

**The Classification of Cubic Rook Polynomials in a Generalized
Board**

**Deepika, S
(14PMA002)**

**Thesis Submitted to
Avinashilingam Institute for Home Science and Higher Education for Women
Coimbatore-641 043**

**In Partial Fulfilment of the Requirements for the Degree of
Master of Science in Mathematics**

April 2016

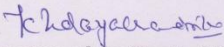
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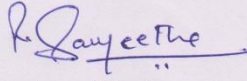
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CONTENT

CHAPTER

TITLE

INTRODUCTION

REVIEW OF LITERATURE

I THE CLASSIFICATION OF QUADRATIC ROOK POLYNOMIALS

- 1.1 Basic terminologies
- 1.2 Boards with $r_3 = 0$
- 1.3 Classification of quadratic rook
Polynomials
- 1.4 Example of finding $r_2(B)$ values

II CUBIC ROOK POLYNOMIALS

- 2.1 Basic Definitions
- 2.2 Rook equivalency of boards contained in
three rows or columns

III THE CLASSIFICATION OF CUBIC ROOK POLYNOMIALS

- 3.1 Classification of cubic rook polynomials

SUMMARY AND CONCLUSION

BIBLIOGRAPHY

PUBLICATION

INTRODUCTION

The term rook polynomial was coined by John Riordan. Despite the name's derivation from chess, the impetus for studying rook polynomials is their connection with counting permutations with restricted positions. A board B that is a subset of the $n \times n$ chess board corresponds to permutation of n objects, which we may take to the numbers $1, 2, \dots, n$ such that the number a_i in the j^{th} position in the permutation must be the column number of an allowed square in row j of B . Famous examples include the number of ways to place n not-attacking rooks on,

- (i) An entire $n \times n$ chess board, which is an elementary combinatorial problem.
- (ii) The same board with its diagonal squares forbidden, this is the derangement or "hat-check" problem.

Rook placement problems arises in pure and applied Combinatorics, group theory, number theory and statistical physics.

The particular value of rook polynomials comes from the utility of the generating function approach, and also from the fact that the zeros of the rook polynomial of a board provide valuable information about its coefficients (i.e.) the number of non-attacking placements of k rooks.

A precursor to the rook polynomial is the classic "Eight rooks problem" by H.E.Dudency in which he shows that the maximum number of an non-attacking rooks on a chess board is eight by placing them on one of the main diagonals. The number of ways of placing eight rooks on 8×8 chess board so that neither of them attacks the other is $8! = 40,320$.

Obviously there must be a rook in every row and every column. Starting with the bottom row, it is clear that the first rook can be put on any one of eight difference squares. Whenever it is placed, there is the option of seven squares for the second rook in the second row. Then there are six squares from which to select the third row, five in the fourth, and so on. Therefore the number of different ways must be $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$

Let us generalize this problem by considering $m \times n$ board, that is, a board with m rows and n columns. The problem is in finding the number of ways we can arrange k rooks on an $m \times n$ board in such a way that they do not attack each other.

It is clear that for the problem to be solvable, k must be less than or equal to the smaller of the numbers m and n . Otherwise one cannot avoid placing a pair of rooks on a row or on a column. The arrangement of rooks can be carried out in two steps. First, choose the set of k rows on which to place the rooks. Since the number of rows is m , of which k must be chosen, this choice can be done in $\binom{m}{k}$ ways. Similarly the set of k columns on which to place the rooks can be chosen in $\binom{n}{k}$ ways.

Because the choice of columns does not depend on the choice of rows, using *product rule* there are $\binom{m}{k} \binom{n}{k}$ ways to choose the square on which the rook can be placed.

Since k rows and k columns intersect in k^2 square, by deleting unused rows and columns and compacting the remaining rows and columns together, we obtain a new board of k rows and k columns. We already know on such a board, k rooks can be placed in $k!$ ways. Such that they do not attack each other. Therefore, the total number of possible non-attacking rooks arrangements is

$$\begin{aligned} r_k &= \binom{m}{k} \binom{n}{k} k! \\ &= \frac{m! n!}{k!(n-k)!(m-k)!} \end{aligned}$$

For example, two rooks can be placed on a 8×8 chess board in

$$\frac{8! 8!}{2! 6! 6!} = 1568 \text{ ways}$$

For $k = m = n$, the above formula gives

$$r_k = \frac{n! n!}{n! 0! 0!} = n!$$

which corresponds to the result obtained for the classical chess problem.

In Chapter I, we provide the necessary background to work with boards that satisfy our desired conditions. We prove several theorems regarding rook equivalency that allow us to minimize the number and types of boards that we need to work with we focus on the classification of all quadratic polynomials. We provide a theorem that introduces a formula that we can use to solve our problem. We conclude with an example that utilizes our formula.

In Chapter II, we provide the necessary background to work with boards having rook numbers less than or equal to 3 for rook polynomials. The theorems proved in chapter I, we redefined and proved regarding rook equivalency that allow us to minimize the number and types of boards that are completely contained in 3×3 board.

In Chapter III, The polynomials for the possibilities of cubic rook polynomials under certain conditions are calculated.

REVIEW OF LITERATURE

Researchers on the field of classification of rook polynomials in generalized board was developed by many authors in the last 15 years. Since the advent of these notions, Several authors contributed to the study of these concepts and several worthwhile research papers have been published. We provide a brief review of literature in some of the important articles published that are related to this topic.

The initiation of the study of "*The problem of Rook and its Applications*" was done by Irving Kaplansky and John Riordan in the year 1946.

The theory was extensively developed in 1997 and several new concepts were studied and investigated.

The study of Two and Three dimensions of "*Rook polynomials for chess boards of Two and Three dimensions*" was done by Benjamin Zindle in the year 2010.

The study of "*A classification of Quadratic Rook polynomials*" was done by Alicia Velek in the year 2014. She introduced Rook Theory in the study of permutations using terminology from the game of chess. In Rook Theory, A generalized board B is any subset of the squares of an $n \times n$ square chess board for some positive n . Rook numbers count the number of ways to place non-attacking rooks on a generalized board, and the rook polynomial is a polynomial that organises the rook numbers for a board B . In this paper, She has classified all quadratic polynomials that are the rook polynomials for some generalized board B .

CHAPTER I

The Classification of Quadratic Rook Polynomials

1.1 Basic Terminologies:

Definition: 1.1.1

We define a *board* to be a square $n \times n$ chessboard for any $n \in \mathbb{N}$. A *generalized board* is any subset of squares of the board. Thus, a generalized board can be any arrangement of squares that is completely contained inside a board, or it can be the board itself.

We use ordered pairs in the first quadrant of \mathbb{R}^2 to label squares on our $n \times n$ board. For example, $(1,1)$ denotes the square in the bottom left corner of the board, $(2,3)$ denotes the square in the second column from the left and third row from the bottom, and (n, n) denotes the squares in the top right corner of the board.

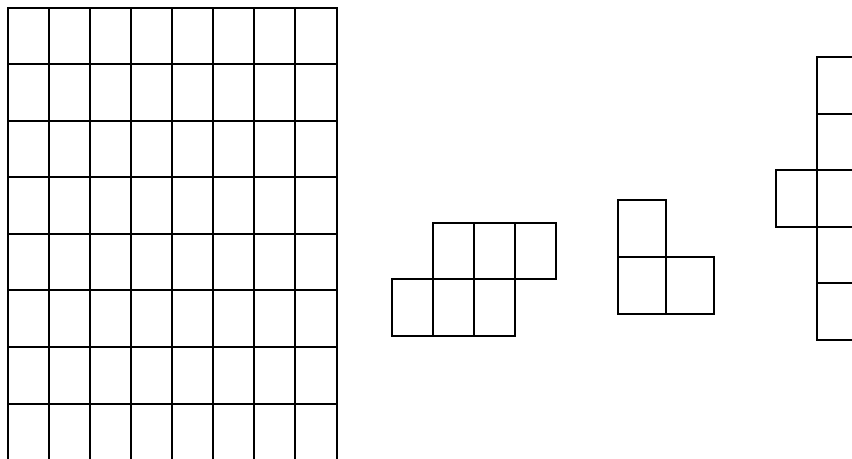


Figure 1.1.1: A board and two generalized boards.

Definition: 1.1.2

A *rook placement* is an arrangement of some number of non –attacking rooks on some board. Note that since rooks attack squares in their row and column, a rook placement cannot have more than one rook in a given row or column.

A placement of n rooks on an $n \times n$ square board can be associated to a permutation $\sigma = \sigma_1 \dots \dots \sigma_n$ of $1,2,\dots,n$ by saying the placement P_σ has a rook on the square (i,j) of the board if and only if $\sigma_i = j$.

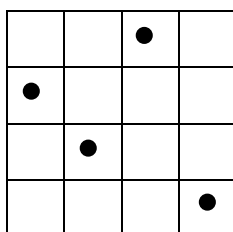


Figure 1.1.2: Example of a permutation.

For example, the board pictured here corresponds to the permutation $\sigma = 3124$

In this way, we see that any theorem about rook placement is a generalization of a theorem about permutations.

Definition: 1.1.3

The k^{th} rook number, $r_k(B)$, counts the number of ways to place k non-attacking rooks on a generalized board B . We will often denote $r_k(B)$ as r_k when B is clear.

Some notes about rook numbers:

1. r_0 is always 1 because there is only one way to place 0 rooks on a generalized board.
2. r_1 is always the number of squares of B because a single rook can be placed in any square of B with no other rook to attack it.
3. Since a rook attacks all squares in its row and column, each rook in a rook placement must be in a different row or column.

4. Once we attain $r_k = 0$, we will always have $r_{k+1}, r_{k+2}, \dots = 0$. For example, if we are unable to place 3 non-attacking rooks on a generalized board, we cannot place 4 or more non-attacking rooks on the generalized board.
5. If B is contained in an $n \times n$ square board and $k > n$, then we have that $r_k = 0$. However, note that r_k could be equal to 0 for smaller values of k as well.

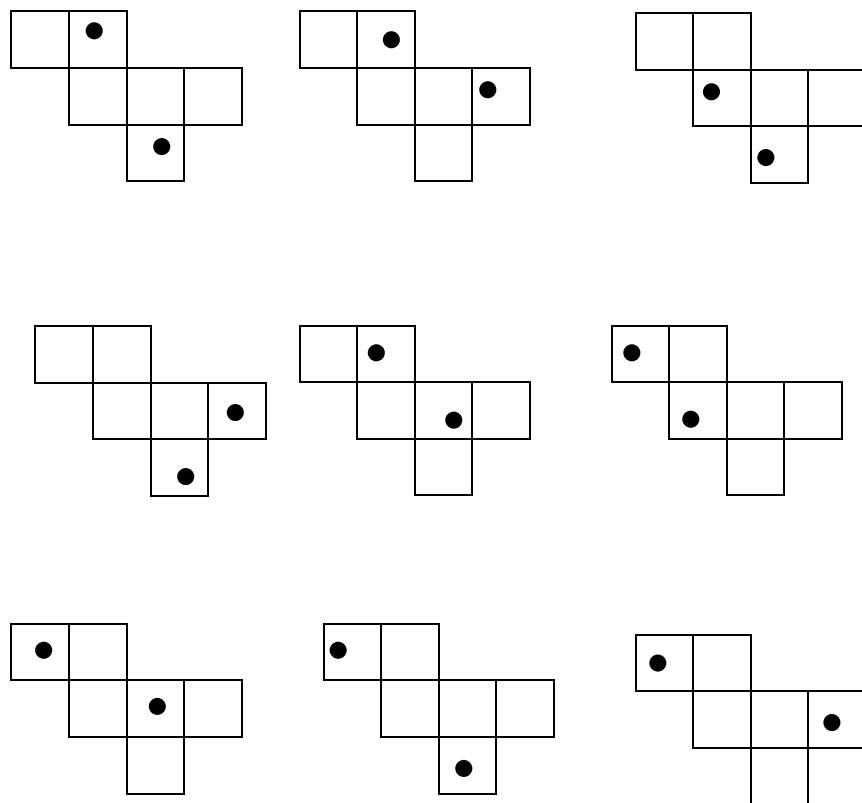
Definition: 1.1.4

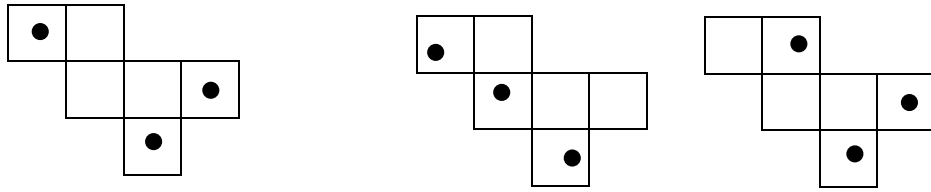
We can construct a polynomial that keeps track of all of the rook numbers of a generalized board at once. Such a polynomial is known as the *rook polynomial in x* and is denoted $R(B, x)$. The r_k 's are the coefficients of the x^k terms, as shown:

$$R(B, x) = r_0(B) + r_1(B)x + r_2(B)x^2 + \dots + r_{n-1}(B)x^{n-1} + r_n(B)x^n$$

