

# **Fuzzy Retrial Queue with Server Breakdown**

**Prema, S**  
**(13PMA007)**

**Thesis Submitted to**  
**Avinashilingam Institute for Home Science and Higher Education for Women**  
**Coimbatore - 641 043**

**In Partial Fulfilment of the Requirements for the Degree of**  
**Master of Science in Mathematics**  
**March, 2015**

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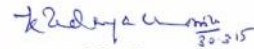
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**Signature of the Head of the Department**



**Signature of the Supervisor**

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*Chapter - 1*

## **Introduction**

*Mathematics is a more powerful instrument of knowledge than any other that has been bequeathed to us by human agency.*

*-Descartes*

Queueing theory is a branch of applied mathematics utilizing concepts from the field of stochastic processes. The queueing theory has its origin in 1909, when A.K.Erlang (1878-1929) published his fundamental paper relative to the study of congestion in telephone traffic. The study of queueing systems finds application in a variety of real life situation like business, industry, engineering, transportation, communication, computer and consumer activities.

### **1.1 Queueing System**

A queue is a waiting line of customers requiring service from one or more servers service facility. In our day-to-day life we have been a unit in one or other of the following waiting lines: registration for the school term, ticket booth at a movie theatre, teller window at a bank and so on.

### **1.2. Characteristics of Queuing System**

The basic characteristics of a queuing system are

- Arrival Process
- Service Mechanism
- Queue Discipline
- System Capacity and
- Service Channels

#### **Arrival Process**

The arrival process characterizes the arrival of units into the queueing system. The calling source can consist of a finite or infinite number of units. Arrival of unit can occur either single or in bulk. The arrival pattern is measured in terms of the mean arrival rate or mean inter-arrival time.

### **Service Mechanism**

Service mechanism describes the manner in which service is rendered. Customers may be served either single or in batches. The queueing system where the service is done in batches is called bulk service queueing system. The time required for servicing a unit (or group) is called service time.

Sometimes, the service rate may depend on the number of customers waiting for service. When the queue becomes longer, a server may work faster or conversely he may become less efficient. The situation in which service depends on the number of waiting customer is known as state dependent service.

### **Queue Discipline**

The most common queue discipline is First In First Out (FIFO) or First Come First Served (FCFS). Another queue discipline is Last In First Out (LIFO) or Last Come First Served (LCFS).

Customers may also be served randomly irrespective of their arrivals into the system as in the selection of Bingo numbers. This type of queue discipline is called Service In Random Order (SIRO). Another discipline is priority queue discipline which allows service to be offered to customer depending on their priority in relation to other customers. There are two types in priority discipline that is pre-emptive priority and non pre-emptive priority. In the pre-emptive case, the customer with high priority is allowed to enter service immediately suspending the service in progress to a customer with lower priority. In non pre-emptive case, the higher priority goes to the head of the queue but gets into service only after the completion of service in progress to the customer with lower priority.

### **System Capacity**

Some of the queueing processes admit the physical limitation to the amount of waiting room, so that when the waiting line reaches the maximum room capacity, no further customer is allowed to enter until space becomes available by a service completion.

## Service Channels

The number of servers in a queueing model may be finite or infinite. The number of servers may be arranged in series or parallel or a combination of both, depending upon the nature of the services required. In parallel channels, all the channels provide identical service facilities so that several customers may be served simultaneously. In series channels, a customer must pass through successively all the ordered channels before service is completed.

### 1.3 Kendall's Notation

This is a useful way to represent different types of queues in a compact and easily understandable fashion. Kendall's Notation describes the nature of arrival process, the nature of service process, the number of servers, maximum number in the queue and some basic queue disciplines. The notation has been considerably extended to allow it to represent a wide variety of queues. A queue is represented by a sequence A/B/C/D/E with the following meaning attached to the letters A to E.

A: This symbolically represents the nature of arrival process to the queue.

Special letters are used to symbolize the nature of inter-arrival time distribution as follows.

M- Exponentially distributed Markovian inter-arrival times.

D- Deterministic inter-arrival time.

$E_k$ - Erlang distribution of order  $k$  for the inter-arrival time.

G- General distribution for the inter-arrival times.

B: This symbolically represents the nature of service-time distribution for the customers getting served in the queue. Here the symbols are similar to inter-arrival time distribution.

C: Number of servers in the queue.

D: System capacity (default is infinity).

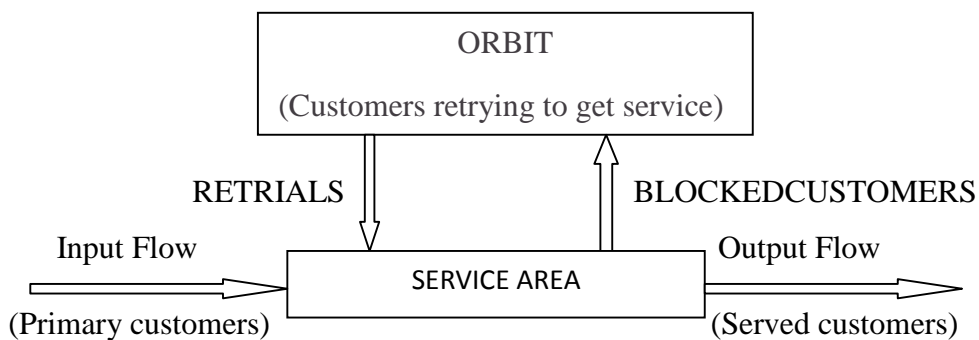
E: Queue discipline (default is FCFS- First Come First Served).

For example, a queueing system with bulk arrival Poisson process, general service distribution and single server with infinite capacity and first in first out queue discipline is denoted by  $M^x/G/1$ .

## 1.4 Retrial Queues

In classical queueing theory it is assumed that a customer who cannot get service immediately after arrival either joins the waiting line or leaves the system forever. Sometimes impatient customers leave the queue, but it is also assumed that they are leaving the system forever. However as a matter of fact the assumption about loss of customers who elected to leave the system is just a first order approximation to a real situation. Usually such a customer after some random period of time returns to the system and tries to get service again. The standard queueing models do not take into account the phenomenon of retrials and therefore cannot be applied in solving a number of practically important problems. Retrial queues have been introduced to solve this deficiency.

The general structure of a retrial queue is shown in the following figure.



**Fig.1 Structure of Retrial Queue**

It is clear that retrial queues can also be regarded as a special type of queueing networks. The following are a few examples for a retrial queues.

### (i) Telephone Systems

Everybody knows from their experience that a telephone subscriber who obtains a busy signal repeats the call until the required condition is made. As a result, the flow of calls circulating in telephone network consist of two parts: the flow of primary calls, which reflects the real wishes of the telephone subscribers and the flow of repeated calls, which is the consequence of the lack of success of previous attempts.

## (ii) Retrial Shopping Queue

In a shop a customer who finds that a queue is too long may wish to do something else and return later on with the hope that the queue dissolves.

## 1.5 Fuzzy Logic

Fuzzy logic is a form of many-valued logic that deals with approximate, rather than fixed and exact reasoning. Compared to traditional binary logic (where variables may take on true or false values), fuzzy logic variables may have a truth value that ranges in degree between 0 and 1.

## 1.6 Fuzzy Set

If  $X$  is a universe of discourse and let  $x$  be any element of  $X$ , then a fuzzy set  $A$  defined on  $X$  is given by

$$A = \{(x, \mu(x)) / x \in X\}$$

where, each pair  $(x, \mu_A(x))$  is called singleton.

## 1.7 Interval-Valued Fuzzy Set

Fuzzy sets defined by the membership function based on the approach that does not assign to each element of the universal set to one real number, but a closed interval of a real numbers between the identified lower and upper bounds, are called interval-valued fuzzy sets. These sets are defined by functions of form

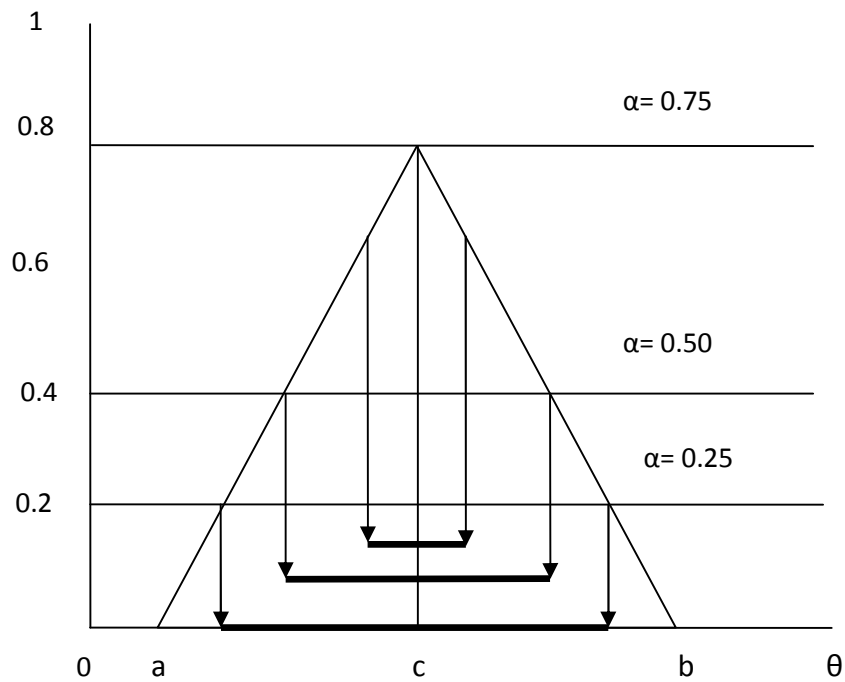
$$A: X \rightarrow \epsilon([0,1])$$

where  $\epsilon([0,1])$  denotes the family of all closed intervals of real number in  $(0,1)$ .

## 1.8 $\alpha$ -cut

An  $\alpha$ -cut of a fuzzy set  $\bar{A}$  is a crisp set  $A_\alpha$  that contains all the elements of the universal set  $x$  that have a membership grade in  $A$  greater than or equal to the specified value of  $\alpha$ . Thus,

$$A_\alpha = \{x \in X : \mu_{\bar{A}}(x) \geq \alpha\}, \quad 0 \leq \alpha \leq 1.$$



**Fig.2 Triangular fuzzy parameters and its  $\alpha$ -cut**

### 1.9 Strong $\alpha$ -cut

The Strong  $\alpha$ -cut of a fuzzy set  $\bar{A}$  is crisp set  $A_\alpha$  that contains all the elements of the universal set  $X$  that have a membership grade in  $A$  greater than specified value of  $\alpha$ . Thus ,

$$A_\alpha = \{x \in X : \mu_{\bar{A}}(x) > \alpha\}, \quad 0 \leq \alpha \leq 1.$$

### 1.10 Fuzzy Number

A fuzzy subset  $A$  of the real line  $R$  with membership function  $\mu_A: R \rightarrow [0,1]$  is called a fuzzy number if

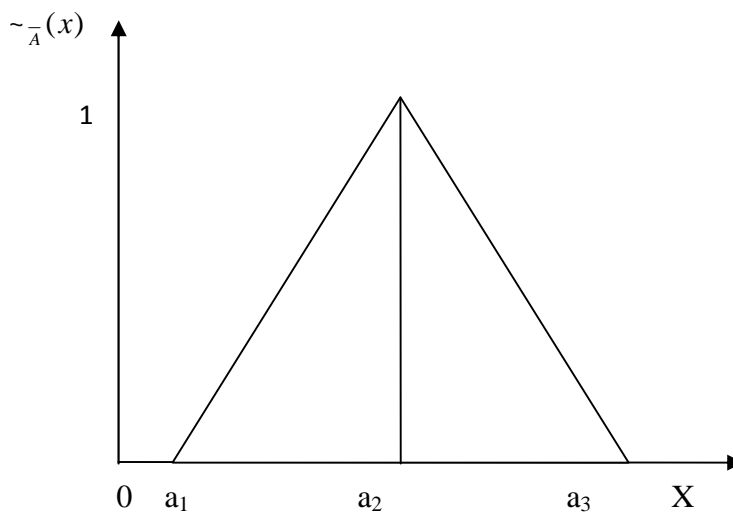
- (i)  $A$  is normal, that is there exists an element  $x_0 \in A$  such that  $\mu_A(x_0) = 1$
- (ii)  $A$  is fuzzy convex, that is  $\mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \{\mu_A(x_1) \wedge \mu_A(x_2)\}$  for all  $x_1, x_2 \in R$  and for all  $\lambda \in [0,1]$
- (iii)  $\mu_A$  is upper semi continuous and
- (iv)  $\text{Sup } A$  is bounded where  $\text{sup } A = \{x \in R : \mu_A(x) > 0\}$

### Triangular fuzzy number

A triangular fuzzy number  $\bar{A}$  is a fuzzy number specified by 3-tuples  $(a_1, a_2, a_3)$  such that  $a_1 \leq a_2 \leq a_3$ , with membership function defined as

$$\mu_{\bar{A}}(x) = \begin{cases} 0 & \text{if } x \leq a_1 \\ (x - a_1)/(a_2 - a_1) & \text{if } a_1 \leq x \leq a_2 \\ (x - a_3)/(a_2 - a_3) & \text{if } a_2 \leq x \leq a_3 \\ 0 & \text{if } x \geq a_3 \end{cases}$$

