CHAPTER - IV

CHAPTER IV

INTEGRATED FUZZY TOPSIS METHOD

This chapter deals with an Integrated Fuzzy TOPSIS method to improve the quality of decision making for ranking alternatives. The proposed fuzzy TOPSIS method mainly accounts for the classification of criteria, the integrated weights of criteria and sub-criteria, and the performance values of decision matrix. In this model, the criteria are classified into subjective criteria and objective ones. The fuzzy analytic hierarchy process approach and the entropy weighting method are used to solve the subjective weights and objective ones. In addition, the adjusted integration weights are measured by combining these two methods. The performance values of subjective criteria and of objective ones will be obtained by linguistic expressions and objective evaluation values, respectively. Furthermore, the graded mean integration representation method and the modified distance method [25] are employed to the Integrated Fuzzy TOPSIS method.

Definition 4.1

Let $\tilde{m} = (c_i, a_i, b_i)$ and $\tilde{n} = (c_j, a_j, b_j)$ be two triangular fuzzy numbers. Then, the Hsieh and Chen's modified distance is denoted as

$$\delta_M(\tilde{m}, \tilde{n}) = \left\{ \frac{1}{4} \left[(c_i - c_j)^2 + 2(a_i - a_j)^2 + (b_i - b_j)^2 \right] \right\}^{\frac{1}{2}} \qquad (1)$$

In this method the weights of the subjective and objective criteria will be obtained by using the fuzzy AHP approach and the entropy weighting method. Finally, the integrated weights of all criteria above the alternatives layer can be computed by combining the subjective weights and objective ones.

Fuzzy AHP Approach

A fuzzy AHP approach is used to measure relative weights for evaluating subjective criteria. The systematic steps for evaluating relative weights using fuzzy AHP to be taken are described below.

Step 1: Develop a Hierarchical Structure

A hierarchy structure is the framework of system structure. It can be skeletonized to evaluate research problem and benefit the context. It is not only useful in studying the interaction amongst the elements involved in each level, but it can also help decision-makers to explore the impact of different elements on the evaluated system. Fig. 1 is an incomplete hierarchical structure with k criteria, and $p_1 + \ldots + p_t + \ldots + p_k$ sub-criteria.

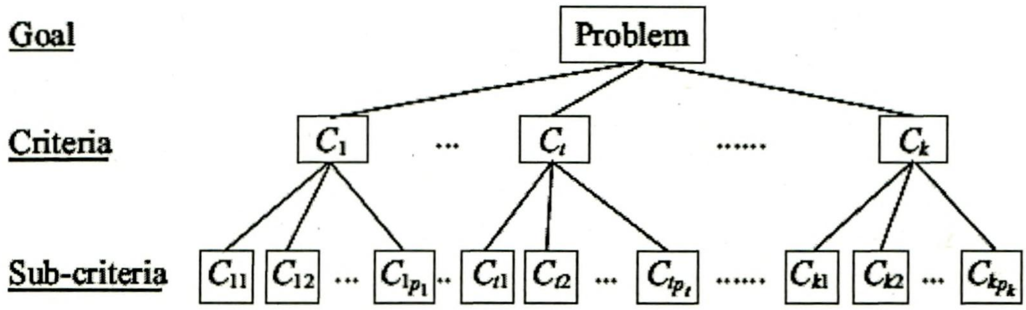

Fig. 1. Hierarchical structure.

Step 2: Build Fuzzy Pair-wise Comparison Matrices

Collecting pair-wise comparison matrices of each layer to represent the relative importance is an important step in fuzzy AHP method. Consequently, these relative importances are evaluated by experts, and these data are transformed into triangular fuzzy numbers using the geometric mean approach [56] (i.e) the triangular fuzzy numbers characterized by using the min, max and geometric mean operations are used to convey the opinions of all experts.

