CHAPTER – 4
INTUITIONISTIC (T, S)-FUZZY CI-ALGEBRAS

SECTION 4.1
INTUITIONISTIC (T, S)-FUZZY SUBALGEBRAS OF CI-ALGEBRAS

Definition : 4.1.1

An **Intuitionistic Fuzzy Set (IFS)** A in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \gamma_A(x)) / x \in X\}$ where the function $\mu_A : X \to [0, 1]$ and $\gamma_A : X \to [0, 1]$ denote the degree of membership and the degree of non-membership, respectively, where $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$, for all $x \in X$.

Note

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) / x \in X\}$ in X can be identified with an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, the symbol $A = \{(\mu_A, \gamma_A) = (x, \mu_A, \gamma_A)\}$ can be used for the IFS
$A = \{(x, \mu_A(x), \gamma_A(x)) / x \in X\}$.

Definition : 4.1.2

Let $A = \{x, \mu_A(x), \gamma_A(x) / x \in X\}$ and $B = \{x, \mu_B(x), \gamma_B(x) / x \in X\}$ be any two IFS in X. Then

(i) $A \subset B$ iff $(\mu_A(x) \leq \mu_B(x)$ and $\gamma_A(x) \geq \gamma_B(x))$, $\forall x \in X$.

(ii) $A = B$ iff $(\mu_A(x) = \mu_B(x)$ and $\gamma_A(x) = \gamma_B(x))$, $\forall x \in X$.

(iii) $\overline{A} = \{(x, \gamma_A(x), \mu_A(x)) / x \in X\}$.

(iv) $A \cap B = \{(x, \min(\mu_A(x), \mu_B(x)), \max(\gamma_A(x), \gamma_B(x))) / x \in X\}$.

(v) $A \cup B = \{(x, \max(\mu_A(x), \mu_B(x)), \min(\gamma_A(x), \gamma_B(x))) / x \in X\}$.

(vi) Let $\{A_i\}_{i \in \Lambda}$ be a family of IFS in X. Then

$$\bigcap_{i \in \Lambda} A_i = \{x, \bigwedge_{i \in \Lambda} \mu_{Ai}(x), \bigvee_{i \in \Lambda} \gamma_{Ai}(x) / x \in X\},$$

$$\bigcup_{i \in \Lambda} A_i = \{x, \bigvee_{i \in \Lambda} \mu_{Ai}(x), \bigwedge_{i \in \Lambda} \gamma_{Ai}(x) / x \in X\}.$$

Definition : 4.1.3

A **t-norm** is a function T : [0, 1] x [0, 1] $\to$ [0, 1] that satisfies

(T 1) T(x, 1) = x

(T 2) T(x, y) = T(y, x)

(T 3) T(x, T(y, z)) = T(T(x, y), z)

(T 4) T(x, y) $\leq$ T(x, z) whenever y $\leq$ z, $\forall$ x, y, z $\in$ [0, 1].

Note

Every t-norm T has the following useful property :

(i) T(α, β) $\leq$ min (α, β), for all α, β $\in$ [0, 1].

Definition : 4.1.4

s-norm S, we mean a function S : [0, 1] x [0, 1] $\to$ [0, 1] satisfying the following conditions :

(S 1) S(x, 0) = x

(S 2) S(x, y) $\leq$ S(x, z) if y $\leq$ z

(S 3) S(x, y) = S(y, x)

(S 4) S(x, S(y, z)) = S(S(x, y), z), $\forall$ x, y, z $\in$ [0, 1].

Note

Every s-norm S has the following useful property :

(i) max (α, β) $\leq$ S(α, β) for all α, β $\in$ [0, 1].

Note

For a t-norm (or s-norm) P on [0, 1] denote by Δ_p the set of elements $\alpha \in$ [0, 1] such that P(α, α) = α. i.e., $\Delta_p = \{\alpha \in [0, 1] / P(\alpha, \alpha) = \alpha\}$.

Definition : 4.1.5

Fuzzy set μ is called a **fuzzy subalgebra of a CI-algebra X with respect to a t-norm** T (a T-fuzzy subalgebra of X) if $\mu(x * y) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$.

Definition : 4.1.6

Fuzzy set μ is called a **fuzzy subalgebra of a CI-algebra X with respect to an s-norm** S (an S-fuzzy subalgebra of X) if $\mu(x * y) \leq S(\mu(x), \mu(y))$ for all $x, y \in X$.

Definition : 4.1.7

Let P be a t-norm (or s-norm). A fuzzy set μ in a CI-algebra X is said to satisfy **the imaginable property** with respect to P if $\mathrm{Im}(\mu) \subseteq \Delta_p$.

Definition : 4.1.8

Let $A = \{\mu_A, \gamma_A\}$ be an IFS in a CI-algebra X. A is called an **intuitionistic (T, S)-fuzzy subalgebra of X** if

(IFSA 1) $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$

(IFSA 2) $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$, for all $x, y \in X$.

Example : 4.1.9

Let X = {1, a, b, c}. Define a binary operation '$*$' on X by the following table :

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

then (X ; $*$, 1) is a CI-algebra.

