CHAPTER - II

FUZZY SOFT MATRICES

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Definition: 2.1

Let U = {c_1, c_2 ...c_m } be the universal set and E = {e_1, e_2, e_3...e_n} be the set of parameters. Let A $\subseteq$ E and (F, A) be a fuzzy soft set in the fuzzy soft class (U, E). Then the fuzzy soft set (F, A) can be represented in matrix form as $S_{m\times n} = [a_{ij}]_{m\times n}$ or simply by S = [a_{ij}], i = 1,2,...,m; j = 1,2,...,n, where

$$a_{ij} = \begin{cases} \mu_j(c_i), & \text{if} \quad e_j \in A \\ 0, & \text{if} \quad e_j \notin A \end{cases}$$

Here μ_j (c_i) represents the membership of c_i in the fuzzy set F(e_j).The matrix $S_{m\times n}$ is called a **fuzzy soft matrix**. One can identify a fuzzy soft set with its fuzzy matrix and use these two concepts interchangeable. The set of all m×n fuzzy soft matrices over U would be denoted by $FSM_{m\times n}$.

Example : 2.2

Let U = {c_1, c_2 ...c_{m}} be the universal set and E be the set of parameters given by E = {e_1 ,e_2 ,e_3,e_4,}. Let P = {e_1 ,e_2 ,e_4} $\subseteq$ E and (F, P) is the fuzzy soft set

(F, P) = {F(e_1) = {(c_1, 0.7), (c_2, 0.6), (c_3, 0.7), (c_4, 0.5)},

F(e_2) = {(c_1, 0.8), (c_2, 0.6), (c_3, 0.1), (c_4, 0.5)},

F(e_4) = {(c_1, 0.1), (c_2, 0.4), (c_3, 0.7), (c_4, 0.3)}}.

The fuzzy soft matrix representing this fuzzy soft set would be represented as

$$S = \begin{bmatrix} 0.7 & 0.8 & 0.0 & 0.1 & 0.0 \\ 0.6 & 0.6 & 0.0 & 0.4 & 0.0 \\ 0.7 & 0.1 & 0.0 & 0.7 & 0.0 \\ 0.5 & 0.5 & 0.0 & 0.3 & 0.0 \end{bmatrix}_{4\times 5}$$

Definition : 2.3

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n} \in FSM_{m\times n}$ be two fuzzy soft matrices.

1) If $m \neq n$ then A is called a **fuzzy soft rectangular matrix**.
2) If m = n, then A is called a **fuzzy soft square matrix**.
3) If m = 1, then A is called a **fuzzy soft row matrix**.
4) If n = 1, then A is called a **fuzzy soft column matrix**.
5) Then A is called **fuzzy soft dioganal matrix** if m = n and $a_{ij} = 0$ for all $i \neq j$.
6) Then A is called **fuzzy soft scalar matrix** if m = n and $a_{ij} = 0$ for all $i \neq j$ and $a_{ij} = \lambda \in [0,1]$ $\forall$ i = j.
7) Then A is called **fuzzy soft upper triangular matrix** if m = n and $a_{ij} = 0$ for all i > j.
8) Then A is called **fuzzy soft lower triangular matrix** if m = n and $a_{ij} = 0$ for all i < j.
9) A fuzzy soft matrix is said to be **triangular** if it is either fuzzy soft lower or soft upper triangular matrix.
10) Then A is called **fuzzy soft null matrix** (or zero matrix) whose elements are all 0, denoted by [0] or 0.
11) Then A is called **fuzzy universal soft matrix** whose elements are all 1, denoted by [1] or U.
12) Then A is called **fuzzy soft submatrix** of B, denoted by $A \subseteq B$, if $a_{ij} \leq b_{ij}$ for all i and j.
13) Then A is called **proper fuzzy soft submatrix** of B, denoted by $A \subseteq B$, if $a_{ij} \leq b_{ij}$ for all i and j and for atleast one term $a_{ij} < b_{ij}$.
14) Then A and B are **fuzzy soft equal matrices**, denoted by A = B, if $a_{ij} = b_{ij}$.

Theorem : 2.4

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [c_{ij}]_{m\times n}$ be three fuzzy soft matrices. Then

1) $0 \subseteq A$
2) $A \subseteq U$
3) $A \subseteq A$
4) $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

where 0 is a m × n fuzzy soft null matrix whose elements are all 0 and U is a m × n fuzzy universal soft matrix whose elements are all 1.

