

Δ^* -Separation Axioms In Topological Spaces

3.1 Introduction

The topological spaces usually have two features which do not seem to reconcile each other usually. One thing is any two points can be separated. Second thing is every point can be approached very closely from other points and if a sequence of points are taken generally they will get to cluster around a point. Topological spaces become interesting precisely because they are good in both above features. So far any conditions which guarantee the existence of sufficient number of open sets are not yet studied. Indeed the only open sets whose existence was absolutely required so far are the empty set and whole set. In case of the indiscrete space they are the only open sets. But most of the theorems reduced to trivialities when specialized to indiscrete space. Since every concept in topology is defined in terms of open sets in order to make non trivial and interesting statements about a space it is necessary that the space possesses a fairly rich collection of open sets.

In this chapter various degrees of such richness are studied. Separation axioms are one among the most common, important and interesting concepts. They can be used to define more restricted classes of topological spaces. Replacing the sets being separated or doing separation in the separation axioms by different types of sets, several extensions of separation axioms have been introduced by mathematicians from time to time. This chapter is committed to the study of separation axioms by Δ^* -closed sets. As applications of Δ^* -closed sets, four new spaces namely, Δ^*T_δ -space, $\Delta^*T_{\delta g^*}$ -space, $g\delta T_{\Delta^*}$ -space and $g^\dagger T_{\Delta^*}$ -space are introduced and their properties are studied.

3.2 Δ^* -Separation Axioms

Definition 3.2.1 A space (X, τ) is said to be a

- 1) Δ^*T_δ -space if every Δ^* -closed set is δ -closed in (X, τ) .
- 2) $\Delta^*T_{\delta g^*}$ -space if every Δ^* -closed set is δg^* -closed in (X, τ) .
- 3) $g\delta T_{\Delta^*}$ -space if every $g\delta$ -closed set is Δ^* -closed in (X, τ) .
- 4) $g^\dagger T_{\Delta^*}$ -space if every δg^\dagger -closed set is Δ^* -closed in (X, τ) .

Δ^*T_δ -space

Proposition 3.2.2 If (X, τ) is a Δ^*T_δ -space then $\Delta^*\text{cl}(B) = \delta\text{cl}(B)$ for each subset B of X .

Proof : Let (X, τ) be a Δ^*T_δ -space. We know that by Proposition 2.2.2 every δ -closed set is Δ^* -closed set. Moreover (X, τ) is a Δ^*T_δ -space. Therefore $\Delta^*\mathcal{C}(X, \tau) = \delta\mathcal{C}(X, \tau)$. Hence by the definition of δ -closure and Δ^* -closure, $\Delta^*\text{cl}(B) = \delta\text{cl}(B)$ for each subset B of X .

Theorem 3.2.3 The following statements are equivalent for a space (X, τ) .

- a) (X, τ) is a Δ^*T_δ -space.
- b) $\tau_\delta = \tau^\#$ holds.
- c) Every singleton $\{x\}$ is either δg -closed or δ -open.
- d) Every singleton $\{x\}$ is either δg -closed or regular open.

Proof : (a) \implies (b) : We claim that $\tau^\# = \tau_\delta$. Let $V \in \tau^\#$.

By the assumption and Proposition 3.2.2, $\Delta^* \text{cl}(U) = \delta \text{cl}(U)$ -----(i) for every subset U of X .

By the definition of $\tau^\#, V \in \tau^\#, \Delta^* \text{cl}(X - V) = (X - V)$. -----(ii).

From (i) and (ii), $V \in \tau_\delta$. Hence $\tau^\# = \tau_\delta$. By Proposition 2.4.6, $\tau_\delta = \tau^\#$.

Therefore $\tau^\# = \tau_\delta$.

(b) \Rightarrow (c) : Let $x \in X$. By Proposition 2.5.14, if $\{x\}$ is not δg -closed then $X - \{x\}$ is Δ^* -closed. Therefore $\{x\} \in \tau^\# = \tau_\delta$ by (b). Therefore $\{x\}$ is δ -open.

(c) \Rightarrow (a) : Let A be any Δ^* -closed set and $x \in \delta \text{cl}(A)$. We prove that $x \in A$ for the following two cases.