That is, let $x_{ij}^h \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1\} \cup \{1, 2, \ldots, 8, 9\} (h = 1, 2, \ldots, n, \forall i, j = 1, 2, \ldots, k)$ be the relative importance given to i^{th} criterion to j^{th} criterion by h^{th} expert on the criteria layer in Fig. 1. Then, the pair-wise comparison matrix is defined as $[x_{ij}^h]_{k \times k}$. After integrating the opinions of all n experts, the triangular fuzzy numbers can be denoted by

$$\tilde{a}_{ij}^{C} = (c_{ij}, a_{ij}, b_{ij}) \text{ where } c_{ij} = \min\{x_{ij}^{1}, x_{ij}^{2}, \ldots, x_{ij}^{n}\}, a_{ij} = \left(\mathop{\pi}_{h=1}^{n} x_{ij}^{h} \right)^{1/n},$$

$$b_{ij} = \max\{x_{ij}^{1}, x_{ij}^{2}, \ldots, x_{ij}^{n}\}$$

The integrated triangular fuzzy numbers are used to build a fuzzy pair-wise comparison matrix (given to i^{th} criterion to j^{th} criterion). For the criteria layer, the fuzzy pair-wise comparison matrix can be denoted by

$$A_{k}^{C} = [\tilde{a}_{ij}^{C}]_{k \times k} = \begin{bmatrix} \tilde{1} & \tilde{a}_{12}^{C} & \cdots & \tilde{a}_{1k}^{C} \\ 1/\tilde{a}_{12}^{C} & \tilde{1} & \cdots & \tilde{a}_{2k}^{C} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\tilde{a}_{1k}^{C} & 1/\tilde{a}_{2k}^{C} & \cdots & \tilde{1} \end{bmatrix} \text{where } \tilde{a}_{ij}^{C} \otimes \tilde{a}_{ji}^{C} \cong 1, \forall i, j = 1,2,\ldots,k.$$

Let $x_{uv}^{sh} \in \{\frac{1}{9}, \frac{1}{8}, \ldots, \frac{1}{2}, 1\} \cup \{1,2,\ldots,8,9\}$

$$(h = 1,2,\ldots,n, \forall u, v = 1,\ldots,p_1;\ \forall u, v = 1,\ldots,p_t;\ldots;u, v = 1,\ldots,p_k)$$

be the relative importance given to u^{th} sub-criterion to v^{th} sub-criterion by h^{th} expert on the sub-criteria layer in Fig. 1. Then, the pair-wise comparison matrices are defined as $[x_{uv}^{sh}]_{p_1 \times p_1}, \ldots, [x_{uv}^{sh}]_{p_t \times p_t}, \ldots, [x_{uv}^{sh}]_{p_k \times p_k}$. Therefore, integrating the opinions of all n experts given to sub-criterion u to sub-criterion v on the sub-criteria layer, the triangular fuzzy numbers can be denoted by

$$\tilde{a}_{uv}^{SC} = (c_{uv}, a_{uv}, b_{uv})\ \ \forall u, v = 1,\ldots,p_1; \forall u, v = 1,\ldots,p_t; \forall u, v = 1,\ldots,p_k$$

where $c_{uv} = \min\{x_{uv}^{s1}, x_{uv}^{s2}, \ldots, x_{uv}^{sn}\}$ $a_{uv} = \left(\mathop{\pi}_{h=1}^{n} x_{uv}^{sh} \right)^{1/n}$ $b_{uv} = \max\{x_{uv}^{s1}, x_{uv}^{s2}, \ldots, x_{uv}^{sn}\}$.

The integrated triangular fuzzy numbers is used to build the fuzzy pair-wise comparison matrices for the sub-criteria layer and can be denoted by

$$A_{p_1}^{SC} = [\tilde{a}_{uv}^{SC}]_{p_1 \times p_1} = \begin{bmatrix} \tilde{1} & \tilde{a}_{12}^{SC} & \cdots & \tilde{a}_{1p_1}^{SC} \\ 1/\tilde{a}_{12}^{SC} & \tilde{1} & \cdots & \tilde{a}_{2p_1}^{SC} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\tilde{a}_{1p_1}^{SC} & 1/\tilde{a}_{2p_1}^{SC} & \cdots & \tilde{1} \end{bmatrix}$$

where $\tilde{a}_{uv}^{SC} \otimes \tilde{a}_{uv}^{SC} \cong 1, \forall u, v = 1,2,\ldots,p_1$.