Definition : 2.5

Let $A = [a_{ij}]_{m\times n} \in FSM_{m\times n}$, where $a_{ij} = \mu_j (c_i)$, then the element $a_{11}, a_{12}, \ldots a_{mm}$ are called the **diagonal elements** and the line along which they lie is called the **principal diagonal** of the fuzzy soft matrix.

Definition : 2.6 [22]

Let $A = [a_{ij}]$, $B = [b_{ij}] \in FSM_{m\times n}$. Then **union** of A, B is defined by $A_{mxn} \cup B_{mxn} = C_{mxn} = [c_{ij}]_{mxn}$, where $c_{ij} = a_{ij} \lozenge b_{ij} = a_{ij} + b_{ij} - a_{ij} b_{ij}$ for all i and j.

Example : 2.7

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.7 \\ 0.0 & 1.0 & 0.3 \end{bmatrix} \quad \text{and } B = \begin{bmatrix} 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.7 & 0.0 \end{bmatrix}$$

Then $A_{3x3} \cup B_{3x3} = C_{3x3} = \begin{bmatrix} 0.55 & 0.36 & 0.72 \\ 0.28 & 0.65 & 0.76 \\ 0.20 & 1.00 & 0.30 \end{bmatrix}$

Definition : 2.8 [22]

Let $A = [a_{ij}]$, $B = [b_{ij}] \in FSM_{mxn}$. Then **intersection** of A, B is defined by $A_{mxn} \cap B_{mxn} = C_{mxn} = [c_{ij}]_{mxn}$, where $c_{ij} = a_{ij} * b_{ij} = a_{ij} b_{ij}$ for all i and j.

Example : 2.9

Let A = $\begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.7 \\ 0.0 & 1.0 & 0.3 \end{bmatrix}$ and B = $\begin{bmatrix} 0.5 & 0.2 & 0.6 \\ 0.1 & 0.3 & 0.2 \\ 0.2 & 0.7 & 0.0 \end{bmatrix}$

Then $A_{3x3} \cap B_{3x3} = C_{3x3} = \begin{bmatrix} 0.05 & 0.04 & 0.18 \\ 0.02 & 0.15 & 0.14 \\ 0.00 & 0.70 & 0.00 \end{bmatrix}$

Definition : 2.10 [10]

Let $[a_{ij}]$, $[b_{ij}] \in FSM_{m\times n}$. Then the fuzzy soft matrix $[c_{ij}]$ is called

1) **Union** of $[a_{ij}]$ and $[b_{ij}]$, denoted $[a_{ij}] \cup [b_{ij}]$, if $c_{ij} = \max \{a_{ij}, b_{ij}\}$ for all i and j.
2) **Intersection** of $[a_{ij}]$ and $[b_{ij}]$, denoted $[a_{ij}] \cap [b_{ij}]$, if $c_{ij} = \min \{a_{ij}, b_{ij}\}$ for all i and j.

Definition : 2.11 [10]

Let $[a_{ij}]$, $[b_{ij}] \in FSM_{m\times n}$. Then $[a_{ij}]$ and $[b_{ij}]$ are **disjoint**, if $[a_{ij}] \cap [b_{ij}] = [0]$ for all i and j.

Example : 2.12

Assume that $[a_{ij}] = \begin{bmatrix} 0 & 0.6 & 0 & 0 \\ 0.1 & 0 & 1 & 0 \\ 0 & 0.3 & 0.8 & 0 \\ 0.7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $[b_{ij}] = \begin{bmatrix} 0 & 0 & 0.7 & 0.4 \\ 0 & 0.2 & 0 & 1 \\ 0 & 0 & 0 & 0.9 \\ 0 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0.3 \end{bmatrix}$

Then $[a_{ij}] \cap [b_{ij}] = [0]$ and

$$[a_{ij}] \cup [b_{ij}] = \begin{bmatrix} 0 & 0.6 & 0.7 & 0.4 \\ 0.1 & 0.2 & 1 & 1 \\ 0 & 0.3 & 0.8 & 0.9 \\ 0.7 & 0 & 0.5 & 1 \\ 0 & 1 & 0 & 0.3 \end{bmatrix}$$

Theorem : 2.13

Let A, B $\in FSM_{mxn}$, Then

1) $A \cup 0 = A$
2) $A \cup U = U$
3) $A \cup B = B \cup A$
4) $(A \cup B) \cup C = A \cup (B \cup C)$

Proof :

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [C_{ij}]_{m\times n}$ be three fuzzy soft matrices.