Example 1.1.1:

Consider the generalized board B in Figure 1.1.3. By our comments above, it follows that $r_0 = 1$ and $r_1 = 6$. Moreover, there are 9 ways to place 2 non-attacking rooks on B , as shown below; therefore, $r_2 = 9$. Similarly, we see $r_3 = 3$ because there are 3 ways to place 3 non-attacking rooks on in B , so $r_4, r_5, r_6, \dots = 0$.





Example of a generalized board.

Thus, after obtaining these rook numbers, the corresponding rook polynomial for this board is: $1 + 6x + 9x^2 + 3x^3$.

Definition: 1.1.5

Two boards are *rook equivalent* if they have the same rook polynomial (and hence, the same rook numbers).

Example 1.1.2:

The board in Figure 1.1.4 is rook equivalent to the board in Example 1.1.1. Verification is left to the reader.

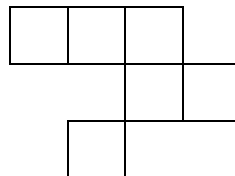


Figure 1.1.4: Example of an equivalent board.

1.2 Boards with $r_3 = 0$

In this section, we will prove several theorems that lead to the classification of all quadratic polynomials that are the rook polynomial of some generalized board B . In order for a board B to have a quadratic rook polynomial, we want $r_3(B) = 0$. Then, by note 4 above, $r_k(B) = 0$ for all $k \geq 3$. Moreover, we also must have $r_2(B) > 0$ to ensure that our polynomial is quadratic and not linear. All boards satisfying these properties are classified by the following theorem.

Theorem 1.2.1

Let B be a nonempty board such that $r_3(B) = 0$. Then B satisfies at least one of the following criteria:

1. B is contained in two or fewer rows;
2. B is contained in two or fewer columns;
3. B is the union of one part of the board contained in one row and another part contained in one column. An example of such a board is shown in Figure 1.1.5.

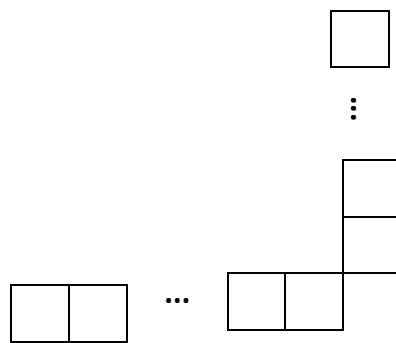


Figure 1.1.5: A board meeting criterion 3.

Proof:

We will prove by contrapositive. Hence, we aim to show that if a board B does not satisfy any of the properties above, then $r_3 \neq 0$.

Assume B does not satisfy any of the listed criteria. So B has at least three rows, at least three columns, does not have one part of the board contained in one row and one part contained in one column. These conditions force B to have three squares each in a different row and column from the squares. Non-attacking rooks can be placed on those squares, so $r_3 \neq 0$, as desired.

Now our goal is to show that for the purposes of our classification of quadratic rook polynomials, we only need to consider boards contained in two rows. We do this in the following two theorems.

Theorem 1.2.2

A generalized board contained in two columns is rook equivalent to one contained in two rows.

Proof:

If B is a generalized board contained in two columns of an $n \times n$ board, let B' be the board obtained by rotating B 90° clockwise, as shown in Figure 1.1.6.

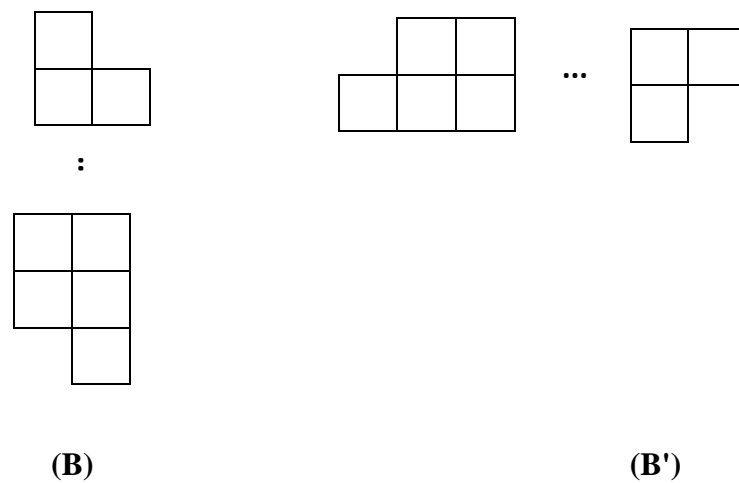


Figure 1.1.6: Equivalent generalized boards

It is clear that B' is contained in two rows, B' and B have the same rook numbers. Hence, a board contained in two columns is rook equivalent to a board contained in two rows.

In order to show that a board satisfying criterion 3 of theorem 1.2.1 is rook equivalent to a board contained in two rows, we will use the following theorem.

Theorem 1.2.3

Any generalized board with one part contained in one row and another part contained in one column has the same rook polynomial as a board contained within two rows.

Proof:

Note that empty rows and columns of the $n \times n$ square board have no effect on rook numbers. Thus, all boards we consider will have all empty rows and columns removed. We have two cases to consider.

Case 1:

The squares in the column cannot attack any square in the row. Note that changing the position of any column has no effect on the rook numbers. In this case, the board pictured below on the left is rook equivalent to the board pictured on the right because $r_0(B) = r_0(B') = 1$ and $r_1(B) = r_1(B') = a + b$, or the number of squares of each board. To find r_2 for either board, we place one rook in any of the a squares in the bottom row. Then there are b choices for a second non-attacking rook. Hence, $r_2(B) = r_2(B') = ab$. Last, $r_k(B) = r_k(B') = 0$ for $k \geq 3$.

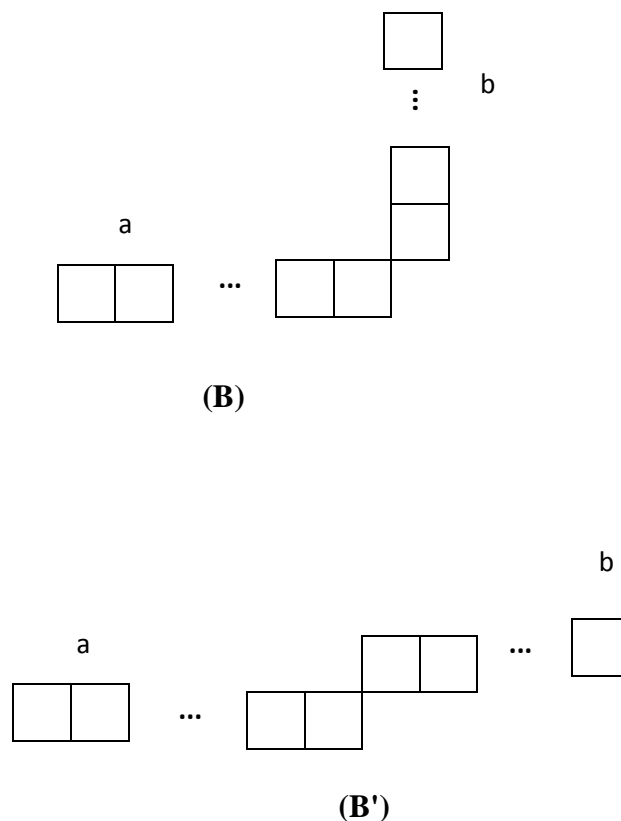
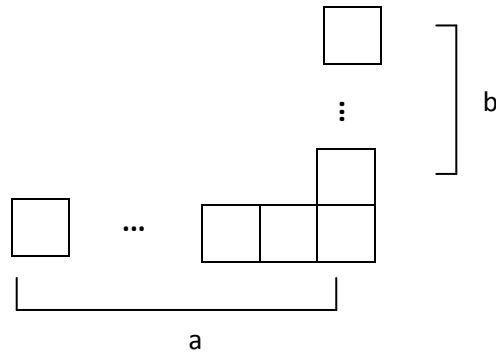


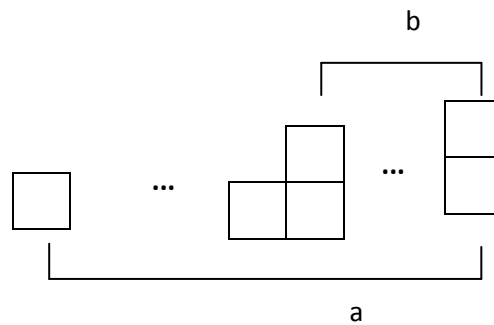
Figure 1.1.7: Rook equivalent boards as in case 1.

Case 2:

The squares in the column can attack squares in the row. Such a board is pictured on the left in Figure 1.1.8. We will show that this board is rook equivalent to the board on the right in Figure 1.1.8, which is contained in two rows



(B)



(B')

Figure 1.1.8: Rook equivalent boards as in case 2.

We see that $r_0(B) = r_0(B') = 1$ and $r_1(B) = r_1(B') = a + b$.

Consider the board on the left in Figure 1.1.8. We can place our first rook in any of the first $a - 1$ squares in the bottom of the row. Then we have b squares to place a second non-attacking rook. Hence $r_2(B) = (a - 1)b = ab - b$.

Now consider the board on the right in Figure 1.1.8. We can see that b is equal to the number of columns where the rows overlap. We can place our first rook in the $a - b$ squares in the bottom row, leaving b squares in which to place a second non-

attacking rook. We could also place our first rook in one of the b squares in the bottom row, leaving $b - 1$ squares in which to place a second non-attacking rook. Hence, $r_2(B) = (a - b)b + b(b - 1) = ab - b$.

Thus, $r_2(B) = r_2(B') = ab - b$. Also note that $r_k(B) = r_k(B') = 0$ for $k \geq 3$. Hence, the two boards are rook equivalent and have the same rook polynomial, as desired.

1.3. Classification of quadratic rook polynomials:

Now we move on to the classification of all quadratic polynomials which are the rook polynomial of some generalized board B . Suppose $r_0(B) + r_1(B)x + r_2(B)x^2$ is a quadratic rook polynomial for some board B . Recall that $r_0(B) = 1$ for any board B and $r_1(B)$ can be any positive integer greater than 1. We must also have $r_2(B) \geq 1$ and $r_k(B) = 0$ for $k \geq 3$. If $r_3(B) \neq 0$ our rook polynomial is of higher degree. Hence the polynomial that we seek has the form $1 + r_1(B)x + r_2(B)x^2$.