Let T : [0, 1] x [0, 1] → [0, 1] be a function defined by $T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ and S : [0, 1] x [0, 1] → [0, 1] be a function defined by $S(\alpha, \beta) = \min(\alpha + \beta, 1)$ for all $\alpha, \beta \in [0, 1]$. Then T is a t-norm and S is an s-norm.

Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ by $\mu_A(1) = \mu_A(b) = \mu_A(c) = 1$, $\mu_A(a) = 0$ and $\gamma_A(1) = \gamma_A(b) = \gamma_A(c) = 0$, $\gamma_A(a) = 1$.

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Theorem : 4.1.10

If $\{A_i\}_{i \in \Lambda}$ is a family of intuitionistic (T, S)-fuzzy subalgebras of a CI-algebra X. Then $\bigcap_{i \in \Lambda} A_i$ is an intuitionistic (T, S)-fuzzy subalgebras of X, where $\bigcap_{i \in \Lambda} A_i = (\wedge \mu_i, \vee \gamma_i)$.

Proof

If $\{A_i\}_{i \in \Lambda}$ is a family of intuitionistic (T, S)-fuzzy subalgebras of a CI-algebra X.

Let x, y ∈ X. Then

$$\wedge \mu_{Ai}(x * y) \geq \wedge(T(\mu_i(x), \mu_i(y))) = T(\wedge \mu_{Ai}(x), \wedge \mu_{Ai}(y))$$

and $\vee \gamma_{Ai}(x * y) \leq \vee(S(\gamma_{Ai}(x), \gamma_{Ai}(y))) = S(\vee \gamma_{Ai}(x), \vee \gamma_{Ai}(y))$

Hence $\bigcap_{i \in \Lambda} A_i = (\wedge \mu_{Ai}, \vee \gamma_{Ai})$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Proposition : 4.1.11

Any subalgebra of a CI-algebra X can be realized as both a μ_A level subalgebra and γ_A of some intuitionistic (T, S)-fuzzy subalgebra of X.

Proof

Let A be a subalgebra of a CI-algebra X and μ_A, γ_A be fuzzy sets in X defined by

$$\mu_A(x) = \begin{cases} \alpha & \text{if } x \in A; \\ 0 & \text{otherwise} \end{cases}$$

and $$\gamma_A(x) = \begin{cases} \beta & \text{if } x \in A; \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in X$ where α and β are fixed numbers in (0, 1) such that $\alpha + \beta < 1$.

If $x, y \in A$ then $x * y \in A$.

Hence $\mu_A(x) = \mu_A(y) = \mu_A(x * y) = \alpha$ and

$\gamma_A(x) = \gamma_A(y) = \gamma_A(x * y) = \beta$.

If atleast one of x or y does not belong to A, then atleast one of $\mu_A(x)$ or $\mu_A(y)$ is equal to 0 and atleast one of $\gamma_A(x)$ or $\gamma_A(y)$ is equal to 1.

Therefore min $(\mu_A(x), \mu_A(y)) = 0$. It follows that $\mu_A(x * y) \geq 0 = \min(\mu_A(x), \mu_A(y))$ and $\gamma_A(x * y) \leq 1 = \max(\gamma_A(x), \gamma_A(y))$.

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic (min, max)-fuzzy subalgebra of X.

Obviously, $U(\mu_A, \alpha) = A = L(\gamma_A, \beta)$.

Theorem : 4.1.12

If A is a subalgebra of a CI-algebra X, then $\overline{A} = (\chi_A, \chi_A^C)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Proof

Let A be a subalgebra of a CI-algebra X. If $x, y \in A$, then $x * y \in A$.

Hence $\chi_A(x * y) = 1 \geq T(\chi_A(x), \chi_A(y))$.

Also, $0 = 1 - \chi_A(x * y) = \chi_A^C(x * y)$

$\leq S(\chi_A^C(x), \chi_A^C(y))$

If $x \in A$ and $y \notin A$, (or $x \notin A$ and $y \in A$), then $\chi_A(x) = 1$ (or) $\chi_A(y) = 0$.

Thus $\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 0) = 0$

And $S(\chi_A^C(x), \chi_A^C(y)) = S(1 - \chi_A(x), 1 - \chi_A(y))$

$= S(0, 1) = 1 \geq \chi_A^C(x * y)$

Hence $\overline{A} = (\chi_A, \chi_A^C)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Theorem : 4.1.13

Let A be a nonempty subset of a CI-algebra X. If $\overline{A} = (\chi_A, \chi_A^C)$ satisfies (IFSA 1) or (IFSA 2), then A is a subalgebra of X.

Proof

Let A be a nonempty subset of a CI-algebra X.

Suppose that $A = (\chi_A, \chi_A^C)$ satisfy (IFSA 1) and $x, y \in A$.

Then $\chi_A(x * y) \geq T(\chi_A(x), \chi_A(y)) = T(1, 1) = 1$. So that $\chi_A(x * y) = 1$

i.e, $(x * y) \in X$.

Hence A is a subalgebra of X.

