Analysis of Retrial Queueing Models with Fluctuating Modes of Service

By

Aarthi P

(20PMA001)

Supervisor

Dr. K.Kirupa

Thesis Submitted to

Avinashilingam Institute for Home Science and Higher Education for Women

Coimbatore-641043

In Partial Fullfilment of the Requirement for the Degree of

Master of Science in Mathematics

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Signature of the Head of the Department | **Signature of the supervisor**

DECLARATION

I declare that the dissertion entitled **"Analysis of Retrial Queueing Models with Fluctuating Modes of Service"** submitted by me for the degree of **Master of Science** is a record of research work carried out by me during the period from December 2021 to May 2022 under the guidance of **Dr. (Tmt) K. Kirupa,** Assistant Professor, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore and has not formed the basis for the award of any Degree, Diploma, Associates, Fellowship or other titles in the University or any other University or Institute of Higher Learning.

P. [signature]

Signature of the candidate

ACKNOWLEDGEMENT

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Every work on its backdrop has the blessing of **LORD ALMIGHTY**. Therefore I submit my reverential gratitude at the feet of Lord Almighty.

I take immense pleasure in thanking **Dr. S.P.THYAGARAJAN**, Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for providing all facilities necessary for the study.

My special thanks to **Dr. (Tmt.) BHARATHI HARISHANKAR**, Vice Chancellor, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the encouragement and for providing the opportunity to develop and establish the knowledge.

I would like to thank **Dr. (Tmt) S. KOWSALYA**, Registrar, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for administrative support and for providing and for providing adequate help required to carry out the work.

My sincere thanks to **Dr. (Tmt.) G.PADMAVATHI,** Dean, School of Physical Sciences and Computational Sciences, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her guidance, affectionate support and expert suggestions throughout the project.

The investigator would like to thank **Dr.(Tmt) N.BALAMANI (SS)** , Assistant Professor and Head (SS), Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her moral support.

I express my heart-felt thanks and sincere gratitude to my guide, **Dr. (Tmt.) K.KIRUPA** Assistant Professor, Department of Mathematics, Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for her guidance, lively discussion, patience and sacrifice for the successful completion of my thesis.

I would like to express my sincere thanks to all the **STAFF MEMBERS OF THE DEPARTMENT OF MATHEMATICS** who rendered their help whenever required.

I extend my thanks to the **LIBRARIAN ,** Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore, for the library facilities rendered.

I owe my special thanks to my **BELOVED PARENTS, LOVING BROTHER, FRIENDS** and **MY WELL WISHERS**, who helped me by providing full strength, support and encouragement to complete my project successfully.

CONTENT

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INTRODUCTION

1. INTRODUCTION

Queueing theory is a branch of applied probability theory utilizing the concepts from the field of stochastic processes. A queueing system can be described as customers arriving for service, waiting for service if it is not immediate and leaving the system after being served. There are many valuable applications of the theory, most of which have been documented in the literature of probability, operations research, management science and industrial engineering.

1.1 CHARACTERISTICS OF QUEUEING SYSTEM

The basic characteristic of a queueing system which provide an adequate description are arrival pattern, service pattern, queue discipline, system capacity and service channels.

Arrival Pattern

Arrival pattern describes the manner in which the units arrive and join the system. The source from which the units come may be finite or infinite. A unit may arrive either singly or in a group. The arrival pattern is often measured in terms of the average number of arrivals per unit time.

Another factor to be considered regarding the arrival pattern is the reaction of the customers in the queue. If the queue is too long, a customer may decide not to enter it upon arrival and i n this situation he is said to have balked. On the other hand, a customer may enter the queue, but after some time lose patience and decide to leave. In this case, he is said to have reneged. In the event that there are more than one queue, customers may Switch from one to another, that is jockey for position.

The arrival pattern of customers that does not change with time is called stationary arrival pattern otherwise, it is called non - stationary.

Service Pattern

Service pattern describe s the manner in which the service is rendered to the arrivals. Customers may be served either singly or in batches. The time required for serving a unit is called service time. The service pattern may be stationary or non - stationary with respect to time and state dependent or independent with respect to number of customers waiting for service.

Queue Discipline

Queue discipline refers to the manner in which customers are selected for service from the queue. The most common disciplines based on the arrivals of customers into the system are first come first served (FCFS) and last come first served (LCFS). Customers may also be served randomly irrespective of their arrivals to the system called service in random order (SIRO).

Another discipline is priority queue discipline. There are two types in priority discipline, preemptive priority and non-preemptive priority. In the preemptive case, the customer with high priority is allowed to enter service immediately suspending the service in progress to a customer with lower priority. In non-preemptive case the higher priority goes to the head of the queue but gets into service only after the completion of service in progress.

System Capacity

In some queueing processes there is a physical limitation to the amount of waiting room so that when the line reaches a certain length, no further customers are allowed to enter until space becomes available as a result of a service completion. These are referred as finite queueing systems. A queue with limited waiting room can be viewed as one with forced balking.

Service Channels

The number of servers in a queueing model may be finite or infinite. The number of servers may be arranged in series, parallel or a combination of both, depending upon the nature of the services required. In parallel channels, all the channels provide identical services so that several customers may be served simultaneously. In series channels, a customer must pass through successively in several ordered channels before service is completed.

1. 2 KENDALL'S NOTATION

A queueing system can be represented by the notation introduced by Kendall (1951) as follows: A/B/X/Y/Z, where A represents inter - arrival time distribution of the customers, B denotes the service time distribution, X is the number of parallel servers, Y represents the capacity of the system and Z denotes the queue discipline. If the queueing system has infinite capacity and the queue discipline is FIFO, then the system is denoted as A/B/C without mentioning X and Y.

For example, a single server queueing system with bulk arrival Poisson process, general service time distribution, infinite capacity and first in first out queue discipline is denoted as $M^x/G/1$.

1.3 MARKOVIAN AND NON - MARKOVIAN QUEUEING MODELS

Queueing models are classified as Markovian queueing models and non - Markovian queueing models. The techniques generally adopted to solve these types of queueing models are given below.

Markovian Queueing Models

Queueing models with exponential inter - arrival time and exponential service time are called Markovian queueing models. Some of the techniques used to solve Markovian queueing models are:

- Difference - differential equation method
- Neuts matrix - geometric algorithm
- Continued fraction method

Non - Markovian Queueing Models

The exponential assumption on queueing models, although very convenient, is not always realistic. There is a practical need for models that do not depend on strict Markov assumptions. Queueing models having the inter -arrival times or service times which are not exponentially distributed are known as non - Markovian queueing models.

The techniques generally used to study non - Markovian queueing models are:

- Embedded Markov chain technique
- Supplementary variable technique

1.4 RETRIAL QUEUEING SYSTEM

In classical queueing theory it is usually assumed that a customer who cannot get service immediately after arrival either joins the waiting line or departs the system forever. Sometimes impatient customers leave the queue but it is also assumed that they are leaving the system forever. Usually, such customers after some random period of time return to the system and try to get service again. The standard queueing models do not consider the phenomenon of retrials and therefore cannot be applied in solving a number of practically important problems. Retrial queues have been introduced to meet this inadequacy.

Retrial queueing systems are characterized by the feature that arriving customers who cannot receive service immediately may join a virtual queue called orbit to try their request after some random time. General structure of a retrial queueing system is presented in Fig. 1.1.

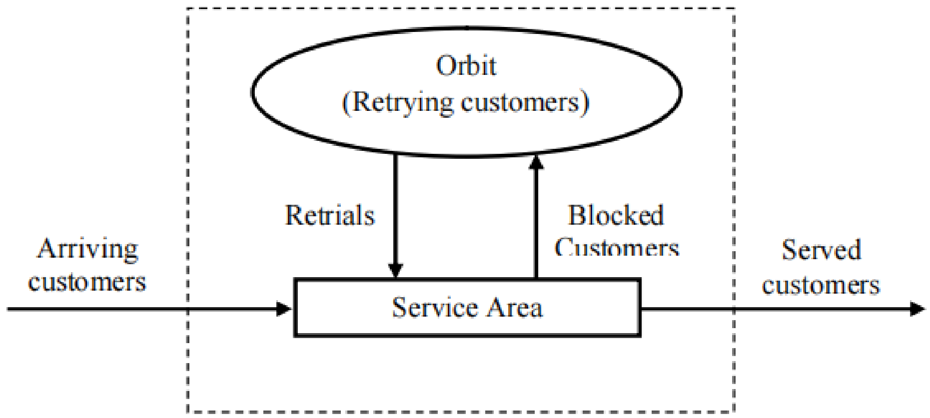

Fig. 1.1. General Structure of a Retrial Queueing System

1.5 REVIEW OF LITERATURE

The theory of queues was initiated by the Danish Mathematician Erlang, who in 1909 published “The theory of probabilities and telephone conversation”.