Thus, taking into account these equivalences and the requirement of $r_3(B) = 0$, we found that it suffices to consider generalized boards that lie within two rows of a board that have spaces which lie consecutively within each row in order to find the desired rook polynomials.

The above discussion reduces the classification in proving the following theorem.

Theorem 1.3.1

Let B be the generalized board contained in two rows whose rook polynomial is quadratic. Given $r_1(B)$, every possible $r_2(B)$ will have the form $r_2(B) = a(r_1(B) - a) - i$, for $1 \leq a \leq \left\lfloor \frac{r_1}{2} \right\rfloor$ and $0 \leq i \leq a$.

Proof:

Given any generalized board B satisfying the conditions stated in the first paragraph of section 2.3 and its analogous value of $r_1(B)$, we want to find all possible corresponding values of $r_2(B)$ by considering each pair of positive integers a and b of $r_1(B)$ such that $a + b = r_1(B)$. Without loss of generality, let's assume that

$a \leq b$. This will imply that $1 \leq a \leq \left\lceil \frac{r_1}{2} \right\rceil$. Note that $b = r_1 - a$. Here a and b represent the number of squares in the two rows that B occupies, respectively.

Then the a and b squares can be arranged such that a squares lie consecutively in one row and b squares lie consecutively in next row. We can rearrange the columns of a generalized board B so that i columns containing squares in both rows and lie in the centre, columns with empty squares in row 1 lie adjacent on the right, and columns with empty squares in row 2 lie adjacent on the left, as illustrated in figure 1.3.1. Such a rearrangement will not change the rook numbers.

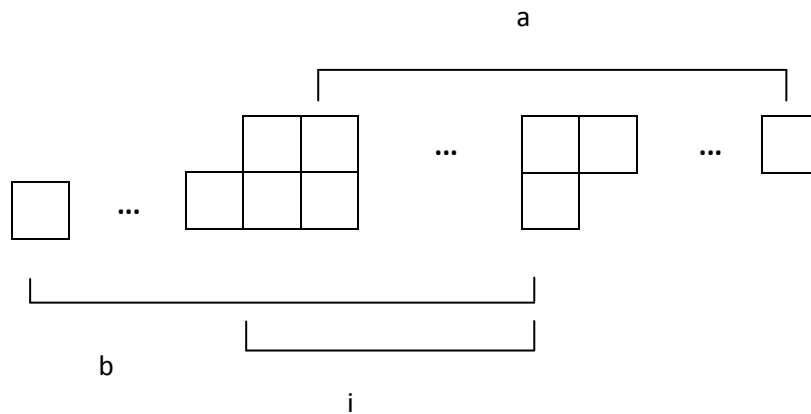


Figure 1.3.1: A generalized board B

By placing one rook in each row, the possible values of $r_2(B)$ can be found for each value $0 \leq i \leq a$. The maximum of r_2 will be when $i = 0$ (a and b are completely disjoint), or $r_2(B) = ab$. The minimum value of $r_2(B)$ will be when $i = a$ (a and b completely overlap), or $r_2(B) = a(b - 1)$.

Every value between $r_2(B) = ab$ and $r_2(B) = a(b - 1)$ can be obtained as well. Beginning with a disjoint pair, shift the row containing a squares one square to the left, which decreases the value of r_2 by one. Continuing this process until a and b are overlapping exhausts all possible values of r_2 . Hence, for a given a and b , each choice of i will yield a different $r_2(B)$ value.

For any particular arrangement, the value of $r_2(B)$ is equal to

$(a - i)(b) + (i)(b - 1)$ which simplifies to $(ab - i)$. Recall that $b = r_2(B) - a$.

1.4 Example of Finding $r_2(B)$ Values:

Suppose we want to classify all quadratic rook polynomials with $r_1(B) = 10$. The possible integer pairs for a and b are :1,9; 2,8; 3,7; 4 ,6; 5, 5. Apply the formula,

$$r_2(B) = a(r_1(B) - a) - i \text{ for each pair}$$

$$(1)(9) - 0 = 9$$

$$(2)(8) - 2 = 14$$

$$(3)(7) - 3 = 18$$

$$(4)(6) - 3 = 21$$

$$(5)(5) - 2 = 23$$

$$(1)(9) - 1 = 8$$

$$(3)(7) - 0 = 21$$

$$(4)(6) - 0 = 20$$

$$(4)(6) - 4 = 20$$

$$(5)(5) - 3 = 22$$

$$(2)(8) - 0 = 16$$

$$(3)(7) - 1 = 20$$

$$(4)(6) - 1 = 23$$

$$(5)(5) - 0 = 25$$

$$(5)(5) - 4 = 21$$

$$(2)(8) - 1 = 15$$

$$(3)(7) - 2 = 19$$

$$(4)(6) - 2 = 22$$

$$(5)(5) - 1 = 24$$

$$(5)(5) - 5 = 20$$

Therefore, a generalized board with 10 squares can obtain every value for $r_2(B)$ between 8 and 25 except for 10, 11, 12 ,13 and 17.

This possible rook polynomial for $r_1(B) = 10$ are as follows:

$1 + 10x + 8x^2$	$1 + 10x + 20x^2$
$1 + 10x + 9x^2$	$1 + 10x + 21x^2$
$1 + 10x + 14x^2$	$1 + 10x + 22x^2$
$1 + 10x + 15x^2$	$1 + 10x + 23x^2$
$1 + 10x + 16x^2$	$1 + 10x + 24x^2$
$1 + 10x + 18x^2$	$1 + 10x + 25x^2$
$1 + 10x + 19x^2$	

This process can be completed for any value of $r_1(B)$.

The following is a table of all quadratic rook polynomial with $r_1(B) < 10$. Note that we must begin with $r_1(B) = 2$.

<u>For $r_1(B) = 2$</u>	<u>For $r_1(B) = 3$</u>
$1 + 2x + x^2$	$1 + 3x + x^2$
	$1 + 3x + 2x^2$
<u>For $r_1(B) = 4$</u>	<u>For $r_1(B) = 5$</u>
$1 + 4x + 3x^2$	$1 + 5x + 3x^2$
$1 + 4x + 4x^2$	$1 + 5x + 4x^2$
	$1 + 5x + 5x^2$
	$1 + 5x + 6x^2$
<u>For $r_1(B) = 6$</u>	<u>For $r_1(B) = 7$</u>
$1 + 6x + 4x^2$	$1 + 7x + 5x^2$
$1 + 6x + 5x^2$	$1 + 7x + 6x^2$
$1 + 6x + 6x^2$	$1 + 7x + 8x^2$

$$1 + 6x + 7x^2$$

$$1 + 6x + 8x^2$$

$$1 + 6x + 9x^2$$

$$\text{For } r_1(B) = 8$$

$$1 + 8x + 6x^2$$

$$1 + 8x + 7x^2$$

$$1 + 8x + 9x^2$$

$$1 + 8x + 10x^2$$

$$1 + 8x + 11x^2$$

$$1 + 8x + 12x^2$$

$$1 + 8x + 13x^2$$

$$1 + 8x + 14x^2$$

$$1 + 8x + 15x^2$$

$$1 + 8x + 16x^2$$

$$1 + 7x + 9x^2$$

$$1 + 7x + 10x^2$$

$$1 + 7x + 11x^2$$

$$1 + 7x + 12x^2$$

$$\text{For } r_1(B) = 9$$

$$1 + 9x + 7x^2$$

$$1 + 9x + 8x^2$$

$$1 + 9x + 12x^2$$

$$1 + 9x + 13x^2$$

$$1 + 9x + 14x^2$$

$$1 + 9x + 15x^2$$

$$1 + 9x + 16x^2$$

$$1 + 9x + 17x^2$$

$$1 + 9x + 18x^2$$

$$1 + 9x + 19x^2$$

$$1 + 9x + 20x^2$$

By theorem 2.3.1, we now see that a quadratic polynomial is a rook polynomial of a generalized board if and only if it has the form:

$$1 + r_1(B)x + [a(r_1(B) - a)i]x^2$$

for positive integers $r_1(B), a, i$, where $r_1 > 1, 1 \leq a \leq \lfloor \frac{r_1}{2} \rfloor$, and

$$0 \leq i \leq a$$

CHAPTER II

Cubic Rook Polynomials

2.1. Basic Definitions:

Definition: 2.1.1

We define a board to be a square $n \times n$ chessboard for any $n \in \mathbb{N}$. A *generalized board* is any subset of squares of the board. Thus, a generalized board can be any argument of squares that is completely contained inside a board (or) it can be the board itself.

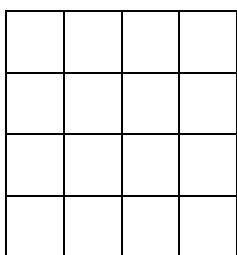


Fig 2.1.1 (a) Board

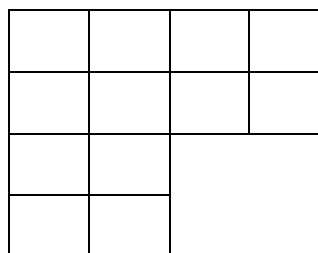


Fig 2.1.1 (b) Generalized board

Fig 2.1.1 (a) is an example of 4x4 board.

Fig 2.1.1 (b) is the generalized board with 12 squares that is σ contained inside the 4x4 board (Fig (a)).

Definition: 2.1.2

A rook polynomial is an arrangement of some numbers of non-attacking rooks on some board. Since rooks attack square in their row and column, a rook placement cannot have more than one rook in a given row (or) column.

A placement of n rooks on an $n \times n$ square board can be associated to a permutation $\sigma = \sigma_1, \sigma_2, \dots, \sigma_n$, of $1, 2, 3, \dots, n$ by saying the placement $P\sigma$ has a rook on the square $n \times n$ of the board if and only if $\sigma_i = j$.

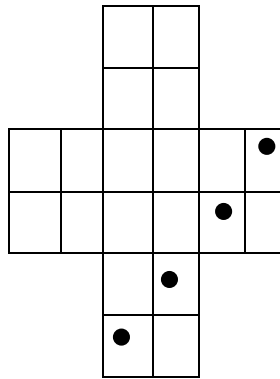


Fig 2.1.2: Example of permutation

In the above generalized board the maximum number of non- attacking rooks that can be placed is four.

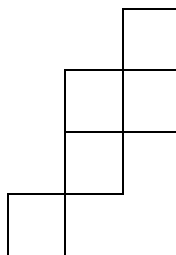
Definition: 2.1.3

The *rook polynomial* $R_B(x)$ of a board B is generating for the number of arrangements of non attacking rooks.

$$R_B(x) = \sum_{k=0}^{\infty} R_k(B) x^k$$

where r_k is the number of ways to place k non attacking rooks on the board. This sum is finite since the board is finite and so there is a maximum number of non-attacking rooks it can hold. Indeed, there cannot be more rooks than the smaller of the number of rows and columns in the board.

Example:



In the above example there are five ways to place one non-attacking rook, seven ways to place two non attacking rooks and three ways to place three non attacking rooks and no way to place four or more non attacking rooks.

Definition: 2.1.4

The k^{th} rook number $r_k(B)$ counts the number of ways to place k non- attacking rooks on a generalized board B . $r_k(B)$ is as r_k when B is clear.

Remark on rook numbers:

1. r_0 is always 1 because there is only one way to place 0 rooks on a generalized board.
2. r_1 is always the number of squares of B because a single rook can be placed in any square of B with no other rook to attack it.
3. Since a rook attacks all squares in its row and column, each rook in a rook placement must be in a different row or column.
4. Once we attain $r_k=0$, we will always have $r_{k+1}, r_{k+2}, \dots = 0$. For example, if we are unable to place 3 non-attacking rooks on a generalized board, we cannot place 4 or more non-attacking rooks on the generalized board.
5. If B is contained in an $n \times n$ square board and $k > n$, then we have $r_k=0$. For example, in an 3×3 square board $r_4=0$ ($k=4$) as we could not place 4 non-attacking rooks in 3×3 square board.

