CHAPTER VII

REPAIRABLE TWO PHASE SERVICE $M^X/G/1$ QUEUEING MODELS WITH FINITE NUMBER OF IMMEDIATE FEEDBACKS UNDER BERNOULLI SCHEDULE VACATION

INTRODUCTION

Queueing systems with various feedback policies have been investigated by many authors. Kalidass and Kasturi (2013) motivated by the transactions through ATM machine, analysed a reliable M/G/1 queueing system, with two-phases of heterogeneous service and a finite number of immediate Bernoulli feedbacks. The first phase consists of single service and the second phase contains **C** multiple optional service facilities. All the arriving customers are provided with the same type of service in the first phase and after completing the first phase service, the customers will choose one of the second optional services, which are available in the second phase of the system. After having completed both phases of service, each customer is allowed to make an immediate feedback. These authors assumed that the feedback service also consists of two phases of service. In the feedback, the first phase of service is of the same type as in the previous service and in the second phase, the customer is permitted to choose an optional service may be different from the one chosen earlier. They have considered a reliable and single Poisson arrival queueing model in which each customer is allowed to make a finite number of such feedbacks before departing from the system.

In the present chapter, the author considers an unreliable $M^X/G, G_{i\ (1 \le i \le c)}/1$ queueing system with two phases of heterogeneous service in which the server operates single service in the first phase and multi-optional heterogeneous service facilities in second phase. The arriving customers have to undergo the first phase service and any one of the second optional services to complete the first round of service. After completing the first round of service, the customers may demand for re-services from the second phase, finitely many times before leaving the system.

The server is subject to unpredictable breakdowns during busy period and sent for repair immediately. In breakdown queueing models, the service interrupted customers may stay in the service facility to complete the remaining service (or) join the head of the queue to repeat the service from the beginning as soon as the server is fixed. In the present chapter, these two cases are considered separately in sections 7.1 and 7.2 respectively and it is verified that the results for both the cases coincide, if the service times follow exponential distribution. It is further assumed that, after a successful completion of a service (including the feedback services) of each customer the server may take a Bernoulli schedule vacation before starting a new service for the next customer.

SECTION : 7.1

M^X/G/1 QUEUE WITH FINITE NUMBER OF FEEDBACKS AND RESUMPTION OF INTERRUPTED SERVICE

7.1.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

7.1.1.1 Model Description

The customers arrive in batches according to the time-homogenous compound Poisson process with group arrival rate λ (as in Chapter II).

Service is provided one by one according to FCFS basis. Every customer has to undergo two stages of services following different general (arbitrary) distributions. The arriving customers first receive the First Phase Service S (FPS), which is followed by any of the second phase services (SPS) S_i ($1 \le i \le C$). Customers after completing the first phase service will opt for a certain i^{th} ($1 \le i \le C$) optional service with probability r_i where $\sum_{i=1}^{C} r_i = 1$. The second phase service commences immediately after the first phase service and all the services are provided by the same server. The service time S in the first phase has a distribution function S(t), density function s(t), LST $S^*(\theta)$ with finite first and second moments $E(S^k)$, k = 1, 2. The i^{th} ($1 \le i \le C$) optional service S_i of the second phase follows a general distribution function

$S_i(t)$, $1 \le i \le C$, density function $s_i(t)$, LST $S_i^*(\theta)$ with finite moments $E(S_i^k)$, $k = 1, 2$. The service time of the first round of service of a customer is thus $S + S_i$ (for some $i = 1$ to C) and it is termed as primary or fresh service. It is assumed that the customer who finishes the first round of service either feeds back immediately into any of the i^{th} type service from the second phase with probability $(f_1 r_i)$ $(1 \le i \le C)$ or leaves the system for ever with probability $(1 - f_1)$. After the completion of the first feedback service, the customer may again go in for a second feedback service with probability $f_2 r_j$ $(1 \le j \le C)$ in the second phase or leave the system with probability $(1 - f_2)$. This process may continue, until the customer obtains at most **m** rounds of services (including the primary service), after which the customer has to leave the system. The next customer in the queue can go into the system only after the completion of all the feedback rounds of the preceding customer.

The server is subject to unpredictable breakdowns while serving the customers. The breakdowns occur according to the Poisson process with rate **a** in first phase and at the rate $\mathbf{a_2^{(i,j)}}$ during the i^{th} type j^{th} feedback service in the second phase. As soon as a breakdown occurs, the server is sent for repair immediately and the customer whose service is interrupted stays in the service facility to complete the remaining service. The repair times R_1, $R_2^{(i,j)}$ $1 \le i \le C$, $0 \le j \le m-1$ are arbitrary distributed with probability distribution functions $R_1(t)$ and $R_2^{(i,j)}(t)$ according as the interruption occurs in phase 1 or phase 2 in the i^{th} type of service due to the j^{th} feedback. Immediately after the server is fixed, it starts to serve the customer, whose service is interrupted and the service time is assumed to be cumulative.

After completing a service to a customer, the server may take Bernoulli schedule vacation with probability **p** or continue to serve the next customer if any or stay idle if the system is empty with probability **(1 – p)**. The vacation time V has general distribution and its distribution function, density function, LST and the first and second moments are respectively denoted by $V(t)$, $v(t)$, $V^*(\theta)$, $E(V^k)$, $k = 1, 2$.

Thus each customer primarily will undergo the first phase of service and at least one of the i^{th} optional services from the second phase before leaving the system. The customers can use the feedback facility only from the second phase service and can feedback atmost **m–1** times. Thus a cycle consists of primary service (phase 1 service and any of the phase 2 service), atmost (m–1) feedback services (from second phase), breakdown period and vacation period. We denote the model by $M^X/(G, G_{i\,(1\le i\le C)})/1$/finite feedbacks/BSV/Breakdown$_{\text{(resume service)}}$, where BSV denotes Bernoulli schedule vacation.

Various stochastic processes involved in the queueing system are assumed to be independent of each other. The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Using supplementary variable technique the steady state system equations under the steady-state condition are analysed and the PGF of the system size is obtained so that various performance measures of the model can be derived from it.

The notations $N_S(t)$, λ, X, g_k, $g_n^{(i)}$, X(z) are the same as in Chapter II.

