = \hat{1}_{bp}$, except for finitely many λ's.

Theorem : 3.3.3

Let $\left(X, \tau_{\mathfrak{B}_1}\right)$ and $\left(Y, \tau_{\mathfrak{B}_2}\right)$ be two first order bipolar fuzzy topological spaces. If $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be the second order bipolar fuzzy topological spaces from $\left(X, \tau_{\mathfrak{B}_1}\right)$ and $\left(Y, \tau_{\mathfrak{B}_2}\right)$ respectively, through the relation R_1, then $\tau_{\mathfrak{B}_1} \widehat{\times} \tau_{\mathfrak{B}_2} = \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$.

Proof :

Let us consider the basis element

$$\left(\hat{A}_{bp}\right)_1 \times \left(\hat{A}_{bp}\right)_2 = \left(\left(\hat{A}_{bp}{}^{+}\right)_1 \times \left(\hat{A}_{bp}{}^{+}\right)_2, \left(\hat{A}_{bp}{}^{-}\right)_1 \times \left(\hat{A}_{bp}{}^{-}\right)_2\right) \text{ of } \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}.$$

For $(x, y) \in X \times Y$ and $\alpha \in I$, consider

$$\begin{aligned}
\left(\left(\hat{A}_{bp}{}^{+}\right)_1 \times \left(\hat{A}_{bp}{}^{+}\right)_2\right)(x, y)(\alpha) &= \left(\hat{A}_{bp}{}^{+}\right)_1 (x)(\alpha) \wedge \left(\hat{A}_{bp}{}^{+}\right)_2 (y)(\alpha) \\
&= \left(A_{bp}{}^{+}\right)_1 (x) \wedge \left(A_{bp}{}^{+}\right)_2 (y) \\
&= \left(\left(A_{bp}{}^{+}\right)_1 \times \left(A_{bp}{}^{+}\right)_2\right)(x, y) \\
&= \left(\left(A_{bp}{}^{+}\right)_1 \widehat{\times} \left(A_{bp}{}^{+}\right)_2\right)(x, y)(\alpha).
\end{aligned}$$

Therefore $\left(\hat{A}_{bp}{}^{+}\right)_1 \times \left(\hat{A}_{bp}{}^{+}\right)_2 = \left(A_{bp}{}^{+}\right)_1 \widehat{\times} \left(A_{bp}{}^{+}\right)_2$.

$$\begin{aligned}
\left(\left(\widehat{A_{bp}}{}^{-}\right)_1 \times \left(\widehat{A_{bp}}{}^{-}\right)_2\right)(x, y)(\alpha) &= \left(\hat{A}_{bp}{}^{-}\right)_1 (x)(\alpha) \vee \left(\hat{A}_{bp}{}^{-}\right)_2 (y)(\alpha) \\
&= \left(A_{bp}{}^{-}\right)_1 (x) \vee \left(A_{bp}{}^{-}\right)_2 (y) \\
&= \left(\left(A_{bp}{}^{-}\right)_1 \times \left(A_{bp}{}^{-}\right)_2\right)(x, y) \\
&= \left(\left(A_{bp}{}^{-}\right)_1 \widehat{\times} \left(A_{bp}{}^{-}\right)_2\right)(x, y)(\alpha).
\end{aligned}$$

Hence $\left(\hat{A}_{bp}{}^{-}\right)_1 \times \left(\hat{A}_{bp}{}^{-}\right)_2 = \left(A_{bp}{}^{-}\right)_1 \widehat{\times} \left(A_{bp}{}^{-}\right)_2$.

Thus $\left(\hat{A}_{bp}\right)_1 \times \left(\hat{A}_{bp}\right)_2 = \left(A_{bp}\right)_1 \widehat{\times} \left(A_{bp}\right)_2 \in \tau_{\mathfrak{B}_1} \widehat{\times} \tau_{\mathfrak{B}_2}$.