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$$A_{p_t}^{SC} = [\tilde{a}_{uv}^{SC}]_{p_t \times p_t} = \begin{bmatrix} \tilde{1} & \tilde{a}_{12}^{SC} & \cdots & \tilde{a}_{1p_t}^{SC} \\ 1/\tilde{a}_{12}^{SC} & \tilde{1} & \cdots & \tilde{a}_{2p_t}^{SC} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\tilde{a}_{1p_t}^{SC} & 1/\tilde{a}_{2p_t}^{SC} & \cdots & \tilde{1} \end{bmatrix},$$

where $\tilde{a}_{uv}^{SC} \otimes \tilde{a}_{uv}^{SC} \cong 1, \forall u, v = 1,2,\ldots,p_t$.

,...........,

$$A_{p_k}^{SC} = [\tilde{a}_{uv}^{SC}]_{p_k \times p_k} = \begin{bmatrix} \tilde{1} & \tilde{a}_{12}^{SC} & \cdots & \tilde{a}_{1p_k}^{SC} \\ 1/\tilde{a}_{12}^{SC} & \tilde{1} & \cdots & \tilde{a}_{2p_k}^{SC} \\ \vdots & \vdots & \ddots & \vdots \\ 1/\tilde{a}_{1p_k}^{SC} & 1/\tilde{a}_{2p_k}^{SC} & \cdots & \tilde{1} \end{bmatrix},$$

where $\tilde{a}_{uv}^{SC} \otimes \tilde{a}_{uv}^{SC} \cong 1, \forall u, v = 1,2,\ldots,p_k$.

Step 3: Calculate the Fuzzy Weights of the Fuzzy Pair-wise Comparison Matrices

Let $\tilde{Z}_i^C = (\tilde{a}_{i1}^C \otimes \tilde{a}_{i2}^C \otimes \ldots \otimes \tilde{a}_{ik}^C)^{1/k} (\forall i = 1,2,\ldots,k)$ be the geometric mean of triangular fuzzy number of i^{th} criterion on the criteria layer. Then, the fuzzy weight of i^{th} criterion is denoted by $\tilde{w}_i^C = \tilde{Z}_i^C \otimes (\tilde{Z}_i^C \otimes \tilde{Z}_2^C \otimes \ldots \otimes \tilde{Z}_k^C)^{-1}$. For being convenient, the fuzzy weight is denoted by $\tilde{w}_i^C \cong (w_{ic}, w_{ia}, w_{ib})$

Let $\widetilde{Z}_u^{SC} = (\widetilde{a}_{u1}^{SC} \otimes \widetilde{a}_{u2}^{SC} \otimes \ldots \otimes \widetilde{a}_{up_1}^{SC})^{1/p_1}$ $\forall u = 1,2,\ldots p_1$ be the geometric mean of triangular fuzzy number of u[th] sub-criterion on the sub-criteria layer. Then, the fuzzy weight of u[th] sub-criterion is denoted by

$$\widetilde{w}_u^{SC} = \widetilde{Z}_u^{SC} \otimes (\widetilde{Z}_i^{SC} \oplus \widetilde{Z}_2^{SC} \oplus \ldots \oplus \widetilde{Z}_{p_1}^{SC})^{-1},$$

where the fuzzy weight is denoted by $\widetilde{w}_u^{SC} \cong (w_{uc}, w_{ua}, w_{ub})$ $\forall u = 1,2,\ldots p_1$.

Step 4: Defuzzify the Fuzzy Weights to Crisp Weights

For solving the problem of defuzzification powerfully, the Graded Mean Integration Representation (GMIR) method is used to defuzzify the fuzzy weights. Let $\widetilde{w}_i^C \cong (w_{ic}, w_{ia}, w_{ib})$ $(\forall i = 1,2,\ldots,k)$ be k triangular fuzzy numbers. By the powerful method, the GMIR of crisp weights k is denoted by $W_i^C = \dfrac{w_{ic} + 4w_{ia} + w_{ib}}{6}$ $(\forall i = 1,2,\ldots,k)$

Step 5: Calculate and Normalize the Weight Vector of Each Layer

For being convenient to compare the relative importance between each layer, these crisp weights are normalized and denoted by $NW_i^C = W_i^C \Big/ \sum_{i=1}^{k} W_i^C$.