The notations of Random Variable (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transforms (LST) and its k^{th} moment of the RVs are listed below.

Table 7.1.1

	RV	**CDF**	**PDF**	**LST**	**k^{th} moments k = 1, 2**
Service time in first phase	S	S(x)	s(x)	$S^*(\theta)$	$E(S^k)$
i^{th} optional service time of the second phase	S_i	$S_i(x)$	$s_i(x)$	$S_i^*(\theta)$	$E(S_i^k)$, i = 1 to C
Repair time in first phase	R_1	$R_1(y)$	$r_1(y)$	$R_1^{*1}(\theta_1)$	$E(R_1^k)$
Repair time in second phase	$R_2^{(i,j)}$	$R_2^{(i,j)}(y)$	$r_2^{(i,j)}(y)$	$R_2^{(i,j)^{*1}}(\theta_1)$	$E((R_2^{(i,j)})^k), 0 \le j \le m-1$
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$

Let $\left(S^{o}(t), S_i^{o}(t), R_1^{o}(t), \left(R_2^{(i,j)}\right)^{o}(t), V^{o}(t)\right)$ respectively denote the remaining times of the random variables ; service time in first stage, second stage, repair time in first stage, second stage and vacation time at time t. Further the server states are denoted by the random variable Y(t) at time t. Then the state space $\{N_S(t), \delta(t)\}$ where $\delta(t) = (0, S^{o}(t), S_i^{o}(t), R_1^{o}(t), \left(R_2^{(i,j)}\right)^{o}(t), V^{o}(t))$ according as Y(t) = 0, 1, 2, 3, 4 and 5 respectively follows bivariate Markov process. The following joint probability functions are defined, for further analysis :

$PI(t)$ = $\Pr\{N_S(t) = 0, Y(t) = 0\}$, when the server is idle.

For $n \geq 1$, at time t,

$P_{1,n}(x, t)\,dt$ = $\Pr\{N_S(t) = n, x < S^{o}(t) \leq x + dt, Y(t) = 1\}$, a customer is being served in the first phase service (Primary)

For $1 \leq i \leq C$, $0 \leq j \leq m-1$

$P_{2,n}^{(i,j)}(x, t)\,dt$ = $\Pr\{N_S(t) = n, x < S_i^{o}(t) \leq x + dt, Y(t) = 2\}$, a customer is being served in the i^{th} optional service of the second phase during the j^{th} feedback.

j = 0 corresponds to fresh or primary i^{th} second phase service.

$BR_{1,n}(x, y, t)\,dt$ = $\Pr\{N_S(t) = n, S^{o}(t) = x, y < R_1^{o}(t) \leq y + dt, Y(t) = 3\}$, a customer is waiting for the first phase service due to breakdown, to complete the remaining service time.

$BR_{2,n}^{(i,j)}(x, y, t)\,dt$ = $\Pr\{N_S(t) = n, S_i^{o}(t) = x, y < \left(R_2^{(i,j)}\right)^{o}(t) \leq y + dt, Y(t) = 4\}$, a customer is waiting for the i^{th} optional service of the second phase during the j^{th} feedback due to breakdown.

$Q_n(x, t)\,dt$ = $\Pr\{N_S(t) = n, x < V^{o}(t) \leq x + dt, Y(t) = 5\}$, when the server is in vacation.

7.1.1.2 System Size Distribution at Random Epoch

Observing the changes of states during the interval $(t, t + \Delta t)$ for any time t, the steady state equations are given by :

Vacation State

$$-\frac{d}{dx}Q_n(x) = -\lambda\, Q_n(x) + \lambda\,(1-\delta_{0,n}) \sum_{k=1}^{n} Q_{n-k}(x)\, g_k$$

$$+ \sum_{i=1}^{C}\sum_{j=0}^{m-1} (1-f_{j+1})\, P_{2,n+1}^{(i,j)}(0)\, p\, v(x), \qquad n \geq 0$$

Busy with First Phase of Service

$$-\frac{d}{dx}P_{1,n}(x) = -(\lambda + a_1)\, P_{1,n}(x) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{1,n-k}(x)\, g_k$$

$$+ (1-p)\, s(x) \sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{2,n+1}^{(i,j)}(0)\ (1-f_{j+1}) + \mathrm{PI}\,\lambda\, g_n\, s(x)$$

$$+ BR_{1,n}(x, 0) + Q_n(0)\, s(x), \qquad n \geq 1$$

Busy with Primary Service in the i^th Phase

$$-\frac{d}{dx}P_{2,n}^{(i,0)}(x) = -(\lambda + a_2^{(i,0)})\, P_{2.n}^{(i,0)}(x) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)}(x)\, g_k$$

$$+ P_{1,n}(0)\, r_i\, s_i(x) + BR_{2,n}^{(i,0)}(x, 0), \qquad 1 \leq i \leq C, \qquad n \geq 1$$

Busy with j^th Feedback Service in the i^th Phase (1 ≤ j ≤ m–1)

$$-\frac{d}{dx}P_{2,n}^{(i,j)}(x) = -(\lambda + a_2^{(i,j)})\, P_{2,n}^{(i,j)}(x) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,j)}(x)\, g_k$$

$$+ \sum_{=1}^{C} P_{2,n}^{(\ ,j-1)}(0)\, f_j\, r_i\, s_i(x) + BR_{2,n}^{(i,j)}(x, 0), \qquad n \geq 1$$

Breakdown State in First Service

$$-\frac{\partial}{\partial y}BR_{1,n}(x, y) = -\lambda\, BR_{1,n}(x, y) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}(x, y)\, g_k$$

$$+ a_1 P_{1,n}(x)\, r_1(y), \qquad n \geq 1$$

Breakdown in Second Service (0 ≤ j ≤ m–1)

$$-\frac{\partial}{\partial y}BR_{2,n}^{(i,j)}(x, y) = -\lambda\, BR_{2,n}^{(i,j)}(x, y) + \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)}(x, y)\, g_k$$

$$+ a_2^{(i,j)}\, r_2^{(i,j)}(y)\, P_{2,n}^{(i,j)}(x), \qquad n \geq 1$$

Idle State

$$\lambda\,\mathrm{PI} = Q_0(0) + (1-p) \sum_{i=1}^{C}\sum_{j=0}^{m-1} (1-f_{j+1})\, P_{2,1}^{(i,j)}(0)$$