Since $\left(\hat{A}_{bp}\right)_1 \times \left(\hat{A}_{bp}\right)_2$ is a basis element of $\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$,

$\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2} \subseteq \widehat{\tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}}$ (1)

Consider $\hat{B}_{bp} \in \widehat{\tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}}$ where $B_{bp} \in \tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}$

$$B_{bp} = \cup\left\{\left(A_{bp}\right)_1 \times \left(A_{bp}\right)_2 / \left(A_{bp}\right)_1 \in \tau_{\mathfrak{B}_1}, \left(A_{bp}\right)_2 \in \tau_{\mathfrak{B}_2} \,\&\, \left(A_{bp}\right)_1 \times \left(A_{bp}\right)_2 < B_{bp}\right\}$$

$$\hat{B}_{bp} = \cup\left\{\widehat{\left(A_{bp}\right)_1 \times \left(A_{bp}\right)_2} / \left(A_{bp}\right)_1 \in \tau_{\mathfrak{B}_1}, \left(A_{bp}\right)_2 \in \tau_{\mathfrak{B}_2} \,\&\, \left(A_{bp}\right)_1 \times \left(A_{bp}\right)_2 < B_{bp}\right\}$$

$$\hat{B}_{bp} = \cup\left\{\left(\hat{A}_{bp}\right)_1 \times \left(\hat{A}_{bp}\right)_2 / \left(\hat{A}_{bp}\right)_1 \in \hat{\tau}_{\mathfrak{B}_1}, \left(\hat{A}_{bp}\right)_2 \in \hat{\tau}_{\mathfrak{B}_2} \,\&\, \left(\hat{A}_{bp}\right)_1 \times \left(\hat{A}_{bp}\right)_2 < \hat{B}_{bp}\right\},$$

then $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$.

Therefore $\widehat{\tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}} \subseteq \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$ (2)

From (1) and (2), we get

$$\widehat{\tau_{\mathfrak{B}_1} \times \tau_{\mathfrak{B}_2}} = \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}.$$

Theorem:3.3.4

Let $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be two second order bipolar fuzzy topological spaces. For $\alpha \in I$, if $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha$ and $\left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$ be the first order bipolar fuzzy topologies from $\hat{\tau}_{\mathfrak{B}_1}$ and $\hat{\tau}_{\mathfrak{B}_2}$ respectively, through the relation R_3, then $\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_\alpha = \left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$.

Proof :

Consider a basis element

$\left(\hat{A}_{bp}\right)_\alpha \times \left(\hat{B}_{bp}\right)_\alpha = \left(\left(\hat{A}_{bp}{}^{+}\right)_\alpha \times \left(\hat{B}_{bp}{}^{+}\right)_\alpha, \left(\hat{A}_{bp}{}^{-}\right)_\alpha \times \left(\hat{B}_{bp}{}^{-}\right)_\alpha\right)$ of $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$.

where $\left(\hat{A}_{bp}\right)_\alpha \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha$ and $\left(\hat{B}_{bp}\right)_\alpha \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$.

Then $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ and $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$.

Therefore $\hat{A}_{bp} \times \hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$.

For $(x, y) \in X \times Y$ and $\alpha \in I$, consider

$$\left(\left(\hat{A}_{bp}{}^{+}\right)_\alpha \times \left(\hat{B}_{bp}{}^{+}\right)_\alpha\right)(x, y) = \left(\hat{A}_{bp}{}^{+}\right)_\alpha (x) \wedge \left(\hat{B}_{bp}{}^{+}\right)_\alpha (y)$$

$$= \left(\widehat{A}_{bp}{}^{+}\right)(x)(\alpha) \wedge \left(\widehat{B}_{bp}{}^{+}\right)(y)(\alpha)$$

$$= \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)(x, y)(\alpha)$$

$$= \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_{\alpha}(x, y).$$

Then, $\left(\widehat{A}_{bp}{}^{+}\right)_{\alpha} \times \left(\widehat{B}_{bp}{}^{+}\right)_{\alpha} = \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_{\alpha}$ and

$$\left(\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha} \times \left(\widehat{B}_{bp}{}^{-}\right)_{\alpha}\right)(x, y) = \left(\widehat{A}_{bp}{}^{-}\right)_{\alpha}(x) \vee \left(\widehat{B}_{bp}{}^{-}\right)_{\alpha}(y)$$

$$= \left(\widehat{A}_{bp}{}^{-}\right)(x)(\alpha) \vee \left(\widehat{B}_{bp}{}^{-}\right)(y)(\alpha)$$