Note:

r_k could be equal to 0 for smaller values of k as well. For example, the generalized board in fig(2) which is completely contained in 6×6 board has $r_5 = r_6 = 0$ as we could not place 5 or more non-attacking rooks in the board. We have considered.

2.2 Rook equivalency of boards contained in three rows or columns:

Lemma 2.2.1

let B be a board of darkened squares that decomposes into disjoint sub boards B_1 and B_2 . Then

$$r_k(B) = r_k(B_1) r_k(B_2) + r_{k-1}(B_1) r_1(B_2) + \dots + r_0(B_1) r_k(B_2)$$

Now we define the rook polynomial $R(X,B)$ of the board B as follows:

$$R(x, B) = r_0(B) + r_1(B)x + r_2(B)x^2 + \dots + r_n(B)x^n + \dots$$

Theorem 2.2.1

Let B be a board of darkened squares that decomposes into disjoint sub boards B_1 and B_2 . Then

$$R(X,B) = R(x, B_1) + R(x, B_2).$$

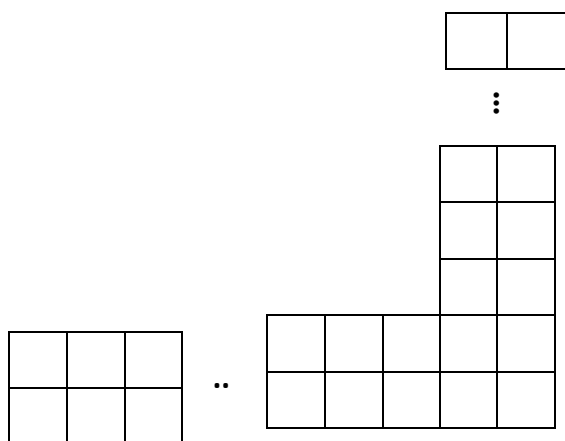
Proof:

$$\begin{aligned} R(X,B) &= r_0(B) + r_1(B)x + r_2(B)x^2 + r_3(B)x^3 + \dots \\ &= 1 + [r_1(B_1) r_0(B_2) + r_0(B_1) r_1(B_2)]x + \\ &\quad [r_2(B_1) r_0(B_2) + r_1(B_1)r_1(B_2) + r_0(B_1)r_2(B_2)]x^2 + \dots \\ &= [r_0(B_1) + r_1(B_1)x + r_2(B_1)x^2 + \dots]x \times [r_0(B_2) + r_1(B_2)x + \\ &\quad r_2(B_2)x^2 + \dots] \\ &= R(x, B_1) + R(x, B_2). \end{aligned}$$

Theorem 2.2.2

Let B be non empty board such that $r_4(B) = 0$. Then B satisfies at least one of the following criteria:

1. B is contained in three or fewer rows;
2. B is contained in three or fewer columns;
3. B is the union of one part of the board contained in two rows and another part contained in two columns. An example of such a board is shown in figure



A board meeting criterion 3.

Proof:

We will prove by contra positive. Hence, we aim to show that if a board does not satisfy any of the properties above, then $r_4 \neq 0$. Assume B does not satisfy any of the listed criteria. So B has at least four rows, at least three columns, and does not have one part of the board contained in two row and one part contained in two column. These conditions force B to have four squares each in a different row and column from the other squares. Non-attacking rooks can be placed on these squares, so $r_4(B) \neq 0$, as desired.

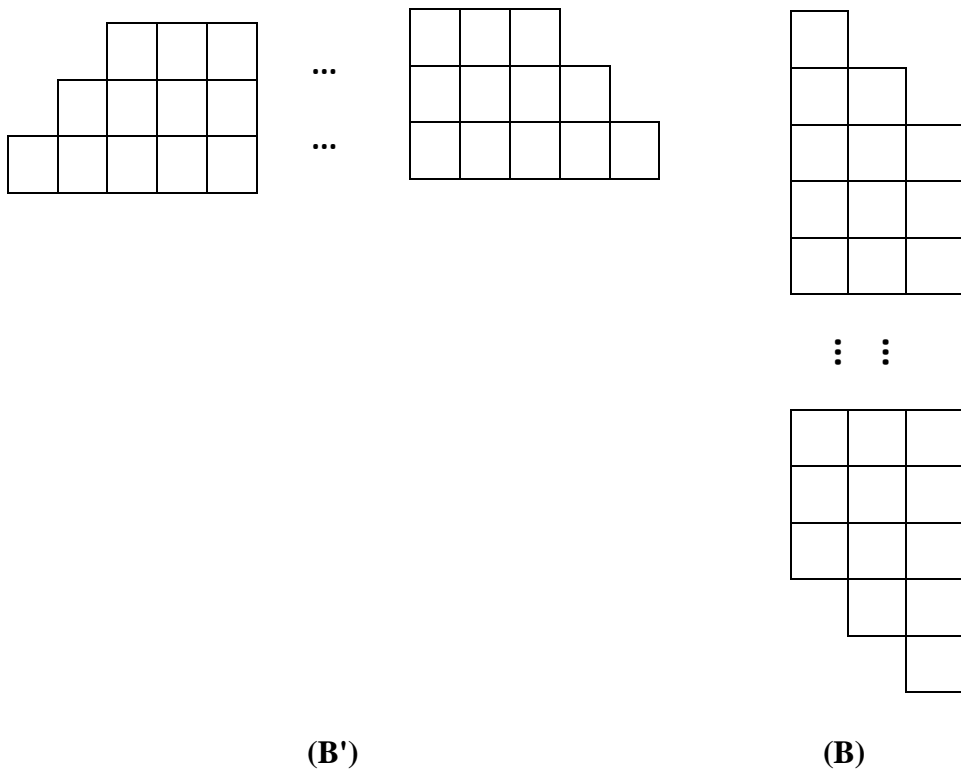
For the purpose of our classification of cubic rook polynomials, we only need to consider boards contained in three rows.

Theorem 2.2.3

A generalized board contained in three columns is rook equivalent to one contained in three rows.

Proof :

If B is generalized board contained in two columns of an $n \times n$ board, Let B' be the board obtained by rotating B 90° clock wise, as shown in the given figure.



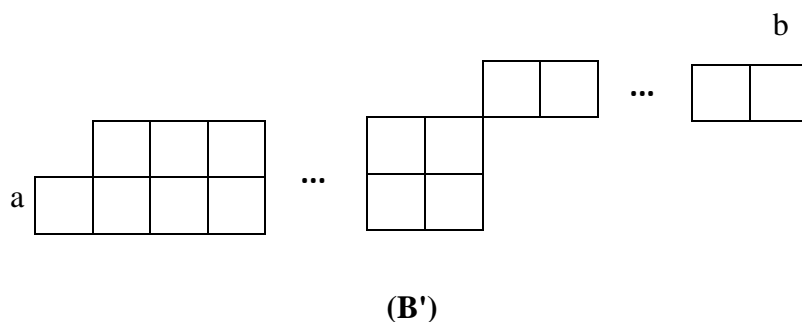
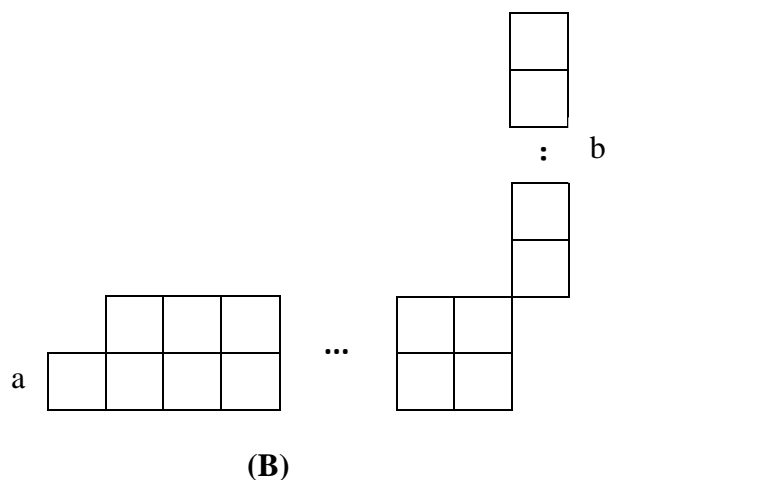
It is clear that a B' contained in three rows, B and B' have the same rook numbers. Hence, a board contained in three columns is rook equivalent to a board contained in three rows.

Theorem 2.2.4

Any generalized board with one part contained in two rows and another part contained in one columns has the same rook polynomial as a board contained within three rows.

Proof

Case 1: The squares in the column cannot attack any square in the row.



$$r_0(B) = r_0(B') = 1$$

Let a_1 be the number of single cell square (row wise) and a_2 be the number of double squares (row wise). Let b be the number of cells across the column.

$$r_1(B) = r_1(B') = a_1 + 2a_2 + b$$

This is the number of squares of each board.

To find $r_2(B)$:

When we place rook in the a_1 cell, then are $(a_2 + b)$ choices in placing the other rook.

When the rook is placed in any one of the a_2 cells (double squares along the row). we have $(a_2 - 1)+b$ choices in placing the other rook.

Similarly, for the other a_2 cells we have $(a_2 - 1)+b$ choices in placing the other rook.

Thus, there are $a_2 + b + a_2 \{(a_2 - 1)+b\}$ rooks ways in placing two non attacking rooks.

$$\begin{aligned} r_2 &= a_1(a_2 + b) + a_2(a_2 - 1) + a_2a_2b. \\ &= a_1a_2 + a_1b + a_2^2 - a_2 + a_2b \\ &= a_2^2 + (a_1 + a_2)b + a_1a_2 - a_2. \end{aligned}$$

To find $r_3(B)$:

When one rook is placed in the single cell a_1 , the other two rooks can be placed in a_1a_2b ways.

When a rook is placed in any one of the two cell piece (a_2) then we have $(a_2 - 1)b$ ways in placing the other two rooks.

Similarly, for the other a_2 cells we have $(a_2 - 1)b$ ways.