Case (1) : Let $\{x\}$ be δg -closed. Assume that $x \notin A$. Then $\delta \text{cl}(A) - A$ contains a δg -closed set $\{x\}$. This is a contradiction to the Theorem 2.3.4. Therefore $x \in A$.

Case (2) : Let $\{x\}$ be δ -open. Then $X - \{x\}$ is δ -closed. If $x \notin A$ then $A \subset X - \{x\}$. Since $x \in \delta \text{cl}(A)$, we get $x \in (X - \{x\})$ which is a contradiction. Hence $x \in A$.

(c) \Rightarrow (d) : The proof follows from the fact that in any space, a singleton set is δ -open if and only if it is regular open.

Proposition 3.2.4 Every $\delta g^\dagger T_\delta$ -space is a $\Delta^* T_\delta$ -space.

Proof : Let A be a Δ^* -closed set in (X, τ) . Since every Δ^* -closed is δg^\dagger -closed, A is δg^\dagger -closed (Proposition 2.2.6) in (X, τ) and (X, τ) is a $\delta g^\dagger T_\delta$ -space, A is δ -closed. Hence (X, τ) is a $\Delta^* T_\delta$ -space.

Proposition 3.2.5 a) Every T_δ -space is a $\Delta^* T_\delta$ -space.

b) Every $g_\delta T_\delta$ -space is a $\Delta^* T_\delta$ -space.

Proof : a) Let A be Δ^* -closed in (X, τ) . Since every Δ^* -closed set is $g\delta$ -closed (Proposition 2.2.8) in (X, τ) and (X, τ) is a T_δ -space, A is δ -closed. Hence (X, τ) is a Δ^*T_δ -space.

b) Let A be a Δ^* -closed set in (X, τ) . Since every Δ^* -closed is $g\delta$ -closed, A is $g\delta$ -closed (Proposition 2.2.8) in (X, τ) and (X, τ) is a $g_\delta T_\delta$ -space, A is δ -closed. Hence (X, τ) is a Δ^*T_δ -space.

The converse of the propositions 3.2.4 and 3.2.5 is not true as seen from the following example.

Counter Example 3.2.6 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then (X, τ) is a Δ^*T_δ -space but not a T_δ -space, not a $\delta g^\dagger T_\delta$ -space and not a $g_\delta T_\delta$ -space as the subset $\{b\}$ is $g\delta$ -closed, g^\dagger -closed and $g\delta$ -closed but not δ -closed in (X, τ) .

Proposition 3.2.7 : Every Δ^*T_δ -space is a δg^*T_δ -space.

Proof : Let (X, τ) be a Δ^*T_δ -space. Let A be a g^* -closed set in (X, τ) . Then A is Δ^* -closed (by Proposition 2.2.4). Since (X, τ) is a Δ^*T_δ -space, A is δ -closed. Hence (X, τ) is a δg^*T_δ -space.

Remark 3.2.8 The spaces T_c and T_d are independent with Δ^*T_δ -space as seen from the following examples.

Counter Example 3.2.9 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a Δ^*T_δ -space but not a T_c -space and T_d -space since the subset $\{b\}$ is gs-closed but not g^* -closed and not g-closed in (X, τ) .

Counter Example 3.2.10 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$. Then (X, τ) is a T_c -space and T_d -space but not a Δ^*T_δ -space since the subset $\{c\}$ is Δ^* -closed but not δ -closed in (X, τ) .

Remark 3.2.11 The Δ^*T_δ -space is independent with T_b -space, αT_b -space, αT_c -space and $T_{1/2}$ -space as seen from the following examples.

Counter Example 3.2.12 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a T_b -space, αT_b -space, αT_c -space and $T_{1/2}$ -space but not a Δ^*T_δ -space since the subset $\{b, c\}$ is Δ^* -closed but not δ -closed in (X, τ) .

Counter Example 3.2.13 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then (X, τ) is a Δ^*T_δ -space but not a T_b -space, αT_b -space, αT_c -space and $T_{1/2}$ -space since the subset $\{a, c\}$ is αg -closed, g-closed and gs-closed but not closed and g^* -closed in (X, τ) .

Proposition 3.2.14 If (X, τ) is a $\delta g T_{\delta g^*}$ -space and Δ^*T_δ -space then it is a $T_{3/4}$ -space.