This is represented diagrammatically as



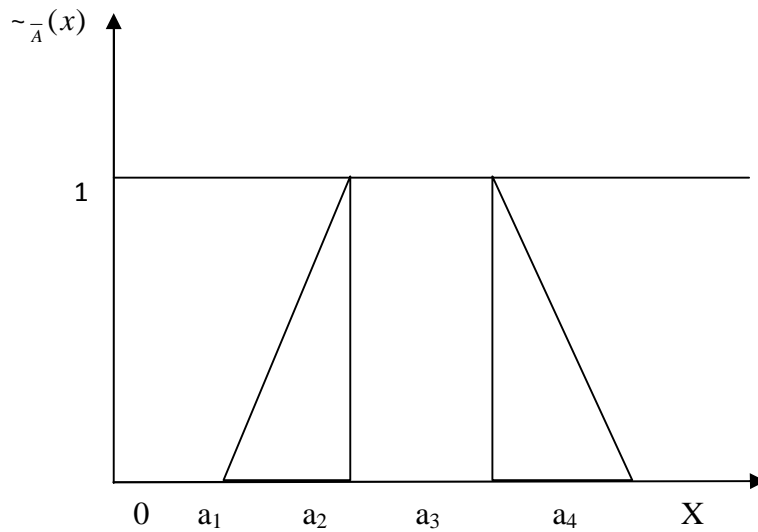
**Fig.3 Membership function of a Trapezoidal Fuzzy number  $\bar{A}$**

### Trapezoidal fuzzy number

A trapezoidal fuzzy number  $\bar{A}$  is a fuzzy number fully specified by 4-tuples  $(a_1, a_2, a_3, a_4)$  such that  $a_1 \leq a_2 \leq a_3 \leq a_4$ , with membership function defined as

$$\mu_{\bar{A}}(x) = \begin{cases} (x - a_1)/(a_2 - a_1) & \text{if } a_1 \leq x \leq a_2 \\ 1 & \text{if } a_2 \leq x \leq a_3 \\ (x - a_4)/(a_3 - a_4) & \text{if } a_3 \leq x \leq a_4 \\ 0 & \text{if } \textit{otherwise} \end{cases}$$

This is represented diagrammatically as



**Fig.4 Membership function of a Trapezoidal Fuzzy number A**

### 1.11 Zadeh's Extension Principle

The membership function of system characteristics of the queuing model is derived by using Zadeh's Principle. Let  $P(x,y)$  denote the system characteristic of interest. Clearly when arrival rate  $\bar{\lambda}$  and service rate  $\bar{\mu}$  are fuzzy numbers, then  $P(\bar{\lambda}, \bar{\mu})$  will be fuzzy as well. On the basis of Zadeh's extension principle, the membership function of the performance measures  $(\bar{\lambda}, \bar{\mu})$  is defined as:

$$\tilde{\mu}_{P(\bar{\lambda}, \bar{\mu})}(z) = \sup_{\substack{x \in X, \\ y \in Y}} \{ \min(\tilde{\lambda}(x), \tilde{\mu}(y)) / z = P(x, y) \}$$

### 1.12 Review of Literature

Queueing theory had its origin in 1909, when Erlang published his fundamental paper relating to the study of congestion in telephone traffic.

#### Rerial Queues

Retrial queueing systems are characterized by the fact that an arriving customer who finds the server occupied is obliged to join an group of blocked customers (called orbit) and reapply after random intervals of time to obtain the service. Retrial queueing

systems are basically motivated by the applications to telephone switching systems, telecommunication networks and computer systems.

One of the earliest papers in retrial queues was on the influence of repeated calls in the theory of probabilities of blocking by Kosten (1947). Kulkani (1983) published the article on queueing system with retrials using the technique of supplementary variables. The first investigation with general retrial queues was done by Falin (2008), in which each customer in the orbit generates a stream of repeated attempts that are independent of the customers in the orbit and the state of the server.

Two extensive survey articles in retrial queues are due to Falin (1990) covering respectively the developments upto mid 80's and late 80's. The only monograph on this topic is by Falin Templeton (1997) which provides an excellent scenario of retrial queues. A comparative analysis of standard and retrial queueing system was presented in Artalejo and Falin (2002). Bocharov et al. (2001) considered retrial queueing system with several input flows.

### **Breakdown Queueing Models**

A marked and unavoidable phenomenon in the service facility of a queueing system is breakdown. A queueing system might suddenly breakdown and hence the server will not be able to continue providing service unless the system is repaired. Queues with server subject to breakdowns and repairs are often encountered in many practical applications. Kulkani and Choi(1990) and Aissani(1998) considered retrial queueing systems with server breakdowns and repair. Aissani and Artalejo(1998) studied a single server retrial queueing system subject to active independent breakdown. Wang et al.(2001) investigated the reliability analysis of the retrial queue with server breakdowns and repairs.

Djellab(2002) obtained explicit expressions of availability, failure frequency and reliability function of the server for M/G/1 retrial queue with server breakdown. Wang et al. (2004) analysed the reliability of the retrial queue with server breakdown and repairs. Li and Zhao(2005) investigated a retrial queue with a constant retrial rate, server breakdowns and impatience customers. Choudhury and Kandarpa Deka(2008) provided M/G/1 retrial queue with two phase of service subject to the server breakdowns and repairs.

## **Fuzzy Queueing Model**

Fuzzy queueing models are more practical and realistic than deterministic queueing models. Within the context of traditional queueing theory the inter-arrival times and service times are required to follow certain distributions. However, in many practical situations, the arrival pattern and service pattern are described by linguistic quantifiers such as fast, slow or moderate, rather than by probability distributions.

The concepts of fuzzy sets was first introduced by Zadeh (1965) in his paperwork entitled “fuzzy sets”. One of the most important tools in fuzzy set theory is Zadeh’s extension principle. Following Zadeh, many researchers have considered the problem of fuzzy queueing systems. Li and Lee (1989) discussed the problem of fuzzy queues by using Zadeh’s extension principle, the possibility concept and fuzzy Markov chain. Negi and Lee (1992) introduced  $\alpha$  – cut and two variable simulation approaches to analyze fuzzy queues.

Chen (2004) studied a fuzzy finite capacity queue with fuzzified exponential arrival rate and service rate. Chen (2005) analyzed bulk service fuzzy queueing system with fuzzy parameters. Jau – Chuanke et al. (2007) discussed retrial queueing model with fuzzy parameters. Stephen Vincent et al. (2011) studied fuzzy retrial queue with priority subscribers. Barak et al. (2012) analyzed the cost of fuzzy queueing systems. Upadhyaya (2013) investigated Bernoulli vacation policy for a bulk retrial queue with fuzzy parameters.

## **Profile of Present Work**

The main objective of the dissertation is to analyze the steady state behavior of retrial queueing systems with server breakdown and repair.

The content of the thesis is presented in three chapters.

In the first chapter introduction and review of literature are presented. M/G/1 retrial queue with server breakdown is analyzed in chapter 2. In the final chapter bulk arrival retrial queue with server breakdown is considered.

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*Chapter - 2*

## M/G/1 Retrial Queue with Server Breakdown

In this chapter retrial queue with single server subject to breakdown and repair is analyzed. The breakdown occurs only during the busy period of the server. After repair, the server continues the service of the interrupted customer. The content of this chapter is divided into three sections. In the first section it is assumed that the arrival follows Poisson distribution and life time of the server follows exponential distribution. The service time, retrial time and repair time are assumed to follow general distribution. The mean number of customers in the orbit, the mean number of customers in the system, availability of the server and steady state failure frequency of the server are derived using supplementary variable technique. In section 2, the model is analyzed under fuzzy environment. In the final section performance measures are calculated numerically by assuming crisp values and fuzzy numbers for the system parameters.

### §1. Crisp M/G/1 Retrial Queue with Server Breakdown

Consider a single server retrial queueing system in which Customers arrive according to a Poisson process with rate  $\lambda$ . If an arriving customer finds the server idle, he may obtain service immediately. If the server is found busy or down, the arriving primary customer joins a retrial queue.

The service times are independently identically distributed with common distribution function  $B(\cdot)$ , probability density function  $b(\cdot)$ , Laplace-Stieltjes transform  $B^*(\cdot)$  and the first two moments  $\bar{b}_1$  and  $\bar{b}_2$ . Retrial times are independent random variables with common distribution function  $A(\cdot)$ , probability density function  $a(\cdot)$  and Laplace-Stieltjes transform  $A^*(\cdot)$ .

It is assumed that the server fails (or breakdown) only when a customer is in service. The time until failure is exponentially distributed with rate  $\gamma$ . The repair time is generally distributed with distribution function  $H(\cdot)$ , the probability density function  $h(\cdot)$ , Laplace-Stieltjes transform  $H^*(\cdot)$  and first two moments  $h_1$  and  $h_2$ .

If the function  $y(x)$ ,  $\tilde{r}(x)$ ,  $s(x)$  are the conditional completion rates (at time  $x$ ) for repeated attempts, for service and for repair time respectively, then

$$y(x) = \frac{a(x)}{1-A(x)}, \quad \tilde{r}(x) = \frac{b(x)}{1-B(x)}, \quad s(x) = \frac{h(x)}{1-H(x)}$$

The state of the system at time  $t$  can be described by the Markov process

$$\{N(t); t \geq 0\} = \{J(t), X(t), \langle_0(t), \langle_1(t), \langle_2(t); t \geq 0\}$$

where  $J(t)$  denotes the server state 0, 1 or 2 according as the server being idle, busy or under repair.  $X(t)$  denotes the number of customers in the retrial queue at time  $t$ .  $\langle_0(t)$  represents the elapsed retrial time;  $\langle_1(t)$  corresponds to the elapsed service time  $\langle_2(t)$  represents the elapsed repair time.

### 2.1.1 Steady State Distribution

For the process  $\{N(t); t \geq 0\}$ ; we define the following probabilities.

$$I_0(t) = P\{J(t) = 0, x(t) = 0\} \text{ for } t > 0; x > 0 \text{ and } y \geq 0$$

$$I_n(t, x) dx = P\{J(t) = 0, x(t) = n, x < \langle_0(t) < x + dx\}, n \geq 0$$

$$W_n(t, x) dx = P\{J(t) = 1, x(t) = n, x < \langle_1(t) < x + dx\}, n \geq 0$$

$$F_n(t, x, y) dy = P\{J(t) = 2, x(t) = n, x < \langle_1(t) < x + dx, y < \langle_2(t) < y + dy\}, n \geq 0$$

The system of equations that governs the model under consideration by supplementary variable method is given below

$$\frac{d}{dt} I_0(t) = -\lambda I_0(t) + \int_0^{\infty} W_0(t, x) \tilde{r}(x) dx \quad (2.1.1)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right] I_n(t, x) = -(\lambda + y(x)) I_n(t, x), \quad n \geq 1 \quad (2.1.2)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right] W_n(t, x) = -(\lambda + \mu(x) + \kappa) W_n(t, x) + \int_0^{\infty} F_n(t, x, y) S(y) dy + \lambda (1 - u_{0n}) W_{n-1}(t, x), \quad n \geq 0 \quad (2.1.3)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right] F_n(t, x, y) = -(\lambda + S(y)) F_n(t, x, y) + \lambda (1 - u_{0n}) F_{n-1}(t, x, y), \quad n \geq 0 \quad (2.1.4)$$

The boundary conditions are

$$I_n(t, 0) = \int_0^{\infty} W_n(t, x) \mu(x) dx, \quad n \geq 1 \quad (2.1.5)$$

$$W_0(t, 0) = \lambda I_0(t) + \int_0^{\infty} I_1(t, x) \gamma(x) dx \quad (2.1.6)$$

$$W_n(t, 0) = \int_0^{\infty} I_{n+1}(t, x) \gamma(x) dx + \lambda \int_0^{\infty} I_n(t, x) dx, \quad n \geq 1 \quad (2.1.7)$$

$$F_n(t, x, 0) = \kappa W_n(t, x), \quad n \geq 0 \quad (2.1.8)$$

Then the steady state equations corresponding to the above equations are

$$\lambda I_0 = \int_0^{\infty} W_0(x) \mu(x) dx \quad (2.1.9)$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \gamma(x)) I_n(x), \quad n \geq 1 \quad (2.1.10)$$

$$\frac{d}{dx} W_n(x) = -(\lambda + \kappa + \mu(x)) W_n(x) + \lambda (1 - u_{0n}) W_{n-1}(x) + \int_0^{\infty} F_n(x, y) S(y) dy, \quad n \geq 0 \quad (2.1.11)$$

$$\frac{d}{dy} F_n(x, y) = -(\lambda + S(y)) F_n(x, y) + \lambda (1 - u_{0n}) F_{n-1}(x, y), \quad n \geq 0 \quad (2.1.12)$$

with boundary conditions

$$I_n(0) = \int_0^{\infty} W_n(x) \mu(x) dx, \quad n \geq 1 \quad (2.1.13)$$

$$W_0(0) = \lambda I_0 + \int_0^{\infty} I_1(x) \gamma(x) dx \quad (2.1.14)$$

$$W_n(0) = \int_0^{\infty} I_{n+1}(x)y(x)dx + \int_0^{\infty} I_n(x)dx, \quad n \geq 1 \quad (2.1.15)$$

$$F_n(x,0) = x W_n(x), \quad n \geq 0 \quad (2.1.16)$$

The normalizing condition is

$$I_0 + \sum_{n=0}^{\infty} \int_0^{\infty} I_n(x)dx + \sum_{n=0}^{\infty} \int_0^{\infty} W_n(x)dx + \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{\infty} F_n(x,y)dxdy = 1 \quad (2.1.17)$$

Define the probability generating functions

$$I(z,x) = \sum_{n=1}^{\infty} I_n(x)z^n$$

$$W(z,x) = \sum_{n=1}^{\infty} W_n(x)z^n$$

$$F(z,x,y) = \sum_{n=1}^{\infty} F_n(x,y)z^n$$

Multiplying equation (2.1.10) by  $z^n$  and summing over  $n$  for  $n = 1, 2, 3, \dots$ , we get

$$\frac{d}{dx} \sum_{n=1}^{\infty} I_n(x)z^n = - \sum_{n=1}^{\infty} (\lambda + y(x))I_n(x)z^n, \quad n \geq 1$$

$$\left[ \frac{d}{dx} + \lambda + y(x) \right] I(z,x) = 0$$

Solution of the above linear differential equation is

$$\begin{aligned} I(z,x) &= C_1 e^{-\int (\lambda + y(x))dx} \\ &= C_1 e^{-\lambda x} e^{-\int y(x)dx} \\ &= C_1 e^{-\lambda x} e^{-\int \frac{d(1-A(x))}{1-A(x)}} \\ &= C_1 e^{-\lambda x} e^{-(-\log(1-A(x)))} \\ &= C_1 e^{-\lambda x} [1-A(x)] \end{aligned}$$

