

CHAPTER I

CHAPTER I

BASIC DEFINITIONS AND PROPERTIES OF FUZZY MATRICES, BIMATRICES, FUZZY BIMATRICES, INTERVAL MATRICES, INTERVAL BIMATRICES AND NEUTROSOPHIC MATRICES

Definition 1.1

A **fuzzy matrix** is a matrix with elements having values in the closed unit interval $[0,1]$.

Example:

$$\text{Let } A = \begin{bmatrix} 0 & .1 & 0 \\ .1 & .2 & .1 \\ .3 & .2 & .1 \end{bmatrix}$$

Then A is called the 3x3 fuzzy matrix.

Definition 1.2

Let $R = [r_{ij}]$ and $S = [s_{ij}]$ be $n \times n$ fuzzy matrices. Then

- (i) $R \vee S = [r_{ij} \vee s_{ij}]$, where $X \vee Y = \max\{X, Y\}$
- (ii) $R \wedge S = [r_{ij} \wedge s_{ij}]$, where $X \wedge Y = \min\{X, Y\}$
- (iii) $R \ominus S = [r_{ij} \ominus s_{ij}]$, where $X \ominus Y = \begin{cases} X & \text{for } X > Y \\ 0 & \text{for } X < Y \end{cases}$
- (iv) $R \times S = [r_{i1} \wedge s_{1j}] \vee [(r_{i2} \wedge s_{2j}) \vee \dots \vee (r_{in} \wedge s_{nj})]$ $R \times S$ can be written as RS . If $R \times R = R$, then R is called idempotent.
- (v) Denote $R^k = [r_{ij}^k]$, $k = 2, 3, \dots$. Then

$$r_{ij}^k = \bigvee_{j_1=1}^n \bigvee_{j_2=1}^n \dots \bigvee_{j_{k-1}=1}^n (r_{ij_1} \wedge r_{j_1 j_2} \wedge \dots \wedge r_{j_{k-1} j})$$

- (vi) $R^{k+1} = R^k \times R$, $k = 1, 2, 3, \dots$

- (vii) $R^{\circ} = I$, where I is the unit $n \times n$ matrix (all diagonal entries 1 and all other 0)
- (viii) R^T (or) $R' = [r_{ij}]$ (the transpose of R)
- (ix) $\Delta R = R' \odot R'$, $\nabla R = R \wedge R'$
- (x) $R \leq S$ iff $(r_{ij} \leq s_{ij})$ for all $i, j \in \{1, 2, \dots, n\}$.

Definition 1.3

A **bimatrix** A_B is defined as the union of two rectangular array of numbers A_1 and A_2 arranged into rows and columns. It is written as follows $A_B = A_1 \cup A_2$ where $A_1 \neq A_2$ with

$$A_1 = \begin{bmatrix} a_{11}^1 & a_{12}^1 & \cdots & a_{1n}^1 \\ a_{21}^1 & a_{22}^1 & \cdots & a_{2n}^1 \\ \cdots & & & \cdots \\ a_{m1}^1 & a_{m2}^1 & \cdots & a_{mn}^1 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 & \cdots & a_{1n}^2 \\ a_{21}^2 & a_{22}^2 & \cdots & a_{2n}^2 \\ \cdots & & & \cdots \\ a_{m1}^2 & a_{m2}^2 & \cdots & a_{mn}^2 \end{bmatrix}$$

‘ \cup ’ is just the notational convenience (symbol) only.

The above array is called a m by n bimatrix since each of A_i ($i = 1, 2$) has m rows and n columns. It is to be noted that a bimatrix has no numerical value associated with it. It is only a convenient way of representing a pair of arrays of numbers.

Example :

The following are bimatrices

$$A_B = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 2 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a 2×3 bimatrix.

Definition 1.4

Consider the bimatrices $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$.

Then we define the following

(1) $A_B = C_B$

Let $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$ be two bimatrices. We say A_B and C_B are equal and written as $A_B = C_B$ if and only if A_1 and C_1 are identical and A_2 and C_2 are identical i.e., $A_1 = C_1$ and $A_2 = C_2$.

(2) $A_B \neq C_B$

If $A_B = A_1 \cup A_2$ and $C_B = C_1 \cup C_2$, we say A_B is not equal to C_B we write as $A_B \neq C_B$ if and only if $A_1 \neq C_1$ or $A_2 \neq C_2$.

Example:

$$\text{Let } A_B = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{and } C_B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

Clearly $A_B \neq C_B$.