Let NW_i^C and NW_u^{SC} be the normalized crisp weights on the criteria and sub-criteria layers, respectively. Then,

(i) The integrated weight of each criterion on the criteria layer is $IW_i^C = NW_i^C, \forall i = 1,2,\ldots,k.$

(ii) The integrated weight of each sub-criterion on the subcriteria layer is

$$\begin{aligned} &IW_u^{SC} = NW_i^C \times NW_u^{SC}, \forall i = 1,2,\ldots,k; \\ &\forall u = 1,2,\ldots,p_1; \forall u = 1,2,\ldots,p_t; \ldots; \forall u = 1,2,\ldots,p_k \end{aligned} \quad (2)$$

Entropy Weighting Method

This section tries to solve the objective weight of objective sub-criteria above the alternative level using the entropy weighting method. Thus, the steps are as follows:

Step 1: Construct a Decision Matrix

Here, let m and q respectively denote the numbers of alternatives and the objective sub-criteria above the alternatives layer. Allow $\tilde{x}_{ij}, i = 1,2,...,m; j = 1,2,...,q$,to be the triangular fuzzy number of original evaluation value of i^{th} alternative under j^{th} sub-criterion. Then, the decision matrix $D = [\tilde{x}_{ij}]_{m\times q}, i = 1,2,...,m; j = 1,2,...,q$ is obtained.

To ensure compatibility between the positive sub-criterion j (the criterion that has positive contribution to the objective, e.g., benefit criterion) and the negative one (the criterion that has negative contribution to the objective, e.g., cost criterion), the original evaluation value is converted to dimensionless index.

Let d_{ij} (i = 1, 2,…, m; j = 1, 2, …, q) denote the normalized evaluation value of i^{th} alternative under j^{th} sub-criterion. Let $\tilde{x}_{ij}$ can be expressed as $R(\tilde{x}_{ij})$. The fuzzy positive value $\tilde{x}_j^P$ and fuzzy negative value $\tilde{x}_j^N$ of each criterion above the alternatives layer are judged and determined by comparing with these representation values $R(\tilde{x}_{ij})$.

Then,

(i) For the positive sub-criterion j:

$$d_{ij} = R(\tilde{x}_{ij}) / R(\tilde{x}_j^P), \text{ where } \tilde{x}_j^N = \min_i \{\tilde{x}_{ij}\} \text{ and } 0 \le d_{ij} \le 1.$$

(ii) For the negative sub-criterion j:

$$d_{ij} = R(\tilde{x}_{ij}^N) / R(\tilde{x}_{ij}), \text{ where } \tilde{x}_j^P = \max_i \{\tilde{x}_{ij}\} \text{ and } 0 \le d_{ij} \le 1.$$

Then the normalized decision matrix $D=[d_{ij}]_{mxq}$, i=1, 2,.., m; j=1,2,…,q are determined.

Step 2: Calculate the Entropy Value of Each Criterion

The entropy value E_j of each objective evaluation subcriterion j is calculated by $E_j = -k\sum_{i=1}^{m}\frac{d_{ij}}{D_j}\ln\frac{d_{ij}}{D_j}$ where $k = \frac{1}{\ln m} > 0$, $0 \le E_j \le 1$ and $D_j = \sum_{i=1}^{m} d_{ij}$, j=1,2,…,q.

Step 3: Compute the Total Entropy Value

The total entropy value E is computed as $E = \sum_{j=1}^{q} E_j$

Step 4: Obtain the Objective Weight of Each Objective Criterion

The objective weight π_j of the j^{th} objective sub-criterion above the alternative level is calculated by $\pi_j = \frac{1-E_j}{\sum_{j=1}^{q}(1-E_j)} = \frac{1-E_j}{q-E}, 0 \le \pi_j \le 1, \sum_{j=1}^{q}\pi_j = 1$. (3)

The Integrated Weights

Here, the incomplete hierarchical structure of Fig. 1 is expanded into a complete one, which has k criteria, $p_1 + \ldots + p_t + \ldots + p_k$ sub-criteria and m alternatives. The weights of subjective and objective sub-criteria above the alternatives layer is obtained by using the fuzzy AHP approach and the entropy weighting method. Next the integration weights of all sub-criteria above the alternatives layer is computed by combining the subjective weights and objective ones.

The three cases appeared in the MCDM problems in terms of the criteria aspects are

Case 1: If all the sub-criteria above the alternatives layer are subjective, then using the fuzzy AHP approach. The integrated weight of each subjective sub-criterion is obtained by using the equation (2).

Case 2: If all the sub-criteria above the alternatives layer are objective, then using the entropy weighting method. The integrated weight of each objective sub-criterion is obtained by using the equation (3).