That is

$$\delta g T_{\delta g^*}\text{-space} \ \& \ \Delta^* T_\delta\text{-space} \ \longrightarrow \ T_{3/4}\text{-space}$$

Proof : Let A be δg -closed in (X, τ) . Since (X, τ) is a $\delta g T_{\delta g^*}$ -space, A is δg^* -closed.

Since every δg^* -closed is Δ^* -closed (Proposition 2.2.4) and (X, τ) is a Δ^*T_δ -space, A is δ -closed. Hence (X, τ) is a $T_{3/4}$ -space.

$\Delta^*T_{\delta g^*}$ -space

Proposition 3.2.15 If (X, τ) is a $\Delta^*T_{\delta g^*}$ -space then $\Delta^*cl(A) = \delta g^*cl(A)$ for each subset A of X .

Proof : Let (X, τ) be a $\Delta^*T_{\delta g^*}$ -space. We know that by Proposition 2.2.4 every δg^* -closed is Δ^* -closed. Moreover (X, τ) is a $\Delta^*T_{\delta g^*}$ -space. Therefore $\Delta^*C(X, \tau) = \delta g^*C(X, \tau)$. Hence by the definition of δg^* -closure and Δ^* -closure, $\Delta^*cl(A) = \delta g^*cl(A)$ for each subset A of X .

Proposition 3.2.16 Every Δ^*T_{δ} -space is a $\Delta^*T_{\delta g^*}$ -space but the converse is not true.

Proof : Let A be Δ^* -closed in (X, τ) . Since (X, τ) is a Δ^*T_{δ} -space, A is δ -closed. Moreover every δ -closed is δg^* -closed in (X, τ) . Hence (X, τ) is a $\Delta^*T_{\delta g^*}$ -space.

Counter Example 3.2.17 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a, b\}\}$. Then (X, τ) is a $\Delta^*T_{\delta g^*}$ -space but not a Δ^*T_{δ} -space since the subset $\{c\}$ is Δ^* -closed but not δ -closed in (X, τ) .

But the above result can overcome by adding a condition, $T_{3/4}$ -space to $\Delta^*T_{\delta g^*}$ -space.

Proposition 3.2.18 If (X, τ) is a $\Delta^*T_{\delta g^*}$ -space and $T_{3/4}$ -space then it is a Δ^*T_{δ} -space.

That is

$$\Delta^*T_{\delta g^*}\text{-space \& } T_{3/4}\text{-space} \rightarrow \Delta^*T_{\delta}\text{-space}$$

Proof : Let (X, τ) be a $\Delta^*T_{\delta g^*}$ -space and $T_{3/4}$ -space. Let A be a Δ^* -closed set in (X, τ) .

Since (X, τ) is a $\Delta^*T_{\delta g^*}$ -space, A is δg^* -closed. As every δg^* -closed is δg -closed

(Sudha, 2012) and (X, τ) is a $T_{3/4}$ -space, A is δ -closed in (X, τ) . Hence (X, τ) is

a Δ^*T_{δ} -space.

Counter Example 3.2.19 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Then in (X, τ) ,

$$\Delta^*C(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\};$$

$$\delta G^*C(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\};$$

$$\delta GC(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\};$$

$$\delta C(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}.$$

The above four collections coincide and substantiating Proposition 3.2.18

Proposition 3.2.20 If (X, τ) is a $g\delta T_{\delta g^*}$ -space then it is a $\Delta^*T_{\delta g^*}$ -space.

Proof : Let (X, τ) be a $g\delta T_{\delta g^*}$ -space. Let A be Δ^* -closed in (X, τ) . We know that every

Δ^* -closed is $g\delta$ -closed (Proposition 2.2.8). Therefore A is $g\delta$ -closed. Since (X, τ) is a

$g\delta T_{\delta g^*}$ -space, A is δg^* -closed. Hence (X, τ) is a $\Delta^*T_{\delta g^*}$ -space.