Suppose that $\overline{A} = (\chi_A, \chi_A^C)$ satisfy (IFSA 2).

If $x, y \in A$, then

$\chi_A^C(x * y) \leq S(\chi_A^C(x), \chi_A^C(y))$

$\leq S(1 - \chi_A(x), 1 - \chi_A(y)) = S(0, 0) = 0$

$\chi_A^C(x * y) = 1 - \chi_A(x * y) = 0$

i.e., $\chi_A(x * y) = 1$.

Theorem : 4.1.14

Let A be a fuzzy subalgebra with membership function μ_A in a CI-algebra X. Then $\overline{A}$ is an intuitionistic (T, S)-fuzzy subalgebra of X, where $\overline{A} = (\mu_A, \mu_A^C)$.

Proof

Let A be a fuzzy subalgebra with membership function μ_A in a CI-algebra X.

To show that $\overline{A} = (\mu_A, \mu_A^C)$ is an intuitionistic (T, S)-fuzzy subalgebra of X, it is sufficient to show that μ_A^C satisfies the condition (IFSA 2).

If x, y $\in$ X, then $\mu^C(x * y) = 1 - \mu_A(x * y)$

$$\leq 1 - T(\mu_A(x), \mu_A(y))$$
$$= S(1 - \mu_A(x), 1 - \mu_A(y))$$
$$= S(\mu^C(x), \mu^C(y))$$

Hence $\overline{A}$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Theorem : 4.1.15

An intuitionistic fuzzy set A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra of a CI-algebra X if and only if the fuzzy sets μ_A and γ_A^C are T-fuzzy subalgebra of X.

Proof

Let A = (μ_A, γ_A) be an intuitionistic (T, S)-fuzzy subalgebra of a CI-algebra X.

Then μ_A is a T-fuzzy subalgebra of X.

For all x, y $\in$ X, $\gamma_A^C(x * y) = 1 - \gamma_A(x * y)$

$$\geq 1 - S(\gamma_A(x), \gamma_A(y))$$
$$= T(1 - \gamma_A(x), 1 - \gamma_A(y))$$
$$= T(\gamma_A^C(x), \gamma_A^C(y)).$$

Then γ_A^C is a T-fuzzy subalgebra of X.

Conversely, assume that μ_A and γ_A^C are T-fuzzy subalgebras of X.

To prove IFS A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra of X, it is enough to prove that $\gamma_A(x * y) \le S(\gamma_A(x), \gamma_A(y))$, for all x, y $\in$ X.

Since γ_A^C is a T-fuzzy subalgebra of X, then

$$\begin{aligned} \gamma_A^C(x * y) &= 1 - \gamma_A(x * y) \\ &\ge 1 - T(\gamma_A^C(x), \gamma_A^C(y)) \\ &= T(1 - \gamma_A(x), 1 - \gamma_A(y)) \\ &= 1 - S(\gamma_A(x), \gamma_A(y)) \end{aligned}$$

Hence for all x, y $\in$ X,

$$\gamma_A(x * y) \le S(\gamma_A(x), \gamma_A(y)).$$

Definition : 4.1.16

An intuitionistic (T, S)-fuzzy subalgebra A = (μ_A, γ_A) is called an **intuitionistic imaginable (T, S)-fuzzy subalgebra** of a CI-algebra X if μ_A and γ_A satisfy the imaginable property with respect to T and S respectively.

Example : 4.1.17

Let X = {1, a, b, c} be a CI-algebra as in example (4.1.9). And A = (μ_A, γ_A) is an intuitionistic imaginable (T, S)-fuzzy subalgebra of X.

Example : 4.1.18

Let X = {1, a, b, c} be a CI-algebra as in example (2.2.16). Define an intuitionistic fuzzy set A = (μ_A, γ_A) by

$$\mu_A(x) = \begin{cases} 0.7 & \text{if } x \in \{1, c\} \\ 0.2 & \text{otherwise} \end{cases}$$

and $$\gamma_A(x) = \begin{cases} 0.2 & \text{if } x \in \{1, c\} \\ 0.7 & \text{otherwise} \end{cases}$$

Let T : [0,1] x [0, 1] → [0, 1] be a function defined by $T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$. And S: [0,1] x [0, 1] → [0, 1] be a function defined by $S(\alpha, \beta) = \min(\alpha + \beta, 1)$ for all $\alpha, \beta \in [0, 1]$.

Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X which is not imaginable.

Because $T(\mu_A(a), \mu_A(a)) = T(0.2, 0.2)$

$= \max(0.2 + 0.2 - 1, 0) = 0.7 \neq \mu_A(a) = 0.2$

Proposition : 4.1.19

If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X, then $\mu_A(x * 1) \geq \mu_A(x)$ and $\gamma_A(x * 1) \leq \gamma_A(x)$ for all $x \in X$.

Proof

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic imaginable (T, S)-fuzzy subalgebra of X.

For any $x \in X$, $\mu_A(x * 1) \geq T(\mu_A(1), \mu_A(x)) \geq T(\mu_A(x * x), \mu_A(x))$

$= T(T(\mu_A(x), \mu_A(x)), \mu_A(x)) = \mu_A(x)$.