The classical M/G/1 queueing system received a good deal of attention in the literature since Kendall (1953). He was the first to study such a system through an imbedded Markov chain technique by using the concept of regeneration points.

Queueing systems with repeated attempts are found suitable for modeling the processes in telephone switching systems, digital cellular mobile networks, packet switching networks, local area networks, stock and flow etc. Review of **retrial** queueing literature can be found in the survey papers of Yang and Templeton (1987) and Falin (1990), the bibliographies of Artalejo (1999a, 1999b) and the books by Falin and Templeton (1997) and Artalejo and Gomez Corral (2008). The applications of retrial queues in science and engineering are given in Kulkarni and Liang (1997).

Retrial queueing system with two phases of service is a class of queueing system in which the server provides the first essential service to all arriving primary customers or retrial customers, whereas some of them receive second optional service. Several contributions have been done in the recent years. Atencia et al. (2006) discussed a two phase retrial queueing system with two types of customers under linear retrial policy by assuming essential service time to follow arbitrary distribution and the optional service time to follow exponential distribution.

Dimitriou and Langaris (2008) analysed a retrial queue with two phase service and server vacation and obtained transient and steady state system probabilities.

The phenomenon of **feedback** in the retrial queueing systems occurs in many practical situations. Takacs (1963) was the first to study feedback queueing models. For instance, in multiple access telecommunication systems, where messages turned out as errors are sent again can be modeled as retrial queue with feedback. Choi and Kulkarni. Discrete time retrial queue with feedback was considered by Atencia et al. (2009) and Chen et al. (2009).Chang et al . (2018) studied an unreliable single server retrial queue with geometric loss and feedback under threshold-based policy to derive the formulae for computing rate matrix and stationary probabilities. Single server bulk arrival retrial queue with M stages of heterogeneous service and feedback was presented by Sangeetha et al. (2020). Varalakshmi and Rajadurai (2021) examined the preemptive priority feedback retrial queueing system with disaster under operating breakdown and vacation to obtain the explicit expressions for the average orbit length, average system length and reliability indices.

A feedback retrial queueing system with starting failures and single vacation was studied by Mokaddis et al. (2007). The authors obtained analytical results for the system size distribution at random points and established the stochastic decomposition law. Arivudainambi and Godhandaraman (2012) suggested a two phase batch arrival retrial queueing system with feedback and K optional vacations with practice justification.

Boualem et al. (2012) proposed stochastic ordering techniques to establish various monotonicity results for an M/G/1 retrial queue with a Bernoulli feedback.

In recent years, retrial queues with **vacation** have drawn more attention. Excellent surveys on early works on vacation models in the queueing literature may be found in Doshi(1986), books by Takagi (1991) and Tian and Zhang (2006). Such type of queueing models occur in many real life situations where the server may be used for other secondary jobs. Allowing server to take vacations makes the queueing model more realistic and flexible in studying real world queueing situations. Li and Yang (1995) studied an M/G/1 retrial queue with a Bernoulli vacations. Zhou (2005) derived the ergodic condition and the probability generating function of the system size for M/G/1 retrial queue with Bernoulli vacation. Choudhury (2008) extended the same model by considering linear retrial policy. Boualem et al. (2009) derived stochastic inequalities for M/G/1 retrial queue with vacations and constant retrial policy. Wang (2012) extended the continuous time M/G/1 retrial queue with Bernoulli vacation to discrete time. Gao and Wang (2019) suggested a single server retrial queue with server vacation, two waiting buffers based on ATM networks and carried out an extensive analysis to give the mean queue lengths in the original buffer, retrial buffer and in the system. Ke and Wang (2021) studied the single server constant retrial queue with Bernoulli vacation and obtained the threshold equilibrium strategies of blocked customer in different information levels.

In the retrial setup, each service is preceded and followed by the server's idle time because of the ignorance of the status of the server and orbital customers by each other. Server's idle time is reduced by the introduction of **search of orbital customers** immediately after a service completion. Artalejo et al. (2002) considered a retrial queue in which immediately after a service completion the server searches for customers from the orbit or remains idle. Krishnamoorthy et al. (2005) analysed M/G/1 retrial queue with non persistent customers and orbital search using supplementary variable method and discussed the structure of the busy period and its analysis in terms of Laplace transform. Deepak et al. (2012) examined mean queue length of an $M^x/G/1$ retrial system with two types of search of customers from the orbit. Gao and Zhang (2019) developed an extensive discussion of the joining probabilities and optimal pricing problems for an M/M/1 retrial queue with orbital search, where arriving customers have the option to decide whether to enter the orbit on finding a busy server.

1.6 PROFILE OF PRESENT WORK

The main objective of the dissertation is to analyse the steady state behaviour of retrial queueing systems with fluctuating modes of service, feedback, single vacation and orbital search.

The concept of the thesis is presented in four chapters.

- Chapter one gives the preliminary results and review of literature.
- In Chapter two, retrial Queue with fluctuating modes of service and feedback is analysed
- In Chapter three, retrial queue with fluctuating modes of service, feedback and single vacation is considered.
- Chapter three is further extended by including the concept of orbital search and is presented in chapter four.

CHAPTER-2

2. Analysis of Single Server Retrial Queue with Fluctuating Modes of Service and Feedback

In this chapter, single server retrial queue with fluctuating modes of service and feedback for single arrival is analyzed. If the server is idle upon the arrival of a customer, then the customer receives any one of services immediately. Otherwise, he joins the orbit. After completion of service, the customer may join the orbit as a feedback customer or depart the system. Using supplementary variable technique expected number of customers in the orbit and expected number of customer in the system are derived. Stochastic decomposition property is verified.

2.1 Model Description

Consider a single server retrial queueing system where the customers arrive according to Poisson process with rate λ. The server provides two types of service – type 1and type 2. Customer opts type 1 service with probability p_1 or type 2 service with probability p_2 ($p_1+p_2=1$). If the customer finds the server free, then the customer receives any one of the services immediately. Otherwise he joins the retrial queue. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltje's transform $A^*(\theta)$ and conditional completion rate $\eta(x) = a(x) / [1- A(x)]$.

The service time follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace-Stieltje's transform $B_i^*(\theta)$, n^{th} factorial moments μ_{in}, i=1,2and conditional completion rate $\mu_i(x) = b_i(x) / [(1 - B_i(x)]$. After receiving service, the customer may again join the orbit as a feedback with probability δ or depart the system with its complementary probability $1 - \delta$ (= $\bar{\delta}$).

2.2 Steady State Distribution

In this section, by treating elapsed service time as supplementary variable, the steady state probability generating function of the orbit size distribution is derived.

Define the states of the server at time t as,

$$C(t) = \begin{cases} 0, & \text{if the server is idle} \\ 1, & \text{if the server is busy in type 1 service} \\ 2, & \text{if the server is busy in type 2 service} \end{cases}$$

For $t \geq 0$, we define the random variable $\xi(t)$ as follows:

(i) If $C(t) = 0$, then $\xi(t)$ represents the elapsed retrial time,

(ii) If $C(t) = 1$ or 2 then $\xi(t)$ represents the elapsed service time,

Then, the process $\{X(t) ; t \geq 0\} = \{C(t), N(t), \xi(t) ; t \geq 0\}$ is a Markov process, where N(t) denotes the number of customers in the orbit. For the process $\{X(t) ; t \geq 0\}$, we define the following probabilities.