The LST of the Steady-State equations are given by :

$$\theta\, Q_n^*(\theta) - Q_n(0) \;=\; \lambda\, Q_n^*(\theta) - \lambda\,(1-\delta_{0,n}) \sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k$$
$$- p\; V^*(\theta) \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1-f_{j+1})\, P_{2,n+1}^{(i,j)}(0), \qquad n \ge 0 \qquad (7.1.1)$$

$$\theta\, P_{1,n}^*(\theta) - P_{1,n}(0) \;=\; (\lambda + a_1)\, P_{1,n}^*(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{1,n-k}^*(\theta)\, g_k$$
$$- (1-p)\; S^*(\theta) \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1-f_{j+1})\, P_{2,n+1}^{(i,j)}(0) - S^*(\theta)\, \lambda\, \mathrm{PI}\, g_n$$
$$- BR_{1,n}^*(\theta,0) - S^*(\theta)\, Q_n(0), \qquad n \ge 1 \qquad (7.1.2)$$

$$\theta\, P_{2,n}^{(i,0)^*}(\theta) - P_{2,n}^{(i,0)}(0) \;=\; (\lambda + a_2^{(i,0)})\, P_{2,n}^{(i,0)^*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)\,*}(\theta)\, g_k$$
$$- S_i^*(\theta)\, r_i\, P_{1,n}(0) - BR_{2,n}^{(i,0)^*}(\theta,0),$$
$$1 \le i \le C,\; n \ge 1 \qquad (7.1.3)$$

$$\theta\, P_{2,n}^{(i,j)^*}(\theta) - P_{2,n}^{(i,j)}(0) \;=\; (\lambda + a_2^{(i,j)})\, P_{2,n}^{(i,j)^*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,j)\,*}(\theta)\, g_k$$
$$- S_i^*(\theta)\, f_j\, r_i \sum_{=1}^{C} P_{2,n}^{(\;,j-1)}(0) - BR_{2,n}^{(i,j)^*}(z,\theta,0),$$
$$n \ge 1,\; 1 \le j \le m-1 \qquad (7.1.4)$$

$$\theta_1\, BR_{1,n}^{**_1}(\theta,\theta_1) - BR_{1,n}^*(\theta,0) \;=\; \lambda\, BR_{1,n}^{**_1}(\theta,\theta_1) - \lambda(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{**_1}(\theta,\theta_1)\, g_k$$
$$- a_1\, P_{1,n}^*(\theta)\, R_1^{*_1}(\theta_1), \qquad n \ge 1 \qquad (7.1.5)$$

$$\theta_1\, BR_{2,n}^{**_1}(\theta,\theta_1) - BR_{2,n}^{(i,j)^*}(\theta,0) \;=\; \lambda\, BR_{2,n}^{(i,j)^{**_1}}(\theta,\theta_1)$$
$$- \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)**_1}(\theta,\theta_1)\, g_k$$
$$- a_2^{(i,j)}\; P_{2,n}^{(i,j)^*}(\theta)\, R_2^{(i,j)*_1}(\theta_1),$$
$$n \ge 1,\;\; 1 \le i \le C,\; 0 \le j \le m-1 \qquad (7.1.6)$$

$$\lambda \mathrm{PI} \;=\; Q_0(0) + \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1-f_{j+1})\, P_2^{(i,j)}(0)\,(1-p) \qquad (7.1.7)$$

7.1.1.3 Probability Generating Functions

The following partial PGF$_S$, are defined to obtain the generating functions of the system size distribution of the model.

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta)\, z^n, \quad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0)\, z^n$$

$$P_1^*(z,\theta) = \sum_{n=1}^{\infty} P_{1,n}^*(\theta)\, z^n, \quad P_1(z,0) = \sum_{n=1}^{\infty} P_{1,n}(0)\, z^n$$

$$P_2^{(i,j)^*}(z,\theta) = \sum_{n=1}^{\infty} P_{2,n}^{(i,j)^*}(\theta)\, z^n, \quad P_2^{(i,j)}(z,0) = \sum_{n=1}^{\infty} P_{2,n}^{(i,j)}(0)\, z^n, \quad 1 \le i \le C, \quad 0 \le j \le m-1$$

$$BR_1^{**_1}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{1,n}^{**_1}(\theta,\theta_1)\, z^n, \quad BR_1^*(z,\theta,0) = \sum_{n=1}^{\infty} BR_{1,n}^*(\theta,0)\, z^n$$

$$BR_2^{(i,j)^{**_1}}(z,\theta,\theta_1) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,j)^{**_1}}(\theta,\theta_1)\, z^n, \quad BR_2^{(i,j)^*}(z,\theta,0) = \sum_{n=1}^{\infty} BR_{2,n}^{(i,j)^*}(\theta,0)\, z^n, \quad 1 \le i \le C, \quad 0 \le j \le m-1$$

Multiplying the corresponding equations by suitable powers of z and adding the equations, partial generating functions are derived, through some algebraic manipulations. Equation (7.1.1) implies

$$(\theta - w_X(z))\, Q^*(z,\theta) = Q(z,0) - \frac{p\, V^*(\theta)}{z} \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1 - f_{j+1})\, P_2^{(i,j)}(z,0) \qquad (7.1.8)$$

where $w_X(z) = \lambda\,(1 - X(z))$.