And $\gamma_A(x * 1) \leq S(\gamma_A(1), \gamma_A(x)) \leq S(\gamma_A(x * x), \gamma_A(x))$

$= S(S(\gamma_A(x), \gamma_A(x)), \gamma_A(x)) = \gamma_A(x)$.

Proposition : 4.1.20

If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X, then $\mu_A(1) \geq \mu_A(x)$ and $\gamma_A(1) \leq \gamma_A(x)$, for all $x \in X$.

Proof

Let $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X.

For every $x \in X$, $\mu_A(1) = \mu_A(x * x) \geq T(\mu_A(x), \mu_A(x)) = \mu_A(x)$.

And $\gamma_A(1) = \gamma_A(x * x) \leq S(\gamma_A(x), \gamma_A(x)) = \gamma_A(x)$.

Proposition : 4.1.21

If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X, then the set $X_A = \{\mu_A(x) = \mu_A(1), \gamma_A(x) = \gamma_A(1) \,/\, x \in X\}$ is a subalgebra of X.

Proof

Let $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X.

If $x, y \in X_A$, then $\mu_A(x) = \mu_A(y) = \mu_A(1)$ and $\gamma_A(x) = \gamma_A(y) = \gamma_A(1)$.

Since $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X, it follows that

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) = T(\mu_A(1), \mu_A(1)) = \mu_A(1)$$
$$\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) = S(\gamma_A(1), \gamma_A(1)) = \gamma_A(1)$$

so that $\mu_A(x * y) = \mu_A(1)$ and $\gamma_A(x * y) = \gamma_A(1)$. Thus $x * y \in X_A$ and consequently X_A is a subalgebra of X.

Theorem : 4.1.22

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of a CI-algebra X and $\alpha \in [0, 1]$. Then

(i) if $\alpha = 1$, then the upper level subset $U(\mu_A ; \alpha)$ is either empty or a subalgebra of X.

(ii) if $\alpha = 0$, then the lower level subset $L(\gamma_A ; \alpha)$ is either empty or a subalgebra of X.

(iii) if $T = \min$, then the upper level subset $U(\mu_A ; \alpha)$ is either empty or a subalgebra of X.

(iv) if $S = \max$, then the lower level subset $L(\gamma_A ; \alpha)$ is either empty or a subalgebra of X.

Proof

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy subalgebra of a CI-algebra X and $\alpha \in [0, 1]$.

To Prove (i)

Suppose that $\alpha = 1$ and $x, y \in U(\mu_A ; \alpha)$ then $\mu_A(x) \geq \alpha = 1$ and $\mu_A(y) \geq \alpha = 1$. It follows that $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) \geq T(1, 1) = 1$. So that $x * y \in U(\mu_A ; \alpha)$. Hence $U(\mu_A ; \alpha)$ is a subalgebra of X when $\alpha = 1$.

To Prove (ii)

Suppose that $\alpha = 0$ and $x, y \in L(\gamma_A ; \alpha)$ then $\gamma_A(x) \leq \alpha = 0$ and $\gamma_A(y) \leq \alpha = 0$. It follows that $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) \leq S(0, 0) = 0$. So that $x * y \in L(\gamma_A ; \alpha)$. Hence $L(\gamma_A ; \alpha)$ is a subalgebra of X when $\alpha = 0$.

To Prove (iii)

Assume that $T = \min$ and $x, y \in U(\mu_A ; \alpha)$ then

$$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y)) = \min(\mu_A(x), \mu_A(y))$$
$$\geq \min(\alpha, \alpha) = \alpha, \text{ for all } \alpha \in [0, 1]$$

Hence $x * y \in U(\mu_A ; \alpha)$ and $U(\mu_A ; \alpha)$ is a subalgebra of X.

To Prove (iv)

Let $S = \max$ and $x, y \in L(\gamma_A ; \alpha)$. Then

$$\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y)) = \max(\gamma_A(x), \gamma_A(y))$$
$$\leq \max(\alpha, \alpha) = \alpha, \text{ for all } \alpha \in [0, 1]$$

Hence $x * y \in L(\gamma_A ; \alpha)$ and so $L(\gamma_A ; \alpha)$ is a subalgebra of X.

Theorem : 4.1.23

If $A_a^1 = \{\mu_A(x) \geq a \ / \ x \in X\}$ and $A_b^2 = \{1 - \gamma_A(x) \geq b \ / \ x \in X\}$ are subalgebras of a CI-algebra of X, then A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra of X.

Proof

Assume that A_a^1 and A_b^2 are subalgebras of X, for any a $\in$ [0, 1].

Let a = $T(\mu_A(x), \mu_A(y))$, for any x, y $\in$ X. Then $\mu_A(x), \mu_A(y) \geq a$, which implies that x, y $\in A_a^1$. But A_a^1 is a subalgebra of X. Therefore, $x * y \in A_a^1$.