Hence the total number of ways of placing 3 non attacking rook is given by,

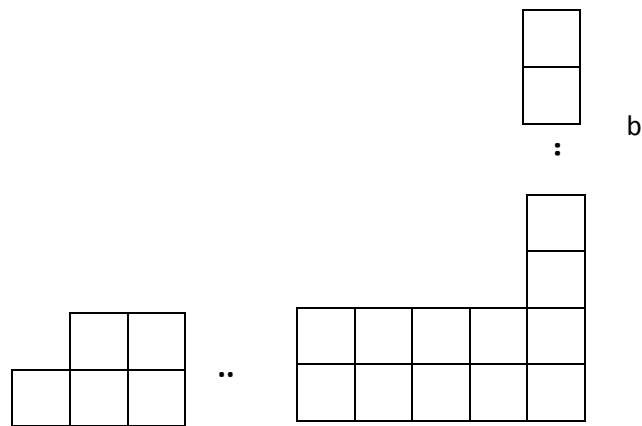
$$\begin{aligned} r_3 &= a_1a_2b + a_2(a_2 - 1)b \\ r_3 &= a_1a_2b + a_2^2b \end{aligned}$$

Case 2:

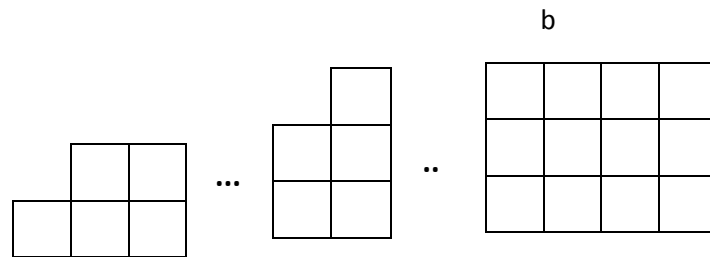
a_1 = number of single cells.

a_2 = number of double cells.

b = number of cells column wise



(B)



(B')

$$r_0(B) = r_0(B') = 1$$

$$r_1(B) = r_1(B') = a_1 + 2a_2 + b$$

To find $r_2(B)$:

When we place rook in any of the a_1 cell, the number of ways in placing the second rook is

$$a_1(a_2 + b) \quad (2.4.1)$$

When we place rook in any of the a_2 cell then number of ways in placing the second rook is

$$\begin{aligned} &= (a_2 - 1)[(a_2 - 1) + b] + a_2 - 1 \\ &= (a_2 - 1)[a_2 + b] \\ &= a_2^2 + a_2b - a_2 - b \end{aligned} \quad (2.4.2)$$

Adding equation (2.4.1) and (2.4.2) we get

$$\begin{aligned} &= a_2^2 + a_2b - a_2 - b + a_1a_2 + a_1b \\ &= a_2^2 + b(a_1 + a_2) + a_1a_2 - a_2 - b \end{aligned} \quad (2.4.3)$$

When a rook is placed in the second row of any a_2 cell, then the number of ways in placing the second rook is given by,

$$(a_2 - 1)b \quad (2.4.4)$$

Adding equation (2.4.3) and (2.4.4) we get

$$\begin{aligned} &= a_2^2 + b(a_1 + a_2) + a_1a_2 - a_2 - b + (a_2 - 1)b \\ &= a_2^2 + b(a_1 + a_2) + a_1a_2 + a_2b - a_2 - 2b \end{aligned} \quad (2.4.5)$$

To find $r_2(B')$:

$$a_1 \text{ cell} \quad a_1(a_2 + b) \quad (2.4.6)$$

$$a_2 \text{ cell} \quad (a_2 - b)[(a_2 - 1) + b] + b[(a_2 - 1) + (b - 1)] \quad (2.4.7)$$

When the first rook is placed in any of the second row a_2 cell, then the other rook can be placed in

$$(a_2 - b)b + b(b - 1) \quad (2.4.8)$$

Adding equations (2.4.6), (2.4.7) and (2.4.8) we get,

$$\begin{aligned} &= a_1a_2 + a_1b + a_2^2 - a_2 + a_2b - a_2b + b - b^2 + a_2b - 2b + b^2 + a_2b - b^2 \\ &\quad + b^2 - b \\ &= a_1a_2 + a_1b + a_2b + a_2^2 - a_2 - 2b + a_2b \\ &= a_2^2 + b(a_1 + a_2) + a_1a_2 + a_2b - a_2 - b \end{aligned} \quad (2.4.9)$$

From the equations (2.4.5) and (2.4.9) we conclude that both the boards B and B' have the same value for r_2 .

To find $r_3(B)$:

When a rook is placed in any of the a_1 cell, the other two rooks the $(a_2 - 1)b$

$$\text{Total ways} = a_1 [(a_2 - 1)b] \quad (2.4.10)$$

When a rook is placed in any of the a_2 cell except the a_2 cell containing the b cells, the other two rooks can be placed in $(a_2 - 1)b$

$$\text{Total ways} = (a_2 - 1)(a_2 - 2)b \quad (2.4.11)$$

Adding equations (3.4.10) and (3.4.11) we get,

$$\begin{aligned} &= a_1[(a_2 - 1)b] + (a_2 - 1)(a_2 - 2)b \\ &= a_1[a_2b - b] + (a_2^2 - 3a_2 + 2)b \\ &= a_1a_2b - a_1b + a_2^2b - 3a_2b + 2b \end{aligned} \quad (2.4.12)$$

To find $r_3(B')$:

When a rook placed in any of the a_1 cells, then the number of ways in placing the other two rooks is

$$(a_2 - b)b + b(b - 1)$$

$$\begin{aligned} \text{Total ways} &= a_1 [(a_2 - b)b + b(b - 1)] \\ &= a_1(a_2b - b) \end{aligned} \quad (2.4.13)$$

When a rook is placed in any of the a_2 cells other than the cells being the b cells, the other two rooks can be placed in

$$(a_2 - b - 1)b + b(b - 1)$$

$$\begin{aligned} \text{Total ways} &= (a_2 - b)[(a_2 - b - 1)b + b(b - 1)] \\ &= (a_2 - b)b(a_2 - 2) \end{aligned} \quad (2.4.14)$$

When the first rook is placed in the board containing both a_2 and b cells, the number of ways of placing the other two rooks is,

$$\begin{aligned} \text{Total ways} &= b[(a_2 - b)(b - 1) + (b - 1)(b - 2)] \\ &= b[a_2b - a_2 - 2b + 2] \end{aligned} \quad (2.4.15)$$

Adding equations (2.4.13),(2.4.14) and (2.4.15) we get,

$$\begin{aligned} &= a_1(a_2b - b) + (a_2 - b)b(a_2 - 2) + b[a_2b - a_2 - 2b + 2] \\ &= a_1a_2b - a_1b + a_2^2b - 3a_2b + 2b \end{aligned} \quad (2.4.16)$$

The equation (2.4.12) and (2.4.16) gives the same value. Thus r_3 is same for B and B' .

Theorem 2.2.5

Let B be a generalized board in the rows, whose rook polynomial is cubic.

Given $r_1(B)$ The possible $r_2(B)$ will have either of the forms,

$$r_2(B) = \frac{5}{16} r_1^2 - \frac{r_1}{4} - 2i, \text{ for } 0 \leq i \leq \left\lfloor \frac{r_1}{4} \right\rfloor \text{ and}$$

$$r_2(B) > 5 \left\lfloor \frac{r_1}{16} \right\rfloor - 11 \left\lfloor \frac{r_1}{4} \right\rfloor + 6 - 2i, \text{ for } 0 \leq i \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right) \text{ for and } r_3(B) \text{ will have of}$$

$$\text{the forms, } r_3(B) = 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor - i^2 - 2i, \text{ } 0 \leq i \leq b \text{ and}$$

$$0 \leq b \leq \left\lfloor \frac{r_1}{4} \right\rfloor \text{ and } r_3 > 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 7 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 3 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 4i \left\lfloor \frac{r_1}{4} \right\rfloor - 2i,$$

$$0 \leq i \leq b \text{ and } 0 \leq i \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right), \text{ for } a_1 = a_2 = b \text{ and } a_1 > a_2 > b$$

respectively.

Proof :

Given the value of, we find all possible corresponding values of $r_2(B)$ and $r_3(B)$ such that $r_1(B) = a_1 + 2a_2 + b$

Case 1:

Let us assume $a_1 = a_2$ and $a_2 = b$

This implies $a_1 = b$

We know $r_1 = a_1 + 2a_2 + b$

$$b = r_1 - a_1 - 2a_2$$

$$= r_1 - b - 2b$$

$$= r_1 - 3b$$

$$\Rightarrow r_1 = 4b$$

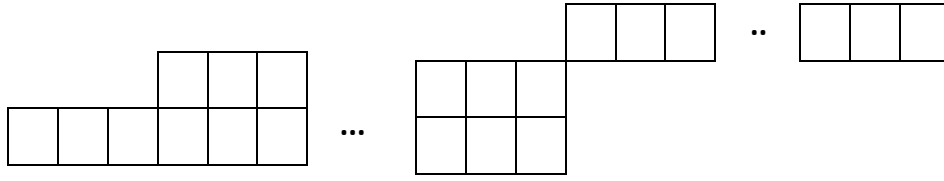
$$b = \left\lfloor \frac{r_1}{4} \right\rfloor \tag{2.5.1}$$

To find r_2 :

By placing one rook in each row, the possible values of $r_2(B)$ can be found for each value of $0 \leq i \leq b$

$$\text{that is, } 0 \leq i \leq \left\lfloor \frac{r_1}{4} \right\rfloor \quad (\text{using (2.5.1)})$$

When $i = 0$ we obtain the maximum value of r_2



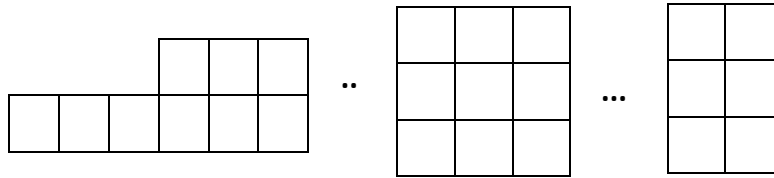
$$r_2 = a_1(a_2 + b) + a_2((a_2 - 1) + b) + a_2b$$

Substitute $a_1 = a_2 = b$, we get

$$\begin{aligned} r_2 &= 2b^2 + b(2b - 1) + b^2 \\ &= 5b^2 - b \end{aligned}$$

$$r_2 = 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - \left\lfloor \frac{r_1}{4} \right\rfloor \tag{2.5.2}$$

When $i = b$ we obtain the maximum value of r_2



$$r_2 = a_1(a_2 + b) + b((a_2 - 1) + (b - 1)) + b(b - 1)$$

Substitute $a_1 = a_2 = b$, we get

$$\begin{aligned} r_2 &= b(2b) + b((b - 1) + (b - 1)) + b(b - 1) \\ &= 2b^2 + b[(2b - 1) + (b - 1)] \\ &= 2b^2 + b[3b - 3] \\ &= 2b^2 + 3b^2 - 3b \\ &= 5b^2 - 3b \end{aligned}$$

$$r_2 = 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 3 \left\lfloor \frac{r_1}{4} \right\rfloor \tag{2.5.3}$$

When $i > 0$ & $i < \left\lfloor \frac{r_1}{4} \right\rfloor$ (the other cases of i)

$$\begin{aligned} r_2 &= a_1(a_2 + b) + (a_2 - i)((a_2 - 1) + b) + i((a_2 - 1) + (b - 1)) + \\ &(a_2 - i)b + i(b - 1) \end{aligned}$$

Substitute $a_1 = a_2 = b$, we get

$$\begin{aligned} r_2 &= b(2b) + (b - i)((b - 1) + b) + i((b - 1) + (b - 1)) + (b - i)b + \\ &i(b - 1) \\ &= 2b^2 + (b - i)(2b - 1) + i(2b - 2) + b^2 - ib + ib - i \end{aligned}$$

$$\begin{aligned}
&= 2b^2 + 2b^2 - b - 2ib + i + 2ib - 2i + b^2 - ib + ib - i \\
&= 4b^2 + b^2 - b - 2i \\
&= 5b^2 - b - 2i
\end{aligned}$$

$$r_2 = 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - \left\lfloor \frac{r_1}{4} \right\rfloor - 2i \quad (2.5.4)$$

From the equations (2.5.2), (2.5.3) & (2.5.4) we obtain the general form of r_2 when our generalized board satisfies the case $a_1 = a_2 = b$. It is given by,

$$r_2 = 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - \left\lfloor \frac{r_1}{4} \right\rfloor - 2i, \text{ Where } 0 \leq i \leq \left\lfloor \frac{r_1}{4} \right\rfloor.$$

To find r_3 :

When $i = 0$,

$$r_3 = a_1 a_2 b + a^2_2 b - a_2 b$$

Substitute $a_1 = a_2 = b$, we get

$$\begin{aligned}
r_3 &= b^3 - b^2 + b^3 \\
&= 2b^3 - b^2
\end{aligned}$$

$$r_3 = 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - \left\lfloor \frac{r_1}{4} \right\rfloor^2 \quad (2.5.5)$$