Counter example 3.2.21 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b, c\}\}$. Then

$$\Delta^*C(X, \tau) = \{\phi, X, \{a\}, \{b, c\}\};$$

$$\delta G^*C(X, \tau) = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\};$$

$$G\delta C(X, \tau) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}.$$

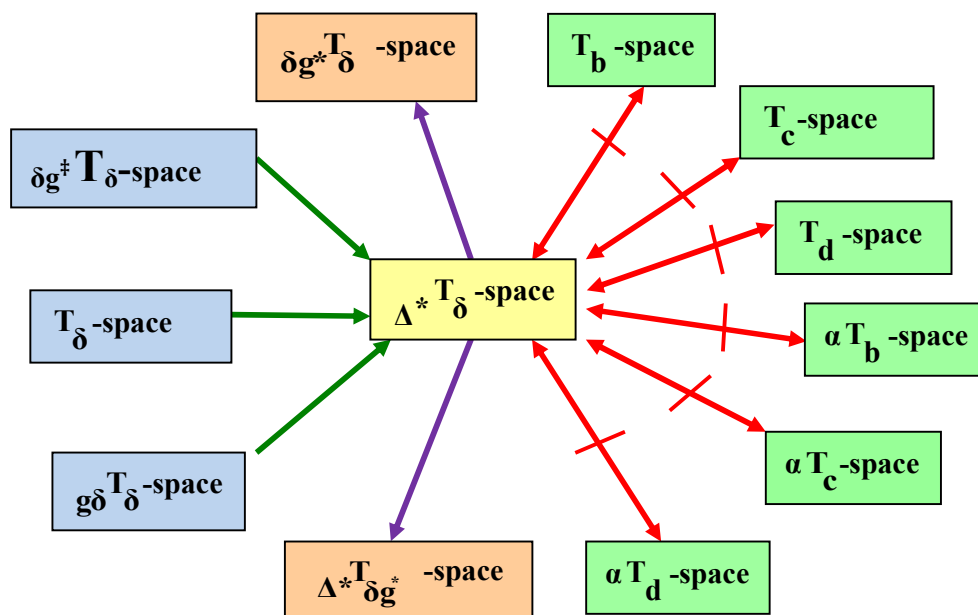
Here (X, τ) is a $\Delta^*T_{\delta g^*}$ -space but not a $g\delta T_{\delta g^*}$ -space.

Remark 3.2.22 The $\Delta^*T_{\delta g^*}$ -space is independent with T_b, T_c and T_d -spaces as seen from the following examples.

Counter Example 3.2.23 Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then (X, τ) is a $\Delta^*T_{\delta g^*}$ -space but not T_b, T_c and T_d -spaces since the subset $\{b\}$ is g -closed but not a closed set, not a g^* -closed set, and not a g -closed set respectively for T_b, T_c and T_d -spaces.

Counter Example 3.2.24 Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is T_b, T_c and T_d -spaces but not a $\Delta^*T_{\delta g^*}$ -space since the subset $\{b\}$ is Δ^* -closed but not a δg^* -closed set in (X, τ) .

Remark 3.2.25 The above results are depicted by the following diagram.



$g\delta T_{\Delta^*}$ -space

Proposition 3.2.26 If (X, τ) is a T_{δ} -space then it is a $g\delta T_{\Delta^*}$ -space.

Proof : Let A be $g\delta$ -closed in (X, τ) . Since (X, τ) is a T_{δ} -space, A is δ -closed and hence it is Δ^* -closed in (X, τ) . Therefore (X, τ) is a $g\delta T_{\Delta^*}$ -space.

Proposition 3.2.27 If (X, τ) is a $g\delta T_{\Delta^*}$ -space and a $\Delta^* T_{\delta}$ -space then it is a T_{δ} -space.

That is

$$\boxed{g\delta T_{\Delta^*}\text{-space} \ \& \ \Delta^* T_{\delta}\text{-space} \ \rightarrow \ T_{\delta}\text{-space}}$$

Proof : Let A be $g\delta$ -closed in (X, τ) . Since (X, τ) is a $g\delta T_{\Delta^*}$ -space, A is Δ^* -closed in (X, τ) . Also (X, τ) is a $\Delta^* T_{\delta}$ -space. Therefore A is δ -closed in (X, τ) . Hence (X, τ) is a T_{δ} -space.

Proposition 3.2.28 If (X, τ) is a $g\delta T_{\Delta^*}$ -space and a $\Delta^* T_{\delta g^*}$ -space then it is a

$g\delta T_{\delta g^*}$ -space. That is

$$\boxed{g\delta T_{\Delta^*}\text{-space} \ \& \ \Delta^* T_{\delta g^*}\text{-space} \ \rightarrow \ g\delta T_{\delta g^*}\text{-space}}$$

Proof : Let A be $g\delta$ -closed in (X, τ) . Since (X, τ) is a $g\delta T_{\Delta^*}$ -space, A is Δ^* -closed in (X, τ) . Also (X, τ) is a $\Delta^* T_{\delta g^*}$ -space. Therefore A is δg^* -closed in (X, τ) .