$I_0(t) = P\{C(t) = 0, N(t) = 0\}$

$I_n(x, t)\,dx = P\{C(t) = 0, N(t) = n, x < \xi(t) \leq x + dx\}, n \geq 1$

$P_n^{(i)}(x, t)\,dx = P\{C(t) = i, N(t) = n, x < \xi(t) \leq x + dx\}, n \geq 0, i=1,2$

The system of equation that governs the model under supplementary variable technique are given below.

$$\frac{d}{dt} I_0(t) = -\lambda I_0(t) + \delta\left[\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx\right] \tag{2.1}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) I_n(x,t) = -(\lambda + \eta(x)) I_n(x,t), n \geq 1 \tag{2.2}$$

$$\left(\frac{d}{dx} + \frac{d}{dt}\right) P_n^{(i)}(x,t) = -(\lambda + \mu_i(x)) P_n^{(i)}(x,t) + (1 - \delta_{0n})\lambda P_{n-1}^{(i)}(x,t), n \geq 0; i = 1,2 \tag{2.3}$$

with boundary conditions

$$I_n(0,t)=\delta\left[\int_0^\infty P_{n-1}{}^{(1)}(x,t)\mu_1(x)dx+\int_0^\infty P_{n-1}{}^{(2)}(x,t)\mu_2(x)dx\right]+\bar{\delta}\left[\int_0^\infty P_n{}^{(1)}(x,t)\mu_1(x)dx+P_n{}^{(2)}(x,t)\mu_2(x)dx\right]$$

$$;n\geq 1$$

(2.4)

$$P_0{}^{(i)}(0,t)=p_i[\lambda I_0+\int_0^\infty I_1(x,t)\eta(x)dx],i=1,2 \quad (2.5)$$

$$P_n{}^{(i)}(0,t)=p_i\left[\int_0^\infty I_{n+1}(x,t)\eta(x)dx+\lambda\int_0^\infty I_n(x,t)dx\right],n\geq 1,i=1,2 \quad (2.6)$$

Taking limit as t $\rightarrow\infty$ on both sides of the equations, we get the following steady state equations

$$\lambda I_\circ \quad =\bar{\delta}\left[\int_0^\infty P_\circ^{(1)}(x)\mu_1(x)dx+\int_0^\infty P_\circ^{(2)}(x)\mu_2(x)dx\right]$$

(2.7)

$$\frac{d}{dx}I_n(x)=-(\lambda+\eta(x))I_n(x) \quad (2.8)$$

$$\frac{d}{dx}P_0{}^{(i)}(x)=-(\lambda+\mu_i(x))P_0{}^{(i)}(x),i=1,2 \quad (2.9)$$

$$\frac{d}{dx}P_n{}^{(i)}(x)=-(\lambda+\mu_i(x))P_n{}^{(i)}(x)+\lambda P_{n-1}(x),n\geq 1,i=1,2$$

(2.10)

with boundary conditions

$$I_n(0)=\delta\left[\int_0^\infty P_{n-1}{}^{(1)}(x)\mu_1(x)dx+\int_0^\infty P_{n-1}{}^{(2)}(x)\mu_2(x)dx\right]+\bar{\delta}\left[\int_0^\infty P_n{}^{(1)}(x)\mu_1(x)dx+\int_0^\infty P_n{}^{(2)}(x)\mu_2(x)dx\right],n\geq 1$$

(2.11)

$$P_0^{(i)}(0) = p_i\left[\lambda I_0 + \int_0^\infty I_n(x)\eta(x)dx\right], i = 1,2 \tag{2.12}$$

$$P_n^{(i)}(0) = p_i\left[\int_0^\infty I_{n+1}(x)\eta(x)dx + \lambda\int_0^\infty I_n(x)dx\right],\ n \geq 1 \tag{2.13}$$

Define the steady state probabilities

$I_0 \quad = \quad \lim_{t\to\infty} I_0(t)$;

$I_n(x) \quad = \quad \lim_{t\to\infty} I_n(x, t)$, $\quad x \geq 0,\ n \geq 1$;and

$P_n^{(i)}(x) \quad = \quad \lim_{t\to\infty} P_n^{(i)}(x, t)$, $x \geq 0,\ n \geq 0;i=1,2$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty}\int_0^{\infty} I_n(x)\,dx + \sum_{n=0}^{\infty}\int_0^{\infty}\sum_{i=1}^{2} P_n^{(i)}(x)dx = 1 \tag{2.14}$$

Define the probability generating functions for $|z| \leq 1$:

$I(x, z) = \sum_{n=1}^{\infty} I_n(x)\, z^n$; $P^{(i)}(x, z) = \sum_{n=0}^{\infty} P_n^{(i)}(x)\, z^n$; i=1,2

Multiplying equation (2.8) by z^n and summing over n, we get

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x)) I_n(x) \tag{2.15}$$

Solving the partial differential equation (2.15), we get

$$I(x, z) = \frac{C}{e^{\lambda x(1-A(x))}}$$

$$= \quad Ce^{-\lambda x}(1-A(x)) \tag{2.16}$$

Eliminating C by taking x = 0, we get

$$I(0,z)=C$$

$$I(x,z)=I(0,z)e^{-\lambda x}(1-A(x)) \tag{2.17}$$

The partial differential equation in (2.10), yield

$$P^{(i)}(x,z)=C_2 e^{-(\lambda-\lambda z)x}(1-B_i(x)) \tag{2.18}$$

Eliminating C_2 by taking x=0, we get

$$P^{(i)}(0,z)=C_2$$

$$P^{(i)}(x,z)=P^{(i)}(0,z)e^{-(\lambda-\lambda z)x}(1-B_i(x)) \tag{2.19}$$

Multiplying equations (2.11) to (2.13) by z^n and summing over n, we get

$$\begin{aligned} I(0,z) &= \delta z\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx+\int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right]-\lambda I_0 \\ &+\bar{\delta}\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx+\int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right] \end{aligned} \tag{2.20}$$

$$P^{(i)}(0,z)=\frac{p_i}{z}\left[\int_0^\infty I(x,z)\eta(x)dx\right]+\lambda p_i\left[\int_0^\infty I(x,z)dx+I_0\right], i=1,2 \tag{2.21}$$

Using equation (2.17) in equation (2.21), we have

$$P^{(i)}(0,z) = \frac{p_i}{z}\left[z\lambda I_0 + I(0,z)(A^*(\lambda) + z(1-A^*(\lambda))\right], i = 1,2 \tag{2.22}$$

Using equations (2.19) and (2.21) in equation (2.20) and simplifying, we get

$$I(0, z) = \frac{z\lambda I_0\left[(T(z)\right]-1}{D(z)} \tag{2.23}$$

Where

$$T(z) = B_1^*(\lambda-\lambda z)p_1(\delta z+\bar{\delta}) + B_2^*(\lambda-\lambda z)p_2(\delta z+\bar{\delta})$$

$$D(z) = (B_1^*(\lambda-\lambda z)p_1 + B_2^*(\lambda-\lambda z)p_2)(A^*(\lambda)+z(1-A^*(\lambda))(1+(\delta z+\bar{\delta}))-z$$

Using equation (2.23), the equation (2.22), yield

$$P^{(i)}(0,z) = \frac{p_i\lambda I_\circ\left[A^*(\lambda)(z-1)\right]}{z-(B_1^*(\lambda-\lambda z)p_1 + B_2^*(\lambda-\lambda z)p_2)(A^*(\lambda)+z(1-A^*(\lambda))(1+\delta(z-1))} \tag{2.24}$$

Substituting the expressions of I(0, z), $P^{(i)}(0,z)$ in equations (2.17) and (2.19) respectively, we get

$$I(x,z) = \frac{z\lambda I_0[(B_1^*(\lambda-\lambda z)p_1(\delta z+\bar{\delta}) + B_2^*(\lambda-\lambda z)p_2(\delta z+\bar{\delta})-1]e^{-\lambda x}(1-A(x))}{z-B_1^*(\lambda-\lambda z)p_1(A^*(\lambda)+z(1-A^*(\lambda))(\bar{\delta}+\delta z)-B_2^*(\lambda-\lambda z)p_2(A^*(\lambda)+z(1-A^*(\lambda))(\bar{\delta}+\delta z))} \tag{2.25}$$

$$P^{(i)}(x,z) = \frac{\lambda I_0 A^*(\lambda)(z-1)e^{-(\lambda-\lambda z)x}(1-B_i(x))}{D(z)}, i = 1.2 \tag{2.26}$$