Put  $x = 0$  then  $C_1 = I(0,z)$

Accordingly we have

$$I(z,x) = I(z,0) e^{-\lambda x} [1-A(x)] \quad (2.1.18)$$

Multiplying equation (2.1.12) by  $z^n$  and summing over  $n$  for  $n=1,2,3,\dots$ , we get

$$\begin{aligned} \frac{d}{dy} \sum_{n=1}^{\infty} f_n(x, y) z^n &= -(\gamma + S(y)) \sum_{n=1}^{\infty} F_n(x, y) z^n + \gamma z \sum_{n=1}^{\infty} F_{n-1}(x, y) z^{n-1} \\ \left[ \frac{d}{dy} + \gamma + S(y) - \gamma z \right] F(z, x, y) &= 0 \\ F(z, x, y) &= C_2 e^{-[\gamma + S(y)]y} e^{-\int S(y) dy} \\ &= C_2 e^{-(1-z)y} e^{-\int \frac{h(y)}{1-H(y)} dy} \\ &= F(z, x, 0) e^{-[\gamma + S(y)]y} [1-H(y)] \end{aligned} \quad (2.1.19)$$

Multiplying equation (2.1.16) by  $z^n$  and summing over  $n$  for  $n = 1, 2, 3, \dots$ , we have

$$\begin{aligned} \sum_{n=1}^{\infty} F_n(x, 0) z^n &= \sum_{n=1}^{\infty} x W_n(x) z^n \\ F(z, x, 0) &= x W(z, x) \end{aligned} \quad (2.1.20)$$

Multiplying equation (2.1.11) by  $z^n$ , summing over  $n$  for  $n=1, 2, 3, \dots$ , and using equations (2.1.19) and (2.1.20), we get

$$\begin{aligned} \frac{d}{dz} W(z, x) &= -(\gamma + x + \sim(x)) W(z, x) + \gamma z W(z, x) + \int_0^{\infty} F(z, x, 0) e^{-\gamma(1-z)y} h(y) dy \\ &= -(\gamma + x + \sim(x)) W(z, x) + \gamma z W(z, x) + x W(z, x) H^*(\gamma - \gamma z) \\ \left[ \frac{d}{dz} + [\gamma + x + \sim(x) - \gamma z] \right] W(z, x) - x H^*(\gamma - \gamma z) W(z, x) &= 0 \\ \left[ \frac{d}{dz} + \gamma(1-z) + x + \sim(x) - x H^*(\gamma - \gamma z) \right] W(z, x) &= 0 \\ W(z, x) &= C_3 e^{-\int [\gamma(1-z) + x + \sim(x) - x H^*(\gamma - \gamma z)] dx} \\ &= C_3 e^{-[\gamma(1-z) + x - x H^*(\gamma - \gamma z)] x} [1 - B(x)] \\ &= W(z, 0) e^{-G(\gamma)(1-z)x} [1 - B(x)] \end{aligned} \quad (2.1.21)$$

where  $G(x) = x + \gamma - \gamma H^*(x)$

Equation (2.1.13), with the use of equations (2.1.9) and (2.1.21) gives

$$I(z, 0) = \int_0^{\infty} \sum_{n=0}^{\infty} W_n(x) z^n \sim(x) dx - \int_0^{\infty} W_0(x) \sim(x) dx$$

$$= W(z,0)K(z)-\lambda I_0 \quad (2.1.22)$$

where  $K(z) = B^*(G(\lambda-\lambda z))$

Multiplying equation (2.1.15) by  $z^n$ , summing over  $n$  for  $n = 1, 2, 3, \dots$ , adding with equation (2.1.14) and using equations (2.1.18) and (2.1.22), we get

$$\begin{aligned} W(z,0) &= \int_0^\infty I(z,x)dx + \frac{1}{z} \int_0^\infty I(z,x)Y(x)dx + \lambda I_0 \\ &= \int_0^\infty I(z,x)dx + \frac{1}{z} \int_0^\infty I(z,0)e^{-\lambda x}[1-A(x)]Y(x)dx + \lambda I_0 \\ &= \int_0^\infty I(z,0)e^{-\lambda x}[1-A(x)]dx + \frac{1}{z} I(z,0)A^*(\lambda) + \lambda I_0 \\ &= \frac{I(z,0)}{z} [z + A^*(\lambda)(1-z)] + \lambda I_0 \\ &= \lambda I_0 + \left[ \frac{z + (1-z)A^*(\lambda)}{z} \right] [W(z,0)K(z) - \lambda I_0] \end{aligned}$$

Solving the above equation, we obtain

$$W(z,0) = \frac{\lambda I_0(1-z)A^*(\lambda)}{[z + (1-z)A^*(\lambda)]K(z) - z} \quad (2.1.23)$$

Substituting equation (2.1.23) in equation (2.1.22), we get

$$\begin{aligned} I(z,0) &= \frac{\lambda I_0(1-z)A^*(\lambda)[K(z)]}{[z + (1-z)A^*(\lambda)]K(z) - z} - \lambda I_0 \\ &= \frac{\lambda I_0\{(1-z)A^*(\lambda)[K(z)] - [[z + (1-z)A^*(\lambda)]K(z) - z]\}}{[z + (1-z)A^*(\lambda)]K(z) - z} \\ &= \frac{\lambda I_0 z[1-K(z)]}{[z + (1-z)A^*(\lambda)]K(z) - z} \quad (2.1.24) \end{aligned}$$

Substituting equation (2.1.21) in equation (2.1.20) and using equation (2.1.23), we get

$$F(z, x,0) = \gamma W(z,x)$$

$$\begin{aligned}
&= \gamma W(z, 0) e^{-G(\cdot)z)x} [1-B(x)] \\
&= x \frac{I_0(1-z)A^*(\cdot)}{[z+(1-z)A^*(\cdot)]K(z)-z} e^{-G(\cdot)z)x} [1-B(x)] \tag{2.1.25}
\end{aligned}$$

Define the partial generating function  $\psi(z)=\int_0^\infty \mathbb{E}(z,x)dx$  for any generating function  $\psi(z,x)$ .

The probability generating function of the orbit size when the server is idle is given by

$$\begin{aligned}
I(z) &= \int_0^\infty I(z,x)dx \\
&= \int_0^\infty I(z,0)e^{-\lambda x} [1-A(x)]dx \\
&= \frac{I_0 z[1-K(z)]}{[z+(1-z)A^*(\cdot)]K(z)-z} [1-A^*(\cdot)] \tag{2.1.26}
\end{aligned}$$

The probability generating function of the orbit size when the server is busy is given by

$$\begin{aligned}
W(z) &= \int_0^\infty W(z,x)dx \\
&= \int_0^\infty W(z,0)e^{-G(\cdot)z)x} [1-B(x)]dx \\
&= \frac{I_0(1-z)A^*(\cdot)}{[z+(1-z)A^*(\cdot)]K(z)-z} \left[ \frac{1-K(z)}{G(\cdot)-z} \right] \tag{2.1.27}
\end{aligned}$$

The probability generating function of the orbit size when the server is under repair is given by

$$F(z) = \int_0^\infty \int_0^\infty F(z,x,0)e^{-\lambda(1-z)y} [1-H(y)] dx dy$$

$$\begin{aligned}
&= \int_0^{\infty} \int_0^{\infty} x W(z, x) e^{-\lambda(1-z)y} [1-H(y)] dx dy \\
&= \int_0^{\infty} \int_0^{\infty} x W(z, 0) e^{-G(\lambda-z)x} [1-B(x)] e^{-\lambda(1-z)y} [1-H(y)] dx dy \\
&= \frac{\lambda I_0 A^*(\lambda)}{[z + (1-z)A^*(\lambda)]K(z) - z} \left[ \frac{1-K(z)}{G(\lambda-z)} \right] [1-H^*(\lambda-z)] \quad (2.1.28)
\end{aligned}$$

The probability generating function for the number of customers in the orbit is

$$\begin{aligned}
P_q(z) &= I_0 + I(z) + W(z) + F(z) \\
&= \frac{I_0(1-z)A^*(\lambda)}{[z + (1-z)A^*(\lambda)]K(z) - z} \quad (2.1.29)
\end{aligned}$$

The probability generating function for the number of customers in the system is

$$\begin{aligned}
P_s(z) &= I_0 + I(z) + zW(z) + zF(z) \\
&= \frac{I_0[(1-z)A^*(\lambda)]K(z)}{[z + (1-z)A^*(\lambda)]K(z) - z} \quad (2.1.30)
\end{aligned}$$

Now we calculate  $I(z)$ ,  $W(z)$  and  $F(z)$  at  $z=1$

$$\begin{aligned}
I &= \lim_{z \rightarrow 1} I(z) \\
&= \frac{I_0[1-A^*(\lambda)]K'(1)}{A^*(\lambda) - K'(1)} \quad (2.1.31)
\end{aligned}$$

$$\begin{aligned}
W &= \lim_{z \rightarrow 1} W(z) \\
&= \lim_{z \rightarrow 1} \frac{\lambda I_0(1-z)A^*(\lambda)}{[z + (1-z)A^*(\lambda)]K(z) - z} \left[ \frac{1-K(z)}{G(\lambda-z)} \right] \\
&= \frac{I_0 A^*(\lambda) \lambda}{A^*(\lambda) - K'(1)} \quad (2.1.32)
\end{aligned}$$

where  $K'(1) = \rho_1[1 + \lambda h_1]$

$$\begin{aligned}
 F &= \lim_{z \rightarrow 1} F(z) \\
 &= \lim_{z \rightarrow 1} \frac{\lambda I_0 A^*(\rho_1)}{[z + (1-z)A^*(\rho_1)]K(z) - z} \left[ \frac{1-K(z)}{G(\rho_1 - \rho_1 z)} \right] [1-H^*(\lambda-\lambda z)] \\
 &= \frac{\lambda I_0 A^*(\rho_1) h_1 \rho_1}{A^*(\rho_1) - K'(1)} \tag{2.1.33}
 \end{aligned}$$

The normalizing condition is  $I_0 + I + W + F = 1$  (2.1.34)

Using equations (3.1.31), (3.1.32) and (3.1.33) in (3.1.34), we get

$$I_0 = \frac{A^*(\rho_1) - K'(1)}{A^*(\rho_1)} \tag{2.1.35}$$

### 2.1.3 Performance Measures

The steady state probability that the server idle in the non-empty system is given by

$$I = \frac{[1 - A^*(\rho_1)]K'(1)}{A^*(\rho_1)} \tag{2.1.36}$$

The steady state probability that the server is busy is given by

$$W = \lambda \mu_1 \tag{2.1.37}$$

The steady state probability that the server is under repair is given by

$$F = \gamma \lambda \mu_1 h_1 \tag{2.1.38}$$

The mean number of customers in the orbit is

$$L_q = \lim_{z \rightarrow 1} \frac{d}{dz} P_q(z)$$

By applying L'Hospital's rule repeatedly we get

$$L_q = P_q'(1) = \frac{I_0 A^*(\rho_1) [2(1 - A^*(\rho_1))K'(1) + K''(1)]}{2[-(A^*(\rho_1) - K'(1))]^2}$$

$$\begin{aligned}
&= \frac{I_0 A^*(\gamma) T_2}{T_1^2} \\
&= \frac{T_2}{T_1}
\end{aligned} \tag{2.1.39}$$

where  $T_1 = A^*(\gamma) - K'(1)$

$$T_2 = \frac{K''(1) + 2K'(1)[1 - A^*(\gamma)]}{2}$$

and  $K''(1) = \gamma^2 [\gamma_2 (1 + \gamma h_1)^2 + \gamma \gamma_1 h_2]$

The mean number of customers in the system is

$$\begin{aligned}
L_s &= \lim_{z \rightarrow 1} \frac{d}{dz} P_s(z) \\
&= \frac{T_1 K'(1) + T_2}{T_1} \\
&= K'(1) + L_q
\end{aligned} \tag{2.1.40}$$

### 2.1.4 Reliability Measures of the System

The availability of the server in steady state system is

$$\begin{aligned}
I_0 + I + W &= I_0 + \frac{[1 - A^*(\gamma)]K'(1)}{A^*(\gamma)} + \gamma \gamma_1 \\
&= 1 - \lambda \mu_1 \gamma h_1
\end{aligned} \tag{2.1.41}$$

The steady state failure frequency of the server is

$$W_f = \gamma W = \gamma \lambda \mu_1 \tag{2.1.42}$$

## §2. Fuzzy M/G/1 Retrial Queue with Server Breakdown

In the queueing model, we assume that group arrival rate  $\bar{\lambda}$ , the service rate  $\bar{\mu}$  and the repair rate  $\bar{s}$  are fuzzy numbers which are approximately known. Then, we represent these fuzzy numbers as

$$\bar{\lambda} = \{(x, \varphi_{\bar{\lambda}}(x)) / x \in S(\bar{\lambda})\} \quad (2.2.1)$$

$$\bar{\mu} = \{(y, \varphi_{\bar{\mu}}(y)) / y \in S(\bar{\mu})\} \quad (2.2.2)$$

$$\bar{s} = \{(w, \varphi_{\bar{s}}(s)) / w \in S(\bar{s})\} \quad (2.2.3)$$

Here,  $\varphi_a(b)$  and  $S(a)$  denote the membership function and support of a where  $a = \bar{\lambda}, \bar{\mu}, \bar{s}$  are the fuzzy numbers and  $b=x, y, w$  are the crisp values corresponding to group arrival rate, service rate and repair rate, respectively.