1) $A \cup 0 = [a_{ij} + 0 - a_{ij}\, 0] = [a_{ij}] = A$
2) $A \cup U = [a_{ij} + 1 - a_{ij}\, 1] = [1] = U$
3) $A \cup B = [a_{ij} + b_{ij} - a_{ij}\, b_{ij}]$

   $= [b_{ij} + a_{ij} - b_{ij}\, a_{ij}]$

   $= B \cup A$
4) $(A \cup B) \cup C = [a_{ij} + b_{ij} - a_{ij}\, b_{ij}] \cup [c_{ij}]$

   $= [(a_{ij} + b_{ij} - a_{ij}\, b_{ij}) + c_{ij} - (a_{ij} + b_{ij} - a_{ij}\, b_{ij})\, c_{ij}\,]$

   $= [a_{ij} + b_{ij} + c_{ij} - a_{ij}\, b_{ij} - a_{ij}\, c_{ij} - b_{ij}\, c_{ij} + a_{ij}\, b_{ij}\, c_{ij}]$

$A \cup (B \cup C) = [a_{ij}] \cup [b_{ij} + c_{ij} - b_{ij}\, c_{ij}]$

$= [a_{ij} + (b_{ij} + c_{ij} - b_{ij}\, c_{ij}) - a_{ij}\, (b_{ij} + c_{ij} - b_{ij}\, c_{ij})]$

$= [a_{ij} + b_{ij} + c_{ij} - a_{ij}\, b_{ij} - a_{ij}\, c_{ij} - b_{ij}\, c_{ij} + a_{ij}\, b_{ij}\, c_{ij}]$

Hence $(A \cup B) \cup C = A \cup (B \cup C)$.

Theorem : 2.14

Let A, B $\in$ FSM_{mxn}. Then

1) $A \cap 0 = 0$
2) $A \cap U = A$
3) $A \cap B = B \cap A$
4) $(A \cap B) \cap C = A \cap (B \cap C)$

Proof :

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$, $C = [C_{ij}]_{m\times n}$ be three fuzzy soft matrices.

1) $A \cap 0 = [a_{ij}\, 0] = [0] = 0$
2) $A \cap U = [a_{ij}\, 1] = [a_{ij}] = A$
3) $A \cap B = [a_{ij}\, b_{ij}]$

$= [b_{ij}\, a_{ij}]$

$= B \cap A$

4) $(A \cap B) \cap C = [a_{ij}\, b_{ij}] \cap [c_{ij}]$

$= [(a_{ij}\, b_{ij})\, c_{ij}]$

$= [a_{ij}\, (b_{ij}\, c_{ij})]$

$= [a_{ij}] \cap [b_{ij}\, c_{ij}]$

$= A \cap (B \cap C)$

Definition : 2.15 [22]

Let $A = [a_{ij}]_{mxn}$, then **complement** of A is denoted by $A^C = [c_{ij}]$, where $c_{ij} = 1 - a_{ij}$, for all i and j.

Example : 2.16

Let $A = \begin{bmatrix} 0.9 & 0.8 & 0.0 \\ 0.8 & 0.5 & 0.0 \\ 1.0 & 0.0 & 0.0 \end{bmatrix}$ then $A^C = \begin{bmatrix} 0.1 & 0.2 & 1.0 \\ 0.2 & 0.5 & 1.0 \\ 0.0 & 1.0 & 1.0 \end{bmatrix}$

Theorem : 2.17

1) $(A^C) = A$
2) $0^C = U$

Proof:

1) Let $A = [a_{ij}]_{m\times n}$ be a fuzzy soft matrix.
   Then $A^C = [1 - a_{ij}]$ for all i and j
   $\therefore (A^C)^C = [1-(1 - a_{ij})] = [a_{ij}] = A$
2) $(0)^C = [1 - 0] = [1] = U$

Theorem : 2.18

Let $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ be two fuzzy soft matrices then De Morgan's Laws are void for all i and j.