Hence $\mu_A(x * y) \geq a = T(\mu_A(x), \mu_A(y))$. Also, if $b = T(1 - \gamma_A(x), 1 - \gamma_A(y))$, in a similar way $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$.

In general, the converse of the above theorem is not true in the following example.

Example : 4.1.24

Let X = {1, a, b, c} be a CI-algebra, as in example (4.1.9) then $A_{0.1}^1 = A_{0.5}^2$ = {1, b, c}, $A_{0.1}^1 = A_{0.5}^2$ are not subalgebras of X because b, c $\in A_{0.1}^1, A_{0.5}^2$ but $b * c = a \notin A_{0.1}^1, A_{0.5}^2$.

Theorem : 4.1.25

If A = (μ_A, γ_A) is an intuitionistic (min, max)-fuzzy subalgebra of a CI-algebra X, then both A_a^1, A_b^2 are subalgebras of X.

Proof

Let A be an intuitionistic (min, max)-fuzzy subalgebra of X.

Let x, y $\in A_a^1$, then $\mu_A(x), \mu_A(y) \geq a$.

By hypothesis, $\mu_A(x * y) \geq \min(\mu_A(x), \mu_A(y)) \geq \min(a, a) = a$.

Hence $x * y \in A_a^1$. Similarly, A_b^2 is a subalgebra of X.

Theorem : 4.1.26

If A is an intuitionistic (min, max)-fuzzy subalgebra of a CI-algebra X, then the lower cut set

$$A_\lambda(x) = \begin{cases} 1 & \text{if } \mu_A(x) \geq \lambda \\ \frac{1}{2} & \text{if } \mu_A(x) < \lambda \leq 1 - \gamma_A(x) \\ 0 & \text{if } \lambda \geq 1 - \gamma_A(x) \end{cases}$$

is a fuzzy subalgebra of X.

Proof

To prove, $A_\lambda(x)$ is a fuzzy subalgebra of X, it is enough to prove that, $A_\lambda(x * y) \geq \min (A_\lambda(x), A_\lambda(y))$.

For this, consider the following cases.

Case (i)

If $A_\lambda(x) = A_\lambda(y) = 1$, then $\mu_A(x) \geq \lambda$ and $\mu_A(y) \geq \lambda$. Now $\mu_A(x * y) \geq \min (\mu_A(x), \mu_A(y)) \geq \lambda$. Therefore, $A_\lambda(x * y) = 1 \geq \min (A_\lambda(x), A_\lambda(y))$.

Case (ii)

If $A_\lambda(x) = 1$ and $A_\lambda(y) = \frac{1}{2}$, then $\mu_A(x) \geq \lambda$ and $\mu_A(y) < \lambda \leq 1 - \gamma_A(x)$.

Now $1 - \gamma_A(x * y) \geq \min (1 - \gamma_A(x), 1 - \gamma_A(y))$

By hypothesis, $1 - \gamma_A(x) \geq \mu_A(x)$, then

$1 - \gamma_A(x * y) \geq \min (1 - \gamma_A(x), 1 - \gamma_A(y)) \geq \min (\lambda, \lambda) = \lambda$.

Then $A_\lambda(x * y) = \frac{1}{2} \geq \min (A_\lambda(x), A_\lambda(y))$.

Case (iii)

If $A_\lambda(x) = A_\lambda(y) = \frac{1}{2}$, then $1 - \gamma_A(x) \geq \lambda$ and $1 - \gamma_A(y) \geq \lambda$.

Similarly, we can show that $A_\lambda(x * y) \geq \min (A_\lambda(x), A_\lambda(y))$.

Therefore A_λ is a fuzzy subalgebra of X.

Theorem : 4.1.27

If A is an intuitionistic (T_1, S_1)-fuzzy subalgebra of a CI-algebra X, $T_1 \geq T_2$ and $S_1 \leq S_2$, then A is an intuitionistic (T_2, S_2)-fuzzy subalgebra of X.

Proof

Since A is an intuitionistic (T_1, S_1)-fuzzy subalgebra of a CI-algebra X, then $\mu_A(x * y) \geq T_1(\mu_A(x), \mu_A(y)) \geq T_2(\mu_A(x), \mu_A(y))$. Thus $\mu_A(x * y) \geq T_2(\mu_A(x), \mu_A(y))$. Similarly $\gamma_A(x * y) \leq S_2(\gamma_A(x), \gamma_A(y))$. Hence A is an intuitionistic (T_2, S_2)-fuzzy subalgebra of X.

In general, the converse of the above theorem is not true can be seen in the following example.

Example : 4.1.28

Let X = {1, a, b, c} be a CI-algebra as in example (4.1.9). An intuitionistic (T, S)-fuzzy subalgebra defined is not intuitionistic (min, max)-fuzzy subalgebra because $\mu_A(b * c) = \mu_A(a) = 0 \ngeq \min (\mu_A(b), \mu_A(c)) = \min (1, 1) = 1$.