Let  $P(x,y,w)$  and  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})$  denote the system performance measures of interest in the crisp and fuzzy environment, respectively. As  $\tilde{\lambda}, \tilde{\mu}, \tilde{s}$  are fuzzy numbers,  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})$  will also be fuzzy. Using Zadeh's extension principle, the membership function of the performance measure  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})$  is defined as

$$\varphi_{\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})}(z) = \sup_{x \in X, y \in Y, w \in W} \min\{\varphi_{\tilde{\lambda}}(x), \varphi_{\tilde{\mu}}(y), \varphi_{\tilde{s}}(w) / z = P(x, y, w)\} \quad (2.2.4)$$

The membership function for  $L_q$  is given by

$$\varphi_{\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})}(z) = \sup_{x \in X, y \in Y, w \in W} \min\{\varphi_{\tilde{\lambda}}(x), \varphi_{\tilde{\mu}}(y), \varphi_{\tilde{s}}(w) / z = L_q\}$$

### The $\alpha$ – cut approach based on extension principle

Using the concept of  $\alpha$  -cuts, we develop the mathematical programming approach for deriving the desired membership function. The definition for the  $\alpha$  -cuts of  $\tilde{\lambda}, \tilde{\mu}$  and  $\tilde{s}$  as crisp intervals are as follows

$$\lambda_{\alpha} = \{x \in X / \varphi_{\tilde{\lambda}}(x) \geq \alpha\}$$

$$\mu_{\alpha} = \{y \in Y / \varphi_{\tilde{\mu}}(y) \geq \alpha\}$$

$$\beta_\alpha = \{w \in W / \varphi_{\bar{s}}(w) \geq \alpha\}$$

It is clear that  $\lambda_\alpha$ ,  $\mu_\alpha$ , and  $\beta_\alpha$  are crisp sets rather than fuzzy sets; these crisp sets can be expressed in the following forms

$$\lambda_\alpha = [x_r^L, x_r^U] = [\text{Min}_{x \in X} \{x \in X / \{\bar{\lambda}\}(x) \geq r\}, \text{Max}_{x \in X} \{x \in X / \{\bar{\lambda}\}(x) \geq r\}]$$

$$\mu_\alpha = [y_r^L, y_r^U] = [\text{Min}_{y \in Y} \{y \in Y / \{\bar{\mu}\}(y) \geq r\}, \text{Max}_{y \in Y} \{y \in Y / \{\bar{\mu}\}(y) \geq r\}]$$

$$\beta_\alpha = [w_r^L, w_r^U] = [\text{Min}_{w \in W} \{w \in W / \{\bar{s}\}(w) \geq r\}, \text{Max}_{w \in W} \{w \in W / \{\bar{s}\}(w) \geq r\}]$$

The intervals defined above provide information that where the group arrival rate, service rate and repair rate lie at possibility  $\alpha$ . By using the concept of  $\alpha$ -cuts, the imbedded fuzzy Markov chain in the M/G/1 retrial queue with server breakdown, can be decomposed into a family of ordinary Markov chains which possesses different transition probability matrices parameterized by  $\alpha$ . The three fuzzy parameters group arrival rate, service rate and repair rate can also be expressed by different levels of confidence intervals. As a result of this, the fuzzy batch arrival retrial queue with server breakdown can be reduced to a family of crisp batch arrival retrial queue with server breakdown with different  $\alpha$ - level sets  $\{\lambda_\alpha / 0 < \alpha \leq 1\}$ ,  $\{\mu_\alpha / 0 < \alpha \leq 1\}$  and  $\{\beta_\alpha / 0 < \alpha \leq 1\}$ . These three sets represent sets of movable boundaries, forming nested structures for expressing the relationship between ordinary sets and fuzzy sets. By the convexity of a fuzzy number, the bounds of these intervals are functions of  $\alpha$  and can be obtained as

$$x_r^L = \text{Min} \{ \bar{\lambda}^{-1}(r) \}, \quad x_r^U = \text{Max} \{ \bar{\lambda}^{-1}(r) \}$$

$$y_r^L = \text{Min} \{ \bar{\mu}^{-1}(r) \}, \quad y_r^U = \text{Max} \{ \bar{\mu}^{-1}(r) \}$$

$$w_r^L = \text{Min} \{ \bar{s}^{-1}(r) \}, \quad w_r^U = \text{Max} \{ \bar{s}^{-1}(r) \}$$

Now as defined in equation (2.2.4), the membership function of  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{s})$  is also parameterized by  $\alpha$ . Consequently its  $\alpha$ -cuts can be used to construct its membership function.

### Construction of membership function

In order to find the membership function  $\varphi_{\bar{L}_q}(z)$ , it is sufficient to find the left shape function (LSF) and the right shape function (RSF) of  $\varphi_{\bar{L}_q}(z)$ , which in turn are used to find the lower bound  $L_{q_r}^L$  and the upper bound  $L_{q_r}^U$  of the  $\alpha$ -cuts of  $\bar{L}_q$  for this model can be written as

$$L_{q_r}^L = \min \left\{ \frac{2x^2}{yw} \left[ \frac{w+s}{x+t} + \frac{(w+s)^2 + s}{w} \right] \right\}$$

such that

$$x_r^L \leq x \leq x_r^U, y_r^L \leq y \leq y_r^U \text{ and } w_r^L \leq w \leq w_r^U$$

$$L_{q_r}^U = \max \left\{ \frac{2x^2}{yw} \left[ \frac{w+s}{x+t} + \frac{(w+s)^2 + s}{w} \right] \right\}$$

such that

$$x_r^L \leq x \leq x_r^U, y_r^L \leq y \leq y_r^U \text{ and } w_r^L \leq w \leq w_r^U$$

This pair of mathematical programs falls into the category of parametric NLPs which facilitates the systematic study of how the optimal solutions change when  $x_r^L, x_r^U, y_r^L, y_r^U, w_r^L, w_r^U$  vary over the interval  $\alpha \in (0,1)$ .

If both the lower bound  $L_{q_r}^L$  and the upper bound  $L_{q_r}^U$  of the  $\alpha$ -cuts of  $\bar{L}_q$  are invertible with respect to  $\alpha$ , then a LSF and a RSF can be obtained as  $L(z) = (L_{q_r}^L)^{-1}$  and  $R(z) = (L_{q_r}^U)^{-1}$ . Further, the membership function  $\varphi_{\bar{L}_q}(z)$  is constructed as

$$\varphi_{\bar{L}_q}(z) = \begin{cases} L(z), & z_1 \leq z \leq z_2 \\ 1, & z_2 \leq z \leq z_3 \\ R(z), & z_3 \leq z \leq z_4 \end{cases}$$

### §3. Numerical Results

### Case(1) :Crisp Queue

Assume that the distributions of service time, retrial time and repair time are exponential with rate  $\mu, \eta$  and  $\beta$  respectively, the combined effect of  $\mu$  and  $\lambda$  on the performance measures are displayed as surface diagram in figures (1)-(8) for the parameters  $(\lambda, \alpha, \mu) = (0.1, 2, 6)$ .

From the figure it is clear that

- As  $\lambda$  increases, the values of  $L_q, L_s, I, W, F$  and  $\mathcal{F}$  increases whereas  $I_0$  and  $\Lambda$  decreases.
- For increasing values of  $\mu$ , the values of  $L_q, L_s, I, W, F$  and  $\mathcal{F}$  increases whereas  $I_0$  and  $\Lambda$  decreases.

### Case(2):Fuzzy Queue

For numerical analysis assume that arrival rate, retrial rate, service rate and repair rate are trapezoidal fuzzy number such that  $\bar{\lambda} = [1 \ 2 \ 3 \ 4]$ ,  $\bar{\eta} = [10 \ 11 \ 12 \ 13]$ ,  $\bar{\mu} = [26 \ 27 \ 28 \ 29]$ ,  $\alpha = 2$ .

Then we get,

$$[x_r^L, x_r^U] = [1+\alpha, 4-\alpha]$$

$$[r_r^L, r_r^U] = [10+\alpha, 13-\alpha]$$

$$[s_r^L, s_r^U] = [26+\alpha, 29-\alpha]$$

Using the MATLAB software, the values of  $(L_q)_r^L$  and  $(L_q)_r^U$  in equations

$$(L_q)_r^L = \min_{\Omega} L_q(x, r, s) \text{ and}$$

$$(L_q)_r^U = \max_{\Omega} L_q(x, r, s) \text{ are calculated as}$$

$$(L_q)_r^L = 4 \frac{[(1+r)^2(4649 - 285r + 4r^2)]}{[(2045 - 493r - 17r^2 + r^3)(-13+r)(-29+r)]} \quad (2.3.1)$$

and

$$(L_q)_r^U = -4 \frac{[(-4+r)^2(3830+261r+4r^2)]}{[(-440-568r+8r^2+r^3)(10+r)(26+r)]} \quad (2.3.2)$$

Table 1 presents fuzzy mean orbit size at different  $\alpha$  level. At extreme end for possibility level  $\alpha=1$ , range of the mean orbit length is approximately [0.1354, 0.4969], indicating that it is definitely possible that the expected number of customers in the system falls between 0.1354 and 0.4969. At another extreme end for possibility level  $\alpha=0$ , the range of the mean orbit length is [0.0241, 2.1427]. This indicates that the orbit length will never exceed 2.1427 or fall below 0.0241. Fig.1 depicts the rough shape of  $\{_{f(\bar{z}, \bar{s})} (z)$  constructed from different  $\alpha$  values. The rough shape turns out rather fine, and looks like a continuous function.

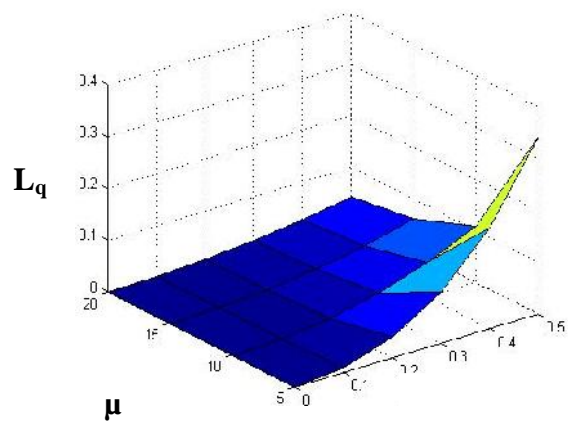


Fig.2.1.  $L_q$  versus ( $\mu$ , )

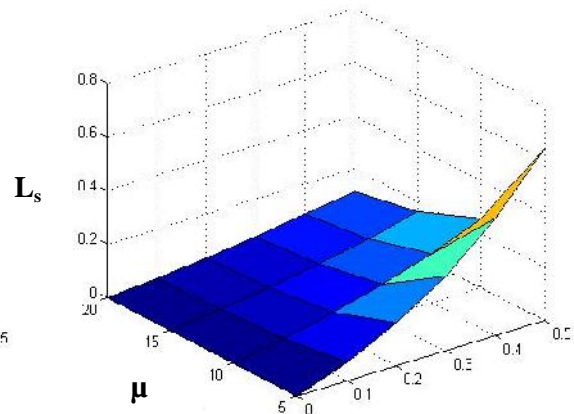


Fig.2.2.  $L_s$  versus ( $\mu$ , )

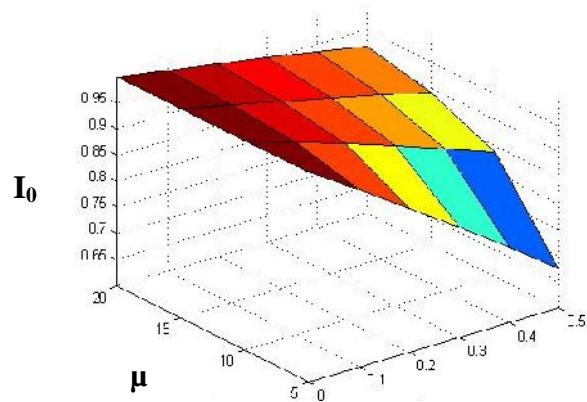


Fig.2.3.  $I_0$  versus ( $\mu$ , )

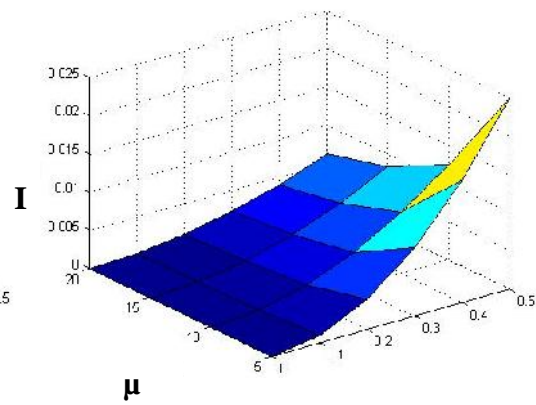
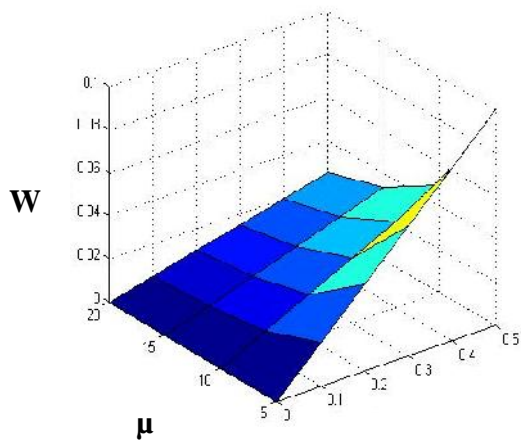
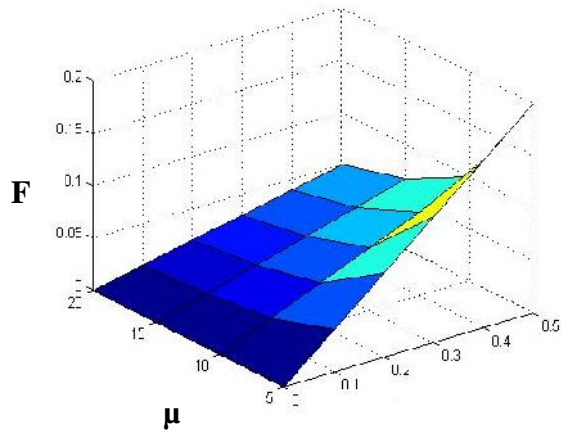