$$= \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)(x, y)(\alpha)$$

$$= \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_{\alpha}(x, y).$$

Thus $\left(\widehat{A}_{bp}{}^{-}\right)_{\alpha} \times \left(\widehat{B}_{bp}{}^{-}\right)_{\alpha} = \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_{\alpha}$,

and $\left(\widehat{A}_{bp}\right)_{\alpha} \times \left(\widehat{B}_{bp}\right)_{\alpha} = \left(\left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_{\alpha}, \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_{\alpha}\right)$

$$= \left(\widehat{A}_{bp} \times \widehat{B}_{bp}\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}.$$

Therefore $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha} \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha} \subseteq \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$. (1)

Consider $\left(\widehat{B}_{bp}\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$, for every $\alpha \in I$

implies $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$

$$\widehat{B}_{bp} = \cup \left\{\left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 / \left(\widehat{A}_{bp}\right)_1 \in \hat{\tau}_{\mathfrak{B}_1}, \left(\widehat{A}_{bp}\right)_2 \in \hat{\tau}_{\mathfrak{B}_2} \text{ and } \left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 < \widehat{B}_{bp}\right\}$$

$$\left(\widehat{B}_{bp}\right)_{\alpha} = \cup\left\{\left(\left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2\right)_{\alpha} / \left(\widehat{A}_{bp}\right)_1 \in \hat{\tau}_{\mathfrak{B}_1}, \left(\widehat{A}_{bp}\right)_2 \in \hat{\tau}_{\mathfrak{B}_2} \text{ and } \left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 < \widehat{B}_{bp}\right\}$$

$$= \cup\left\{\left(\left(\widehat{A}_{bp}\right)_1\right)_{\alpha} \times \left(\left(\widehat{A}_{bp}\right)_2\right)_{\alpha} / \left(\left(\widehat{A}_{bp}\right)_1\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha}, \left(\left(\widehat{A}_{bp}\right)_2\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha},\right.$$

$$\left. \& \left(\left(\widehat{A}_{bp}\right)_1\right)_{\alpha} \times \left(\left(\widehat{A}_{bp}\right)_2\right)_{\alpha} < \left(\widehat{B}_{bp}\right)_{\alpha}\right\}$$

$$\left(\widehat{B}_{bp}\right)_{\alpha} \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_{\alpha} \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_{\alpha}$$

$\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_\alpha \subseteq \left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$. (2)

From (1) and (2), we get

$\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_\alpha = \left(\hat{\tau}_{\mathfrak{B}_1}\right)_\alpha \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_\alpha$.

Theorem:3.3.5

Let $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be two second order bipolar fuzzy topological spaces. If $\left(X, \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c\right)$ and $\left(Y, \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c\right)$ be the two second order bipolar fuzzy topological spaces from $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ respectively, through the relation R_5, then $\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c = \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$.

Proof :

Consider a basis element

$\left(\widehat{A}_{bp}\right)_c \times \left(\widehat{B}_{bp}\right)_c = \left(\left(\widehat{A}_{bp}{}^{+}\right)_c \times \left(\widehat{B}_{bp}{}^{+}\right)_c, \left(\widehat{A}_{bp}{}^{-}\right)_c \times \left(\widehat{B}_{bp}{}^{-}\right)_c\right)$ of $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$.

where $\left(\widehat{A}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c$ and $\left(\widehat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$.

Then $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ and $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$.

Therefore $\widehat{A}_{bp} \times \widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$.

For $(x, y) \in X \times Y$ and $\alpha \in I$, consider

$$\begin{aligned}\left(\left(\widehat{A}_{bp}{}^{+}\right)_c \times \left(\widehat{B}_{bp}{}^{+}\right)_c\right)(x, y)(\alpha) &= \left(\widehat{A}_{bp}{}^{+}\right)_c (x)(\alpha) \wedge \left(\widehat{B}_{bp}{}^{+}\right)_c (y)(\alpha) \\ &= \left(\widehat{A}_{bp}{}^{+}\right)(x)(1-\alpha) \wedge \left(\widehat{B}_{bp}{}^{+}\right)(y)(1-\alpha) \\ &= \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)(x, y)(1-\alpha) \\ &= \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_c (x, y)(\alpha).\end{aligned}$$