1) $(A \cup B)^C = A^C \cap B^C$
2) $(A \cap B)^C = A^C \cup B^C$

Proof :

1) $(A \cup B)^C = ([a_{ij}] \cup [b_{ij}])^C$

$$= [a_{ij} + b_{ij} - a_{ij}\, b_{ij}]^C$$

$$= [1 - a_{ij} - b_{ij} + a_{ij}\, b_{ij}]$$

$$= [(1 - a_{ij})(1 - b_{ij})]$$

$$= [1 - a_{ij}] \cap [\ 1 - b_{ij}]$$

$$= A^C \cap B^C$$

2) $(A \cap B)^C = ([a_{ij}] \cap [b_{ij}]$

$$= [a_{ij}\, b_{ij}]^C$$

$= [1 - a_{ij} b_{ij}]$

$A^C \cup B^C = [1 - a_{ij}] \cup [1 - b_{ij}]$

$= [1 - a_{ij} + 1 - b_{ij} – (1 - a_{ij})(1 - b_{ij})]$

$= [1 - a_{ij} b_{ij}]$

Hence $(A \cap B)^C = A^C \cup B^C$

Theorem : 2.19 [10]

1) $[a_{ij}] \subseteq [1]$
2) $[0] \subseteq [a_{ij}]$
3) $[a_{ij}] \subseteq [a_{ij}]$
4) $[a_{ij}] \subseteq [b_{ij}]$ and $[b_{ij}] \subseteq [c_{ij}] \Rightarrow [a_{ij}] \subseteq [c_{ij}]$
5) $[a_{ij}] \subseteq [b_{ij}] \Leftrightarrow [a_{ij}] \cap [b_{ij}] = [a_{ij}] \Leftrightarrow [a_{ij}] \cup [b_{ij}] = [b_{ij}]$

Theorem : 2.20 [10]

Let $[a_{ij}], [b_{ij}], [c_{ij}] \in FSM_{m \times n}$. Then

1) $[a_{ij}] = [b_{ij}]$ and $[b_{ij}] = [c_{ij}] \Leftrightarrow [a_{ij}] = [c_{ij}]$
2) $[a_{ij}] \subseteq [b_{ij}]$ and $[b_{ij}] \subseteq [c_{ij}] \Leftrightarrow [a_{ij}] = [b_{ij}]$

Theorem : 2.21 [10]

Let $[a_{ij}], [b_{ij}] \in FSM_{m \times n}$. Then De Morgan's Laws are valid

1) $([a_{ij}] \cup [b_{ij}])^C = [a_{ij}]^C \cap [b_{ij}]^C$
2) $([a_{ij}] \cap [b_{ij}])^C = [a_{ij}]^C \cup [b_{ij}]^C$

Proof :

For all i and j

1) $([a_{ij}] \cup [b_{ij}])^C = [\max \{a_{ij}, b_{ij}\}]^C$

$= [1- \max\{a_{ij}, b_{ij}\}]$

$= [\min\{1- a_{ij}, 1- b_{ij}\}]$

$= [a_{ij}]^C \cap [b_{ij}]^C$

2) It can be proved similarly.

Definition : 2.22

Let $A = [a_{ij}] \in FSM_{mxn}$, and k, $0 \leq k \leq 1$ any number called scalar. Then **scalar multiple** of A by k is denoted by $kA = [ka_{ij}]_{mxn}$.

Example : 2.23

Let $A = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.7 \\ 0.0 & 1.0 & 0.3 \end{bmatrix}$

$\therefore (0.5)A = \begin{bmatrix} 0.05 & 0.10 & 0.15 \\ 0.10 & 0.25 & 0.35 \\ 0.00 & 0.50 & 0.15 \end{bmatrix}$

Theorem : 2.24

If s and t are two scalars such that $0 \leq s, t \leq 1$and $A = [a_{ij}]_{mxn}$ is any fuzzy soft matrix, then

1) $s(tA) = (st)A$
2) $s \leq t \Rightarrow sA \subseteq tA$
3) $A \subseteq B \Rightarrow sA \subseteq sB$

Proof :

we only prove (i) and others follow the similar lines

Let $A = [a_{ij}]_{m\times n}$

$s(tA) = s(t[a_{ij}]_{m\times n})$

$= s[ta_{ij}]_{m\times n}$

$= [s(ta_{ij})]_{m\times n}$

$= [(st)a_{ij}]_{m\times n}$

$= (st)[a_{ij}]_{m\times n}$

Definition : 2.25

Let $A = [a_{ij}]_{m\times n}$ be a **fuzzy soft square matrix**. Then $\text{tr}\, A = \sum_{i=1}^{m} a_{ii} = a_{11}+a_{22}+a_{33}+\ldots\ldots+a_{nn}$.