At $\theta = w_X(z)$,

$$Q(z,0) = \frac{p\, V^*(w_X(z))}{z} \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1 - f_{j+1})\, P_2^{(i,j)}(z,0) \qquad (7.1.9)$$

$$\text{Therefore, } Q^*(z,\theta) = \frac{\frac{p}{z}\left(\sum_{i=1}^{C} \sum_{j=0}^{m-1} (1 - f_{j+1})\, P_2^{(i,j)}(z,0) \right)(V^*(w_X(z)) - V^*(\theta))}{(\theta - w_X(z))} \qquad (7.1.10)$$

The partial probability generating functions of the system size, when the server is in breakdown state during the first and second stage of service are obtained respectively by using the equations (7.1.5) and (7.1.6) and are given by,

$$BR_1^*(z,\theta,0) = a_1\, P_1^*(z,\theta)\, R_1^{*_1}(w_X(z)) \qquad (7.1.11)$$

$$BR_1^{**_1}(z,\theta,\theta_1) = \frac{a_1\, P_1^*(z,\theta)(R_1^{*_1}(w_X(z)) - R_1^{*_1}(\theta_1))}{(\theta_1 - w_X(z))} \qquad (7.1.12)$$

$$BR_2^{(i,j)^*}(z,\theta,0) = a_2^{(i,j)}\; P_2^{(i,j)^*}(z,\theta)\; R_2^{(i,j)^{*_1}}(w_X(z)) \qquad (7.1.13)$$

$$BR_2^{(i,j)^{**1}}(z,\theta,\theta_1) = \frac{a_2^{(i,j)}\,P_2^{(i,j)^*}(z,\theta)\,(R_2^{(i,j)^{*1}}(w_X(z)) - R_2^{(i,j)^{*1}}(\theta_1))}{(\theta_1 - w_X(z))}$$

$$1 \le i \le C,\ 0 \le j \le m-1 \qquad (7.1.14)$$

Equation (7.1.3) gives the generating functions of the system size when the server is busy with second stage i^{th} fresh service, at the service completion epoch and at arbitrary epoch. For $1 \le i \le C$,

$$P_2^{(i,0)}(z,0) \quad = \quad r_i\ S_i^*(h_{a_2(i,0)}(w_X(z)))\,P_1(z,0) \qquad (7.1.15)$$

$$P_2^{(i,0)^*}(z,\theta) \quad = \quad \frac{r_i\,P_1(z,0)\,(S_i^*(h_{a_2(i,0)}(w_X(z))) - S_i^*(\theta))}{(\theta - h_{a_2(i,0)}(w_X(z)))} \qquad (7.1.16)$$

where $h_{a_2(i,0)}(w_X(z)) = w_X(z) + a_2^{(i,0)}(1 - R_2^{(i,0)^{*1}}\,w_X(z)))$ (7.1.17)

Similarly equation (7.1.4) gives the generating functions of the system size when the server is busy with second stage i^{th} optional service at the j^{th} feedback, at the service completion epoch and at arbitrary epoch,

$$P_2^{(i,j)}(z,0) \quad = \quad f_j\,r_i\ S_i^*(h_{a_2(i,j)}(w_X(z)))\left(\sum_{=1}^{C} P_2^{(\ ,j-1)}(z,0)\right) \qquad (7.1.18)$$

$$P_2^{(i,j)^*}(z,\theta) \quad = \quad \frac{f_j\,r_i\,(S_i^*(h_{a_2(i,j)}(w_X(z))) - S_i^*(\theta))\sum_{=1}^{C} P_2^{(\ ,j-1)}(z,0)}{(\theta - h_{a_2(i,j)}(w_X(z)))},$$

$$1 \le i \le C,\ 1 \le j \le m-1 \qquad (7.1.19)$$

where $h_{a_2(i,j)}(w_X(z)) = w_X(z) + a_2^{(i,j)}(1 - R_2^{(i,j)^{*1}}\,w_X(z)))$ (7.1.20)

By recursion, the equation (7.1.18) together with (7.1.15) lead to

$$P_2^{(\ ,j)}(z,0) \quad = \quad r_\ell\ S^*(h_{a_2(\ ,j)}(w_X(z)))\left(\prod_{S=0}^{j-1} k_S(z)\,f_S\right) f_j\,P_1(z,0),$$

$$1 \le j \le m-1,\ 1 \le \ell \le C \qquad (7.1.21)$$

where $k_s(z) \quad = \quad \sum_{=1}^{C} r\ \ S^*(h_{a_2(\ ,s)}(w_X(z))), \qquad 0 \le s \le m-1,$ (7.1.22)

Adding the equations in (7.1.21) over ℓ = 1 to C and so also the equations in (7.1.15), over i = 1 to C, we have,

$$\sum_{=1}^{C} P_2^{(\ ,j)}(z,0) = \left(\prod_{s=0}^{j} k_s(z)\,f_s\right) P_1(z,0), \quad 0 \le j \le m-1 \qquad (7.1.23)$$

The PGF of the system size when the server is busy in first stage is obtained by using the equation (7.1.2),

$$\theta\, P_1^*(z,\theta) - P_1(z, 0)$$

$$= (\lambda + a_1)\, P_1^*(z,\theta) - \lambda X(z)\, P_1^*(z,\theta) - \mathrm{PI}\,\lambda\, X(z) S^*(\theta) - S^*(\theta)(Q(z, 0) - Q_0(0))$$

$$-\frac{(1-p)\,S^*(\theta)}{z}\left(\sum_{i=1}^{C}\sum_{j=0}^{m-1}(P_2^{(i,j)}(z,0) - P_{2,1}^{(i,j)}(0)\,z)(1-f_{j+1})\right) - BR_1^*(z,\theta,0) \quad (7.1.24)$$

Substituting for Q(z, 0), $P_2^{(i,j)}(z,0)$, $BR_1^*(z,\theta,0)$ and PI from equations (7.1.9), (7.1.18), (7.1.11) and (7.1.7) and simplifying the equation (7.1.24),

$$(\theta - h_{a_1}(w_X(z)))\, P_1^*(z,\theta)$$

$$= P_1(z,0)\left[1 - \frac{S^*(\theta)(1-p+pV^*(w_X(z)))}{z}\left(\sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j}k_s(z)\,f_s\right)\right] + S^*(\theta)\,\mathrm{PI}\,w_X(z)$$

(7.1.25)

where $h_{a_1}(w_X(z)) = w_X(z) + a_1\,(1 - R_1^{*_1}(w_X(z)))$ (7.1.26)

Therefore at $\theta = h_{a_1}(w_X(z))$, the equation (7.1.25) gives,

$$P_1(z, 0) = \frac{z\,\mathrm{PI}\,(-w_X(z))\,S^*(h_{a_1}(w_X(z)))}{D_{1,FF}^{BV}(z)} \quad (7.1.27)$$

where

$$D_{1,FF}^{BV}(z) = [z - S^*(h_{a_1}(w_X(z)))(1 - p + p\,V^*(w_X(z)))\sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j}k_s(z)\,f_s]$$