When $i = b$,

$$r_3 = a_1 a_2 b - a_1 b + a^2_2 b - 3a_2 b + 2b$$

Substitute $a_1 = a_2 = b$, we get

$$\begin{aligned}
r_3 &= b^3 - b^2 + b^3 - 3b^2 + 2b \\
r_3 &= 2b^3 - 4b^2 + 2b
\end{aligned}$$

$$(2.5.6)$$

When $0 < i < b$, $1 \leq b \leq \left\lfloor \frac{r_1}{4} \right\rfloor$,

$$\begin{aligned}
r_3 &= a_1 [(a_2 - i)b + i(b - 1)] + (a_2 - i)[(a_2 - i - 1)b + i(b - 1)] + \\
&\quad i[(a_2 - i)(b - 1)(i - 1)(b - 2)]
\end{aligned}$$

Substitute $a_1 = a_2 = b$

$$\begin{aligned}
r_3 &= b[(b - i)b + i(b - 1)] + (b - i)[(b - i - 1)b + i(b - 1)] + \\
&\quad i[(b - i)(b - 1) + (i - 1)(b - 2)] \\
&= b[b^2 - ib + ib - i] + (b - i)[b^2 - ib - b + ib - b] + i[b^2 - ib - \\
&\quad b + i + ib - 2i - b + 2]
\end{aligned}$$

$$r_3 = 2b^3 - b^2 - 2ib - i^2 + 2i$$

put $b = \left\lfloor \frac{r_1}{4} \right\rfloor$, we get

$$r_3 = 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor - i^2 - 2i \quad (2.5.7)$$

From the equations (2.5.5), (2.5.6) & (2.5.7) we obtain the general form of r_3 when our generalized board satisfies the case $a_1 = a_2 = b$, it is given by,

$$r_3 = 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor - i^2 - 2i, \text{ Where } 0 \leq i \leq b, 0 < i < \frac{r_1}{4}.$$

Case 2:

Let us assume $a_1 > a_2 > b$,

This implies $a_1 > b$,

$$\text{We know } r_1 = a_1 + 2a_2 + b$$

$$b < r_1 - a_1 - 2a_2$$

$$< r_1 - b - 2b$$

$$< r_1 - 3b$$

$$\Rightarrow r_1 < 4b$$

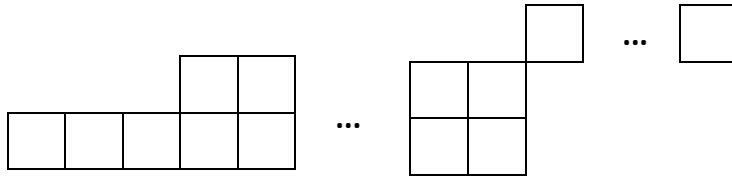
$$b < \left\lfloor \frac{r_1}{4} \right\rfloor \quad (2.5.8)$$

To find r_2 :

By placing one rook in each row, the possible values of $r_2(B)$ can be found for each value of $0 \leq i \leq b$

$$\text{that is, } 0 \leq b \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right) \quad (\text{using (2.5.8)})$$

$$\text{When } i = 0, \quad b = \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right)$$



$$r_2 > a_1(a_2 + b) + a_2((a_2 - 1) + b) + a_2b$$

Substitute $a_1 > a_2 > b$, we get

$$r_2 > b(2b) + b((b - 1) + (b - 1)) + b^2$$

$$> 2b^2 + b^2 - b + b^2 + b^2$$

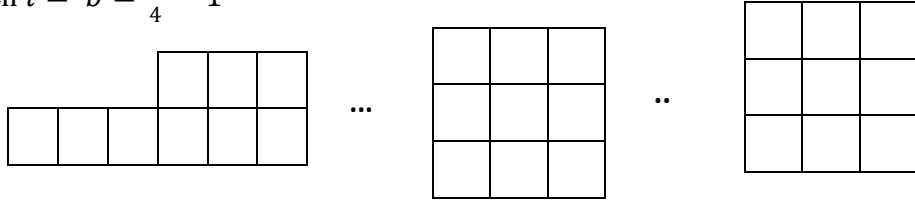
$$> 2b^2 + 3b^2 - b$$

$$> 5b^2 - b$$

Put $b = \left\lfloor \frac{r_1}{4} \right\rfloor - 1$ we get,

$$\begin{aligned}
 r_2 &> 5 \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right)^2 - \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right) \\
 &> 5 \left[\left(\left\lfloor \frac{r_1}{4} \right\rfloor \right)^2 - 2 \left(\left\lfloor \frac{r_1}{4} \right\rfloor \right) + 1 \right] - \left(\left\lfloor \frac{r_1}{4} \right\rfloor \right) + 1 \\
 &> 5 \left(\left\lfloor \frac{r_1}{4} \right\rfloor \right)^2 - 10 \left(\left\lfloor \frac{r_1}{4} \right\rfloor \right) + 5 - \left(\left\lfloor \frac{r_1}{4} \right\rfloor \right) + \\
 r_2 &> 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 11 \left\lfloor \frac{r_1}{4} \right\rfloor + 6 \tag{2.5.9}
 \end{aligned}$$

When $i = b = \frac{r_1}{4} - 1$



$$\begin{aligned}
 r_2 = a_1(a_2 + b) + (a_2 - i)((a_2 - 1) + b) + i((a_2 - 1) + (b - 1)) \\
 + (a_2 - i)b + i(b - 1)
 \end{aligned}$$

Substitute $a_1 > a_2 > b$, we get

$$r_2 > 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 13 \left\lfloor \frac{r_1}{4} \right\rfloor + 8 \tag{2.5.10}$$

When i lies between 0 and $1 \leq b \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right)$,

$$\begin{aligned}
 r_2 = a_1(a_2 + b) + (a_2 - i)((a_2 - 1) + b) + i((a_2 - 1) + (b - 1)) + (a_2 - i)b + \\
 i(b - 1)
 \end{aligned}$$

Substitute $a_1 > a_2 > b$, we get,

$$r_2 > 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 11 \left\lfloor \frac{r_1}{4} \right\rfloor + 6 - 2i \tag{2.5.11}$$

From the equations (2.5.9), (2.5.10) & (2.5.11) we obtain the general form of r_2 when our generalized board satisfies the case $a_1 > a_2 > b$. It is given by,

$$r_2 > 5 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 11 \left\lfloor \frac{r_1}{4} \right\rfloor + 6 - 2i, \text{ Where } 0 \leq i \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right).$$

To find r_3 :

When $i = 0$

$$r_3 = a_1 a_2 b + a^2_2 b - a_2 b$$

Substitute $a_1 > a_2 > b$, and $b = \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1 \right)$, we get

$$r_3 > b^3 + b^3 - b^2$$

$$\begin{aligned}
&> 2b^3 - b^2 \\
r_3 &> 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 7 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 3
\end{aligned} \tag{2.5.12}$$

When $i = b$

$$r_3 = a_1 a_2 b - a_1 b + a^2_2 b - 3a_2 b + 2b$$

Substitute $a_1 > a_2 > b$ and $b = \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)$, we get

$$\begin{aligned}
&> b^3 - b^2 + b^3 - 3b^2 + 2b \\
&> 2b^3 - 4b^2 + 2b \\
&> 2 \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)^3 - 4 \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)^2 + 2 \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right) \\
&> 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 6 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 6 \left\lfloor \frac{r_1}{4} \right\rfloor - 2 - 4 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 4 + \\
r_3 &> 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 10 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 16 \left\lfloor \frac{r_1}{4} \right\rfloor - 8
\end{aligned} \tag{2.5.13}$$

When $0 < i < b$, $1 \leq b \leq \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)$,

$$\begin{aligned}
r_3 = a_1 [(a_2 - i)b + i(b - 1)] + (a_2 - i)[(a_2 - i - 1)b + i(b - 1)] + \\
i[(a_2 - i)(b - 1) + (i - 1)(b - 2)]
\end{aligned}$$

Substitute $a_1 > a_2 > b$ and $b = \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)$, we get

$$\begin{aligned}
&> 2b^3 - b^2 - 2ib - i^2 + 2i \\
&> 2 \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)^3 - \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right)^2 - 2i \left(\left\lfloor \frac{r_1}{4} \right\rfloor - 1\right) - i^2 + 2i \\
&> 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 7 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 3 - i \left[2 \left\lfloor \frac{r_1}{4} \right\rfloor^2 - 4 \left\lfloor \frac{r_1}{4} \right\rfloor + 4 \right] + 2i \\
r_3 &> 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 7 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 3 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 4i \left\lfloor \frac{r_1}{4} \right\rfloor - 2i
\end{aligned} \tag{2.5.14}$$

From the equations (2.5.12), (2.5.13) & (2.5.14) we obtain the general form of r_2 when our generalized board satisfies the case $a_1 > a_2 > b$. It is given by,

$$r_3 > 2 \left\lfloor \frac{r_1}{4} \right\rfloor^3 - 7 \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 8 \left\lfloor \frac{r_1}{4} \right\rfloor - 3 - 2i \left\lfloor \frac{r_1}{4} \right\rfloor^2 + 4i \left\lfloor \frac{r_1}{4} \right\rfloor - 2i.$$

CHAPTER III

Classification of Cubic Rook Polynomials

3.1 Classification of cubic Rook polynomials:

The possible cubic rook polynomials for the generalized board under consideration when $r_1(B) = 4, 5, 6, 7, 8, 9, 10$.

For $r_1(B) = 4$

$$1 + 4x + 4x^2 + x^3$$

For $r_1(B) = 6$

$$1 + 6x + 6x^2 + x^3$$

$$1 + 6x + 8x^2 + 2x^3$$

$$1 + 6x + 8x^2 + 3x^3$$

$$1 + 6x + 9x^2 + 3x^3$$

$$1 + 6x + 10x^2 + 3x^3$$

$$1 + 6x + 10x^2 + 4x^3$$

For $r_1(B) = 8$

$$1 + 8x + 10x^2 + 6x^3$$

$$1 + 8x + 13x^2 + 3x^3$$

$$1 + 8x + 14x^2 + 3x^3$$

$$1 + 8x + 14x^2 + 4x^3$$

$$1 + 8x + 14x^2 + 14x^3$$

$$1 + 8x + 15x^2 + 8x^3$$

$$1 + 8x + 15x^2 + 9x^3$$

$$1 + 8x + 16x^2 + 5x^3$$

For $r_1(B) = 5$

$$1 + 5x + 5x^2 + x^3$$

$$1 + 5x + 6x^2 + 2x^3$$

$$1 + 5x + 7x^2 + 2x^3$$

For $r_1(B) = 7$

$$1 + 7x + 10x^2 + 2x^3$$

$$1 + 7x + 10x^2 + 4x^3$$

$$1 + 7x + 11x^2 + 3x^3$$

$$1 + 7x + 12x^2 + 4x^3$$

$$1 + 7x + 12x^2 + 5x^3$$

$$1 + 7x + 12x^2 + 6x^3$$

$$1 + 7x + 13x^2 + 4x^3$$

$$1 + 7x + 13x^2 + 6x^3$$

$$1 + 7x + 14x^2 + 6x^3$$

$$1 + 7x + 14x^2 + 8$$

For $r_1(B) = 9$

$$1 + 9x + 16x^2 + 4x^3$$

$$1 + 9x + 17x^2 + 5x^3$$

$$1 + 9x + 18x^2 + 6x^3$$

$$1 + 9x + 18x^2 + 8x^3$$

$$1 + 9x + 18x^2 + 10x^3$$

$$1 + 9x + 19x^2 + 6x^3$$

$$1 + 9x + 20x^2 + 8x^3$$

$$1 + 9x + 20x^2 + 10x^3$$

$$\begin{aligned}
&1 + 8x + 16x^2 + 6x^3 \\
&1 + 8x + 16x^2 + 7x^3 \\
&1 + 8x + 16x^2 + 8x^3 \\
&1 + 8x + 17x^2 + 9x^3 \\
&1 + 8x + 18x^2 + 8x^3 \\
&1 + 8x + 18x^2 + 9x^3 \\
&1 + 8x + 19x^2 + 12x^3
\end{aligned}$$