Then $\left(\widehat{A}_{bp}{}^{+}\right)_c \times \left(\widehat{B}_{bp}{}^{+}\right)_c = \left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_c$ and

$$\begin{aligned}
\left(\left(\widehat{A}_{bp}{}^{-}\right)_c \times \left(\widehat{B}_{bp}{}^{-}\right)_c\right)(x,y)(\alpha) &= \left(\widehat{A}_{bp}{}^{-}\right)_c(x)(\alpha) \vee \left(\widehat{B}_{bp}{}^{-}\right)_c(y)(\alpha) \\
&= \left(\widehat{A}_{bp}{}^{-}\right)(x)(1-\alpha) \vee \left(\widehat{B}_{bp}{}^{-}\right)(y)(1-\alpha) \\
&= \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)(x,y)(1-\alpha) \\
&= \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_c(x,y)(\alpha).
\end{aligned}$$

Then $\left(\widehat{A}_{bp}{}^{-}\right)_c \times \left(\widehat{B}_{bp}{}^{-}\right)_c = \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_c$.

$$\begin{aligned}
\text{Thus } \left(\widehat{A}_{bp}\right)_c \times \left(\widehat{B}_{bp}\right)_c &= \left(\left(\widehat{A}_{bp}{}^{+} \times \widehat{B}_{bp}{}^{+}\right)_c, \left(\widehat{A}_{bp}{}^{-} \times \widehat{B}_{bp}{}^{-}\right)_c\right) \\
&= \left(\widehat{A}_{bp} \times \widehat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c.
\end{aligned}$$

Hence $\left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c \subseteq \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c$. (1)

Consider $\left(\widehat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c$. Then $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}$

$$\widehat{B}_{bp} = \cup\left\{\left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 / \left(\widehat{A}_{bp}\right)_1 \in \hat{\tau}_{\mathfrak{B}_1}, \left(\widehat{A}_{bp}\right)_2 \in \hat{\tau}_{\mathfrak{B}_2} \text{ and } \left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 < \widehat{B}_{bp}\right\}$$

$$\begin{aligned}
\left(\widehat{B}_{bp}\right)_c &= \cup\{\left(\left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2\right)_c / \left(\widehat{A}_{bp}\right)_1 \in \hat{\tau}_{\mathfrak{B}_1}, \left(\widehat{A}_{bp}\right)_2 \in \hat{\tau}_{\mathfrak{B}_2} \,\&\, \\
&\qquad \left(\widehat{A}_{bp}\right)_1 \times \left(\widehat{A}_{bp}\right)_2 < \widehat{B}_{bp}\} \\
&= \cup\{\left(\left(\widehat{A}_{bp}\right)_1\right)_c \times \left(\left(\widehat{A}_{bp}\right)_2\right)_c / \left(\left(\widehat{A}_{bp}\right)_1\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c, \left(\left(\widehat{A}_{bp}\right)_2\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c \,\&\, \\
&\qquad \left(\left(\widehat{A}_{bp}\right)_1\right)_c \times \left(\left(\widehat{A}_{bp}\right)_2\right)_c < \left(\widehat{B}_{bp}\right)_c\}.
\end{aligned}$$

Then $\left(\widehat{B}_{bp}\right)_c \in \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$

Thus $\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c \subseteq \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$ (2)

From (1) and (2) , we get $\left(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2}\right)_c = \left(\hat{\tau}_{\mathfrak{B}_1}\right)_c \times \left(\hat{\tau}_{\mathfrak{B}_2}\right)_c$.

Theorem : 3.3.6

If $(X, \hat{\tau}_{\mathfrak{B}_1})$ and $(Y, \hat{\tau}_{\mathfrak{B}_2})$ be two second order bipolar fuzzy topological spaces, then $S_2(\hat{\tau}_{\mathfrak{B}_1}) \times S_2(\hat{\tau}_{\mathfrak{B}_2}) \subseteq S_2(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$

Proof:

For any $\widehat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$, $\widehat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$, consider an element $S_2(\widehat{A}_{bp}) \times S_2(\widehat{B}_{bp})$ of $S_2(\hat{\tau}_{\mathfrak{B}_1}) \times S_2(\hat{\tau}_{\mathfrak{B}_2})$.