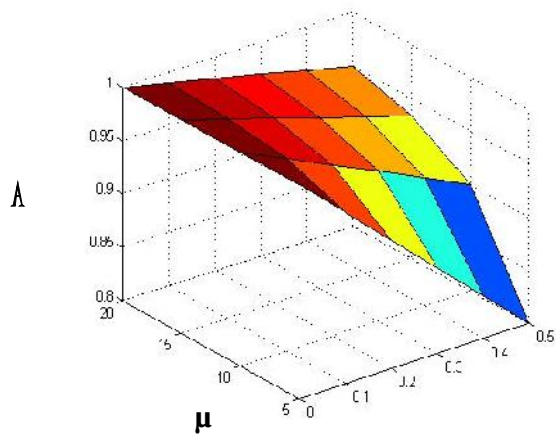
Fig.2.4.  $I$  versus ( $\mu$ , )



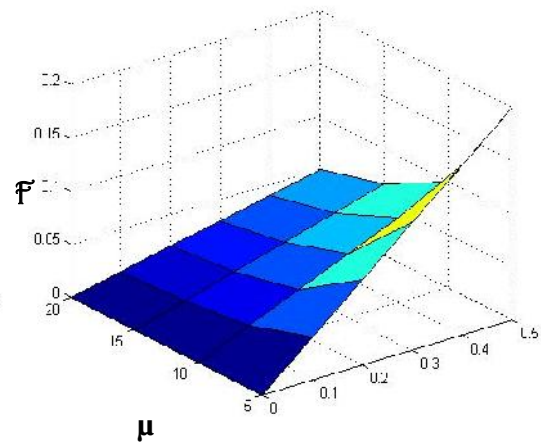
**Fig.2.5. W versus ( $\mu$ , )**



**Fig.2.6. F versus ( $\mu$ , )**



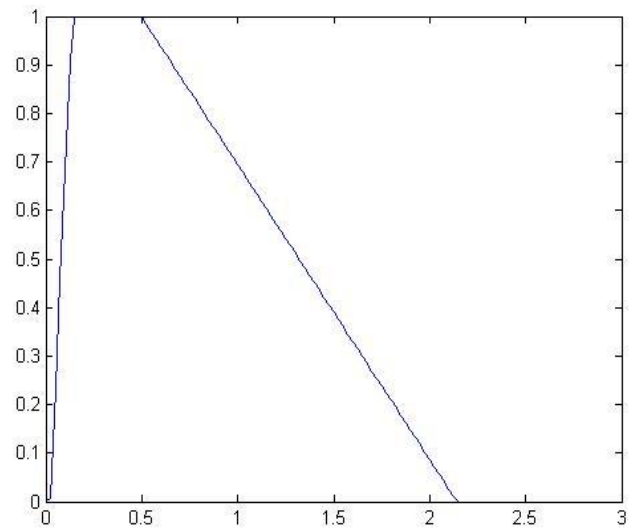
**Fig.2.7. Availability versus ( $\mu$ , )**



**Fig.2.8. Failure Frequency versus ( $\mu$ , )**

**Table. The  $\alpha$ - cuts of Mean Orbit Size**

$\alpha$	$(L_q)_r^L$	$(L_q)_r^U$
0.0	0.0241	2.1427
0.1	0.0301	1.7917
0.2	0.0369	1.5166
0.3	0.0447	1.2960
0.4	0.0535	1.1158
0.5	0.0635	0.9664
0.6	0.0747	0.8410
0.7	0.0874	0.7345
0.8	0.1016	0.6435
0.9	0.1175	0.5650
1.0	0.1354	0.4969



**Fig.2.9 Mean Orbit Size**

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*Chapter - 3*

## Bulk Arrival Retrial Queue with Server Breakdown

Batch arrival retrial queue with server breakdown is analyzed. If an arriving batch finds the server free, then one of the arrivals receives the service immediately and the remaining customers in the arriving batch join the orbit. Otherwise the arriving customers join the orbit. The server is subject to breakdown and repair. The repair of the failed server starts immediately. After the completion of repair the server resumes the service for the interrupted customer. The content of this chapter is given in three sections. In section 1, bulk arrival retrial queue with server breakdown having certain probability distributions for system parameters is considered. Using supplementary variable technique various performance measures are derived. In section 2 the same model is analyzed and the membership function for expected number in the queue is constructed by taking system parameters as fuzzy numbers. In the final section numerical results are given for both crisp and fuzzy queues.

### §1. Crisp Bulk Arrival Retrial Queue with Server Breakdown

Consider a single server queueing system where in customers arrive in batches according to Poisson process with rate  $\lambda$ . The batch size  $Y$  is a random variable with distribution function  $P(Y=k)=C_k$ ,  $k=1,2,3,\dots$ , probability generating function  $C(z)$  and the first two moments  $m_1$  and  $m_2$ . If an arriving batch finds the server idle, then one of the customers obtains the service immediately and others join the orbit. Otherwise all the customer in the arriving batch join the orbit. The retrial time of the customer in the retrial queue is generally distributed with distribution function  $A(x)$ , density function  $a(x)$ , Laplace Stieltjes transformation  $A^*(x)$  and the conditional completion rate is

$$y(x) = \frac{a(x)}{1 - A(x)}.$$

The service time of customers are independently and identically distributed with common distribution function  $B(x)$ , density function  $b(x)$ , Laplace Stieltjes transform  $B^*(x)$ , first two moments  $\mu_1$  and  $\mu_2$  and the conditional completion rate

$$\tilde{~}(x) = \frac{b(x)}{1 - B(x)}.$$

The server is subject to break down while it is working. It is assumed that life time of the server is exponential with rate  $\gamma$ . When the server fails, it stops providing service and send for repair immediately. As soon as the repair of the server is completed the server enters the working state immediately and continues the service to the interrupted customer.

The repair time is generally distributed with common distribution function  $H(x)$ , the density function  $h(x)$ , the Laplace Stieltjes transform  $H^*(x)$  and first two moments  $h_1$  and  $h_2$ .

### 3.1.1 Analysis of The Steady State Distribution

The state of the system at time  $t$  can be described by the Markov process  $\{X(t), t \geq 0\} = \{C(t), N(t), \kappa(t), t \geq 0\}$  where  $C(t)$  denotes the server state 0,1,2 according as the server being idle, busy and under repair.  $N(t)$  denotes the number of customers in the orbit at time  $t$ .  $\kappa(t)$  represents the elapsed retrial time, service time and repair time.

Define the following probabilities for  $t \geq 0, x \geq 0$ .

$$I_0(t) = P\{C(t) = 0, N(t) = 0\}$$

$$I_n(t, x)dx = P\{C(t) = 0, N(t) = n, x < \kappa(t) < x + dx\}, n \geq 1$$

$$W_n(t, x)dx = P\{C(t) = 1, N(t) = n, x < \kappa(t) < x + dx\}, n \geq 0$$

$$F_n(t, x)dx = P\{C(t) = 2, N(t) = n, x < \kappa(t) < x + dx\}, n \geq 0$$

Let  $I_0, I_n(x), W_n(x), F_n(x)$  be the corresponding steady state probability.

The system of equations that governs the model under consideration by supplementary variable technique are given below.

$$\frac{d}{dt} I_0(t) = -\lambda I_0(t) + \int_0^{\infty} W_0(t, x) \gamma(x) dx \quad (3.1.1)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right] I_n(t, x) = -(\lambda + \gamma(x)) I_n(t, x), \quad n \geq 1 \quad (3.1.2)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \right] W_n(t, x) = -(\lambda + \mu(x) + \kappa) W_n(t, x) + \int_0^{\infty} F_n(t, x, y) S(y) dy + \sum_{k=1}^n C_k (1 - u_{0n}) W_{n-k}(t, x), \quad n \geq 1 \quad (3.1.3)$$

$$\left[ \frac{\partial}{\partial t} + \frac{\partial}{\partial y} \right] F_n(t, x, y) = -(\lambda + S(y)) F_n(t, x, y) + \sum_{k=1}^n C_k (1 - u_{0n}) F_{n-k}(t, x, y), \quad n \geq 0 \quad (3.1.4)$$

with boundary conditions

$$I_n(t, 0) = \int_0^{\infty} W_n(t, x) \mu(x) dx, \quad n \geq 1 \quad (3.1.5)$$

$$W_0(t, 0) = \lambda C_1 I_0(t) + \int_0^{\infty} I_1(t, x) \gamma(x) dx \quad (3.1.6)$$

$$W_n(t, 0) = \lambda C_{n+1} I_0(t) + \int_0^{\infty} I_{n+1}(t, x) \gamma(x) dx + \sum_{k=1}^n C_k \int_0^{\infty} I_{n-k}(t, x) dx, \quad n \geq 1 \quad (3.1.7)$$

$$F_n(t, x, 0) = \kappa W_n(t, x), \quad n \geq 0 \quad (3.1.8)$$

Then the steady state equations corresponding to the above equations are

$$\lambda I_0 = \int_0^{\infty} W_0(x) \mu(x) dx \quad (3.1.9)$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \gamma(x)) I_n(x), \quad n \geq 1 \quad (3.1.10)$$

$$\frac{d}{dx} W_n(x) = -(\lambda + \kappa + \mu(x)) W_n(x) + \sum_{k=1}^n C_k (1 - u_{0n}) W_{n-k}(x) + \int_0^{\infty} F_n(x, y) S(y) dy, \quad n \geq 0 \quad (3.1.11)$$

$$\frac{d}{dy} F_n(x, y) = -(\lambda + S(y)) F_n(x, y) + \sum_{k=1}^n C_k (1 - u_{0n}) F_{n-k}(x, y), \quad n \geq 0 \quad (3.1.12)$$

with boundary conditions

$$I_n(0) = \int_0^{\infty} W_n(x) \sim(x) dx, \quad n \geq 1 \quad (3.1.13)$$

$$W_0(0) = \int_0^{\infty} C_1 I_0 + \int_0^{\infty} I_1(x) y(x) dx \quad (3.1.14)$$

$$W_n(0) = \int_0^{\infty} C_{n+1} I_0 + \int_0^{\infty} I_{n+1}(x) y(x) dx + \int_{k=1}^n C_k \int_0^{\infty} I_{n-k}(x) dx, \quad n \geq 1 \quad (3.1.15)$$

$$F_n(x,0) = \lambda W_n(x), \quad n \geq 0 \quad (3.1.16)$$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty} \int_0^{\infty} I_n(x) dx + \sum_{n=0}^{\infty} \int_0^{\infty} W_n(x) dx + \sum_{n=0}^{\infty} \int_0^{\infty} \int_0^{\infty} F_n(x, y) dx dy = 1 \quad (3.1.17)$$

Define the probability generation function

$$I(z,x) = \sum_{n=1}^{\infty} I_n(x) z^n$$

$$W(z,x) = \sum_{n=0}^{\infty} W_n(x) z^n$$

$$F(z,x,y) = \sum_{n=1}^{\infty} F_n(x, y) z^n$$

Multiplying equation (3.1.10) by  $z^n$  and summing over  $n$  for  $n = 1, 2, 3, \dots$ , we get

$$\frac{d}{dx} \sum_{n=1}^{\infty} I_n(x) z^n = - \sum_{n=1}^{\infty} (\lambda + y(x)) I_n(x) z^n, \quad n \geq 1$$

$$\left[ \frac{d}{dx} + \lambda + y(x) \right] I(z, x) = 0$$

Solution of the above equation is given by

$$\begin{aligned} I(z,x) &= D_1 e^{-\int (\lambda + y(x)) dx} \\ &= D_1 e^{-\lambda x} e^{-\int y(x) dx} \\ &= D_1 e^{-\lambda x} e^{-\int \frac{d(1-A(x))}{1-A(x)}} \end{aligned}$$

$$\begin{aligned}
&= D_1 e^{-\lambda x} e^{-(-\log(1-A(x)))} \\
&= D_1 e^{-\lambda x} [1 - A(x)]
\end{aligned}$$

Put  $x = 0$  then  $D_1 = I(0, z)$

Accordingly we have

$$I(z, x) = I(z, 0) e^{-\lambda x} [1 - A(x)] \quad (3.1.18)$$

Multiplying equation (2.1.12) by  $z^n$  and summing over  $n$  for  $n=1, 2, 3, \dots$ , we get

$$\frac{d}{dy} \sum_{n=1}^{\infty} F_n(x, y) z^n = -(\lambda + S(y)) \sum_{n=1}^{\infty} F_n(x, y) z^n + \lambda z^k \sum_{k=1}^n \sum_{n=1}^{\infty} (1 - u_{0n}) F_{n-k}(x, y) z^{n-k}$$

$$\left[ \frac{d}{dy} + \lambda + S(y) - \lambda C(z) \right] F(z, x, y) = 0$$

$$\begin{aligned}
F(z, x, y) &= D_2 e^{-[\lambda + S(y)]y} e^{-\int S(y) dy} \\
&= D_2 e^{-\lambda(1-C(z))y} e^{-\int \frac{h(y)}{1-H(y)} dy} \\
&= F(z, x, 0) e^{-[\lambda + S(y)]y} [1 - H(y)]
\end{aligned} \quad (3.1.19)$$