Example : 2.26

$$\text{Let } A = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ 0.2 & 0.5 & 0.7 \\ 0.0 & 1.0 & 0.3 \end{bmatrix}$$

$\therefore$ tr A = 0.1 + 0.5 +0.3 = 0.9

Theorem : 2.27

Let A and B be two fuzzy soft square matrices of order m and k be a scalar. Then tr (kA) = k tr A.

Proof :

Let $A = [a_{ij}]_{m\times m}$

we have, $kA = [ka_{ij}]_{m\times m}$

$$\therefore \text{tr}\,(kA) = \sum_{i=1}^{m} ka_{ii} = k\sum_{i=1}^{m} a_{ij} = k\, \text{tr}\, A$$

Theorem : 2.28

If A be a fuzzy soft matrix of order m × n, then $(kA)^T = kA^T$, k being any scalar.

Proof :

Let $A = [a_{ij}]_{m\times n}$ be fuzzy soft matrix.

We have $kA = [ka_{ij}]_{m\times n}$

$\therefore\ (kA)^T = [ka_{ji}]_{n\times m} = k[a_{ji}]_{n\times m} = kA^T$

Theorem : 2.29

Let A, B $\in$ FSM_{mxn}, Then

1) $(A\cup B)^T = A^T \cup B^T$
2) $(A\cap B)^T = A^T \cap B^T$
3) $(A^C)^T = (A^T)^C$

Remark : 2.30

Let A, B $\in$ FSM_{mxn}. Then the following distributive laws are valid for max and min operations only.

1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Definition : 2.31

A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous t-norm if $*$ satisfies the following conditions.

1) $*$ is commutative and associative

2) $*$ is continuous
3) $a * 1 = a \ \forall \ a \in [0,1]$
4) $a * b \leq c * d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$

A few examples of continuous t-norm are $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a+b-1, 0\}$.

Definition : 2.32

A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm if $\Diamond$ satisfies the following conditions.

1) $\Diamond$ is commutative and associative
2) $\Diamond$ is continuous
3) $a \Diamond 0 = a \ \forall \ a \in [0,1]$
4) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$, $b \leq d$ and $a, b, c, d \in [0,1]$

A few examples of continuous t-conorm are $a \Diamond b = a + b - ab$, $a \Diamond b = \max\{a, b\}$, $a \Diamond b = \min\{a+b, 1\}$.

Definition : 2.33 [22]

Let $A_k = [a^k_{ij}] \in FSM_{mxn}$, $k = 1,2,3,\ldots\ldots,\ell$. Then the **T - product** of fuzzy soft matrices, denoted by $\prod_{k=1}^{l} A_k = A_1 \times A_2 \times A_3 \times \ldots\ldots \times A_\ell$ is defined by $\prod_{k=1}^{l} A_k = [c_i]_{mx1}$, where $c_i = \sum_{j=1}^{n} \underset{k=1}{\overset{l}{T}} a_{ij}^{k}$, $i = 1,2,3,\ldots\ldots,m$.

Here $T = *$ or $T = \Diamond$ according to the type of the problems.

Example : 2.34

We assume that $A_1, A_2, A_3 \in FSM_{mxn}$ are given as follows.

$$A_1 = \begin{bmatrix} 0.2 & 0.7 \\ 0.6 & 0.4 \\ 0.3 & 0.6 \\ 0.1 & 0.9 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0.6 & 0.2 \\ 0.1 & 0.6 \\ 0.8 & 0.5 \\ 0.8 & 0.2 \end{bmatrix} A_3 = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.1 \\ 0.3 & 0.2 \\ 0.5 & 0.7 \end{bmatrix}$$

Then the $*$ product is $\prod_{k=1}^{3} A_k = A_1 \times A_2 \times A_3$

$$= \begin{bmatrix} 0.2*0.6*0.4+0.7*0.2*0.3 \\ 0.6*0.1*0.3+0.4*0.6*0.1 \\ 0.3*0.8*0.3+0.6*0.5*0.2 \\ 0.1*0.8*0.5+0.9*0.2*0.7 \end{bmatrix}$$

$$= \begin{bmatrix} 0.090 \\ 0.042 \\ 0.132 \\ 0.166 \end{bmatrix}$$

Theorem : 2.35

Let A, B, C $\in FSM_{mxn}$. Then

1) A × B = B × A
2) (A × B) × C = A× (B × C)
3) A × B $\subseteq$ A × C

Proof :

1) Let $A = [a_{ij}]$, $B = [b_{ij}]$ be two fuzzy soft matrices.