So, $S_2(\widehat{A}_{bp} \times \widehat{B}_{bp})$

$$= \{(x,y) \in X \times Y \,/\, \left({\widehat{A}_{bp}}^{+} \times {\widehat{B}_{bp}}^{+}\right)(x,y)(\alpha) > 0, \left({\widehat{A}_{bp}}^{-} \times {\widehat{B}_{bp}}^{-}\right)(x,y)(\alpha) < 0, \text{ for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/ {\widehat{A}_{bp}}^{+}(x)(\alpha) \wedge {\widehat{B}_{bp}}^{+}(y)(\alpha) > 0, {\widehat{A}_{bp}}^{-}(x)(\alpha) \vee {\widehat{B}_{bp}}^{-}(y)(\alpha) < 0, \text{ for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/ {\widehat{A}_{bp}}^{+}(x)(\alpha) > 0 \text{ and } {\widehat{B}_{bp}}^{+}(y)(\alpha) > 0, {\widehat{A}_{bp}}^{-}(x)(\alpha) < 0 \text{ or } {\widehat{B}_{bp}}^{-}(y)(\alpha) < 0, \text{ for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/ {\widehat{A}_{bp}}^{+}(x)(\alpha) > 0 \,, {\widehat{A}_{bp}}^{-}(x)(\alpha) < 0 \text{ and } {\widehat{B}_{bp}}^{+}(y)(\alpha) > 0, {\widehat{B}_{bp}}^{-}(y)(\alpha) < 0, \text{ for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/ x \in S_2(\widehat{A}_{bp}), y \in S_2(\widehat{B}_{bp})\}$$

$$= S_2(\widehat{A}_{bp}) \times S_2(\widehat{B}_{bp}).$$

Then $S_2(\widehat{A}_{bp}) \times S_2(\widehat{B}_{bp}) = S_2(\widehat{A}_{bp} \times \widehat{B}_{bp}) \in S_2(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

Thus $S_2(\hat{\tau}_{\mathfrak{B}_1}) \times S_2(\hat{\tau}_{\mathfrak{B}_2}) \subseteq S_2(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

Theorem:3.3.7

If $\left(X, \hat{\tau}_{\mathfrak{B}_1}\right)$ and $\left(Y, \hat{\tau}_{\mathfrak{B}_2}\right)$ be two second order bipolar fuzzy topological spaces, then

(i) $i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1}) \times i_\varepsilon(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

(ii) $i^*(\hat{\tau}_{\mathfrak{B}_1}) \times i^*(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i^*(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

Proof :

(i) It is enough to observe that for any $\hat{A}_{bp} \in \hat{\tau}_{\mathfrak{B}_1}$ and $\hat{B}_{bp} \in \hat{\tau}_{\mathfrak{B}_2}$,

$$\left(L_{\hat{A}_{bp}}\right)_\varepsilon \times \left(L_{\hat{B}_{bp}}\right)_\varepsilon = \left(L_{(\hat{A}_{bp} \times \hat{B}_{bp})}\right)_\varepsilon.$$

Consider $\left(L_{\hat{A}_{bp}}\right)_\varepsilon \times \left(L_{\hat{B}_{bp}}\right)_\varepsilon$

$$= \{(x,y) \in X \times Y \,/\, \left(\left(\hat{A}_{bp}{}^{+}(x)\right)^{-1}(\varepsilon,1] = I, \left(\hat{A}_{bp}{}^{-}(x)\right)^{-1}[-1,-\varepsilon) = I\right) \text{ and }$$

$$\left(\left(\hat{B}_{bp}{}^{+}(y)\right)^{-1}(\varepsilon,1] = I, \left(\hat{B}_{bp}{}^{-}(y)\right)^{-1}[-1,-\varepsilon) = I\right), \text{ for every } \varepsilon \in (0,1)\}$$

$$= \{(x,y) \in X \times Y \,/\, \left(\hat{A}_{bp}{}^{+}(x)(\alpha) > \varepsilon, \hat{A}_{bp}{}^{-}(x)(\alpha) < -\varepsilon\right) \text{ and }$$