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \int_0^\infty I(x, z)\, dx$$

$$= \frac{zI_0\left\lfloor 1-(B_1^*(\lambda-\lambda z)p_1(\delta z+\bar{\delta})+B_2^*(\lambda-\lambda z)p_2(\delta z+\bar{\delta}))(1-A^*(\lambda)\right\rfloor}{D(z)} \qquad (2.27)$$

The partial probability generating function of the orbit size when the server is busy is given by

$$P^{(i)}(z) = \int_0^\infty P^{(i)}(x,z)\; dx$$

$$= \frac{p_i I_0 A^*(\lambda)\left\lfloor(1-B_i^*(\lambda-\lambda z))\right\rfloor}{D(z)}, i=1,2 \qquad (2.28)$$

The partial probability generating function of the orbit size is,

$$P_q(z) = I_0 + I(z) + P^{(1)}(z) + P^{(2)}(z)$$

$$= N(z) / D(z) \qquad (2.29)$$

where

$$N(z) = I_0 A^*(\lambda)(1-z)\left\lfloor 1-\delta(p_1 B_1^*(\lambda-\lambda z)+p_2 B_2^*(\lambda-\lambda z))\right\rfloor$$

$$D(z) = (B_1^*(\lambda-\lambda z)p_1 + B_2^*(\lambda-\lambda z)p_2)(A^*(\lambda)+z(1-A^*(\lambda))(1+(\delta z+\bar{\delta}))-z$$

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + zP^{(1)}(z) + zP^{(2)}(z)$$

$$= N_1(z) / D(z) \qquad (2.30)$$

where

$$N_1(z) = I_0 A^*(\lambda)(p_1 B_1^*(\lambda-\lambda z)+p_2 B_2^*(\lambda-\lambda z))((1-z)(1-\delta))$$

2.3 Performance Measures

The probability that the server is idle during retrial time is given by

$$I = \lim_{z \to 1} I(z)$$

$$= \frac{I_0(1-(p_1(\delta z+\bar{\delta})+p_2(\delta z+\bar{\delta}))(1-A^*(\lambda))}{D'(1)} \qquad (2.31)$$

Where

$$D'(1) = A^*(\lambda) - \delta - \lambda(p_1\mu_{11} + p_2\mu_{12})$$

The probability that the server is busy in type i (i = 1, 2) service is given by

$$P^{(i)} = \lim_{z \to 1} P^{(i)}(z)$$

$$= \frac{p_i I_0 A^*(\lambda)\left[1-B_i^*(\lambda-\lambda z)\right]}{D'(1)} \qquad (2.32)$$

The normalizing equation (2.14) is equivalent to

$$I_0 + I + P^{(1)} + P^{(2)} = 1 \qquad (2.33)$$

Using equations equation (2.31) and (2.32) in (2.33), become

$$I_0 = \frac{A^*(\lambda) - \delta - \lambda(p_1\mu_{11} + p_2\mu_{21})}{A^*(\lambda)\left[1-\delta(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))\right]} \qquad (2.34)$$

The mean number of customers in the orbit is given by

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{D'(1)\,N''(1) - N'(1)\,D''(1)}{2\,D''(1)^2} \tag{2.35}$$

Where N(z) and D(z) are the Numerator and Denominator of $P_q(z)$.

$N'(1) = I_0 A^*(\lambda)(\delta - 1)$

$N''(1) = 2 I_0 A^*(\lambda)\delta(p_1\mu_{12} + p_2\mu_{22})$

$D'(1) = A^*(\lambda) - \delta - \lambda(p_1\mu_{11} + p_2\mu_{21})$

$D''(1) = (p_1\mu_{11} + p_2\mu_{21})(2\delta\lambda + 2(1 - A^*(\lambda))) + \lambda^2(p_1\mu_{12} + p_2\mu_{22}) + 2(1 - A^*(\lambda))\delta$

The mean number of customer in the system is given by

$$L_S = \lim_{z \to 1} \frac{d}{dz} P_s(z)$$

$$= \frac{N_1''(1)\,D'(1) - N_1'(1)\,D''(1)}{2\,D''(1)^2} \tag{2.36}$$

Where $N_1(z)$ denotes the Numerator of $P_s(z)$.

$N_1'(1) = I_0 A^*(\lambda)(\delta - 1)$

$N_1''(1) = 2 I_0 A^*(\lambda)(\delta - 1)\lambda(p_1\mu_{11} + p_2\mu_{21})$

2.4 Stochastic Decomposition

Theorem 2.1

The decomposition property states that the number of customers in the system (L_S) can be expressed as the sum of two independent random variables, one of which is

L - the mean number of customers in single arrival queue and feedback and the other is

L_I - the mean number of customers in the system when the service is idle

Proof

The probability generating function π(z) of the system size in the classical single arrival queue with feedback is given by

$$\pi(z) = \frac{I_0(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))((1-z)(1-\delta))}{(B_1^*(\lambda-\lambda z)p_1+B_2^*(\lambda-\lambda z)p_2)(1+\delta(z-1))-z} \qquad (2.37)$$

The probability generating function χ(z) of the number of customers in the orbit when the system is idle is given by

$$\chi(z) = \frac{I_0+I(z)}{I_0+I}$$

$$= \frac{[z(\lambda-\lambda z)-p_1T(z)+p_2T(z)]D'(1)}{D(z)[\lambda[1-\delta(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))]]} \qquad (2.38)$$

It is observed that probability generating function of the number of customers in the system

$P_s(z)$ is decomposed as $P_s(z) = \pi(z)\,\chi(z)$ and hence $L_S = L + L_I$.

CHAPTER-3

3. Analysis of Single Server Retrial Queue with Fluctuating Modes of Service, Feedback, Single Vacation

In this chapter, single server retrial queue with fluctuating modes of service feedback is analyzed for single arrival. If the server is idle upon the arrival of a customer, then the customer receives any one of the service immediately. Otherwise, he joins the orbit. After completion of service, the customer may join the orbit as a feedback customer or depart the system. At the completion epoch of each service, the server may take a single vacation. Using supplementary variable technique expected number of customers in the orbit and expected numbers of customers in the system are derived. Stochastic decomposition property is verified.

3.1 Model Description

Consider a single server retrial queueing system where customers arrive according to Poisson process with λ. The server provides two types of service type 1 and type 2.Customer opts type 1 service with probability p_1 or type 2 service with probability p_2 ($p_1+p_2=1$). If the customer finds the server free, then the customer receives any one of the services immediately. Otherwise he joins the orbit. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltje's transform $A^*(\theta)$ and conditional completion rate $\eta(x) = a(x) / [1 – A(x)]$.

The service time follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace- Stieltje's transform $B_i^*(\theta)$, n^{th} factorial moments μ_{in}, i=1,2 and conditional completion rate $\mu(x) = b_i(x) / [1 – B_i(x)]$. After receiving service, the customer may again join the orbit as a feedback with probability δ or depart the system with its complementary probability $1 – \delta$ (= $\bar{\delta}$).

At the completion of each service the server takes a single vacation with probability τ or waits for the next customer with complementary probability $\bar{\tau}$. The vacation time is generally distributed with distribution function V(x), density function v(x), Laplace - Stieltje's transform $V^*(\theta)$, n^{th} factorial moments v_n and conditional completion rate $\gamma(x) = v(x) / [1 – V(x)]$.

3.2 Steady State Distribution

In this section, by treating elapsed service time, elapsed vacation time of the server as supplementary variables, the steady state probability generating functions of the orbit size distribution are derived.