Case 3: If some sub-criteria above the alternatives layer are subjective, and others are objective. Then, the adjusted integration weights of objective sub-criterion is obtained by using the equation (4).

That is, let $O = \{o_1, \ldots, o_t, \ldots, o_q\}$ be the set of all q objective sub-criteria above the alternatives layer. And η_t, $t = 1, 2, \ldots, q$, to be the subjective integration weights of objective sub-criteria o_t, then Case 1 is used. By the same concept, let λ_t, $t = 1, 2, \ldots, q$, to be the objective weights of objective sub-criteria o_t, then Case 2 is used. By combining the objective weights λ_t, and the subjective integrated weights η_t, the adjusted integration weights w_t^+ of all q objective sub-criteria can be obtained

$$w_t^+ = \frac{\lambda_t \eta_t}{\sum_{t=1}^{q} \lambda_t \eta_t} \times \sum_{t=1}^{q} \eta_t, t = 1,2,\ldots,q \quad (4)$$

THE INTEGRATED FUZZY TOPSIS METHOD

The systematic steps for ranking alternatives based on the proposed fuzzy TOPSIS method are described below.

1. Form a committee of Decision Makers to identify the appropriate alternatives and adopt the evaluation criteria and sub-criteria are formed.
2. Classify the sub-criteria above the alternatives layer into the subjective and objective categories.
3. Compute the subjective integration weights of all subcriteria above the alternatives layer.

4. Estimate the superiority of alternatives versus all subcriteria.
5. Utilize the entropy weighting method to adjust the subjective integration weights of objective sub-criteria above the alternatives layer.
6. Calculate the fuzzy ideal solution and anti-ideal solution.
7. Compute the distance of different alternatives versus the fuzzy ideal solution and anti-ideal solution.
8. Calculate the relative approximation value of different alternatives versus ideal solution.
9. Rank the alternatives to select the best one.

1. Estimating the Superiority of Alternatives versus All Sub-criteria

The sub-criteria above the alternatives layer are classified into the subjective and objective categories. Let $S = \{s_1, \ldots, s_r, \ldots, s_p\}$ and $O = \{o_1, \ldots, o_t, \ldots, o_q\}$ be the sets of all p subjective sub-criteria and q objective ones above the alternatives layer.

Case 1: For the Subjective Sub-criteria

The superiority of all alternatives versus all subjective sub-criteria above the alternatives layer is obtained by linguistic values.

Subsequently, the arithmetic mean method is used to solve the average superiority of evaluation value for each alternative versus all subjective sub-criteria. Let be the fuzzy superiority of the i^{th} alternative versus the r^{th} subjective sub-criterion evaluated by the h^{th} expert. Then, the average fuzzy superiority value of the i^{th} alternative versus the r^{th} subjective sub-criterion is expressed as $\left(\frac{\sum_{h=1}^{n} c_{ir}^{h}}{n}, \frac{\sum_{h=1}^{n} a_{ir}^{h}}{n}, \frac{\sum_{h=1}^{n} b_{ir}^{h}}{n}\right)$.

Case 2: For the Objective Sub-criteria

The fuzzy ratings of all alternatives versus all objective sub-criteria above the alternatives layer are tackled by the following method [38,56].

(a) When the appropriateness rating of alternative is estimated effectively in values, the triangular fuzzy numbers is used directly. For example, if the return on investment (ROI) per year is about 10%, it is subjectively expressed as (9.4%, 10%, 10.6%).

(b) If there are historical data, e.g. let $x_1, x_2, \ldots, x_k$ represent the ROI of past k periods, the fuzzy rating of the ROI is used the geometric mean method to express as (L,M,U), where $L = \min_i\{x_i\}, M = \left(\prod_{i=1}^{k} x_i\right)^{1/k}, U = \max_i\{x_i\}$.

For example, if the current four historical data of the ROI of alternative A_1 are 6%, 9%, 3%, and 8%, then the evaluation value is transformed into triangular fuzzy number as (3%, $\sqrt[4]{3 \times 6 \times 8 \times 9}$ %, 9%) = (3%, 6%, 9%).

2. Calculating the Fuzzy Ideal Solution and Anti-ideal Solution

The ideal and anti-ideal solutions [37] are based on the concept of relative closeness in compliance with the shorter (longer) the distance of alternative i to ideal (anti-ideal), the higher the priority is ranked.