Hence (X, τ) is a $g\delta T_{\delta g^*}$ -space.

Proposition 3.2.29 If (X, τ) is a T_{δ} -space then it is a $T_{3/4}$ -space, a $\Delta^* T_{\delta}$ -space, a $\Delta^* T_{\delta g^*}$ -space and a $g\delta T_{\Delta^*}$ -space.

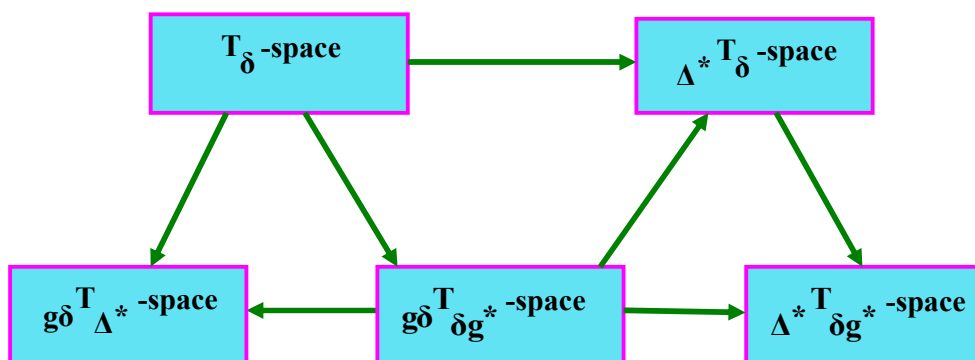
Proof : In T_{δ} -space every $g\delta$ -closed is δ -closed.

Hence $\tau_\delta = \Delta^* \# = g\delta \#$ -----(i) Also $\tau_\delta = \delta g^* \# = \delta g \# = g\delta \#$ -----(ii).

From (i) and (ii), we get the following results.

- 1) Every δg -closed is δ -closed. Therefore (X, τ) is a $T_{3/4}$ -space.
- 2) Every Δ^* -closed is δ -closed. Therefore (X, τ) is a $\Delta^* T_\delta$ -space.
- 3) Every Δ^* -closed is δg^* -closed. Therefore (X, τ) is a $\Delta^* T_{\delta g^*}$ -space.
- 4) Every $g\delta$ -closed is Δ^* -closed. Therefore (X, τ) is a $g\delta T_{\Delta^*}$ -space.

Remark 3.2.30 From the above results we have the following diagram.



$g\delta T_{\Delta^*}$ -Space

Remark 3.2.31 The spaces T_b , T_c and T_d are independent with $g\delta T_{\Delta^*}$ -space as seen from the following examples.

Counter example 3.2.32 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then (X, τ) is a $g\delta T_{\Delta^*}$ -space but not T_b, T_c and T_d -spaces since the subset $\{a\}$ is $g\delta$ -closed but not a

closed set, not a g^* -closed set and not a g -closed set in (X, τ) for T_b, T_c and T_d -spaces respectively.

Counter example 3.2.33 Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$. Then (X, τ) is a T_b, T_c and T_d but not a $g^\dagger T\Delta^*$ -space since the subset $\{a, b\}$ is g^\dagger -closed but not Δ^* -closed in (X, τ) .

Proposition 3.2.34 Every $g^\dagger T\delta g^*$ -space (resp., $g^\dagger T\delta$ -space) is a $g^\dagger T\Delta^*$ -space.

Proof : Let A be a g^\dagger -closed (resp., δ -closed) in (X, τ) . Since (X, τ) is a $g^\dagger T\delta g^*$ -space (resp., $g^\dagger T\delta$ -space), A is δg^* -closed (resp., δ -closed) in (X, τ) . Since every δg^* -closed (resp., δ -closed) is Δ^* -closed (Proposition 2.2.4 & 2.2.2), A is Δ^* -closed. Hence (X, τ) is a $g^\dagger T\Delta^*$ -space.

Remark 3.2.35 From the above results we have the following diagram.