(7.1.28)

Substituting for $P_1(z, 0)$ in equation (7.1.25),

$$P_1^*(z,\theta) = \frac{z\,\mathrm{PI}\,w_X(z)\,(S^*(\theta) - S^*(h_{a_1}(w_X(z)))}{(\theta - h_{a_1}(w_X(z)))\;D_{1,FF}^{BV}(z)} \quad (7.1.29)$$

Thus the partial generating functions, when the system is in different states at arbitrary epochs are obtained using the respective equations and are given by

$$P_1^*(z,0) = \frac{\mathrm{PI}\;\, z\,w_X(z)\,(S^*(h_{a_1}(w_X(z))) - 1)}{D_{1,FF}^{BV}(z)\;h_{a_1}(w_X(z))} \quad (7.1.30)$$

For $1 \le i \le C$, $0 \le j \le m-1$

$P_2^{(i,j)^*}(z,0)$

$$= \frac{\mathrm{PI}\ z\, r_i\, (w_X(z))\, S^*(h_{a_1}(w_X(z)))\ (S_i^*(h_{a_{2\,(i,j)}}(w_X(z)))\ -1)}{D_{1,FF}^{BV}(z)\ h_{a_{2\,(i,j)}}(w_X(z))} \frac{\prod_{s=0}^{j} k_s(z)\, f_s}{k_j(z)}$$

(7.1.31)

$$Q^*(z,0) = \frac{\mathrm{PI}\ p\,(V^*(w_X(z))-1)\, S^*(h_{a_1}(w_X(z))) \sum_{j=0}^{m-1} (1-f_{j+1}) \prod_{s=0}^{j} k_s(z)\, f_s}{D_{1,FF}^{BV}(z)} \qquad (7.1.32)$$

$$BR_1^{**_1}(z,0,0) \quad = \quad \frac{a_1\, P_1^*(z,0)\,(1-R_1^{*_1}(w_X(z)))}{w_X(z)} \qquad (7.1.33)$$

$$BR_2^{(i,j)^{**_1}}(z,0,0) \ = \ \frac{a_2^{(i,j)}\, P_2^{(i,j)^*}(z,0)\,(1-R_2^{(i,j)^{*_1}}(w_X(z)))}{w_X(z)},\ \ 1 \le i \le C \qquad (7.1.34)$$

where $D_{1,FF}^{BV}(z)$ and $k_s(z)$ are given by the equations (7.1.28) and (7.1.22) respectively with $f_0 = 1$ and $f_m = 0$.

To derive the total PGF of the system size distribution, the following generating functions are considered.

$P_{Idle}(z)$ = Probability generating function of the system size when the server is in idle state

= $\mathrm{PI} + Q^*(z,0) =$

(7.1.35)

$P_{Comp}(z)$ = The PGF of the system size when server is busy or in breakdown state

$$= \ P_1^*(z,0) + BR_1^{**_1}(z,0,0) + \sum_{j=0}^{m-1} \sum_{i=1}^{C} (BR_2^{(i,j)^{**_1}}(z,0,0) + P_2^{(i,j)^*}(z,0))$$

$$= \ \frac{\mathrm{PI}\ z}{D_{1,FF}^{BV}(z)} \left[\sum_{j=0}^{m-1} (1-f_{j+1})\, S^*(h_{a_1}(w_X(z))) \prod_{s=0}^{j} k_s(z)\, f_s - 1 \right] \qquad (7.1.36)$$

Thus the total PGF of the system size distribution is given by

$$P_{FF}^{BV}(z) = P_{Idle}(z) + P_{Comp}(z) = \frac{\mathrm{PI}\,(z-1)\, S^*(h_{a_1}(w_X(z))) \left(\sum_{j=0}^{m-1} (1-f_{j+1}) \prod_{s=0}^{j} k_s(z)\, f_s \right)}{D_{1,FF}^{BV}(z)}$$

(7.1.37)

PI can be calculated by using the normalizing condition $P_{FF}^{BV}(1) = 1$ and found to be PI = $1 - \rho_{1,FF}^{BV}$, where, (7.1.38)

$$\rho_{1,FF}^{BV} = \lambda\, E(X)\, [p\, E(V) + E(H_1) + \sum_{j=0}^{m-1} \sum_{i=1}^{C} r_i\, E(H_2^{(i,j)}) \prod_{s=0}^{j} f_s] \tag{7.1.39}$$

Hence

$$P_{FF}^{BV}(z) = \frac{(1-\rho_{1,FF}^{BV})(z-1)\, S^*(h_{a_1}(w_X(z)))\left(\sum_{j=0}^{m-1} (1-f_{j+1}) \prod_{s=0}^{j} (k_s(z) f_s)\right)}{D_{1,FF}^{BV}(z)} \tag{7.1.40}$$

The measures E(H)s' are obtained by considering the following LST of the random variables ;

$H_1^*(z) = S^*(h_{a_1}(w_X(z)))$ and $H_2^{(i,j)^*}(z) = S_i^*(h_{a_{2\,(i,j)}}(w_X(z)))$ ($1 \le i \le C$ and $0 \le j \le m-1$), and are given by :

$$E(H_1) = E(S)\,(1 + a_1\, E(R_1)) \tag{7.1.41}$$

$$E(H_2^{(i,j)}) = E(S_i)\,(1 + a_2^{(i,j)}\, E(R_2^{(i,j)})) \tag{7.1.42}$$

$$E(H_1^2) = E(S)\, a_1\, E(R_1^2) + E(S^2)\,(1 + a_1\, E(R_1))^2 \tag{7.1.43}$$

$$E((H_2^{(i,j)})^2) = E(S_i)\, a_2^{(i,j)}\, E((R_2^{(i,j)})^2) + E(S_i^2)(1 + a_2^{(i,j)}\, E(R_2^{(i,j)}))^2 \tag{7.1.44}$$

7.1.1.4 Decomposition Property

Using equation (7.1.35), equation (7.1.40) can be re-written as

$$P_{FF}^{BV}(z) = \left(\frac{(z-1)(1-\rho_{1,FF})\, S^*(h_{a_1}(w_X(z))) \sum_{j=0}^{m-1} (1-f_{j+1}) \prod_{s=0}^{j} (k_s(z) f_s)}{z - S^*(h_{a_1}(w_X(z))) \sum_{j=0}^{m-1} (1-f_{j+1}) \prod_{s=0}^{j} (k_s(z) f_s)}\right)\left(\frac{P_{Idle}(z)}{P_{Idle}(1)}\right) \tag{7.1.45}$$

where $\rho_{1,FF} = \lambda\, E(X)\, [E(H_1) + \sum_{j=0}^{m-1} \sum_{i=1}^{C} r_i\, E(H_2^{(i,j)}) \prod_{s=0}^{j} f_s]$ (7.1.46)

and $E(H_1)$ and $E(H_2^{(i,j)})$ are given by (7.1.41) and (7.1.42) respectively.