   Then $A \times B = [\sum_{j=1}^{n} a_{ij} \, T b_{ij}] = [\sum_{j=1}^{n} b_{ij} \, T a_{ij}] = B \times A$, where T= $*$ or $\Diamond$.

Algorithim for fuzzy soft matrix decision making method by using fuzzy soft "$*$" product.

Input : Fuzzy soft sets with m objects, each of which has n parameters.

Output : An optimum set.

Step 1 : Choose the set of parameters,

Step 2 : Construct the fuzzy soft matrices for each set of parameters,

Step 3 : Compute "∗" product of the fuzzy soft matrices,

Step 4 : Find the optimum subscript set O_s,

Step 5 : Find the optimum decision set O_d.

Example: 2.36

Suppose U = $\{c_1, c_2, c_3, c_4, c_5\}$ be the five candidates appearing in an interview for appointment in managerial level in a company and E = $\{e_1$(enterprising), e_2(confident), e_3(willing to take risk)$\}$ be the set of parameters. Suppose three experts, Mr. A, Mr. B, and Mr. C, take interview of the five candidates and the following fuzzy soft matrices are constructed accordingly.

$$A = \begin{bmatrix} 0.3 & 0.2 & 0.1 \\ 0.5 & 0.4 & 0.2 \\ 0.6 & 0.5 & 0.7 \\ 0.4 & 0.6 & 0.8 \\ 0.8 & 0.6 & 0.3 \end{bmatrix} \quad B = \begin{bmatrix} 0.7 & 0.2 & 0.5 \\ 0.6 & 0.4 & 0.9 \\ 0.7 & 0.8 & 0.6 \\ 0.5 & 0.6 & 1.0 \\ 0.4 & 0.5 & 0.7 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.4 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.5 \\ 0.8 & 0.6 & 0.4 \\ 0.5 & 0.6 & 0.5 \end{bmatrix}$$

$$\text{we have, } A \times B \times C = \begin{bmatrix} 0.3*0.7*0.5+0.2*0.2*0.4+0.1*0.5*0.6 \\ 0.5*0.6*0.4+0.4*0.4*0.7+0.2*0.9*0.6 \\ 0.6*0.7*0.6+0.5*0.8*0.5+0.7*0.6*0.5 \\ 0.4*0.5*0.8+0.6*0.6*0.6+0.8*1.0*0.4 \\ 0.8*0.4*0.5+0.6*0.5*0.6+0.3*0.7*0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 0.151 \\ 0.340 \\ 0.662 \\ 0.696 \\ 0.445 \end{bmatrix}$$

It is clear that the maximum score is 0.696, scored by c_4 and the decision is in favor of selecting c_4.

Definition : 2.37 [10]

Let $[a_{ij}]$, $[b_{ik}] \in FSM_{m\times n}$. Then **And - product** of $[a_{ij}]$ and $[b_{ik}]$ is defined by $\wedge : FSM_{m\times n} \times FSM_{m\times n} \to FSM_{m\times n^2}$, $[a_{ij}] \wedge [b_{ik}] = [c_{ip}]$ where $c_{ip} = \min \{a_{ij}, b_{ik}\}$ such that $p = n\,(j-1) + k$.

Definition : 2.38 [10]

Let $[a_{ij}]$, $[b_{ik}] \in FSM_{m\times n}$. Then **Or - product** of $[a_{ij}]$ and $[b_{ik}]$ is defined by $\vee : FSM_{m\times n} \times FSM_{m\times n} \to FSM_{m\times n^2}$, $[a_{ij}] \vee [b_{ik}] = [c_{ip}]$ where $c_{ip} = \max \{a_{ij}, b_{ik}\}$ such that $p = n\,(j-1) + k$.

Definition : 2.39 [10]

Let $[a_{ij}]$, $[b_{ik}] \in FSM_{m\times n}$. The **And - Not - product** of $[a_{ij}]$ and $[b_{ik}]$ is defined by $\overline{\wedge} : FSM_{m\times n} \times FSM_{m\times n} \to FSM_{m\times n^2}$, $[a_{ij}] \overline{\wedge} [b_{ik}] = [c_{ip}]$ where $c_{ip} = \min \{a_{ij}, 1 - b_{ik}\}$ such that $p = n\,(j-1) + k$.