Define the states of the server at time t as,

$$C(t) = \begin{cases} 0, \; if \;\; the \; server \; is \; idle \\ 1, \; if \;\; the \; server \; is \; busy \; in \; type \, 1 \; service \\ 2, \; if \;\; the \; server \; is \; busy \; in \; type \; 2 \; service \\ 3, \; if \;\; the \; server \; is \; on \; vacation \end{cases}$$

For $t \geq 0$, we define the random variable $\xi(t)$ as follows :

(i) If $C(t) = 0$, then $\xi(t)$ represents the elapsed retrial time at time t,

(ii) If $C(t) = 1$ or 2 then $\xi(t)$ represents the elapsed service time at time t,

(iii) If $C(t) = 3$, then $\xi(t)$ represents the elapsed vacation time at time t.

Then, the process $\{X(t) ; t \geq 0\} = \{C(t), N(t), \xi(t) ; t \geq 0\}$ is a Markov process, where N(t) denotes the number of customers in the orbit. For the process $\{X(t) ; t \geq 0\}$, we define the following probability densities.

$I_0(t) = P\{C(t) = 0, N(t) = 0\}$

$I_n(x, t)\, dx = P\{C(t) = 0, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 1$

$P_n^{(i)}(x, t)\, dx = P\{C(t) = i, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0, i=1,2$

$V_n(x, t)\, dx = P\{C(t) = 3, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 0$

The steady state equations governing the model under consideration are

$$\lambda I_0 = \bar{\delta}\, \bar{\tau} \left[\int_0^\infty P_0^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_0^{(2)}(x)\mu_2(x)dx \right] + \int_0^\infty V_0(x)\gamma(x)dx \qquad (3.1)$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x))\, I_n(x), \quad n \geq 1 \tag{3.2}$$

$$\frac{d}{dx} P_n^{(i)}(x) = -(\lambda + \mu_{(i)}(x)) P_n^{(i)}(x) + (1-\delta_{0n})\lambda P_{n-1}^{(i)}(x); n \geq 0, i=1,2 \tag{3.3}$$

$$\frac{d}{dx} V_n(x) = -(\lambda + \gamma(x))\, V_n(x) + (1-\delta_{0n})\, \lambda V_{n-1}(x),\ n \geq 0 \tag{3.4}$$

with boundary conditions

$$I_n(0) = \begin{array}{l} \bar{\tau}\delta\left[\int_0^\infty P_{n-1}^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_{n-1}^{(2)}(x)\mu_2(x)dx\right] + \\ \bar{\delta}\left[\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x)\mu_2(x)dx\right] + \int_0^\infty V_n(x)\gamma(x)dx, n \geq 1 \end{array} \tag{3.5}$$

$$P_0^{(i)}(0) = p_i\, [\lambda I_0 + \int_0^\infty I_1(x)\, \eta(x)\,]dx, i=1,2 \tag{3.6}$$

$$P_n^{(i)}(0) = p_i\, [\int_0^\infty I_{n+1}(x)\, \eta(x)dx + \lambda \int_0^\infty I_n(x)\, dx]; n\geq 1, i=1,2 \tag{3.7}$$

$$V_n(0) = \tau\bar{\delta}\left[\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x)\mu_2(x)dx\right] + (1-\delta_{0n})\tau\delta\left[\int_0^\infty P_{n-1}^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_{n-2}^{(2)}(x)\mu_2(x)dx\right] \quad n\geq 1 \tag{3.8}$$

The normalizing condition is

$$I_0 + \sum_{n=1}^{\infty} \int_0^\infty I_n(x)\, dx + \sum_{n=0}^{\infty} \left[\sum_{i=1}^{2} \int_0^\infty P_n^{(i)}(x)dx + \int_0^\infty V_n(x)dx\right] = 1 \tag{3.9}$$

Define the probability generating functions as,

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n ; \quad P^{(i)}(x,z) = \sum_{n=0}^{\infty} P_n^{(i)}(x) z^n ; i=1,2$$

and $$V(x, z) = \sum_{n=0}^{\infty} V_n(x) z^n$$

Multiplying equation (3.2) by z^n and summing over n, we get

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x)) I_n(x) \tag{3.10}$$

Solving the partial differential equation (3.10), we get

$$I(x, z) = \frac{C}{e^{\lambda x (1 - A(x))}}$$

$$= Ce^{-\lambda x}(1 - A(x)) \tag{3.11}$$

Eliminating C by taking x = 0, we get

$$I(0, z) = C$$

$$I(x, z) = I(0, z) e^{-\lambda x}(1 - A(x)) \tag{3.12}$$

Applying generating function technique (3.7) and (3.8), yield the following equations

$$P^{(i)}(x, z) = p^{(i)}(0, z) e^{-(\lambda - \lambda z)x}(1 - B_i(x)), i = 1,2 \tag{3.13}$$

$$V(x, z) = V(0, z) \; e^{-(\lambda - \lambda z)x} \; (1 - V(x)) \tag{3.14}$$

Multiplying equations (3.5) to (3.8) and summing over n, we get

$$I(0, z) \quad = \quad \bar{\tau}\ \delta z \left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx \right]$$

$$+ \bar{\tau}\ \bar{\delta} \left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx \right] - \lambda I_0 + \int_0^\infty V(x,z)\gamma(x)dx$$

(3.15)

$$P^{(i)}(0, z) = \frac{p_i}{z}\left[\int_0^\infty I(x,z)\eta(x)dx \right] + p_i\ \lambda \left[\int_0^\infty I(x,z)dx + I_0 \right], i = 1,2 \qquad (3.16)$$

$$V(0,z) = \tau\bar{\delta} \left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx \right] +$$

$$\tau\delta z \left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx \right] \qquad (3.17)$$

Using equation (3.12) in equation (3.16), we obtain

$$P^{(i)}(0,z) = \frac{p_i}{z}\left[z\lambda I_0 + I(0,z)(A^*(\lambda) + z(1 - A^*(\lambda))) \right], i = 1,2 \qquad (3.18)$$

Using equation (3.13) in equation (3.17), we get,

$$V(0,z) = (\tau\bar{\delta} + z\,\tau\delta)[\,p_1(0,z)B_1^*(\lambda - \lambda z) + p_2(0,z)B_2^*(\lambda - \lambda z)] \qquad (3.19)$$

Using equation (3.13) , (3.14),(3.18) and (3.19) in equation (3.15) simplifying, we get

$$I(0,z)=\frac{z\lambda I_0\left\{(\tau+\tau\delta(z-1))\left[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z)\right]+V^*(\lambda-\lambda z)(\tau+\tau\delta(z-1))-1\right\}}{z-(A^*(\lambda)+z(1-A^*(\lambda)))\{(\tau+\tau\delta(z-1)[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z)]+V^*(\lambda-\lambda z)(\tau+\tau\delta(z-1)}$$

$$=\frac{z\lambda I_0(T_1(z)+T_2(z))}{D(z)} \qquad (3.20)$$

Where

$T_1(z)= p_1B_1^*(\lambda-\lambda z)[(\tau+\tau\delta(z-1))+V^*(\lambda-\lambda z)(\tau+\tau\delta(z-1))]$

$T_2(z)= p_2B_2^*(\lambda-\lambda z)[(\tau+\tau\delta(z-1))+V^*(\lambda-\lambda z)(\tau+\tau\delta(z-1))$

$D(z)= z-(A^*(\lambda)+z(1-A^*(\lambda)))(T(z))$

Using equation (3.20), the equation (3.18), we get,

$$P^{(i)}(0,z)=\frac{p_i\lambda I_0A^*(\lambda)(z-1)}{D(z)}, i=1,2 \qquad (3.21)$$

Substituting the result in equation (3.21) in equation (3.19), we get,

$$V(0,z)=\frac{(\tau+\tau\delta(z-1))\lambda I_0A^*(\lambda)(z-1)[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z)]}{D(z)} \qquad (3.22)$$

Solving the equations (3.12) to (3.14), we get

$$I(x,z) = \frac{z\lambda I_0(T(z))e^{-\lambda x}[1-A(x)]}{D(z)} \qquad (3.23)$$

$$P^{(i)}(x,z) = \frac{p_i \lambda I_0 A^*(\lambda)(z-1)e^{-(\lambda-\lambda z)x}[1-B_i(x)]}{D(z)}, i=1,2 \quad (3.24)$$

$$V(x, z) = \frac{\tau + \tau\delta(z-1))\lambda I_0 A^*(\lambda)(z-1)[p_1 B_1^*(\lambda-\lambda z) + p_2 B_2^*(\lambda-\lambda z)]e^{-(\lambda-\lambda z)x}[1-V(x)]}{D(z)}$$