Let m and $p_1 + \ldots + p_t + \ldots + p_k = P_{sc}$ respectively denote the numbers of alternatives and the sub-criteria above the alternatives layer. And let $\tilde{x}_{ij} = (c_{ij}, a_{ij}, b_{ij})$ (i=1,2,…m; j=1,2,…, P_{sc}) be the average fuzzy superiority value of i^{th} alternative under j^{th} sub-criterion. To ensure compatibility between fuzzy ratings of objective criteria and linguistic ratings of subjective criteria, fuzzy superiority values are converted to dimensionless indices. The fuzzy ideal values with minimum values in negative sub-criteria or maximum values in positive sub-criteria should have the maximum rating.

Based on the principle stated as above, let $\alpha_j = \max_i\{b_{ij}\}, \beta_j = \min_i\{c_{ij}\}$ then the normalized fuzzy superiority value S_{ij} of i^{th} alternative under j^{th} sub-criterion is defined as:

(i) For the positive sub-criterion j (the sub-criteria that have positive contribution to the objective, e.g., benefit subcriterion)

$$S_{ij} = (p_{ij}, o_{ij}, q_{ij}) = (\frac{c_{ij}}{\alpha_j}, \frac{a_{ij}}{\alpha_j}, \frac{b_{ij}}{\alpha_j})$$

(ii) For the negative sub-criterion j (the sub-criteria that have negative contribution to the objective, e.g., cost sub-criterion)

$$S_{ij} = (p_{ij}, o_{ij}, q_{ij}) = (\frac{\beta_j}{b_{ij}}, \frac{\beta_j}{a_{ij}}, \frac{\beta_j}{c_{ij}})$$

By using GMIR method S_{ij} is expressed as $R(S_{ij})$.The fuzzy ideal value S_j^+ and fuzzy anti-ideal value S_j^- of each sub-criterion above the alternatives layer are determined by comparing with these representation values $R(S_{ij})$. Then,

(i) if $R(S_{tj}) = \max_i R(S_{ij})$, then the fuzzy ideal value $S_j^+ = S_{tj}$

(ii) if $R(S_{kj}) = \min_i R(S_{ij})$, then the fuzzy anti-ideal value $S_j^- = S_{kj}$

The fuzzy ideal solution $I^+ = (S_1^+, S_2^+, ..., S_j^+, ..., S_{P_{SC}}^+)$ and fuzzy anti-ideal solution $AI^- = (S_1^-, S_2^-, ..., S_j^-, ..., S_{P_{SC}}^-)$ are defined.

3. Computing the Distance of Different Alternatives versus the Fuzzy Ideal Solution and Anti-ideal Solution

Let ρ_j^* (j = 1, 2,…, P_{sc}) be the integrated weights of j^{th} sub-criterion above the alternatives layer. Then, the distance of different alternatives versus I^+ and AI^- which were denoted by D_i^+ and D_i^- are calculated, respectively. Define

$$D_i^+ = \sqrt{\sum_{j=1}^{P_{SC}}[(\rho_j^*)^2 \times (\delta_M(S_j^+, S_{ij}))^2]}, i = 1,2,...,m$$

$$D_i^- = \sqrt{\sum_{j=1}^{P_{SC}}[(\rho_j^*)^2 \times (\delta_M(S_j^-, S_{ij}))^2]}, i = 1,2,...,m$$

where $\delta_M(\bullet)$ can be obtained by using the equation (1) .

4. Calculating the Relative Approximation Value of Different Alternatives versus Ideal Solution and Ranking the Alternatives

The relative approximation value of different alternatives A_i versus ideal solution I^+, denoted as RAV_i^+ is calculated as

$$RAV_i^+ = \frac{D_i^-}{D_i^+ + D_i^-}, i = 1,2,...,m ; 0 \le RAV_i^+ \le 1.$$

Suppose alternative A_i is an ideal solution (i.e. $D_i^+ = 0$), then $RAV_i^+ = 1$; otherwise, if A_i is an anti-ideal solution (i.e. $D_i^- = 1$), then $RAV_i^+ = 0$. The nearer the value RAV_i^+ close to 1 implies a closer alternative A_i approach to the ideal solution, i.e. the maximum value of RAV_i^+ , then the optimal alternative is ranked by a decision maker. Finally, the best alternative is selected.