Multiplying equation (3.1.16) by  $z^n$  and summing over  $n$  for  $n = 1, 2, 3, \dots$ , we have

$$\begin{aligned}
\sum_{n=1}^{\infty} F_n(x, 0) z^n &= \sum_{n=1}^{\infty} \lambda W_n(x) z^n \\
F(z, x, 0) &= \lambda W(z, x)
\end{aligned} \quad (3.1.20)$$

Multiplying equation (3.1.11) by  $z^n$ , summing over  $n$  for  $n=1, 2, 3, \dots$ , and using equations (3.1.19) and (3.1.20), we get

$$\begin{aligned}
\frac{d}{dz} W(z, x) &= -(\lambda + \lambda + \lambda(x)) W(z, x) + \lambda C(z) W(z, x) + \int_0^{\infty} F(z, x, 0) e^{-\lambda(1-C(z))y} [1 - H(y)] dy \\
&= -(\lambda + \lambda + \lambda(x)) W(z, x) + \lambda C(z) W(z, x) + \lambda W(z, x) H^*(\lambda - \lambda C(z)) \\
\frac{d}{dz} + [\lambda + \lambda + \lambda(x) - \lambda C(z)] W(z, x) - \lambda H^*(\lambda - \lambda C(z)) W(z, x) &= 0
\end{aligned}$$

$$\left[ \frac{d}{dz} + \lambda(1-C(z)) + \lambda - \lambda C(z) H^*(\lambda - \lambda C(z)) \right] W(z, x) = 0$$

$$\begin{aligned} W(z, x) &= D_3 e^{-\int [\lambda(1-C(z)) + \lambda - \lambda C(z) H^*(\lambda - \lambda C(z))] dx} \\ &= D_3 e^{-[\lambda(1-C(z)) + \lambda - \lambda C(z) H^*(\lambda - \lambda C(z))]x} [1 - B(x)] \\ &= W(z, 0) e^{-(w(z))x} [1 - B(x)] \end{aligned} \quad (3.1.21)$$

where  $w(z) = \lambda - \lambda C(z) + \lambda - \lambda C(z) H^*(\lambda - \lambda C(z))$

Equation (3.1.13), with use of equations (3.1.9) and (3.1.21) gives

$$\begin{aligned} I(z, 0) &= \int_0^\infty \sum_{n=1}^\infty W_n(x) z^n \sim(x) dx - \int_0^\infty W_0(x) \sim(x) dx \\ &= \int_0^\infty W(z, x) \sim(x) dx - I_0 \\ &= \int_0^\infty W(z, 0) e^{-(w(z))x} b(x) dx - I_0 \\ &= W(z, 0) B^*(w(z)) - \lambda I_0 \end{aligned} \quad (3.1.22)$$

Multiplying equation (3.1.15) by  $z^n$ , summing over  $n$  for  $n = 1, 2, 3, \dots$ , and using equation (3.1.22), we obtain

$$\begin{aligned} W(z, 0) &= \frac{1}{z} \int_0^\infty I(z, 0) e^{-\lambda x} [1 - A(x)] \frac{a(x)}{[1 - A(x)]} dx + \frac{1}{z} C(z) \int_0^\infty I(z, 0) e^{-\lambda x} [1 - A(x)] dx + \frac{1}{z} C(z) I_0 \\ &= \frac{I(z, 0)}{z} [(1 - C(z)) A^*(\lambda) + C(z)] + \frac{1}{z} C(z) I_0 \\ &= \frac{W(z, 0) B^*(w(z)) - I_0}{z} [(1 - C(z)) A^*(\lambda) + C(z)] + \frac{1}{z} C(z) I_0 \end{aligned}$$

Solving for  $W(z, 0)$  we get

$$= \frac{I_0(C(z)-1)A^*(z)}{z - [(1-C(z))A^*(z) + C(z)]B^*(w(z))} \quad (3.1.23)$$

Substituting equation (3.1.23) in equation (3.1.22), we get

$$\begin{aligned} I(z,0) &= \frac{I_0(C(z)-1)A^*(z)B(w(z))}{z - [(1-C(z))A^*(z) + C(z)]B^*(w(z))} - I_0 \\ &= \frac{I_0[C(z)B^*(w(z)) - z]}{z - [(1-C(z))A^*(z) + C(z)]B^*(w(z))} \end{aligned} \quad (3.1.24)$$

Substituting equation (3.1.21) in equation (3.1.20) and using equation (3.1.23), we get

$$\begin{aligned} F(z, x,0) &= \gamma W(z,x) \\ &= \gamma W(z, 0) e^{-(w(z))x} [1-B(x)] \\ &= x \frac{I_0(C(z)-1)A^*(z)}{z - [(1-C(z))A^*(z) + C(z)]B^*(w(z))} e^{-(w(z))x} [1-B(x)] \end{aligned} \quad (3.1.25)$$

Define the partial generating function  $\psi(z) = \int_0^{\infty} F(z, x) dx$  for any generating function  $\psi(z, x)$ .

and take  $D(z) = [(1-C(z))A^*(z) + C(z)]B^*(w(z))$ . Then

The partial probability generating function of the orbit size when the server is idle is given by

$$\begin{aligned} I(z) &= \int_0^{\infty} I(z, x) dx \\ &= \int_0^{\infty} I(z,0) e^{-\lambda x} [1-A(x)] dx \\ &= \int_0^{\infty} \frac{I_0[C(z)B^*(w(z)) - z]}{z - [(1-C(z))A^*(z) + C(z)]B^*(w(z))} e^{-\lambda x} [1-A(x)] dx \\ &= \frac{I_0}{D(z)} [z - C(z)B^*(w(z))] [1-A^*(z)] \end{aligned} \quad (3.1.26)$$

The partial probability generating function of the orbit size when the server is busy is given by

$$\begin{aligned}
W(z) &= \int_0^{\infty} W(z, x) dx \\
&= \int_0^{\infty} W(z, 0) e^{-(w(z))x} [1-B(x)] dx \\
&= \int_0^{\infty} \frac{I_0(C(z)-1)A^*(\cdot)}{z - [(1-C(z))A^*(\cdot) + C(z)]B^*(w(z))} e^{-(w(z))x} [1-B(x)] dx \\
&= \frac{I_0(C(z)-1)A^*(\cdot)}{z - [(1-C(z))A^*(\cdot) + C(z)]B^*(w(z))} \left[ \frac{1-B^*(w(z))}{w(z)} \right] \\
&= \frac{I_0(C(z)-1)A^*(\cdot)}{[(1-C(z))A^*(\cdot) + C(z)]B^*(w(z)) - z} \left[ \frac{1-B^*(w(z))}{w(z)} \right] \tag{3.1.27}
\end{aligned}$$

The partial probability generating function of the orbit size when the server is under repair is given by

$$\begin{aligned}
F(z) &= \int_0^{\infty} \int_0^{\infty} F(z, x, y) dx dy \\
&= \int_0^{\infty} \int_0^{\infty} F(z, x, 0) e^{-\lambda(1-C(z))y} [1-H(y)] dx dy \\
&= \frac{\lambda I_0[C(z)-1]A^*(\cdot)}{z - [(1-C(z))A^*(\cdot) + C(z)]B^*(w(z))} \int_0^{\infty} e^{-(w(z))x} [1-B(x)] dx \int_0^{\infty} e^{-\lambda(1-C(z))y} [1-H(y)] dy \\
&= \frac{\lambda I_0 A^*(\cdot)}{[(1-C(z))A^*(\cdot) + C(z)]B^*(w(z)) - z} \left[ \frac{1-B^*(w(z))}{w(z)} \right] [1-H^*(\cdot) - \lambda C(z)] \tag{3.1.28}
\end{aligned}$$

Let I, W and F be the limiting values of I(z), W(z) and F(z) at z=1 then

$$\begin{aligned}
I &= \lim_{z \rightarrow 1} I(z) \\
&= \lim_{z \rightarrow 1} \frac{I_0 z - C(z)B^*(w(z))(1-A^*(\cdot))}{[A^*(1-C(z)) + C(z)]B^*(w(z)) - z}
\end{aligned}$$

$$= \frac{I_0(1-A^*(z))[1-m_1-m_1^{-1}(1+\chi h_1)]}{m_1(1-A^*(z))+m_1^{-1}(1+\chi h_1)-1} \quad (3.1.29)$$

$$\begin{aligned} W &= \lim_{z \rightarrow 1} W(z) \\ &= \lim_{z \rightarrow 1} \frac{I_0(1-C(z))A^*(z)}{[(1-C(z))A^*(z)+C(z)]B^*(W(z))-z} \left[ \frac{1-B^*(W(z))}{W(z)} \right] \\ &= \frac{m_1^{-1}I_0A^*(z)}{1-m_1(1-A^*(z))-m_1^{-1}(1+\chi h_1)} \end{aligned} \quad (3.1.30)$$

$$\begin{aligned} F &= \lim_{z \rightarrow 1} F(z) \\ &= \lim_{z \rightarrow 1} \frac{\chi I_0A^*(z)}{[(1-C(z))A^*(z)+C(z)]B^*(W(z))-z} \left[ \frac{1-B^*(W(z))}{W(z)} \right] [1-H^*(\lambda-\lambda C(z))] \\ &= \frac{\chi I_0A^*(z)h_1 m_1^{-1}}{1-m_1(1-A^*(z))-m_1^{-1}(1+\chi h_1)} \end{aligned} \quad (3.1.31)$$

$$\text{The normalizing equation (3.1.17) becomes } I_0 + I + W + F = 1 \quad (3.1.32)$$

Using equations (3.1.29), (3.1.30) and (3.1.31) in (3.1.32), we get

$$I_0 = \frac{1-m_1(1-A^*(z))-m_1^{-1}(1+\chi h_1)}{A^*(z)} \quad (3.1.33)$$

### 3.1.2 Performance Measures

The steady state probability that the server idle for the non-empty system is given by

$$I = \frac{[1-A^*(z)]K'(1)}{A'(z)} \quad (3.1.34)$$

The steady state probability that the server is busy is given by

$$W = \lambda m_1 \mu_1 \quad (3.1.35)$$

The steady state probability that the server is under repair is given by

$$F = \gamma \lambda m_1 \mu_1 h_1 \quad (3.1.36)$$

The partial probability generating function for the number of customers in the orbit is

$$\begin{aligned}
 P_q(z) &= I_0 + I(z) + W(z) + F(z) \\
 &= \frac{[1 - m_1(1 - A^*(z)) - \lambda m_1^{-1}(1 + \lambda h_1)](1 - z)}{[1 - C(z)A^*(z) + C(z)]B^*(W(z)) - z}
 \end{aligned} \tag{3.1.37}$$

The mean number of customers in the orbit is

$$\begin{aligned}
 L_q &= \lim_{z \rightarrow 1} \frac{d}{dz} P_q(z) \\
 &= \lim_{z \rightarrow 1} \frac{d}{dz} \left[ \frac{[1 - m_1(1 - A^*(z)) - \lambda m_1^{-1}(1 + \lambda h_1)](1 - z)}{[1 - C(z)A^*(z) + C(z)]B^*(W(z)) - z} \right]
 \end{aligned}$$

By applying L'Hospitals rule repeatedly we get

$$\begin{aligned}
 L_q &= P_q'(1) \\
 &= \frac{[m_1^{-2}]^2 / 2 [(1 + \lambda h_1)^2 \lambda^{-2} + \lambda h_2 \lambda^{-1}] + \lambda m_1^2 (1 + \lambda h_1) (1 - A^*(z)) \lambda^{-1} + [m_2 (1 - A^*(z)) / 2]}{[1 - m_1(1 - A^*(z)) - \lambda m_1^{-1}(1 + \lambda h_1)]}
 \end{aligned} \tag{3.1.38}$$

The partial probability generating function for the number of customers in the system is

$$\begin{aligned}
 P_s(z) &= I_0 + I(z) + zW(z) + zF(z) \\
 &= \frac{[1 - m_1(1 - A^*(z)) - \lambda m_1^{-1}(1 + \lambda h_1)](1 - z)B^*(W(z))}{[1 - C(z)A^*(z) + C(z)]B^*(W(z)) - z} \\
 &= P_q(z) B^*(W(z))
 \end{aligned} \tag{3.1.39}$$

The mean number of customers in the system is

$$\begin{aligned}
 L_s &= \lim_{z \rightarrow 1} \frac{d}{dz} P_s(z) \\
 &= \lim_{z \rightarrow 1} \frac{d}{dz} \left[ \frac{[1 - m_1(1 - A^*(z)) - \lambda m_1^{-1}(1 + \lambda h_1)](1 - z)B^*(W(z))}{[1 - C(z)A^*(z) + C(z)]B^*(W(z)) - z} \right] \\
 &= \lambda m_1 (1 + \lambda h_1) \lambda^{-1} + [m_2 (1 - A^*(z)) / 2] + L_q
 \end{aligned} \tag{3.1.40}$$

### 3.1.3 Reliability Measures of The System

The availability of the server in steady state system is

$$\begin{aligned} I_0 + I + W &= I_0 + \frac{[1 - A^*(\lambda)](\lambda m_1 \mu_1 (1 + \lambda h_1) + m_1 - 1)}{A^*(\lambda)} + \lambda m_1 \mu_1 \\ &= 1 - \lambda m_1 \mu_1 \gamma h_1 \end{aligned} \quad (3.1.41)$$

The steady state failure frequency of the server is

$$W_f = \gamma W = \gamma \lambda m_1 \mu_1 \quad (3.1.42)$$

## §2. Fuzzy Bulk Arrival Retrial Queue with Server Breakdown

Consider bulk arrival retrial queue with server breakdown with the assumption that arrival rate  $\bar{\lambda}$ , the service rate  $\bar{\mu}$  and the repair rate  $\bar{\beta}$  are fuzzy numbers. Then, we represent these fuzzy numbers as

$$\bar{\lambda} = \{(x, \varphi_{\bar{\lambda}}(x)) / x \in S(\bar{\lambda})\} \quad (3.2.1)$$

$$\bar{\mu} = \{(y, \varphi_{\bar{\mu}}(y)) / y \in S(\bar{\mu})\} \quad (3.2.2)$$

$$\bar{\beta} = \{(w, \varphi_{\bar{\beta}}(w)) / w \in S(\bar{\beta})\} \quad (3.2.3)$$

$$\lambda_{\alpha} = \{x \in X / \varphi_{\bar{\lambda}}(x) \geq \alpha\}$$

$$\mu_{\alpha} = \{y \in Y / \varphi_{\bar{\mu}}(y) \geq \alpha\}$$

$$\beta_{\alpha} = \{w \in W / \varphi_{\bar{\beta}}(w) \geq \alpha\}$$

Here,  $\varphi_a(b)$  and  $S(a)$  denote the membership function and support of a where  $a = \bar{\lambda}, \bar{\mu}, \bar{\beta}$  are the fuzzy numbers and  $b = x, y, w$  are the crisp values corresponding to group arrival rate, service rate and repair rate, respectively.