Theorem : 4.1.29

Let f be an endomorphism of a CI-algebra X. If $A = (\mu_A, \gamma_A)$ is an intuitionistic imaginable (T, S)-fuzzy subalgebra of a CI-algebra X, then $B = (\mu_A^f, \gamma_A^f)$ is an intuitionistic (T, S)-fuzzy subalgebra of X, where $\mu_A^f(x) = \mu_A(f(x))$ and $\gamma_A^f(x) = \gamma_A(f(x))$.

Proof

Let f be an endomorphism of a CI-algebra X.

For any x, y $\in$ X, $\mu_A^f(x * y) = \mu_A(f(x * y)) = \mu_A(f(x) * f(y))$

$$\geq T(\mu_A(f(x)), \mu_A(f(y))) = T(\mu_A^f(x), \mu_A^f(y)).$$

Similarly, for any x, y $\in$ X

$\gamma_A^f(x * y) = \gamma_A(f(x * y)) = \gamma_A(f(x) * f(y))$

$$\leq S(\gamma_A(f(x)), \gamma_A(f(y))) = S(\gamma_A^f(x), \gamma_A^f(y)).$$

Theorem : 4.1.30

Let f : X $\rightarrow$ Y be an epimorphism of CI-algebra. If A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy set in Y. If B = (μ_A^f, γ_A^f) is an intuitionistic (T, S)-fuzzy subalgebra of X, then A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra in Y.

Proof

Let f : X $\rightarrow$ Y be an epimorphism of CI-algebra.

For any $y_1, y_2 \in$ Y, there exists $x_1, x_2 \in$ X such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Then $\mu_A(y_1 * y_2) = \mu_A(f(x_1) * f(x_2))$

$$= \mu_A(f(x_1 * x_2)) = \mu_A^f(x_1 * x_2)$$

$$\geq T(\mu_A^f(x_1), \mu_A^f(x_2))$$

$$= T(\mu_A(f(x_1)), \mu_A(f(x_2)))$$

$$= T(\mu_A(y_1), \mu_A(y_2)).$$

Similarly, $\gamma_A(y_1 * y_2) = \gamma_A(f(x_1) * f(x_2))$

$$= \gamma_A(f(x_1 * x_2)) = \gamma_A^f(x_1 * x_2)$$

$$\leq S(\gamma_A^f(x_1), \gamma_A^f(x_2))$$

$$= S(\gamma_A(f(x_1)), \gamma_A(f(x_2)))$$

$$= S(\gamma_A(y_1), \gamma_A(y_2)).$$

Theorem : 4.1.31

Let A = (μ_A, γ_A) be an IFS in a CI-algebra X such that the non empty sets U(μ_A ; α) and L(γ_A ; α) are subalgebras of X, for all $\alpha \in [0, 1]$. Then A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy subalgebra of X.

Proof

Suppose that there exist $x_0, y_0 \in X$ such that

$\mu_A(x_0 * y_0) < T(\mu_A(x_0), \mu_A(y_0))$.

Taking $\alpha_0 = \frac{1}{2}(\mu_A (x_0 * y_0) + T(\mu_A(x_0), \mu_A(y_0)))$, then

$\min (\mu_A(x_0), \mu_A(y_0)) \geq T(\mu_A(x_0), \mu_A(y_0)) \geq \alpha_0 > \mu_A(x_0 * y_0)$.

It follows that $x_0, y_0 \in U(\mu_A ; \alpha_0)$ and $x_0 * y_0 \notin U(\mu_A ; \alpha_0)$. This is a contradiction and hence μ_A sastisfies the inequality

$\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$ for all $x, y \in X$ (1)

Similarly, suppose that there exist $x_0, y_0 \in X$ such that

$\gamma_A(x_0 * y_0) > S(\gamma_A(x_0), \gamma_A(y_0))$.

Taking $\beta_0 = \frac{1}{2} (\gamma_A(x_0 * y_0) + S(\gamma_A(x_0), \gamma_A(y_0)))$ then

$\max (\gamma_A(x_0), \gamma_A(y_0)) \leq S(\gamma_A(x_0), \gamma_A(y_0))$

$\leq \beta_0 < \gamma_A(x_0 * y_0)$

It follows that $x_0, y_0 \in L(\gamma_A ; \beta_0)$ and $x_0 * y_0 \notin L(\gamma_A ; \beta_0)$. Hence γ_A satisfies the inequality $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$ for all $x, y \in X$ (2)

By (1) and (2), A = (μ_A, γ_A) is an intuitionistic fuzzy subalgebra of X.

SECTION 4.2
INTUITIONISTIC (T, S)-FUZZY (CLOSED) FILTERS OF CI-ALGEBRAS

Definition : 4.2.1

Let A = (μ_A, γ_A) be an IFS in X. Then A is called an **intuitionistic (T, S)-fuzzy closed filter** of CI-algebra X if it satisfies the following conditions :

(IFCF 1) $\mu_A(x * 1) \geq \mu_A(x)$ and $\gamma_A(x * 1) \leq \gamma_A(x)$

(IFCF 2) $\mu_A(y) \geq T(\mu_A(x), \mu_A(x * y))$ and

$\gamma_A(y) \leq S(\gamma_A(x), \gamma_A(x * y))$, for all x, y $\in$ X.