For $r_1(B) = 10$

$$\begin{aligned}
&1 + 10x + 16x^2 + 7x^3 \\
&1 + 10x + 19x^2 + 5x^3 \\
&1 + 10x + 20x^2 + 6x^3 \\
&1 + 10x + 21x^2 + 12x^3 \\
&1 + 10x + 22x^2 + 7x^3 \\
&1 + 10x + 22x^2 + 8x^3 \\
&1 + 10x + 22x^2 + 10x^3 \\
&1 + 10x + 24x^2 + 10x^3 \\
&1 + 10x + 24x^2 + 12x^3 \\
&1 + 10x + 24x^2 + 14x^3 \\
&1 + 10x + 24x^2 + 15x^3 \\
&1 + 10x + 25x^2 + 14x^3 \\
&1 + 10x + 26x^2 + 12x^3 \\
&1 + 10x + 26x^2 + 15x^3 \\
&1 + 10x + 26x^2 + 16x^3 \\
&1 + 10x + 26x^2 + 18x^3 \\
&1 + 10x + 26x^2 + 20x^3 \\
&1 + 10x + 27x^2 + 18x^3 \\
&1 + 10x + 27x^2 + 19x^3 \\
&1 + 10x + 28x^2 + 15x^3 \\
&1 + 10x + 28x^2 + 16x^3
\end{aligned}$$

$$\begin{aligned}
&1 + 9x + 20x^2 + 11x^3 \\
&1 + 9x + 20x^2 + 12x^3 \\
&1 + 9x + 21x^2 + 9x^3 \\
&1 + 9x + 22x^2 + 10x^3 \\
&1 + 9 + 22x^2 + 12x^3 \\
&1 + 9x + 22x^2 + 14x^3 \\
&1 + 9x + 22x^2 + 16x^3 \\
&1 + 9x + 23x^2 + 12x^3 \\
&1 + 9x + 24x^2 + 16x^3 \\
&1 + 9x + 24x^2 + 18x^3
\end{aligned}$$

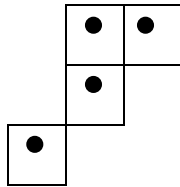
$$\begin{aligned}
&1 + 10x + 29x^2 + 24x^3 \\
&1 + 10x + 30x^2 + 24x^3 \\
&1 + 10x + 30x^2 + 27x^3 \\
&1 + 10x + 29x^2 + 20x^3 \\
&1 + 10x + 28x^2 + 24x^3 \\
&1 + 10x + 28x^2 + 18x^3
\end{aligned}$$

For example:

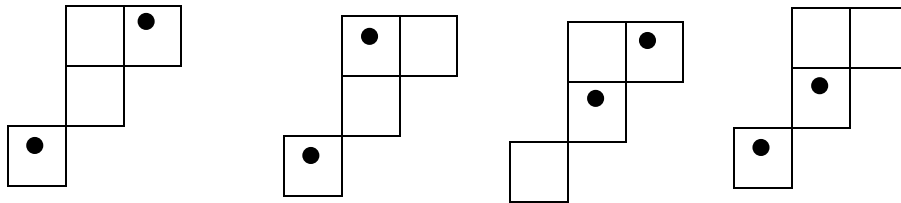
When $r_1(B) = 4$, one of the possible rook polynomial is $1 + 4x + 4x^2 + x^3$,

Here $r_0 = 1$

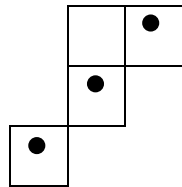
$r_1 = 4$



$r_2 = 4$



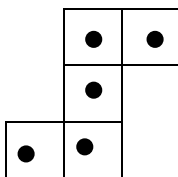
$r_3 = 1$



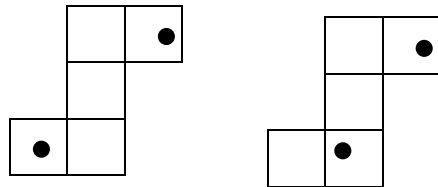
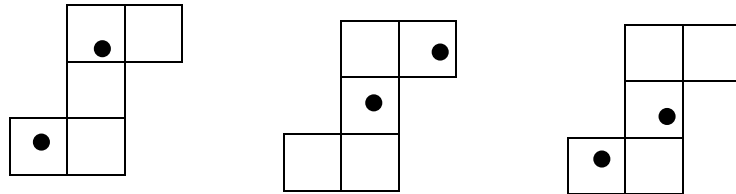
When $r_1(B) = 5$, one of the possible rook polynomial is $1 + 5x + 5x^2 + x^3$,

Here $r_0 = 1$

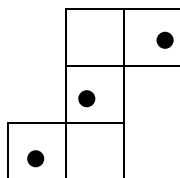
$r_1 = 5$



$$r_2 = 5$$



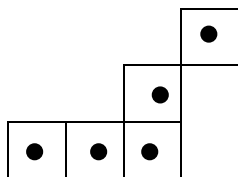
$$r_3 = 1$$



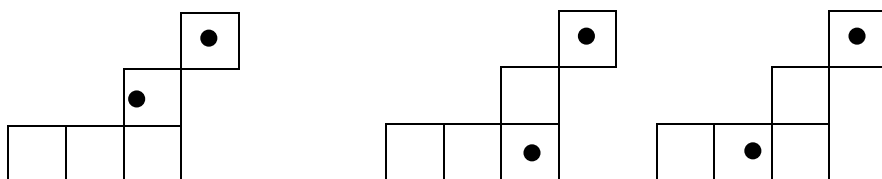
When $r_1(B) = 5$, one of the possible rook polynomial is $1 + 5x + 6x^2 + 2x^3$,

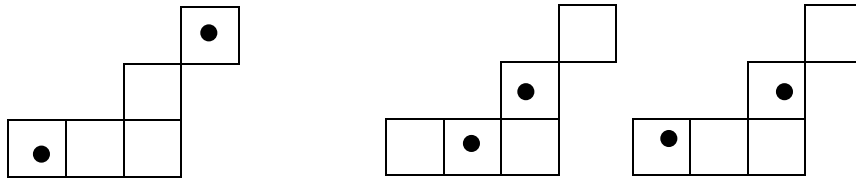
Here $r_0 = 1$

$$r_1 = 5$$

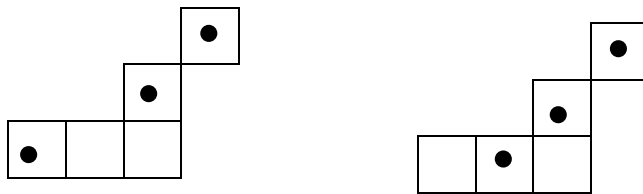


$$r_2 = 6$$





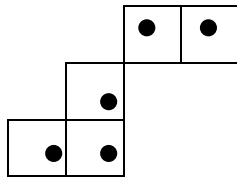
$$r_3 = 2$$



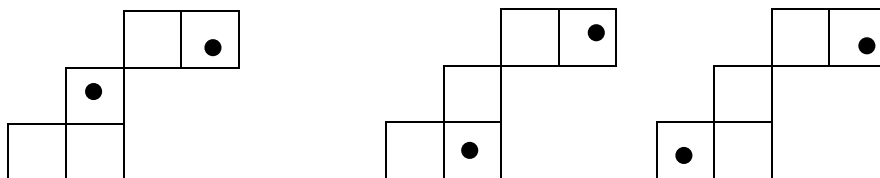
When $r_1(B) = 5$, one of the possible rook polynomial is $1 + 5x + 6x^2 + 2x^3$,

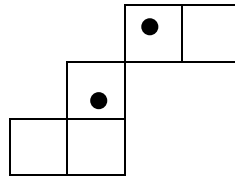
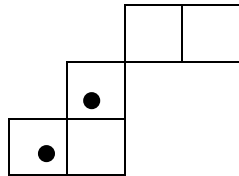
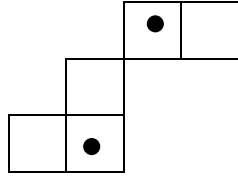
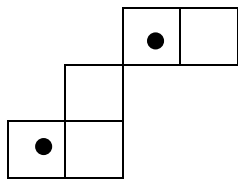
Here $r_0 = 1$

$$r_1 = 5$$

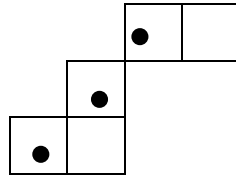
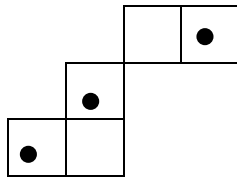


$$r_2 = 7$$





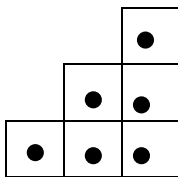
$$r_3 = 2$$



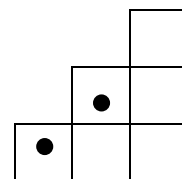
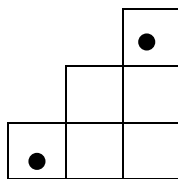
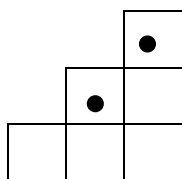
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 6x^2 + x^3$,

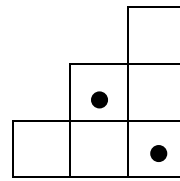
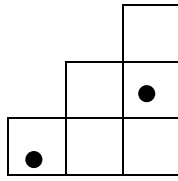
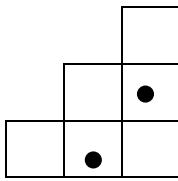
Here $r_0 = 1$

$$r_1 = 6$$

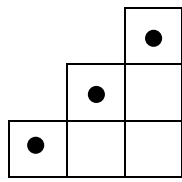


$$r_2 = 6$$





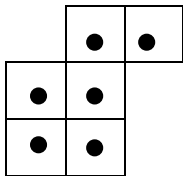
$$r_3 = 1$$



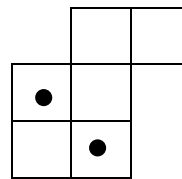
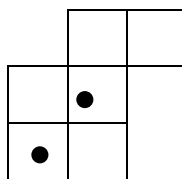
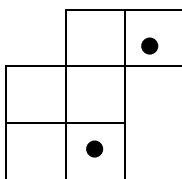
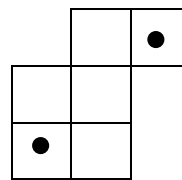
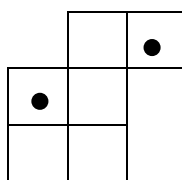
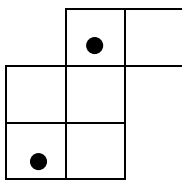
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 8x^2 + 2x^3$,

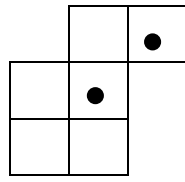
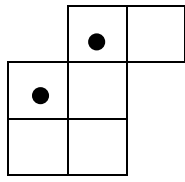
Here $r_0 = 1$

$$r_1 = 6$$

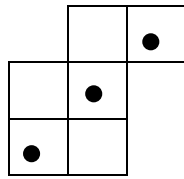
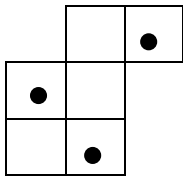


$$r_2 = 8$$





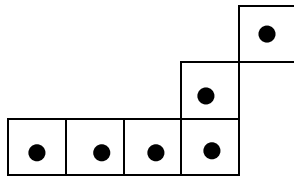
$$r_3 = 2$$



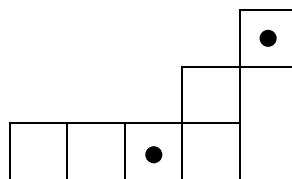
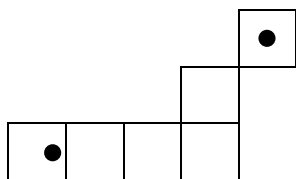
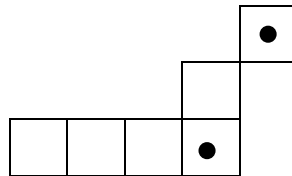
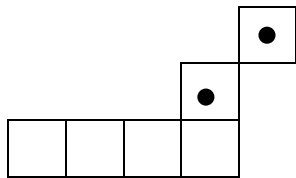
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 8x^2 + 3x^3$,