Let  $P(x, y, w)$  and  $\tilde{P}(\bar{\lambda}, \bar{\mu}, \bar{\beta})$  denote the system performance measures of interest in the crisp and fuzzy environment, respectively. As  $\bar{\lambda}, \bar{\mu}, \bar{\beta}$  are fuzzy

numbers,  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{\beta})$  will also be fuzzy. Using Zadeh's extension principle, the membership function of the performance measure  $\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{\beta})$  is defined as

$$\mu_{\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{\beta})}(z) = \sup_{x \in X, y \in Y, w \in W} \min\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y), \mu_{\tilde{\beta}}(w) / z = P(x, y, w)\} \quad (3.2.4)$$

The membership function for  $L_q$  is given by

$$\mu_{\tilde{P}(\tilde{\lambda}, \tilde{\mu}, \tilde{\beta})}(z) = \sup_{x \in X, y \in Y, w \in W} \min\{\mu_{\tilde{\lambda}}(x), \mu_{\tilde{\mu}}(y), \mu_{\tilde{\beta}}(w) / z = L_q\}$$

### The $\Gamma$ – cut approach based on extension principle

The mathematical programming approach for deriving the desired membership function is developed on  $\Gamma$  -cut concept. The definition for the  $\Gamma$  -cuts of  $\tilde{\lambda}$ ,  $\tilde{\mu}$  and  $\tilde{\beta}$  as crisp intervals are as follows

$$\lambda_\alpha = \{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\}$$

$$\mu_\alpha = \{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\}$$

$$\beta_\alpha = \{w \in W / \mu_{\tilde{\beta}}(w) \geq \alpha\}$$

It is worthwhile to note that  $\lambda_\alpha$ ,  $\mu_\alpha$ , and  $\beta_\alpha$  are crisp sets rather than fuzzy sets; these crisp sets can be expressed in the following forms

$$\lambda_\alpha = [x_r^L, x_r^U] = [\min_{x \in X}\{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\}, \max_{x \in X}\{x \in X / \mu_{\tilde{\lambda}}(x) \geq \alpha\}]$$

$$\mu_\alpha = [y_r^L, y_r^U] = [\min_{y \in Y}\{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\}, \max_{y \in Y}\{y \in Y / \mu_{\tilde{\mu}}(y) \geq \alpha\}]$$

$$\beta_\alpha = [w_r^L, w_r^U] = [\min_{w \in W}\{w \in W / \mu_{\tilde{\beta}}(w) \geq \alpha\}, \max_{w \in W}\{w \in W / \mu_{\tilde{\beta}}(w) \geq \alpha\}]$$

The intervals defined above provide information that where the group arrival rate, service rate and repair rate lie at possibility  $\alpha$ . By using the concept of  $\alpha$ -cuts, the imbedded fuzzy Markov chain in the model under consideration, can be decomposed into a family of ordinary Markov chains which possesses different transition probability matrices parameterized by  $\alpha$ . The three fuzzy parameters arrival rate, service rate and repair rate can also be expressed by different levels of confidence intervals. As a result of this, the fuzzy batch arrival retrial queue with server

breakdown can be reduced to a family of crisp batch arrival retrial queue with server breakdown with different  $\alpha$ - level sets  $\{\lambda_\alpha/0 < \alpha \leq 1\}$ ,  $\{\mu_\alpha/0 < \alpha \leq 1\}$  and  $\{\beta_\alpha/0 < \alpha \leq 1\}$ . These three sets represent sets of movable boundaries, forming nested structures for expressing the relationship between ordinary sets and fuzzy sets. By the convexity of a fuzzy number, the bounds of these intervals can be obtained as

$$x_r^L = \text{Min} \left\{ \frac{-1}{\tilde{y}}(r) \right\}, \quad x_r^U = \text{Max} \left\{ \frac{-1}{\tilde{y}}(r) \right\}$$

$$y_r^L = \text{Min} \left\{ \frac{-1}{\tilde{z}}(r) \right\}, \quad y_r^U = \text{Max} \left\{ \frac{-1}{\tilde{z}}(r) \right\}$$

$$w_r^L = \text{Min} \left\{ \frac{-1}{\tilde{s}}(r) \right\}, \quad w_r^U = \text{Max} \left\{ \frac{-1}{\tilde{s}}(r) \right\}$$

Now as defined in equation (3.2.4), the membership function of  $\tilde{P}(\tilde{y}, \tilde{z}, \tilde{s})$  is also parameterized by  $\alpha$ . Consequently its  $\alpha$ -cuts can be used to construct its membership function.

### Construction of membership function

The lower bound  $L_{q_r}^L$  and the upper bound  $L_{q_r}^U$  of the  $\alpha$ -cuts of  $\bar{L}_q$  for this model are obtained by solving the following problem.

$$L_{q_r}^L = \min \left\{ \frac{\frac{m_1^2 x^2}{2} \left[ \frac{2 \left( 1 + \frac{s}{w} \right)^2}{y^2} + \frac{s^2}{yw^2} \right] + \frac{m_1^2 x}{y} \left[ \left( 1 + \frac{s}{w} \right) \left( \frac{x}{x+t} \right) \right] + \left[ m_2 \left( \frac{x}{x+t} \right) \right]}{\left[ 1 - m_1 \left( \frac{x}{x+t} \right) - \frac{xm_1}{y} \left( 1 + \frac{s}{w} \right) \right]} \right\}$$

such that

$$x_r^L \leq x \leq x_r^U, \quad y_r^L \leq y \leq y_r^U \quad \text{and} \quad w_r^L \leq w \leq w_r^U$$

$$L_{qr}^U = \max \left\{ \left[ \frac{m_1^2 x^2}{2} \left[ \frac{2 \left( 1 + \frac{s}{w} \right)^2}{y^2} + \frac{s^2}{yw^2} \right] + \frac{m_1^2 x}{y} \left[ \left( 1 + \frac{s}{w} \right) \frac{x}{x+t} \right] + \left[ m_2 \frac{\left( \frac{x}{x+t} \right)}{2} \right] \right] \right\}$$

$$\left[ 1 - m_1 \left( \frac{x}{x+t} \right) - \frac{xm_1}{y} \left( 1 + \frac{s}{w} \right) \right]$$

such that

$$x_r^L \leq x \leq x_r^U, y_r^L \leq y \leq y_r^U \text{ and } w_r^L \leq w \leq w_r^U$$

This pair of mathematical programs falls into the category of parametric NLPs which facilitates the systematic study of how the optimal solutions change when  $x_r^L, x_r^U, y_r^L, y_r^U, w_r^L, w_r^U$  vary over the interval  $\alpha \in (0,1)$ .

If both the lower bound  $L_{qr}^L$  and the upper bound  $L_{qr}^U$  of the  $\alpha$ -cuts of  $\bar{L}_q$  are invertible with respect to  $\alpha$ , then a LSF and a RSF can be obtained as  $L(z) = (L_{qr}^L)^{-1}$  and  $R(z) = (L_{qr}^U)^{-1}$ . Further, the membership function  $\varphi_{\bar{L}_q}(z)$  is constructed as

$$\varphi_{\bar{L}_q}(z) = \begin{cases} L(z), & z_1 \leq z \leq z_2 \\ 1, & z_2 \leq z \leq z_3 \\ R(z), & z_3 \leq z \leq z_4 \end{cases}$$

### §3. Numerical Results

#### Case(1):Crisp Queue

Assume that the distributions of service time, retrial time and repair time are exponential with rate  $\mu, \eta$  and  $\beta$  respectively, the combined effect of  $\mu$  and  $\lambda$  on the performance measures are displayed as surface diagram in figures (1)-(8) for the parameters  $(\lambda, \alpha, \eta) = (0.1, 2, 6)$ .

The figures reveals the following

- $L_q$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .
- $L_s$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .
- $I_0$  decreases for decreasing values of  $\lambda$  and increases for increasing  $\mu$ .
- $I$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .
- $W$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .
- $F$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .
- $\Lambda$  decreases for increasing values of  $\lambda$  and increases for increasing  $\mu$ .
- $\mathfrak{F}$  increases for increasing values of  $\lambda$  and decreases for increasing  $\mu$ .

### Case(2):Fuzzy Queue

For numerical analysis assume that arrival rate, retrial rate, service rate and repair rate are trapezoidal fuzzy number such that  $\bar{\lambda} = [1 \ 2 \ 3 \ 4]$ ,  $\bar{r} = [18 \ 19 \ 20 \ 21]$ ,  $\bar{s} = [26 \ 27 \ 28 \ 29]$ ,  $\chi = 2$ ,  $m_1 = 1.5$ ,  $m_2 = 1$ .

Then we get,

$$[x_r^L, x_r^U] = [1+\alpha, 4-\alpha]$$

$$[r_r^L, r_r^U] = [18+\alpha, 21-\alpha]$$

$$[s_r^L, s_r^U] = [26+\alpha, 29-\alpha]$$

Using the MATLAB software, the values of  $(L_q)_r^L$  and  $(L_q)_r^U$  in equations

$$(L_q)_r^L = \min_{\Omega} L_q(x, r, s) \text{ and}$$

$$(L_q)_r^U = \max_{\Omega} L_q(x, r, s) \text{ are calculated as}$$

$$(L_q)_r^L = \frac{1}{2} \frac{[487431 + 535863r + 45040r^2 - 3384r^3 + 9r^4 + r^5]}{[(3024 - 941r - 4r^2 + r^3)(-21+r)(-29+r)]} \quad (3.3.1)$$

and

$$(\mathbf{L}_q)_r^u = \left(\frac{1}{2}\right) \frac{[-2409984 + 716112r - 15340r^2 - 3186r^3 - 24r^4 + r^5]}{[(-192 - 938r - 5r^2 + r^3)(18+r)(26+r)]} \quad (3.3.2)$$

Table 3.1 presents fuzzy mean orbit size at different  $\alpha$  level. At extreme end for possibility level  $\alpha=1$ , range of the mean orbit length is approximately [0.4571, 1.4718], indicating that it is definitely possible that the expected number of customers in the system falls between 0.4571 and 1.4718. At another extreme end for possibility level  $\alpha=0$ , the range of the mean orbit length is [0.1323, 13.4103]. This indicates that the orbit length will never exceed 13.4103 or fall below 0.1323. Figure depicts the rough shape of  $\{_{f(\bar{z}, \bar{s})}(\mathbf{z})$  constructed from different  $\alpha$  values. The rough shape turns out rather fine, and looks like a continuous function.