Definition : 2.40 [10]

Let $[a_{ij}]$, $[b_{ik}] \in FSM_{m\times n}$. The **Or - Not - product** of $[a_{ij}]$ and $[b_{ik}]$ is defined by $\underline{\vee} : FSM_{m\times n} \times FSM_{m\times n} \to FSM_{m\times n^2}$, $[a_{ij}] \underline{\vee} [b_{ik}] = [c_{ip}]$ where $c_{ip} = \max \{a_{ij}, 1 - b_{ik}\}$ such that $p = n\,(j-1) + k$.

Example : 2.41

Assume that $[a_{ij}]$, $[b_{ik}] \in FSM_{2\times3}$ are given as follows

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0.3 \\ 0 & 1 & 0.7 \end{bmatrix} \quad [b_{ik}] = \begin{bmatrix} 0.5 & 0 & 0.2 \\ 0.2 & 0 & 0 \end{bmatrix}$$

To calculate $[a_{ij}] \wedge [b_{ik}] = [c_{ip}]$, we have to find c_{ip} for all i = 1,2,3,....,9.

Let us find c_{17}.

Since n = 3, i = 1 and p = 7, we get j = 3 and k = 1 from 7 = 3 (j-1) + k.

Hence $c_{17} = \min\{a_{13}, b_{11}\} = \min\{0.3, 0.5\} = 0.3$.

If the other entries of $[c_{ip}]$ can be found similarly, obtain the matrix as follows:

$$[a_{ij}] \wedge [b_{ik}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 & 0 & 0 & 0.2 & 0 & 0 \end{bmatrix}$$

Similarly, we can also find products $[a_{ij}] \vee [b_{ik}]$, $[a_{ij}] \overline{\wedge} [b_{ik}]$ and $[a_{ij}] \underline{\vee} [b_{ik}]$.

Note that the commutativity is not valid for the products of fuzzy soft matrices.

Theorem : 2.42

Let $[a_{ij}]$, $[b_{ik}] \in FSM_{m\times n}$. Then the following De Morgan's types of results are true.

1) $([a_{ij}] \vee [b_{ik}])^C = [a_{ij}]^C \wedge [b_{ik}]^C$
2) $([a_{ij}] \wedge [b_{ik}])^C = [a_{ij}]^C \wedge [b_{ik}]^C$
3) $([a_{ij}] \underline{\vee} [b_{ik}])^C = [a_{ij}]^C \overline{\wedge} [b_{ik}]^C$
4) $([a_{ij}] \overline{\wedge} [b_{ik}])^C = [a_{ij}]^C \underline{\vee} [b_{ik}]^C$

Definition : 2.43

Let $[c_{ip}] \in FSM_{m\times n^2}$. $I_k = \{ p : \exists_i, c_{ip} \neq 0, (k-1)n < p \leq kn \}$ for all $k \in I = \{1,2, 3,\ldots,n\}$. Then **fs - max - min decision function**, denoted M_m, is defined as follows:

$$M_m : FSM_{m\times n^2} \rightarrow FSM_{m\times 1},\ M_m [c_{ip}] = [d_{i1}] = [\max_k \{t_{ik}\}], \text{ where}$$

$$t_{ik} = \begin{cases} \min_{p\in I_k} \{c_{ip}\}, & \text{if } I_k \neq 0 \\ 0, & \text{if } I_k = 0 \end{cases}$$

The one column fs - matrix $M_m [c_{ip}]$ is called max - min decision fs - matrix.

Definition : 2.44

Let $U = \{u_1, u_2, u_3,\ldots,u_n,\}$ be an initial universe and $M_m [c_{ip}]=[d_{i1}]$. Then a subset of U can be obtained by using $[d_{i1}]$ as in the following way

$opt_{[di1]}(U) = \{d_{i1}/u_i : u_i \in U\ d_{i1} \neq 0\}$ which is called an **optimum fuzzy set** on U.

Algorithm for fs - max-min decision making (FSMmDM) method by using fs -max - min decision function.

Step 1 : choose feasible subsets of the set of parameters,

Step 2 : construct the fs - matrix for each set of parameters,

Step 3 : find a convenient product of the fs-matrices,

Step 4 : find a max - min decision fs – matrix,

Step 5 : find an optimum fuzzy set on U.