(3.25)

The partial probability generating function of the orbit size when the server is idle is

$$I(z)= \frac{zI_0(1-A^*(\lambda))(\tau+\tau\delta(z-1))\left[p_1 B_1^*(\lambda-\lambda z)+p_2 B_2^*(\lambda-\lambda z)\right]+V^*(\lambda-\lambda z)(\tau+\tau\delta(z-1))-1}{D(z)}$$

(3.26)

The partial probability generating function of the orbit size when the server is busy is

$$P^{(i)}(z) = \frac{p_i I_0 A^*(\lambda)(z-1)(1-B_i^*(\lambda-\lambda z))}{D(z)}, i=1,2 \quad (3.27)$$

The partial probability generating function of the orbit size when the server is on vacation is,

$$V(z) = -(\tau+\tau\delta(z-1))A^*(\lambda) I_0[p_1 B_1^*(\lambda-\lambda z) + p_2 B_2^*(\lambda-\lambda z)][1-V^*(\lambda-\lambda z)] / D(z) \quad (3.28)$$

The partial probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P^{(1)}(z)+P^{(2)}(z)+ V(z)$$
$$= N(z) / D(z) \quad (3.29)$$

Where

$$N(z) = I_0 A^*(\lambda)(z-1)\{1-\delta(p_1 B_1^*(\lambda-\lambda z) + p_2 B_2^*(\lambda-\lambda z)\}$$

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + zP^{(1)}(z)+zP^{(2)}(z)+V(z)$$

$$= N_1(z) / D(z) \qquad (3.30)$$

Where

$$N_1(z) = I_0 A^*(\lambda)\ [p_1 B_1^*(\lambda-\lambda z)+p_2 B_2^*(\lambda-\lambda z)]((z-1)(1-\delta))$$

3.3 Performance Measures

The probability that the server is idle during retrial time is given by

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1-A^*(\lambda))\left[(\delta+\tau v)(p_1\mu_{11}+p_2\mu_{21})+\delta\right]}{D'(1)} \qquad (3.31)$$

where

$$D'(1) = A^*(\lambda)-\delta+\lambda\tau v_1(p_1\mu_{11}+p_2\mu_{21})$$

The probability that the server is busy in service is given by

$$P^{(i)} = \lim_{z\to 1} P^{(i)}(z)$$

$$= p_i I_0 A^*(\lambda)(1-B_i^*(\lambda-\lambda z))/D'(1),\ i=1,2 \qquad (3.32)$$

The probability that the server is on vacation is given by

$$V = \lim_{z\to 1} V(z)$$

$$= I_0 A^*(\lambda)\ \tau\ v_1\ (p_1 B_1^*(\lambda-\lambda z)+p_2 B_2^*(\lambda-\lambda z))/ D'(1)$$

$$(3.33)$$

The normalizing equation is equivalent to

$$I_0 + I + P^{(1)}+P^{(2)} + V = 1 \qquad (3.34)$$

Using above equation ,we get,

$$I_0 = \frac{\lambda[A^*(\lambda) - (\delta+\tau v)(p_1 B_1^*(\lambda-\lambda z) + p_2 B_2^{\;*}(\lambda-\lambda z))}{A^*(\lambda)[1-\delta(p_1 B_1^*(\lambda-\lambda z) + p_2\, B_2^{\;*}(\lambda-\lambda z))]} \qquad (3.35)$$

The mean number of customers in the orbit is given by

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2D''(1)^2} \qquad (3.36)$$

where N(z) and D(z) are the numerator and denominator of $P_q(z)$.

$$N'(1) = I_0 A^*(\lambda)(\delta - 1)$$

$$N''(1) = 2I_0 A^* \lambda\delta(P_1\mu_{12} + P_2\mu_{22})$$

$$D'(1) = A^*(\lambda) - \delta + \lambda\tau v_1(p_1\mu_{11} + p_2\mu_{21})$$

$$D''(1) = \begin{bmatrix}(p_1\mu_{11} + p_2\mu_{21})[A^*(\lambda) + (1-A^*(\lambda))]2\delta\lambda + 2(1-A^*(\lambda)) + (1-A^*(\lambda))\lambda^2(p_1\mu_{21} + p_2\mu_{22}) + \\ 2(1-A^*(\lambda))\delta - (\lambda v_1\tau\delta + \lambda v_1\tau\delta + \lambda^2 v_2(\tau\overline{\delta} + \tau\delta))\end{bmatrix}$$

The mean number of customers in the system is given by

$$L_S = \lim_{z\to 1}\frac{d}{dz}P_s(z)$$

$$= L_q + P^{(1)} + P^{(2)} \qquad (3.37)$$

3.4 Stochastic Decomposition

Theorem 3.1

The decomposition property states that the number of customers in the system (L_S) can be expressed as the sum of two independent random variables, one of which is

L - the mean number of customers in single arrival queue, feedback and the other is

L_I - the mean number of customers in the orbit given that the server is idle or on vacation.

Proof

The probability generating function π(z) of the system size in the classical single arrival queue with fluctuating modes of service feedback and orbital search is given by

$$\pi(z) = \frac{I_0(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))((1-z)(1-\delta))}{(B_1^*(\lambda-\lambda z)p_1+B_2^*(\lambda-\lambda z)p_2)(1+\delta(z-1))-z} \tag{3.38}$$

The probability generating function χ(z) of the number of customers in the orbit when the system is idle or on vacation is given by

$$\chi(z) = \frac{I_0+I(z)+V(z)}{I_0+I+V}$$

$$= \frac{A^*(\lambda)\{z(\lambda-\lambda z)-(p_1T_1(z)+p_2T_2(z))-(\lambda-\lambda z)(1-V^*(\lambda-\lambda z))(\tau\bar{\delta}+\tau\delta z)(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))\}D'(1)}{D(z)\{A^*(\lambda)[1-\delta(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))]\}} \tag{3.39}$$

It is observed that probability generating function of the number of customers in the system is decomposed as $P_s(z) = \pi(z)\,\chi(z)$ and hence $L_S = L + L_I$.

CHAPTER-4

4. Analysis of Single Server Retrial Queue with Fluctuating Modes of Service, Feedback, Single Vacation and Orbital Search

In this chapter, single server retrial queue with fluctuating modes of service feedback is analyzed. If the server is idle upon the arrival of a customer, then the customer receives any one of the service immediately. Otherwise, he joins the orbit. After completion of service, the customer may join the orbit as a feedback customer or depart the system. At the completion epoch of each service, the server may take a single vacation. If the orbit is non-empty during the idle period, the server searches for customer in the orbit or remains idle. Using supplementary variable technique expected number of customers in the orbit and expected numbers of customers in the system are derived. Stochastic decomposition property is verified.

4.1 Model Description

Consider a single server retrial queueing system where the customers arrive according to Poisson process with rate λ.The server provides two types of service - type 1and type 2. Customer opts type 1 service with probability p_1 or type 2 service with probability p_2 ($p_1+p_2=1$). If the customer finds the server free, then the customer receives any one of the services immediately. Otherwise he joins the orbit. The retrial time is generally distributed with distribution function A(x), density function a(x), Laplace-Stieltje's transform $A^*(\theta)$ and conditional completion rate $\eta(x) = a(x) / [1- A(x)]$.

The service time follows a general distribution with distribution function $B_i(x)$, density function $b_i(x)$, Laplace-Stieltje's transform $B_i^*(\theta)$, n[th] factorial moments μ_{in} ,i=1,2and conditional completion rate $\mu_i(x) = b_i(x) / [(1 - B_i(x)]$. After receiving service, the customer may again join the orbit as a feedback with probability δ or depart the system with its complementary probability $1 - \delta$ (= $\bar{\delta}$).