(3) λA_B

Given a bimatrix $A_B = A_1 \cup B_1$ and a scalar λ , the product of λ and A_B written λA_B is defined to be

$$\lambda A_B = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1n} \\ \cdots & & \cdots \\ \lambda a_{m1} & \cdots & \lambda a_{mn} \end{bmatrix} \cup \begin{bmatrix} \lambda b_{11} & \cdots & \lambda b_{1n} \\ \cdots & & \cdots \\ \lambda b_{m1} & \cdots & \lambda b_{mn} \end{bmatrix}$$

Example:

$$\text{Let } A_B = \begin{bmatrix} 2 & 0 & 1 \\ 3 & 3 & 1 \end{bmatrix} \cup \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and $\lambda=3$ then

$$3A_B = \begin{bmatrix} 6 & 0 & 3 \\ 9 & 9 & 3 \end{bmatrix} \cup \begin{bmatrix} 0 & 3 & 3 \\ 6 & 3 & 0 \end{bmatrix}$$

(4) $A_B + C_B$

Let $A_B = A_1 \cup B_1$ and $C_B = A_2 \cup B_2$ be any two $m \times n$ bimatrices. The sum D_B of the bimatrices A_B and C_B is defined as

$D_B = A_B + C_B = [A_1 \cup B_1] + [A_2 \cup B_2] = (A_1 + A_2) \cup (B_1 + B_2)$; where $A_1 + A_2$ and $B_1 + B_2$ are the usual addition of matrices i.e., if

$$A_B = (a_{ij}^1) \cup (b_{ij}^1)$$

and $C_B = (a_{ij}^2) \cup (b_{ij}^2)$

$$\text{then } A_B + C_B = D_B = (a_{ij}^1 + a_{ij}^2) \cup (b_{ij}^1 + b_{ij}^2) \quad (\forall ij).$$

Thus two bimatrices are added by adding the corresponding elements only when compatibility of usual matrix addition exists.

Example:

$$\text{Let } A_B = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\text{and } C_B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\text{then } D_B = A_B + C_B$$

$$= \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} \cup \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

(5) $A_B - C_B$

Subtraction is defined in terms of operations already considered for if

$$A_B = A_1 \cup A_2$$

and $C_B = C_1 \cup C_2$

$$\begin{aligned}
\text{then } A_B - C_B &= A_B + (-C_B) \\
&= (A_1 \cup A_2) + (-C_1 \cup -C_2) \\
&= (A_1 - C_1) \cup (A_2 - C_2) \\
&= [A_1 + (-C_1)] \cup [A_2 + (-C_2)].
\end{aligned}$$

(6) Bimatrix multiplication is not defined when the bimatrices are not square bimatrices. Secondly in case of mixed bimatrices, multiplication is defined when both A_1 and A_2 are square matrices or when $A_B = A_1^{m \times n} \cup A_2^{p \times s}$ and $C_B = B_1^{n \times u} \cup B_2^{s \times t}$.

Definition 1.5

Let $A = A_1 \cup A_2$ where A_1 and A_2 are two distinct fuzzy matrices with entries from the interval $[0,1]$. Then $A = A_1 \cup A_2$ is called the **fuzzy bimatrix**.

Example:

$$\text{Let } A_F = \begin{bmatrix} 0 & .1 & 0 \\ .1 & .2 & .1 \\ .3 & .2 & .1 \end{bmatrix} \cup \begin{bmatrix} .2 & .1 & .1 \\ .1 & 0 & .1 \\ .2 & .1 & .2 \end{bmatrix}$$

A_F is the 3×3 fuzzy bimatrix.

Definition 1.6

Given matrices $B = (b_{ij})$ and $C = (c_{ij})$ of order n such that $b_{ij} \leq c_{ij}$, $i, j = 1, 2, \dots, n$. Then the **interval matrix** $A_I = [B, C]$ is defined by

$A_I = [B, C] = \{A = (a_{ij}) \mid b_{ij} \leq a_{ij} \leq c_{ij}; i, j = 1, 2, \dots, n\}$. (interval vectors and matrices are vectors and matrices whose elements are interval numbers. The superscript I being used to indicate such a vector or matrix)

Example:

Let $A_I = [B, C]$ where

$$B = \begin{bmatrix} 2 & 7 \\ 0 & -5 \end{bmatrix} \text{ and } C = \begin{bmatrix} 12 & 20 \\ 15 & 0 \end{bmatrix}$$

All matrices $D = (d_{ij})$ with

$$2 \leq d_{11} \leq 12,$$

$$7 \leq d_{12} \leq 20,$$

$$0 \leq d_{21} \leq 15 \text{ and}$$

$$-5 \leq d_{22} \leq 0$$

are in the interval matrix $A_I = [B, C]$

Definition 1.7

Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where $[A_1, B_1]$ and $[A_2, B_2]$ interval matrices defined on the same interval $[a, b]$ or on different intervals $[a_1, b_1]$ and $[a_2, b_2]$. We call $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ to be the **interval bimatrix** defined on the same interval $[a, b]$ or on different intervals $[a_1, b_1] \cup [a_2, b_2]$; with no mathematical meaning attached to the symbol ‘ \cup ’; we say $[A, B]$ is the interval bimatrix defined on the bi-interval $[a_1, b_1] \cup [a_2, b_2]$ or on the bi-interval $[a, b] \cup [a, b]$.