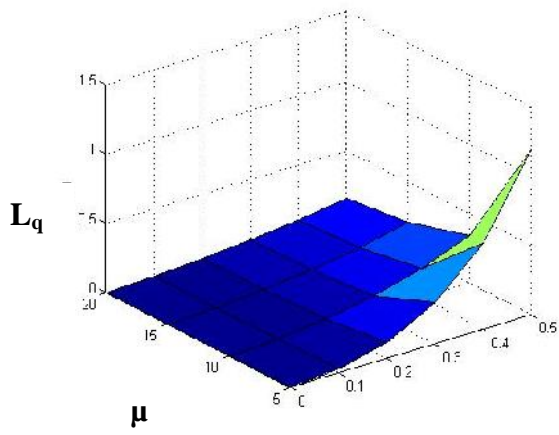


Fig.3.1.  $L_q$  versus  $(\mu, )$

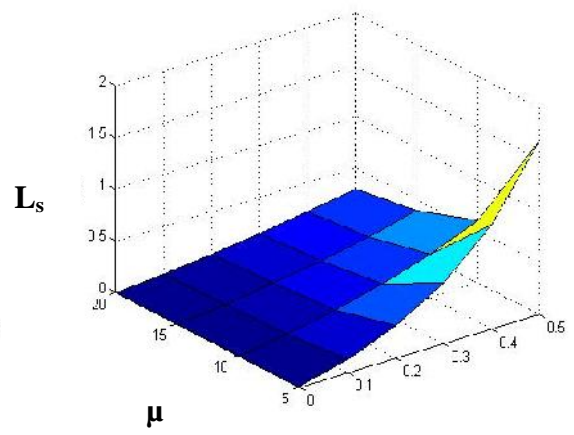


Fig.3.2.  $L_s$  versus  $(\mu, )$

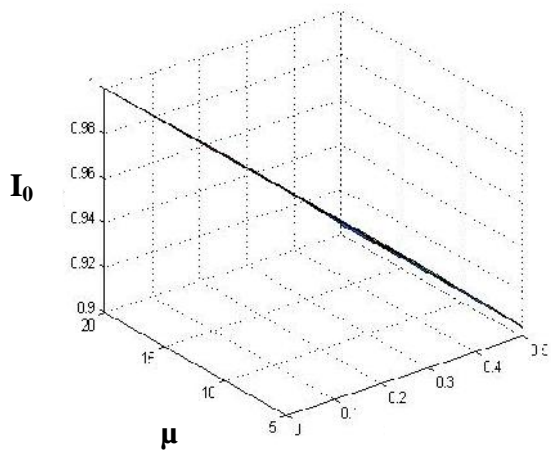


Fig.3.3.  $I_0$  versus  $(\mu, )$

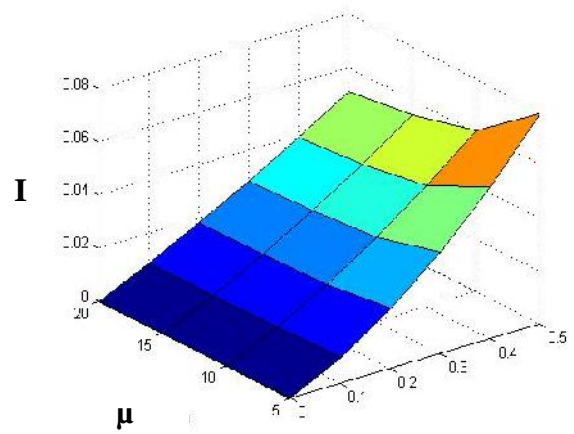
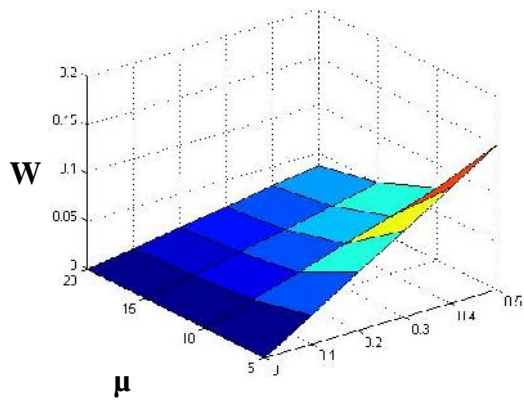
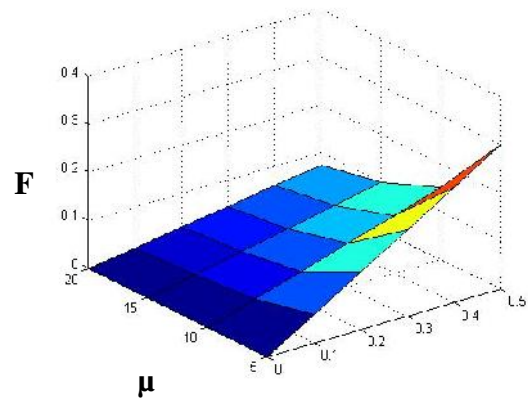


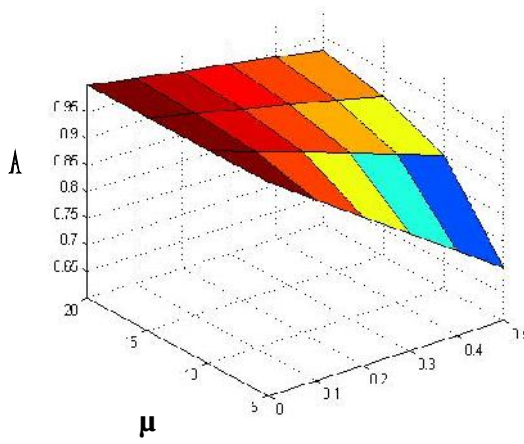
Fig.3.4.  $I$  versus  $(\mu, )$



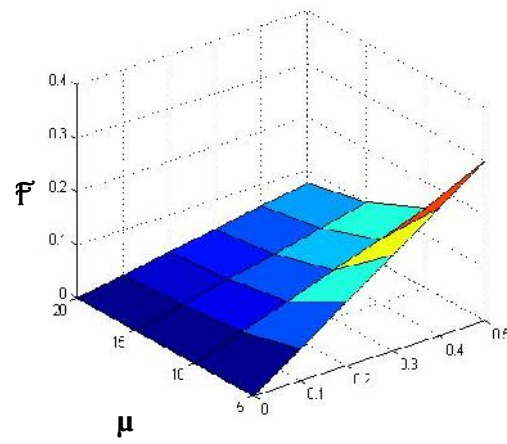
**Fig.3.5. W versus ( $\mu, \lambda$ )**



**Fig.3.6. F versus ( $\mu, \lambda$ )**



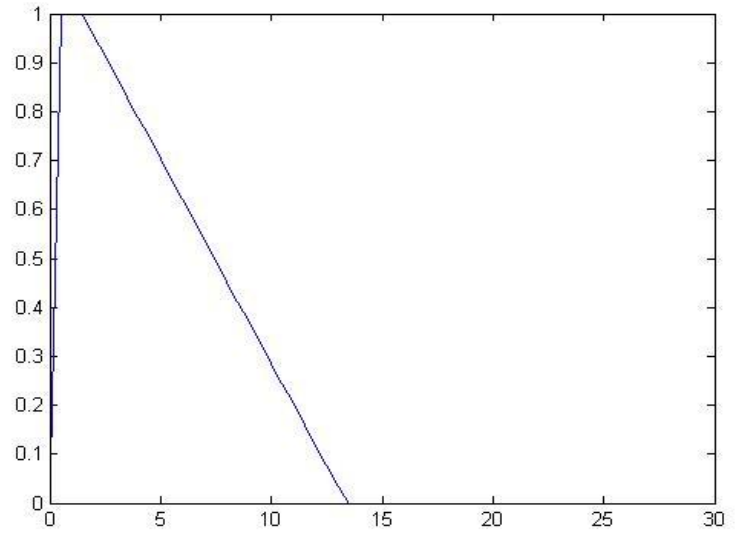
**Fig.3.7. Availability versus ( $\mu, \lambda$ )**



**Fig.3.8. Failure Frequency versus ( $\mu, \lambda$ )**

**Table. The  $\alpha$ - cuts of Mean Orbit Size**

$\alpha$	$(L_q)_r^L$	$(L_q)_r^U$
0.0	0.1323	13.4103
0.1	0.1530	8.6588
0.2	0.1755	6.2601
0.3	0.2002	4.8162
0.4	0.2272	3.8535
0.5	0.2569	3.1672
0.6	0.2894	2.6542
0.7	0.3253	2.2520
0.8	0.3649	1.9411
0.9	0.4086	1.6842
1.0	0.4571	1.4718



**Fig 3.9 Mean Orbit Size**

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## *Summary and Conclusion*

## Summary and Conclusion

In this dissertation, M/G/1 retrial queue with server breakdown and Bulk arrival retrial queue with server breakdown are investigated under crisp environment and fuzzy environment.

For all the above systems the mathematical models are framed using supplementary variable technique and are analyzed under both crisp and fuzzy environments. Membership functions for expected number in the system and expected number of customer in the queue are calculated.  $\alpha$ -cuts of the membership functions are found and their interval limits inverted to attain closed form expressions for the system characteristics. Some special cases are deduced. Numerical analyses are carried out to analyze the effect of parameters on the system performance.

It is assumed that arrival follows Poisson distribution, life time of the server follows exponential distribution and the service time, retrial time and repair time follow general distribution both the models. For crisp case the mean number of customers in the orbit, the mean number of customers in the system, availability of the server and steady state failure frequency of the server are derived using supplementary variable technique.

Under fuzzy environment the membership function of the system performance, expected number of customers in the queue is constructed by assuming arrival rate, service rate and retrial rate as fuzzy numbers.

Finally numerical results are provided to analyse the effect of parameters on the system performance measures.

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## *Bibliography*

## Bibliography

- 1) **Abramov, V. M.**, (2006), Analysis of multiserver retrial queueing system: A martingale approach and an algorithm of solution, *Ann. Oper. Res.* 141 .
- 2) **Aissani, A.**, (1988), On the M/G/1 Queueing System with Repeated Orders and Unreliable Server, *Journal of Technology*, 6,98-123.
- 3) **Aissani, A. and Artalejo, J.R.**, (1998), On the single server retrial queue subject to breakdowns, *Queueing Systems* 30(3-4): 309-321.
- 4) **Artalejo, J.R.**, (1994), New results in retrial queueing systems with breakdown of the servers, *Statistica Neerlandica* 48(1): 23-36.
- 5) **Artalejo, J. R. and Falin, G. I.**, (2002), Standard and retrial queueing systems, A comparative analysis, *Mat. Complut.* 15.
- 6) **Artalejo, J.R. and Gómez-Corral, A.**, (2008), *Retrial queueing systems: A computational approach*, Berlin:Springer-Verlag.
- 7) **Atencia, I. and Moreno, P.**, (2005), A single-server retrial queue with general retrial times and Bernoulli Schedule, *Applied Mathematics and Computation* 162(2): 855-880.
- 8) **Bocharov, P. P., Phong, N. H. and Atencia, I.**, (2001), Retrial queueing systems with several input flows, *Investig. Oper.* 22.
- 9) **Barak, S. and Fallahnezhad, M.S.**, (2012), Cost Analysis of Fuzzy Queueing Systems, 25-36.
- 10) **Chen, S.P.**, (2004), Parametric Nonlinear Programming for Analyzing Fuzzy Queues with Finite Capacity *Eur. J. Oper. Res.*, 157, 429-438.
- 11) **Chen, S.P.**, (2005), Parametric Nonlinear Programming Approach to Fuzzy Queues with Bulk Service, *Eur. J. Oper. Res.*, 163, 434-444.

- 12) **Choi, B.D. and Park, K.K.**, (1990), The M/G/1 Retrial Queue for Bernoulli Schedule, Queueing Systems, 7, 219-228.
- 13) **Choudhury, G. and Kandarpa Deka**, (2008), An M/G/1 Retrial Queue with Two Phases of Service Subject to the Server Breakdowns and Repair, Performance Evaluation,  
Vol. 65, pp. 714724.
- 14) **Djellab, N.V.**, (2002), On the M/G/1 retrial queue subjected to breakdowns, RAIRO Operations Research 36(4): 299–310.
- 15) **Dong-Yuh Yang and Po-Kai Chang**, (2013), A Parametric Programming Solution to the F-Policy Queue with Fuzzy Parameters, 37-41.
- 16) **Falin, G. I.**, (1990), A survey on retrial queues, Queueing Syst. 7.
- 17) **Falin, G.I. and Templeton, J.G.C.**, (1997), Retrial queues. London: Chapman & Hall.
- 18) **Falin, G.I.**, (2008), The M/M/1 retrial queue with retrials due to failures, Queueing Systems 58: 155–160.
- 19) **Jau-ChuanKe, Hsin- I Huang and Chuen- Horng Lin**, (2007), On Retrial Queueing Model with Fuzzy Parameters, 374, 272-280.
- 20) **Kao, C., Chang-Chung Li and Shih-Pin Chen**, (1999), Parametric Programming to the Analysis of Fuzzy Queues, Fuzzy Sets and Systems, 107, 93-100.
- 21) **Kosten, L.**, (1947), On the influence of Repeated Calls in the Theory of probabilities of Blocking, De Ingenieur, Vol. 59, pp.1-25.
- 22) **Kulkarni, V.G.**, (1983), On Queueing Systems with Retrials, Journal of Applied Probability, 20, 380-389.
- 23) **Kulkarni, V.G. and Choi, B.D.**, (1990), Retrial queues with server subject to breakdowns and repairs, Queueing Systems,7, 191-208.
- 24) **Li, R.S and Lee, E.S.**, (1989), Analysis of Fuzzy Queues, Comput. Math. Appl., 17, 1143-1147.

- 25) **Li, H. and Zhao, Y.Q.**, (2005), A retrial queue with a constant retrial rate, server break downs and impatient customers, *Stochastic Models* 21(2–3): 531–550.
- 26) **Medhi, J.**, (1994), *Stochastic Processes*(Second Edition), Wiley Eastern Ltd.
- 27) **Negi, D.S and Lee, E.S.**, (1992). *Analysis and Simulation of Fuzzy Queues*, *Fuzzy Sets and Systems*, 46, 321-330.
- 28) **Prade, H.W.**, (1980), An Outline of Fuzzy (or) Possibilistic Models for Queueing Systems, in P.P. Wong, S.K. Chang (Eds), *Fuzzy Sets* Plenum Press.
- 29) **Saaty, T.L.**, (1962), *Elements of Queueing Theory*, McGraw Hill.
- 30) **Stanford, R.E.**, (1982), The set of Limiting Distributions for a Markov Chain with Fuzzy Transition Probabilities. *Fuzzy Sets and Systems*, 7, 71-78.
- 31) **Stephen Vincent, G and Bhuvanewari, S.**, (2011), On Fuzzy Retrial Queue with Priority Subscribers, 4.
- 32) **Upadhyaya, S.**, (2013), Bernoulli Vacation Policy for a Bulk Retrial Queue with Fuzzy Parameters, 3,1-14.
- 33) **Wang, J., Cao, J. and Li, Q.**, (2001), Reliability analysis of the retrial queue with server breakdowns and repairs, *Queueing Systems* 38(4): 363–380.
- 34) **Wang, K. H., Kao, H. T. and Chen, G.**, (2004), Reliability analysis of the retrial queue with server breakdowns, and repairs *Quality Technology and Quantitative Management*,1(2),325-339.
- 35) **Yager, R.R.**, (1980), On a General Class of Fuzzy Connectives. *Fuzzy Sets and Systems*, 4(6), 235-242.
- 36) **Zadeh, L.A.**, (1978), Fuzzy Sets as a Basis for a Theory of Possibility, *Fuzzy Sets and Systems*, 1, 3-28.