4.2 Steady State Distribution

In this section, by treating elapsed service time, elapsed vacation time of the server as supplementary variables, the steady state probability generating functions of the orbit size distribution are derived.

Define the states of the server at time t as,

$$C(t) = \begin{cases} 0, & \text{if the server is idle} \\ 1, & \text{if the server is busy in type 1 service} \\ 2, & \text{if the server is busy in type 2 service} \\ 3, & \text{if the server is on vacation} \end{cases}$$

For $t \geq 0$, we define the random variable $\xi(t)$ as follows :

(i) If C(t) = 0, then $\xi(t)$ represents the elapsed retrial time at time t,

(ii) If C(t) = 1 or 2 then $\xi(t)$ represents the elapsed service time at time t,

(iii) If C(t) = 3, then $\xi(t)$ represents the elapsed vacation time at time t.

Then, the process $\{X(t) ; t \geq 0\} = \{C(t), N(t), \xi(t) ; t \geq 0\}$ is a Markov process, where N(t) denotes the number of customers in the orbit. For the process $\{X(t) ; t \geq 0\}$, we define the following probability densities.

$I_0(t) = P\{C(t) = 0, N(t) = 0\}$

$I_n(x, t)\,dx = P\{C(t) = 0, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 1$

$P_n^{(i)}(x, t)\,dx = P\{C(t) = i, N(t) = n, x \leq \xi(t)5 < x + dx\}, x \geq 0, n \geq 0, i=1,2$

$V_n(x, t)\,dx = P\{C(t) = 3, N(t) = n, x \leq \xi(t) < x + dx\}, x \geq 0, n \geq 1$

The steady state equations governing the model under consideration are

$$\lambda I_0 = \bar{\delta}\bar{\tau}\left[\int_0^\infty P_0^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_0^{(2)}(x)\mu_2(x)dx\right] + \int_0^\infty V_0(x)\gamma(x)dx \qquad (4.1)$$

$$\frac{d}{dx} I_n(x) = -(\lambda + \eta(x) I_n(x) \qquad n \ge 1 \tag{4.2}$$

$$\frac{d}{dx} P_n^{(i)}(x) = -(\lambda + \mu_i(x)) P_n^{(i)}(x) + (1 - \delta_{0n}) \lambda P_{n-1}^{(i)}(x), n \ge 0, i = 1,2 \tag{4.3}$$

$$\frac{d}{dx} V_n(x) = -(\lambda + \gamma(x)) V_n(x) + (1 - \delta_{0n}) \lambda V_{n-1}(x), \ n \ge 0 \tag{4.4}$$

with boundary conditions

$$I_n(0) = \bar{\tau}\delta\left[\int_0^\infty P_{n-1}^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_{n-1}^{(2)}(x)\mu_2(x)dx\right] + \bar{\delta}\left[\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x)\mu_2(x)dx\right]$$
$$+ \bar{\theta}\int_0^\infty V_n(x)\gamma(x)dx, n \ge 1 \tag{4.5}$$

$$P_0^{(i)}(0) = p_i\left[\lambda I_0 + \int_0^\infty I_1(x)\eta(x)dx\right] + \theta\int_0^\infty V_1(x)\eta(x)dx, i = 1,2 \tag{4.6}$$

$$P_n^{(i)}(0) = p_i\left[\lambda\int_0^\infty I_n(x)dx + \int_0^\infty I_{n+1}(x)\eta(x)dx\right] + \theta\int_0^\infty V_{n+1}(x)\gamma(x)dx, i = 1,2; n \ge 1 \tag{4.7}$$

$$V_n(0) = \tau\bar{\delta}\left[\int_0^\infty P_n^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x)\mu_2(x)dx\right] + (1 - \delta_{0n})\tau\delta\left[\int_0^\infty P_{n-1}^{(1)}(x)\mu_1(x)dx + \int_0^\infty P_{n-2}^{(2)}(x)\mu_2(x)dx\right] ; n \ge 0 \tag{4.8}$$

The normalizing condition is

$$I_0 + \sum_{n=1}^\infty \int_0^\infty I_n(x)\,dx + \sum_{n=0}^\infty \int_0^\infty \sum_{i=1}^2 P_n^{(i)}(x)dx + \sum_{n=0}^\infty \int_0^\infty V_n(x)\,dx = 1 \tag{4.9}$$

Define the probability generating functions for $|z| \le 1$ as

$$I(x, z) = \sum_{n=1}^{\infty} I_n(x) z^n ; \qquad P^{(i)}(x, z) = \sum_{n=0}^{\infty} P_n^{(i)}(x) z^n ; i=1,2$$

and $$V(x, z) = \sum_{n=0}^{\infty} V_n(x) z^n$$

Applying generating function technique equations (4.2) to (4.5) yields, the following equations

$$I(x, z) = I(0, z)\, e^{-\lambda x} (1 - A(x)) \tag{4.10}$$

$$P^{(i)}(x, z) = P^{(i)}(0,z) e^{-(\lambda-\lambda z)x} (1-B_i(x)), i=1,2 \tag{4.11}$$

$$V(x, z) = V(0, z)\, e^{-(\lambda-\lambda z)x} (1 - V(x)) \tag{4.12}$$

Multiplying equations (4.5) to (4.8) and summing over n, we get

$$I(0,z) = \bar{\tau}\delta z\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right] + \bar{\tau}\bar{\delta}\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right]$$
$$+ \bar{\theta}\int_0^\infty V(x,z)\gamma(x)dx - \lambda I_0 \tag{4.13}$$

$$P^{(i)}(0,z) = \frac{p_i}{z}\left[\int_0^\infty I(x,z)\eta(x)dx + \theta\int_0^\infty V(x,z)\gamma(x)dx\right] + \lambda p_i\left[\int_0^\infty I(x,z)dx + I_0\right] \tag{4.14}$$

$$V(0,z) = \tau\delta\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right] + \tau\delta z\left[\int_0^\infty P^{(1)}(x,z)\mu_1(x)dx + \int_0^\infty P^{(2)}(x,z)\mu_2(x)dx\right] \tag{4.15}$$

Using equation (4.11),(4.12),(4.14),(4.15) in equation (4.13), we get,

$$I(0,z) = \frac{\lambda I_0 z[p_1 T_1(z) + p_2 T_2(z) - 1] + \theta[p_1 T_3(z) + p_2 T_4(z)]}{D(z)} \tag{4.16}$$

where

$T_1(z)= B_1^*(\lambda-\lambda z)(\bar{\tau}\delta z+\bar{\tau}\bar{\delta}+\bar{\theta}V^*(\lambda-\lambda z)(\tau\bar{\delta}+\tau\delta z))$

$T_2(z)= B_2^*(\lambda-\lambda z)(\bar{\tau}\delta z+\bar{\tau}\bar{\delta}+\bar{\theta}V^*(\lambda-\lambda z)($ 31

$T_3(z)=(\tau\bar{\delta}+\tau\delta z)B_1^*(\lambda-\lambda z)V^*(\lambda-\lambda z)$

$T_4(z)=(\tau\bar{\delta}+\tau\delta z)B_2^*(\lambda-\lambda z)V^*(\lambda-\lambda z)$

$D(z)= z-[A^*(\lambda)(1-z)+z][p_1T_1(z)+p_2T_2(z)]-\theta[p_1T_3(z)+p_2T_4(z)]$

Using equation (4.16), the equation (4.14) and (4.15),we obtain,

$$P^{(i)}(0,z)=\frac{\lambda I_0 p_i A^*(\lambda)(z-1)}{D(z)}, i=1,2 \qquad (4.17)$$

$$V(0,z) = \frac{\lambda I_0 A^*(\lambda)(z-1)(\tau\bar{\delta}+\tau\delta z)(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))}{D(z)} \qquad (4.18)$$