Equation (7.1.45) shows that, under the steady state condition $\rho_{1,FF}^{BV} < 1$, (hence $\rho_{1,FF} < 1$) the PGF of the stationary system size of the queueing model under consideration is, the product of the PGF of the system size of $M^X/G,G_{i(1 \le i \le C)}/1$ queueing system with finite number of feedbacks and service interruption (without vacation) and the distribution of the number of customers during the idle period given that the server is idle.

7.1.1.5 Queue Size Distribution at Departure Epoch

If π_n^+ denotes the probability that there are n customers in the system at departure epoch, then $\pi_n^+ = D_1 \left[\sum_{j=0}^{m-1} (1-f_{j+1}) \sum_{i=1}^{C} P_{2,n+1}^{(i,j)}(0)\right]$,with normalizing constant D_1.

The PGF $\pi^+(z)$ of the queue size distribution $\{\pi_n^+ ; n \geq 0\}$ at departure epoch is given by

$$\pi^+(z) = \sum_{n=0}^{\infty} \pi_n^+ z^n = \frac{D_1}{z}\left(\sum_{j=0}^{m-1} (1-f_{j+1}) \sum_{i=1}^{C} P_2^{(i,j)}(z,0)\right)$$

$$= \frac{D_1}{(z-1)} \lambda\,(X(z)-1)\, P_{FF}^{BV}(z) \text{(from 7.1.23 and 7.1.27)}$$

Evaluating D_1 using normalizing condition, $\pi^+(z) = \frac{(X(z)-1)}{E(X)(z-1)} P_{FF}^{BV}(z)$

7.1.1.6 Performance Measures

The system size probabilities corresponding to different states of the system are given by :

(i) The probability that the server is on vacation state P_V is

$$P_V = \lim_{z\to 1} Q^*(z,0) = p\,\lambda\, E(X)\, E(V)$$

(ii) The probability that the server is busy is

$$P_{Busy} = P_1 + \sum_{i=1}^{C}\sum_{j=0}^{m-1} P_2^{(i,j)}$$

where $P_1 = \lim_{z\to 1} P_1^*(z,0) = \lambda\, E(X)\, E(S)$

$$P_2^{(i,j)} = \lim_{z\to 1} P_2^{(i,j)^*}(z,0) = \lambda\, E(X) \prod_{s=0}^{j} f_s\, r_i\, E(S_i), 1 \leq i \leq C,\ 0 \leq j \leq m-1$$

(iii) The probability that the server is in breakdown state P_{br} is

$$P_{br} = P_{br_1} + \sum_{i=1}^{C}\sum_{j=0}^{m-1} P_{br_2}^{(i,j)}$$

where $P_{br_1} = a_1\, E(R_1)\, P_1$ and $P_{br_2}^{(i,j)} = a_2^{(i,j)}\, E(R_2^{(i,j)})\, P_2^{(i,j)}$

Mean System Size

(iv) The expected system size of the model is given by

$$L_{FF}^{BV} = \left[\frac{d}{dz}(P_{FF}^{BV}(z))\right]_{z=1} = \frac{(-D_{1,FF}^{BV\,''}(1))}{2\,(1-\rho_{1,FF}^{BV})} + \rho_{1,FF} \qquad (7.1.47)$$

where $\rho_{1,FF}^{BV}$ and $\rho_{1,FF}$ are given by the equations (7.1.39) and (7.1.46),

$$(-D_{1,FF}^{BV\,''}(1))$$

$$= \lambda E(X\,(X-1))\,[pE(V) + E(H_1)] + (\lambda\, E(X))^2\,[\,pE(V^2) + E(H_1^2) + 2pE(V)\,E(H_1)]$$

$$+ \sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j} f_s \left[\left(\prod_{s=0}^{j} k_s(z)\right)''_{z=1} + 2\lambda E(X)\left(\prod_{s=0}^{j} k_s(z)\right)'_{z=1}(p\,E(V)+E(H_1))\right]$$

with $\left(\prod_{s=0}^{j} k_s(z)\right)'_{z=1} = \sum_{s=0}^{j} k_s'(1)$ (Since $k_s(1) = 1$ for every s)

$$k_s'(1) = \lambda\, E(X) \sum_{i=1}^{C} r_i\, E(H_2^{i,s})$$

$$\left(\prod_{s=0}^{j} k_s(z)\right)''_{z=1} = \sum_{s=0}^{j}\left(k_s''(1) - (k_s'(1))^2\right) + \left(\sum_{s=0}^{j} k_s'(1)\right)^2$$

$$k_s''(1) = \lambda\, E(X\,(X-1)) \sum_{i=1}^{C} r_i\, E(H_2^{i,s}) + (\lambda\, E(X))^2 \sum_{i=1}^{C} r_i\, E((H_2^{i,s})^2)$$

and $E(H_1)$, $E(H_2^{(i,j)}), E(H_1^2), E((H_2^{(i,j)})^2)$ are given by the equations (7.1.41), (7.1.42), (7.1.43) and (7.1.44) respectively.

SECTION : 7.2

M^X/G/1 QUEUE WITH FINITE NUMBER OF FEEDBACKS AND REPETITION OF INTERRUPTED SERVICE

7.2.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

7.2.1.1 Model Description

The model of the present Section 7.2 differs from that of 7.1, in the behaviour of the customers during breakdown period. Whenever a service is interrupted due to breakdowns, it was assumed in Section 7.1 that, the service interrupted customers will resume their service, from where they got interrupted. In the present section it is considered that the service gets started from the very beginning independently of the earlier amount of service.