$$\left(\hat{B}_{bp}{}^{+}(y)(\alpha) > \varepsilon\,,\ \hat{B}_{bp}{}^{-}(y)(\alpha) < -\varepsilon\right) \text{ for every } \varepsilon \in (0,1), \text{for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/ \hat{A}_{bp}{}^{+}(x)(\alpha) \wedge \hat{B}_{bp}{}^{+}(y)(\alpha) > \varepsilon, \hat{A}_{bp}{}^{-}(x)(\alpha) \vee \hat{B}_{bp}{}^{-}(y)(\alpha) < -\varepsilon,$$

$$\text{for every } \varepsilon \in (0,1), \text{for every } \alpha \in I\}$$

$$= \{(x,y) \in X \times Y \,/\, \left(\hat{A}_{bp}{}^{+} \times \hat{B}_{bp}{}^{+}\right)(x,y)(\alpha) > \varepsilon, \left(\hat{A}_{bp}{}^{-} \times \hat{B}_{bp}{}^{-}\right)(x,y)(\alpha) < -\varepsilon\}$$

$$= \{(x,y) \in X \times Y \,/\, \left(\left(\hat{A}_{bp}{}^{+} \times \hat{B}_{bp}{}^{+}\right)(x,y)\right)^{-1}(\varepsilon,1] = I,$$

$$\left(\left(\hat{A}_{bp}{}^{-} \times \hat{B}_{bp}{}^{-}\right)(x,y)\right)^{-1}[-1,-\varepsilon) = I\}$$

$$= \left(L_{(\hat{A}_{bp} \times \hat{B}_{bp})}\right)_\varepsilon.$$

Therefore $i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1}) \times i_\varepsilon(\hat{\tau}_{\mathfrak{B}_2}) \subseteq i_\varepsilon(\hat{\tau}_{\mathfrak{B}_1} \times \hat{\tau}_{\mathfrak{B}_2})$.

(ii) The proof is similar as (i).

Theorem : 3.3.8

If (X, τ) and (Y, τ') be two topological spaces, then

(i) For $\varepsilon \in (0,1), \widehat{\omega_\varepsilon(\tau)} \times \widehat{\omega_\varepsilon(\tau')} \subseteq \omega_\varepsilon\widehat{(\tau \times \tau')}$.

(ii) $\widehat{\omega_*(\tau)} \times \widehat{\omega_*(\tau')} \subseteq \omega_*\widehat{(\tau \times \tau')}$.

Proof :

(i) Let $\widehat{\omega_\varepsilon(\tau)}$ and $\widehat{\omega_\varepsilon(\tau')}$ be two second order bipolar fuzzy topologies generated by

$\left(\left(\widehat{K}_1\right)_{bp}\right)_\varepsilon = \{\widehat{A}_{bp} \in SBPF(X) / \left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau\}$ and

$\left(\left(\widehat{K}_2\right)_{bp}\right)_\varepsilon = \{\widehat{B}_{bp} \in SBPF(X) / \left(L_{\widehat{B}_{bp}}\right)_\varepsilon \in \tau'\}$.

Consider a basis element $\widehat{A}_{bp} \times \widehat{B}_{bp}$ of $\widehat{\omega_\varepsilon(\tau)} \times \widehat{\omega_\varepsilon(\tau')}$

implies $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \in \tau$ and $\left(L_{\widehat{B}_{bp}}\right)_\varepsilon \in \tau'$

implies $\left(L_{\widehat{A}_{bp}}\right)_\varepsilon \times \left(L_{\widehat{B}_{bp}}\right)_\varepsilon = \left(L_{(\widehat{A}_{bp} \times \widehat{B}_{bp})}\right)_\varepsilon \in \tau \times \tau'$.

Then $\widehat{A}_{bp} \times \widehat{B}_{bp} \in \omega_\varepsilon\widehat{(\tau \times \tau')}$.

Thus $\widehat{\omega_\varepsilon(\tau)} \times \widehat{\omega_\varepsilon(\tau')} \subseteq \omega_\varepsilon\widehat{(\tau \times \tau')}$.

(ii) The proof is similar as (i).