Solving the equations (4.16) to (4.18), we get

$$I(x,z) = \frac{\lambda I_0 z[p_1T_1(z)+p_2T_2(z)-1]+\theta[p_1T_3(z)+p_2T_4(z)](1-A(x))e^{-\lambda x}}{D(z)} \qquad (4.19)$$

$$P^{(i)}(x,z)=\frac{\lambda I_0 p_i A^*(\lambda)(z-1)(1-B_i(x))e^{-(\lambda-\lambda z)x}}{D(z)}, i=1,2 \qquad (4.20)$$

$$V(x,z)=\frac{\lambda I_0 A^*(\lambda)(z-1)(\tau\bar{\delta}+\tau\delta z)[p_1B_1^*(\lambda-\lambda z)p_2B_2^*(\lambda-\lambda z)](1-V(x))e^{-(\lambda-\lambda z)x}}{D(z)} \qquad (4.21)$$

The partial probability generating function of the orbit size when the server is idle is

$$I(z) = \frac{I_0(1-A^*(\lambda))[z(p_1T_1(z)+p_2T_2(z)-1)+\theta(p_1T_3(z)+p_2T_4(z))]}{D(z)} \tag{4.22}$$

The partial probability generating function of the orbit size when the server is busy is

$$P^{(i)}(z) = \frac{I_0 p_i A^*(\lambda)\left\lfloor B_i^*(\lambda-\lambda z)-1\right\rfloor}{D(z)}, i=1,2 \tag{4.23}$$

The partial probability generating function of the orbit size when the server is on vacation is,

$$V(z) = \frac{I_0 A^*(\lambda)(\tau\bar{\delta}+\tau\delta z)[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z)[V^*(\lambda-\lambda z)-1]}{D(z)} \tag{4.24}$$

The partial probability generating function of the orbit size is

$$P_q(z) = I_0 + I(z) + P^{(1)}(z)+P^{(2)}(z)+ V(z)$$

$$= N(z) / D(z) \tag{4.25}$$

Where

$$N(z) = I_0A^*(\lambda)(z-1)[1-\delta(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))]$$

The partial probability generating function of the system size is

$$P_s(z) = I_0 + I(z) + z\, P^{(1)}(z) + z\, P^{(2)}(z)+ V(z)$$

$$= N_1(z) / D(z) \tag{4.26}$$

where

$$N_1(z) = I_0A^*(\lambda)(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))(z-1)(1-\delta) \tag{4.27}$$

4.3 Performance Measures

The probability that the server is idle during retrial time is given by

$$I = \lim_{z\to 1} I(z)$$

$$= \frac{I_0(1-A^*(\lambda))\{\tau\theta(\delta-1)+\lambda v_1\tau+\lambda(p_1\mu_{11}+p_2\mu_{21})\}}{D'(1)} \quad (4.28)$$

Where

$$D'(1) = -\lambda v_1\tau+\lambda(p_1\mu_{11}+p_2\mu_{21})(1-2\tau\theta)+A^*(\lambda)(1-\tau\theta)-\tau\theta\delta$$

The probability that the server is busy in service is given by

$$P^{(i)} = \lim_{z\to 1} P^{(i)}(z)$$

$$= \frac{I_0 p_i A^*(\lambda)B_i^*(\lambda-\lambda z)}{D'(1)} \quad , i=1,2 \quad (4.29)$$

The probability that the server is on vacation is given by

$$V = \lim_{z\to 1} V(z)$$

$$= \frac{I_0 A^*(\lambda)(\tau\overline{\delta}+\tau\delta z)[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z)][V^*(\lambda-\lambda z)-1]}{D'(1)}$$

(4.30)

The normalizing equation (4.11) is equivalent to

$$I_0 + I + P^{(i)} + V = 1, \quad i=1,2 \quad (4.31)$$

Using equations (4.28) and (4.29) in equation (4.30), yields

$$I_0 = \frac{1-\delta+(1-\tau\theta)(A^*(\lambda)-1)-\lambda(v_1\tau+p_1\mu_{11}+p_2\mu_{21})}{A^*(\lambda)(1-\delta-\tau)} \quad (4.32)$$

The mean number of customers in the orbit is given by

$$L_q = \lim_{z \to 1} \frac{d}{dz} P_q(z)$$

$$= \frac{D'(1)N''(1) - N'(1)D''(1)}{2D''(1)^2} \qquad (4.33)$$

where N (z) and D(z) are the numerator and denominator of $P_q(z)$.

$$N'(1) = I_0 A^*(\lambda)(1-\delta)$$

$$N''(1 = -I_0 A^*(\lambda)\delta\lambda(p_1\mu_{11} + p_2\mu_{21})$$

$$D'(1) = -\lambda v_1 \tau + \lambda(p_1\mu_{11} + p_2\mu_{21})(1-2\tau\theta) + A^*(\lambda)(1-\tau\theta) - \tau\theta\delta$$

$$D''(1) = \begin{bmatrix} 2(A^*(\lambda)-1)(\bar{\tau}\delta + \bar{\theta}\tau\delta + \bar{\theta}\lambda v_1(\tau\bar{\delta} + \tau\delta)) + 2(A^*(\lambda)-1)(\bar{\tau}\delta + \bar{\tau}\bar{\delta} + \bar{\vartheta}(\tau\bar{\delta} + \tau\delta))(p_1\lambda\mu_{11} + p_2\lambda\mu_{21}) \\ -(\bar{\theta}\lambda v_1\tau\delta + \bar{\theta}\lambda v_1\tau\delta + \bar{\theta}\lambda^2 v_2(\tau\bar{\delta} + \tau\delta)) - 2(\bar{\tau}\delta + \bar{\theta}\tau\delta + \bar{\theta}\lambda v_1(\tau\bar{\delta} + \tau\delta))(p_1\lambda\mu_{11} + p_2\lambda\mu_{21}) - \\ (\tau\delta + \bar{\tau}\bar{\delta} + \bar{\theta}(\tau\bar{\delta} + \tau\delta))(p_1\lambda^2\mu_{12} + p_2\lambda^2\mu_{22}) - 2\tau\delta\bar{\theta}\lambda v_1 \end{bmatrix}$$

The mean number of customers in the system is given by

$$L_S = \lim_{z \to 1} \frac{d}{dz} P_s(z) \qquad (4.34)$$

$$= L_q + P^{(1)} + P^{(2)}$$

4.5 Stochastic Decomposition

Theorem 4.1

The decomposition property states that the number of customers in the system (L_S) can be expressed as the sum of two independent random variables, one of which is

L - the mean number of customers in single arrival queue and feedback and the other is

L_I - the mean number of customers in the orbit given that the server is idle or on vacation.

Proof

The probability generating function π(z) of the system size in the classical single arrival queue with negative arrivals feedback and orbital search is given by

$$\pi(z) = \frac{I_0(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))(z-1)(1-\delta)}{z-[(1-z)+z][p_1T_1(z)+p_2T_2(z)]-\theta[p_1T_3(z)+p_2T_4(z)]} \qquad (4.35)$$

The probability generating function χ(z) of the number of customers in the orbit when the system is idle or on vacation is given by

$$\chi(z) = \frac{I_0 + I(z) + V(z)}{I_0 + I + V}$$

$$= \frac{\left\{A^*(\lambda)(\lambda-\lambda z)(p_1B_1^*(\lambda-\lambda z)+p_2B_2^*(\lambda-\lambda z))[1-\delta+\delta z]+A^*(\lambda)z(\lambda-\lambda z)\right\}D'(1)}{D(z)A^*(\lambda)-\delta[p_1B_1^*(\lambda-\lambda z)+p_2B_2^*((\lambda-\lambda z))]}$$

(4.36)

It is observed that probability generating function of the number of customers in the system $P_s(z)$ is decomposed as $P_s(z) = \pi(z)\,\chi(z)$ and hence $L_S = L + L_I$.

SUMMARY AND CONCLUSION

Summary and Conclusion

In Chapter 2, single server retrial queue with fluctuating modes of service and feedback facility is considered. In Chapter 3, the model described in Chapter 2 is extended to accommodate server single vacation. In the final chapter the model considered in Chapter 3 is analysed with orbital search.

For all the above models

i. single arrival of customers are considered.

ii. The retrial time, service time and vacation time are assumed to follow general distribution.

iii. Mathematical formulation and equilibrium analysis are carried out using supplementary variable technique.

The explicit expressions of expected number of customers in the orbit and expected number of customers in the system are derived. Lastly, stochastic decomposition property is verified.

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