The other assumptions regarding batch arrival pattern, finite number of feedbacks from the multi second optional services, occurrence of the break downs and the Bernoulli schedule vacation that the server takes between services are as similar as in Section 7.1. Thus the states of the system Y(t) = 0, 1, 2, 3, 4 and 5 represents the server is idle, busy in first phase, second phase, under repair in first phase and second phase and in vacation respectively.

The notations of Random Variables (RV) Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments of the RVs are similar to Section 7.1 Table 7.1.1.

The definitions of the state dependent probabilities such as PI(t), $P_{1,n}(x, t)$, $P_{2,n}^{(i,j)}(x,t)$ and $Q_n(x, t)$ are the same as in Section 7.1. The joint probabilities corresponding to the breakdown states are explained below :

$BR_{1,n}(y, t)\,dt$ = $\Pr\{N_S(t) = n,\ y < R_1^o(t) \le y + dt,\ Y(t) = 3\}$, $n \ge 1$, the customer whose service is interrupted joins the head of the queue to start a new service and the remaining repair time lies in the interval (y, y + dt).

$BR_{2,n}^{(i,j)}(y,t)\,dt$ = $\Pr\{N_S(t) = n,\ y < (R_2^{(i,j)})^o(t) \le y + dt,\ Y(t) = 4\}$, $n \ge 1$, $0 \le j \le m-1$, $1 \le i \le C$, the customer whose service is interrupted repeats a new second service due to (j^{th} feedback i^{th} service).

The LST of the steady state equations are directly given by :

$$\theta\, Q_n^*(\theta) - Q_n(0) = \lambda\, Q_n^*(\theta) - \lambda\,(1 - \delta_{0,n}) \sum_{k=1}^{n} Q_{n-k}^*(\theta)\, g_k - \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1 - f_{j+1})\, P_{2,n+1}^{(i,j)}(0)\, p\, V^*(\theta), \quad n \ge 0 \qquad (7.2.1)$$

$$\theta\, P_{1,n}^{*}(\theta) - P_{1,n}(0) = (\lambda + a_1)\, P_{1,n}^{*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{1,n-k}^{*}(\theta)\, g_k$$

$$- S^{*}(\theta)(1-p) \sum_{i=1}^{C} \sum_{j=0}^{m-1} P_{2,n+1}^{(i,j)}(0)\,(1-f_{j+1})$$

$$- PI\,\lambda\, g_n S^{*}(\theta) - BR_{1,n}(0)\, S^{*}(\theta) - Q_n(0)\, S^{*}(\theta),$$

$$n \geq 1,\ f_m = 0 \qquad (7.2.2)$$

$$\theta\, P_{2,n}^{(i,0)^*}(\theta) - P_{2,n}^{(i,0)}(0) = (\lambda + a_2^{(i,0)})\, P_{2,n}^{(i,0)^*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,0)\ *}(\theta)\, g_k$$

$$- P_{1,n}(0)\, r_i\, S_i^{*}(\theta) - BR_{2,n}^{(i,0)}(0)\, S_i^{*}(\theta),$$

$$1 \leq i \leq C,\ \ n \geq 1 \qquad (7.2.3)$$

$$\theta\, P_{2,n}^{(i,j)^*}(\theta) - P_{2,n}^{(i,j)}(0) = (\lambda + a_2^{(i,j)})\, P_{2,n}^{(i,j)^*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} P_{2,n-k}^{(i,j)\ *}(\theta)\, g_k$$

$$- \sum_{=1}^{C} P_{2,n}^{(\ ,j-1)}(0)\, f_j\, r_i\, S_i^{*}(\theta) - BR_{2,n}^{(i,j)}(0)\, S_i^{*}(\theta),$$

$$1 \leq i \leq C\ ;\ 1 \leq j \leq m-1\ ;\ n \geq 1 \qquad (7.2.4)$$

$$\theta\, BR_{1,n}^{*}(\theta) - BR_{1,n}(0) = \lambda\, BR_{1,n}^{*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{*}(\theta)\, g_k$$

$$- \left(\int_0^{\infty} P_{1,n}(w)\, dw \right) a_1\, R_1^{*}(\theta) \qquad (7.2.5)$$

$$\theta\, BR_{2,n}^{(i,j)^*}(\theta) - BR_{2,n}^{(i,j)}(0) = \lambda\, BR_{2,n}^{(i,j)^*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)\ *}(\theta)\, g_k$$

$$- \left(\int_0^{\infty} P_{2,n}^{(i,j)}(w)\, dw \right) a_2^{(i,j)}\, R_2^{(i,j)^*}(\theta),$$

$$1 \leq i \leq C\ ;\ 0 \leq j \leq m-1 \qquad (7.2.6)$$

Equations (7.2.5) and (7.2.6) can be re-written as

$$\theta\, BR_{1,n}^{*}(\theta) - BR_{1,n}(0) = \lambda\, BR_{1,n}^{*}(\theta) - \lambda\,(1-\delta_{1,n}) \sum_{k=1}^{n-1} BR_{1,n-k}^{*}(\theta)\, g_k$$

$$- P_{1,n}^{*}(0)\, a_1\, R_1^{*}(\theta) \qquad (7.2.7)$$

$$\theta\, BR_{2,n}^{(i,j)^*}(\theta) - BR_{2,n}^{(i,j)}(0) = \lambda\, BR_{2,n}^{(i,j)^*}(\theta) - (1-\delta_{1,n})\,\lambda \sum_{k=1}^{n-1} BR_{2,n-k}^{(i,j)*}(\theta)\, g_k$$

$$- P_{2,n}^{*}(0)\ a_2^{(i,j)}\, R_2^{(i,j)^*}(\theta), \qquad 1 \leq i \leq C\ ;\ 0 \leq j \leq m-1$$

$$(7.2.8)$$

$$\lambda\, PI = Q_0(0) + \sum_{i=1}^{C} \sum_{j=0}^{m-1} (1-f_{j+1})\, P_{2,1}^{(i,j)}(0)\,(1-p) \qquad (7.2.9)$$