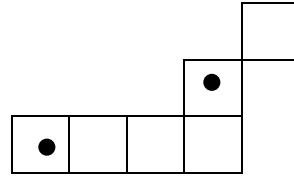
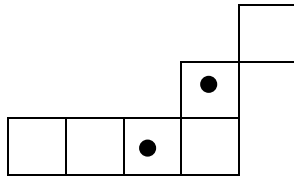
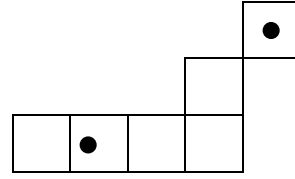
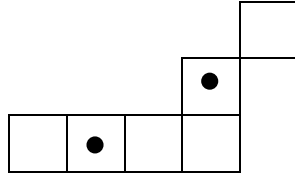
Here $r_0 = 1$

$$r_1 = 6$$

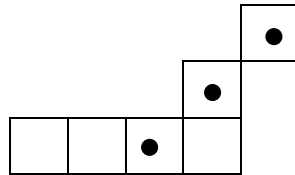
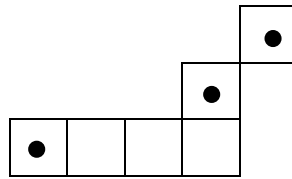
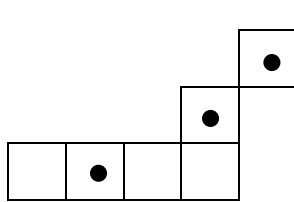


$$r_2 = 8$$





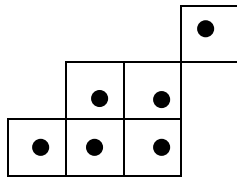
$$r_3 = 3$$



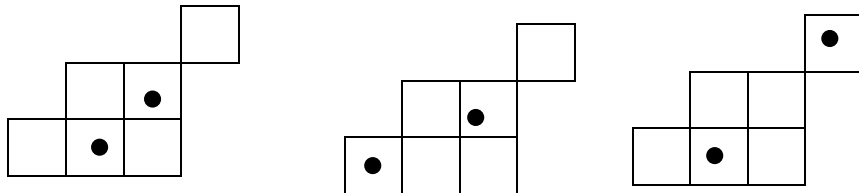
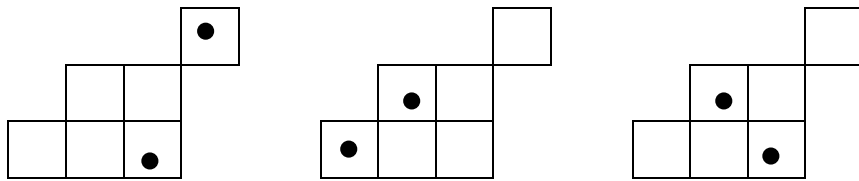
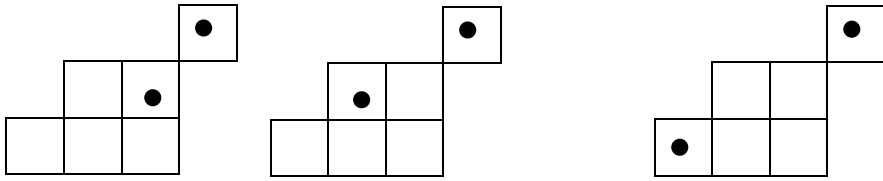
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 9x^2 + 4x^3$,

Here $r_0 = 1$

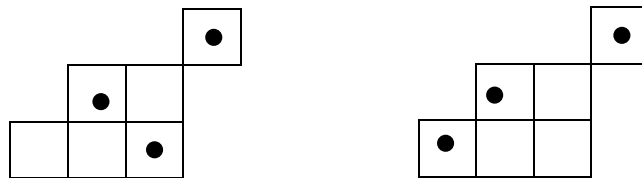
$r_1 = 6$



$$r_2 = 9$$



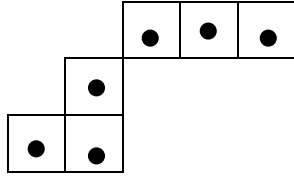
$$r_3 = 4$$



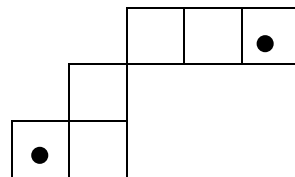
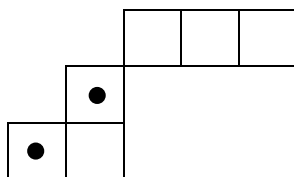
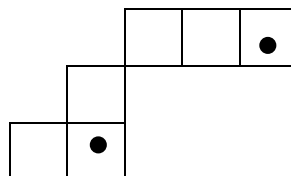
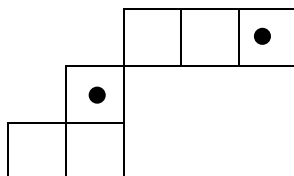
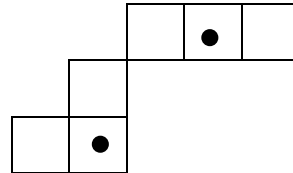
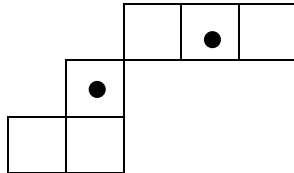
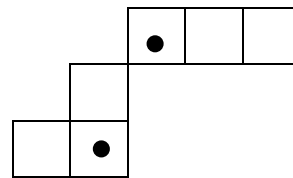
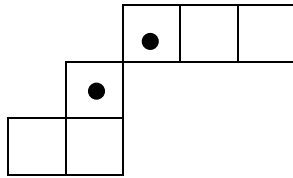
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 10x^2 + 4x^3$,

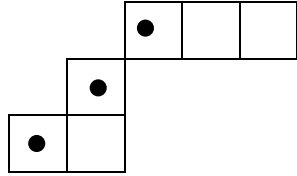
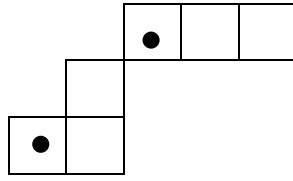
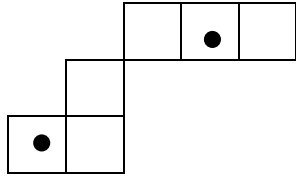
Here $r_0 = 1$

$r_1 = 6$

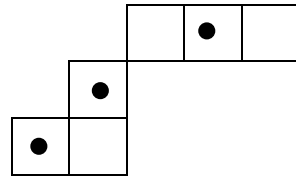
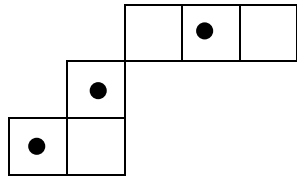


$r_2 = 10$





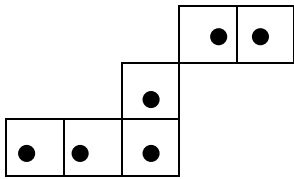
$$r_3 = 3$$



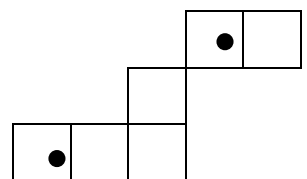
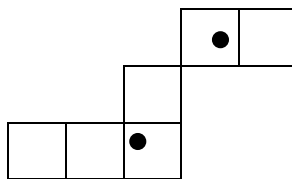
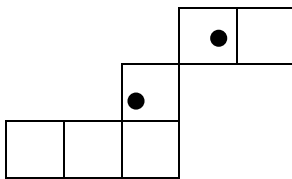
When $r_1(B) = 6$, one of the possible rook polynomial is $1 + 6x + 10x^2 + 4x^3$,

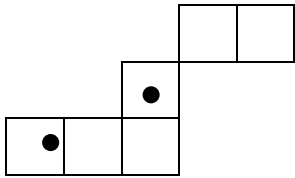
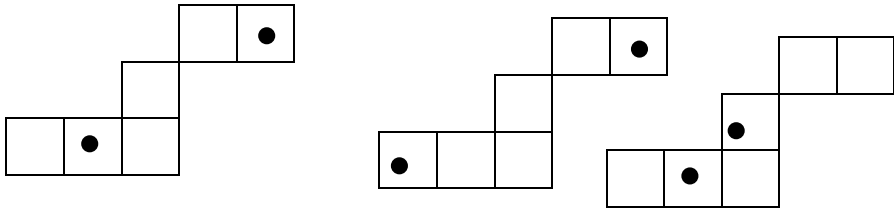
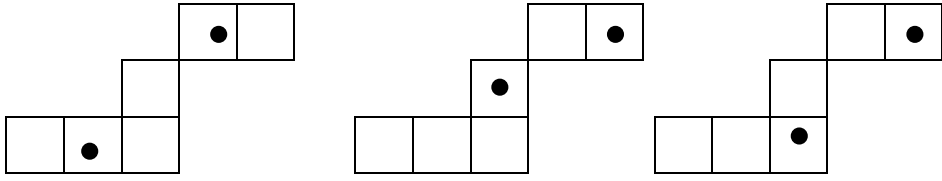
Here $r_0 = 1$

$$r_1 = 6$$

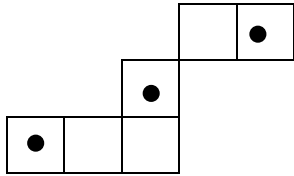
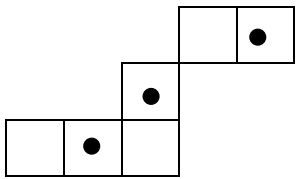
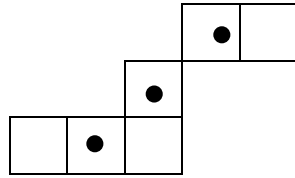
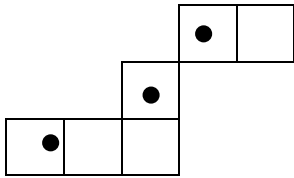


$$r_2 = 10$$





$r_3 = 4$



Summary and Conclusion

First Chapter provided the necessary back ground to work with boards that satisfy our desired conditions. We have proved several theorems regarding rook equivalency that allows us to minimize the number and types of boards that we need to work with we focus on the classification of all quadratic polynomials. We provide a theorem that introduces a formula that helped us to solve our problem, we have conclude with an example that utilizes our formula.

In second chapter a generalized board B was introduced that has rook numbers ≤ 3 . The theorem proved in chapter I were redefined and proved for cubic case and the concept of the rook equivalency allowed us to minimize the number and types of boards that are completely contained in 3×3 board.

All the possible cubic rook polynomials were enumerated for our generalized board under some specific cases say when $a_1 = a_2 = b$ and when $a_1 > a_2 > b$. Where a_1 is the number of single cell square (row wise) , a_2 is the number of double squares (row wise) and b is the number of cells across the column.

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PUBLICATION

1. S.Deepika, M.Uma and R.Sangeetha, "*The Classification of Cubic Rook polynomials*". (to appear)