Definition 1.8

Let $[A, B]$ be an interval matrix with entries from $[0, 1]$ or $[a, b]$ with $0 \leq a < b \leq 1$. $[A, B]$ is called the **fuzzy interval matrix** defined on the interval $[a, b]$ or $[0, 1]$.

Example :

Let $[A, B]$ be a interval of 2×2 fuzzy square matrices where

$$A = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0.5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0.8 \\ 0.9 & 1 \end{bmatrix}$$

$[A, B]$ is the fuzzy interval 2×2 square matrix

Definition 1.9

Let $[A_1, B_1]$ and $[A_2, B_2]$ be any two interval fuzzy matrices defined on the fuzzy intervals say $[a_1, b_1]$ and $[a_2, b_2]$ respectively. Let $[A, B] = [A_1, B_1] \cup [A_2, B_2]$ where ‘ \cup ’ is just a symbol used for notational convenience, then $[A, B]$ contains all fuzzy bimatrices of the form $M_1 \cup M_2$ where M_1 is the fuzzy matrix from the fuzzy interval matrix $[A_1, B_1]$ defined on the interval $[a_1, b_1]$ and M_2 is the fuzzy matrix from the fuzzy interval matrix $[A_2, B_2]$ defined on the interval $[a_2, b_2]$, $M = M_1 \cup M_2$ is a fuzzy bimatrix defined on the fuzzy bi-interval $[a_1, b_1] \cup [a_2, b_2]$ where M is called the **fuzzy bimatrix of the fuzzy interval bimatrix** $[A, B]$.

Definition 1.10

In the neutrosophic logic every logical variable x is described by an ordered triple $x = (T, I, F)$ where T is the degree of truth, F is the degree of false and I the level of indeterminacy.

(A). To maintain consistency with the classical and fuzzy logics and with probability there is the special case where

$$T + I + F = 1.$$

(B). But to refer to intuitionistic logic, which means incomplete information on a variable proposition or event one has

$$T + I + F < 1.$$

(C). Analogically referring to Paraconsistent logic, which means contradictory sources of information about a same logical variable, proposition or event one has

$$T + I + F > 1.$$

Example

From a pool of refugees, waiting in a political refugee camp in Turkey to get the American visa, $a\%$ have the chance to be accepted – where a varies in the

set A, r% to be rejected – where r varies in the set R, and p% to be in pending (not yet decided) – where p varies in P.

Say, for example, that the chance of someone Popescu in the pool to emigrate to USA is (between) 40-60% (considering different criteria of emigration one gets different percentages, we have to take care of all of them), the chance of being rejected is 20-25% or 30-35%, and the chance of being in 15 pending is 10% or 20% or 30%. Then the neutrosophic probability that Popescu emigrates to the Unites States is

NP (Popescu) = ((40-60) (20-25) \cup (30-35), {10,20,30}), closer to the life.

Definition 1.11

Let K be the field of reals. We call the field generated by $K \cup I$ to be the **neutrosophic field** for it involves the indeterminacy factor in it. We define $I^2 = I$, $I + I = 2I$ i.e., $I + \dots + I = nI$, and if $k \in K$ then $k.I = kI$, $0I = 0$. We denote the neutrosophic field by $K(I)$ which is generated by $K \cup I$ that is $K(I) = \langle K \cup I \rangle$.

Example

Let R be the field of reals. The neutrosophic field is generated by $\langle R \cup I \rangle$ i.e. $R(I)$ clearly $R \cup \langle R \cup I \rangle$.

Example

Let Q be the field of rationals. The neutrosophic field is generated by Q and I i.e. $Q \cup I$ denoted by $Q(I)$.

Definition 1.12

Let $M_{n \times m} = \{(a_{ij})/a_{ij} \in K(I)\}$, where $K(I)$, is a neutrosophic field. We call $M_{n \times m}$ to be the **neutrosophic matrix**.

Example

Let $Q(I) = \langle Q \cup I \rangle$ be the neutrosophic field.

$$M_{3 \times 2} = \begin{pmatrix} 0 & 1 \\ -2 & 4I \\ 1 & -I \end{pmatrix}$$

is the neutrosophic matrix, with entries from rationals and the indeterminacy I .