Proceeding as in Section 7.1, the partial generating functions corresponding to different states at arbitrary epochs are calculated using the respective equations and are given by

$$P_1^*(z,0) \;=\; \frac{PI\; z\,(w_X(z))\,(S^*(g_{a_1}(w_X(z)))-1)}{D_{2,ff}^{BV}(z)} \tag{7.2.10}$$

where $g_{a_1}(w_X(z)) = a_1 + w_X(z)$ (7.2.11)

For $1 \le i \le C$ and $0 \le j \le m-1$,

$$P_2^{(i,j)^*}(z,0)$$
$$= \frac{PI\; z\,w_X(z)\,g_{a_1}(w_X(z))\,S^*(g_{a_1}(w_X(z)))\,r_i\,(S_i^*(g_{a_2^{(i,j)}}(w_X(z)))-1)}{D_{2,ff}^{BV}(z)\,(h_{a_2^{(i,j)}}(w_X(z))+S_i^*(g_{a_2^{(i,j)}}(w_X(z))\,a_2^{(i,j)}\,R_2^{(i,j)^*}(w_X(z)))}\left(\prod_{s=0}^{j}\frac{K_s(z)\,f_s}{K_j(z)}\right) \tag{7.2.12}$$

$$Q^*(z,0)$$
$$= \frac{PI\; g_{a_1}(w_X(z))\,S^*(g_{a_1}(w_X(z)))\,p\,(V^*(w_X(z))-1)}{D_{2,ff}^{BV}(z)}\sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j}K_s(z)\,f_s \tag{7.2.13}$$

$$BR_1^*(z,0) \;=\; \frac{a_1\,(1-R_1^*(w_X(z))}{w_X(z)}\,P_1^*(z,0) \tag{7.2.14}$$

$$BR_2^{(i,j)^*}(z,0) \;=\; \frac{a_2^{(i,j)}\,(1-R_2^{(i,j)*}(w_X(z)))}{w_X(z)}\,P_2^{(i,j)^*}(z,0) \tag{7.2.15}$$

where

$$D_{2,ff}^{BV}(z)$$
$$= z\,[h_{a_1}(w_X(z)) + S^*(g_{a_1}(w_X(z)))\;a_1\,R_1^*(w_X(z))]$$
$$-\,[(1-p+p\,V^*(w_X(z)))\;g_{a_1}(w_X(z))\;S^*(g_{a_1}(w_X(z)))\sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j}K_s(z)\,f_s] \tag{7.2.15.1}$$

$$K_s(z) \;=\; \sum_{i=1}^{C}\left(\frac{r_i\,S_i^*(g_{a_2^{(i,s)}}(w_X(z)))\,g_{a_2^{(i,s)}}(w_X(z))}{h_{a_2^{(i,s)}}(w_X(z))+S_i^*(g_{a_2^{(i,s)}}(w_X(z)))\,a_2^{(i,s)}\,R_2^{(i,s)^*}(w_X(z))}\right) \tag{7.2.15.2}$$

$g_{a_2^{(i,s)}}(w_X(z)) = a_2^{(i,s)} + w_X(z)$, $\quad 0 \le s \le m-1$

$h_{a_2^{(i,s)}}(w_X(z)) = g_{a_2(i,s)}(w_X(z)) - a_2^{(i,s)}\,R_2^{(i,s)^*}(w_X(z))$

The total PGF of the system size distribution is obtained by adding equations (7.2.10) to (7.2.15) and

$$P_{ff}^{BV}(z) = \frac{\mathrm{PI}\ g_{a_1}(w_X(z))(S^*(g_{a_1}(w_X(z)))[(z-1)\sum_{j=0}^{m-1}(1-f_{j+1})\prod_{s=0}^{j}K_s(z)\,f_s]}{D_{2,ff}^{BV}(z)} \tag{7.2.16}$$

The constant PI can be calculated by using the normalizing condition

$P_{ff}^{BV}(1) = 1$ and found to be $\mathrm{PI} = 1 - \rho_{2,ff}^{BV}$

where

$$\rho_{2,ff}^{BV} = \lambda\, E(X)\, [\frac{(1-S^*(a_1))}{a_1\, S^*(a_1)}(1 + a_1\, E(R_1)) + p\, E(V)$$
$$+ \sum_{j=0}^{m-1}\prod_{s=0}^{j} f_s \sum_{i=1}^{C} r_i\, \frac{(1-S_i^*(a_2^{(i,j)}))}{a_2^{(i,j)}\, S_i^*(a_2^{(i,j)})}(1 + a_2^{(i,j)} E(R_2^{(i,j)}))]$$

7.2.2 PARTICULAR CASE

When the service times S and S_i ($1 \le i \le C$) follow exponential (Markovian) distribution with parameters μ and μ_i ($1 \le i \le C$) respectively, it is verified that the steady state results for the models in which the service interrupted customers resume service or repeat service of Sections 7.1 and 7.2 coincide. The following observations show that the PGF of the system size of both the models become equal :

For the Markovian service times,

$S^*(\theta) = \dfrac{\mu}{\mu+\theta}$; $S_i^*(\theta) = \dfrac{\mu_i}{\mu_i+\theta}$. This implies

$$K_s(z) = k_s(z) = \sum_{i=1}^{C} r_i\, \frac{\mu_i}{\mu_i + h_{a_2^{(i,s)}}(w_X(z))} \quad (s = 0 \text{ to } m-1)$$

$$\rho_{2,ff}^{BV} = \rho_{1,FF}^{BV} = \lambda E(X)[\frac{1}{\mu}(1 + a_1 E(R_1)) + pE(V) + \sum_{j=0}^{m-1}\prod_{s=0}^{j} f_s \sum_{i=1}^{C}\frac{r_i}{\mu_i}(1 + a_2^{(i,j)}\, E(R_2^{(i,j)}))]$$

and

$$\frac{g_{a_1}(w_X(z))\,S^*(g_{a_1}(w_X(z)))}{D_{2,ff}^{BV}(z)} = \frac{S^*(h_{a_1}(w_X(z)))}{D_{1,FF}^{BV}(z)} = \frac{\mu}{[\mu + h_{a_1}(w_X(z))]\,D_{1,FF}^{BV}(z)}$$