Definition : 4.2.2

An intuitionistic (T, S)-fuzzy closed filter A = (μ_A, γ_A) is called an **intuitionistic imaginable (T, S)-fuzzy closed filter of X** if μ_A and γ_A satisfy the imaginable property with respect to T and S respectively.

Example : 4.2.3

Let X = {1, a, b, c} be a CI-algebra as in example (4.1.9) and A = (μ_A, γ_A) is an intuitionistic (T, S)-fuzzy closed filter of X.

Proposition : 4.2.4

Every intuitionistic imaginable (T, S)-fuzzy subalgebra satisfying (IFCF 2) is an intuitionistic imaginable (T, S)-fuzzy closed filter.

Proof : Obvious.

Theorem : 4.2.5

Every intuitionistic (T, S)-fuzzy closed filter is an intuitionistic (T, S)-fuzzy subalgebra.

Proof

Let $A = (\mu_A, \gamma_A)$ be an intuitionistic (T, S)-fuzzy closed filter of X and $x, y \in X$.

$$
\begin{aligned}
\text{Then } \mu_A(x * y) &\geq T(\mu_A(y), \mu_A(y * (x * y))) \\
&= T(\mu_A(y), \mu_A(x * (y * y))) \\
&= T(\mu_A(y), \mu_A(x * 1)) \\
&\geq T(\mu_A(y), \mu_A(x))
\end{aligned}
$$

$$
\begin{aligned}
\text{And } \gamma_A(x * y) &\leq S(\gamma_A(y), \gamma_A(y * (x * y))) \\
&= S(\gamma_A(y), \gamma_A(x * (y * y))) \\
&= S(\gamma_A(y), \gamma_A(x * 1)) \\
&\leq S(\gamma_A(y), \gamma_A(x))
\end{aligned}
$$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X.

Note

The converse of the theorem (4.2.5) may not be true which can be shown in the following example.

Example : 4.2.6

Let X = {1, a, b, c, d}. Define a binary operation "∗" on X by the following table.

∗	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	c	c
b	1	1	1	c	c
c	c	d	1	1	a
d	c	c	c	1	1

Then (X ; $*$, 1) is a CI-algebra. Let T : [0, 1] x [0, 1] $\rightarrow$ [0, 1] be a function defined by $T(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$ for all $\alpha, \beta \in [0, 1]$ and S : [0, 1] x [0, 1] $\rightarrow$ [0, 1] be a function defined by $S(\alpha, \beta) = \min(\alpha + \beta, 1)$ for all $\alpha, \beta \in [0, 1]$. Then T is a t-norm and S is a s-norm.

Define an IFS $A = (\mu_A, \gamma_A)$ by $\mu_A(1) = \mu_A(d) = 0.7$, $\mu_A(a) = \mu_A(b) = \mu_c = 0.07$, and $\gamma_A(a) = \gamma_A(b) = \gamma_A(c) = 0.7$, $\gamma_A(1) = \gamma_A(d) = 0.07$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy subalgebra of X, but it is not an intuitionistic (T, S)-fuzzy closed filter because $\mu_A(d * 1) = \mu_A(c) = 0.07 \not\geq \mu_A(d) = 0.7$

Theorem : 4.2.7

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (min, max)-fuzzy closed filter of X. If $x \leq y$, then $\mu_A(y) \geq \mu_A(x)$ and $\gamma_A(y) \leq \gamma_A(x)$, for any $x, y \in X$.

Proof

A is an intuitionistic (min, max)-fuzzy filter of X, so if $x \leq y$ then $x * y = 1$.

Hence $\mu_A(y) \geq \min(\mu_A(x), \mu_A(x * y))$

$= \min(\mu_A(x), \mu_A(1)) = \mu_A(x)$

Similarly, $\gamma_A(y) \leq \max(\gamma_A(x), \gamma_A(x * y))$

$= \max(\gamma_A(x), \gamma_A(1)) = \gamma_A(y)$

Theorem : 4.2.8

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (min, max)-fuzzy closed filter of X. If $x \leq z * y$, then $\mu_A(y) \geq \min(\mu_A(x), \mu_A(z))$ and $\gamma_A(y) \leq \max(\gamma_A(x), \gamma_A(z))$, for any $x, y, z \in X$.

Proof

From the assumption that,

$\mu_A(y) \geq \min(\mu_A(x), \mu_A(x * y))$. Put $y = x * y$

Then $\mu_A(x * y) \geq \min(\mu_A(z), \mu_A(z * (x * y)))$

Therefore $\mu_A(y) \geq \min(\mu_A(x), \mu_A(x * y))$

$\geq \min(\mu_A(x), \min(\mu_A(z), \mu_A(z * (x * y))))$

$= \min(\mu_A(x), \min(\mu_A(z), \mu_A(1)))$

$= \min(\mu_A(x), \mu_A(z))$.