Assume that a real estate agent has a set of different types of houses $U = \{u_1, u_2, u_3, u_4, u_5\}$ which may be characterized by a set of parameters

E = {x_1, x_2, x_3, x_4}. For j = 1, 2, 3, 4 the parameters x_j stand for "in good location", "cheap", "modern", "large", respectively. Then we can give the following example.

Example : 2.45

Suppose that a married couple, Mr. X and Mrs. X, come to the real estate agent to buy a house. If each parameter has to consider their own set of parameters, then we select a house on the basis of the sets of partners' parameters by using the FSMmDM as follows.

Assume that U = {u_1, u_2, u_3, u_4, u_5} is a universal set and E = {x_1, x_2, x_3, x_4} is a set of all parameters.

Step 1: First, Mr. X and Mrs. X have to chose the sets of their parameters, A = { x_2, x_3, x_4} and B = { x_1, x_3, x_4}, respectively.

Step 2: Then we can write the following fs-matrices which are constructed according to their parameters.

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0.2 & 0.4 \\ 0 & 0.6 & 0.9 & 0.4 \\ 0 & 0.8 & 0.7 & 0.5 \\ 0 & 0.5 & 0 & 0 \\ 0 & 1 & 0 & 0.8 \end{bmatrix} \qquad [b_{ik}] = \begin{bmatrix} 1 & 0 & 0.9 & 0.7 \\ 0.2 & 0 & 0 & 0.9 \\ 0.7 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0.5 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 3: Now, we can find a product of the fs-matrices $[a_{ij}]$ and $[b_{ik}]$ by using And-product as follows

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.6 & 0.2 & 0 & 0 & 0.9 & 0.2 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0.4 & 0.3 & 0.7 & 0 & 0.4 & 0.3 & 0.5 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \end{bmatrix}$$

Here, we use And-product since both Mr. X and Mrs. X's choices have to be considered.

Step 4: To calculate $Mm([a_{ij}] \wedge [b_{ik}]) = [d_{i1}]$, we have to find d_{i1} for all $i \in \{1, 2, 3, 4, 5\}$. To demonstrate, let us find d_{31}. Since $i = 1$ and $k \in \{1, 2, 3, 4\}$,

$$d_{31} = \max_k\{t_{3k}\} = \max\{t_{31}, t_{32}, t_{33}, t_{34}\}$$

Here, we have to find t_{3k} for all $k \in \{1, 2, 3, 4\}$. To demonstrate, let us find t_{31} and t_{32}.

$$I_1 = \{p : c_{ip} \neq 0, 0 < p \leq 4\} = 0 \text{ for } k = 1 \text{ and } n = 4 \text{ and}$$

$$I_2 = \{p : c_{ip} \neq 0, 4 < p \leq 8\} = \{5, 7, 8\} \text{ for } k = 2 \text{ and } n = 4.$$

Hence $t_{31} = 0$ and $t_{32} = \min\{c_{35}, c_{37}, c_{38}\} = \min\{0.7, 0.4, 0.3\} = 0.3$

Similarly, we can find as $t_{33} = 0.3$ and $t_{34} = 0.3$. Thus,

$$d_{31} = \max\{0.0, 0.3, 0.3, 0.3\} = 0.3$$

Similarly we can find $d_{11} = 0.4$, $d_{21} = 0.0$, $d_{41} = 0.0$ and $d_{51} = 0.0$. Finally, we can obtain the fs-max-min decision fs-matrix as

$$Mm([a_{ij}] \wedge [b_{ik}]) = [d_{i1}] = \begin{bmatrix} 0.4 \\ 0 \\ 0.3 \\ 0 \\ 0 \end{bmatrix}$$

Step 5: Finally, we can find an optimum fuzzy set on U according to $Mm([a_{ij}] \wedge [b_{ik}])$.

$$Opt_{Mm([a_{ij}] \wedge [b_{ik}])}(U) = \{0.4/u_1, 0.3/u_3\}$$

where u_1 is an optimum house to buy for Mr. X and Mrs. X.

Similarly, we can also use products $[a_{ij}] \vee [b_{ik}]$, $[a_{ij}] \overline{\wedge} [b_{ik}]$ and $[a_{ij}] \underline{\vee} [b_{ik}]$ for the other convenient problems.