Similarly, $\gamma_A(y) \leq (\max \gamma_A(x), \gamma_A(x * y))$ put $y = x * y$

Then $\gamma_A(x * y) \leq \max(\gamma_A(z), \gamma_A(z * (x * y)))$

Therefore $\gamma_A(y) \leq \max(\gamma_A(x), \gamma_A(x * y))$

$\leq \max(\gamma_A(x), \max(\gamma_A(z), \gamma_A(z * (x * y))))$

$= \max(\gamma_A(x), (\gamma_A(z))$

Theorem : 4.2.9

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (T, S)-fuzzy subalgebra of X. If $A = (\mu_A, \gamma_A)$ satisfies the imaginable property and inequalities $\mu_A(x * y) \leq \mu_A(y * x)$ and $\gamma_A(x * y) \geq \gamma_A(y * x)$ for all $x, y \in X$, then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X.

Proof

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (T, S)-fuzzy subalgebra of X which satisfies the inequalities.

$\mu_A(x * y) \leq \mu_A(y * x)$ and $\gamma_A(x * y) \geq \gamma_A(y * x)$ for all $x, y \in X$.

From the proposition (4.1.19), that $\mu_A(x * 1) \geq \mu_A(x)$ and $\gamma_A(x * 1) \leq \gamma_A(x)$ for all $x, y \in X$.

Then $\mu_A(y) = \mu_A(1 * y) \geq \mu_A(y * 1) = \mu_A(y * (x * x))$

$= \mu_A(x * (y * x)) \geq T(\mu_A(x), \mu_A(y * x))$

$\geq T(\mu_A(x), \mu_A(x * y))$

And $\gamma_A(y) = \gamma_A(1 * y) \leq \gamma_A(y * 1) = \gamma_A(y * (x * x))$

$= \gamma_A(x * (y * x)) \leq S(\gamma_A(x), \gamma_A(y * x))$

$\leq S(\gamma_A(x), \gamma_A(x * y))$

Hence $A = (\mu_A, \gamma_A)$ is an intuitionisitc (T, S)-fuzzy closed filter of X.

Theorem : 4.2.10

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (T, S)-fuzzy subalgebra of X such that the non-empty sets $U(\mu_A ; \alpha)$ and $L(\gamma_A ; \alpha)$ are closed filters of X and $\alpha \in [0, 1]$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic (T, S)-fuzzy closed filter of X.

Proof

Let $A = (\mu_A, \gamma_A)$ be an intuitionisitc (T, S)-fuzzy subalgebra of X such that the non-empty sets $U(\mu_A ; \alpha)$ and $L(\gamma_A ; \alpha)$ are closed filters of X and $\alpha \in [0, 1]$. Suppose that there exist $x_0, y_0 \in X$ such that $\mu_A(y_0) < T(\mu_A(x_0), \mu_A(x_0 * y_0))$.

Taking $\alpha_0 = \frac{1}{2}(\mu_A(x_0) + T(\mu_A(x_0 * y_0), \mu_A(x_0)))$, then

$\min(\mu_A(x_0 * y_0), \mu_A(x_0)) \geq T(\mu_A(x_0 * y_0), \mu_A(x_0)) \geq \alpha_0 > \mu_A(y_0)$.

It follows that $x_0 * y_0, x_0 \in U(\mu_A ; \alpha_0)$ and $y_0 \notin U(\mu_A ; \alpha_0)$.

This is a contradiction and hence μ_A satisfies the inequality $\mu_A(x * y) \geq T(\mu_A(x), \mu_A(y))$, for all $x, y \in X$. Similarly, suppose that there exist $x_0, y_0 \in X$ such that $\gamma_A(y_0) > S(\gamma_A(x_0), \gamma_A(x_0 * y_0))$.

Taking $\beta_0 = \frac{1}{2}(\gamma_A(x_0) + S(\gamma_A(x_0 * y_0), \gamma_A(x_0)))$, then

$\max(\gamma_A(x_0), \gamma_A(x_0 * y_0)) \leq S(\gamma_A(x_0), \gamma_A(x_0 * y_0)) \leq \beta_0 \subset \gamma_A(y_0)$

It follows that $x_0, x_0 * y_0 \in L(\gamma_A ; \beta_0)$ and $y_0 \notin L(\gamma_A ; \beta_0)$.

This is a contradiction and hence γ_A satisfies the inequality $\gamma_A(x * y) \leq S(\gamma_A(x), \gamma_A(y))$, for all $x, y \in X$.

Now assume that there exists $x_0 \in X$ such that $\mu_A(x * 1) < \mu_A(x_0)$.

Taking $\alpha_0 = \frac{1}{2}(\mu_A(x_0 * 1) + \mu_A(x_0))$ then $\mu(x_0 * 1) \leq \alpha_0$ and $\mu_A(x_0) \geq \alpha_0$. It follows that $x_0 \in U(\mu_A ; \alpha_0)$ but $x_0 * 1 \notin U(\mu_A ; \alpha_0)$. This is a contradiction.

Hence $\mu_A(x * 1) \geq \mu_A(x)$, for all $x \in X$. Similarly, we get that $\gamma_A(x * 1) \leq \gamma_A(x)$ for all $x \in X$.