Bulk Arrival Queueing Models with Second Optional Service Facility under Bernoulli Vacation

Lavanya, T

(12PMA010)

Thesis submitted to

Avinashilingam Institute for Home Science and Higher Education for Women, Coimbatore - 641 043

In Partial Fulfilment of the Requirements for the

Degree of Master of Science in Mathematics

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Parvathi
Signature of the Head of the Department

Begum 31/3/14
Signature of the Supervisor

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INTRODUCTION

Queueing theory is the mathematical study of **queues** or **waiting lines**. A queue is formed whenever the demand for service exceeds the capacity to provide service at that point of time. A classical queueing system may be described as one having a service facility, at which customers arrive for service and whenever there are more customers in the system than the service facility can handle simultaneously, a queue or waiting line is developed. The waiting customers take their turn for service according to a preassigned rule and leave the system after availing service. Thus, the input to the system consists of the customers demanding service and the output refers to the served customers. Queueing systems have many applications, most of which have been well documented in the literature of Probability, Operations Research, Management Science and Industrial Engineering. Some examples are traffic flow (vehicles, aircraft, and communications), scheduling (patients in hospitals, jobs on machine) and facility design (banks, post offices, amusement parks).

To design a novel queueing system, a balance between service to customers and economic considerations needs to be established. For analysis, the informations related to arrival process (how customers arrive), service mechanism (how long the service will take), queue discipline (how to choose the next customer) are required. Depending on the interarrival time and service time distributions, the number of servers and service discipline, the queueing systems lead to different mathematical problems and form an important area of Applied Mathematics.

To evaluate the performance of queueing systems, the following measures are typically considered :

- The average length of the queue.
- The mean waiting time of customers in queue or system.
- The probability of encountering the system in certain states.
- The probability that the queue exceeds certain length.
- The expected utilization of the server and the expected time during which the server will be fully occupied.
- The optimal stationary operating policy of the system, etc.

1.1 CHARACTERISTICS OF QUEUEING SYSTEMS

Some of the basic characteristics of the classical queueing systems are the following:

Arrival Pattern of Customers

The arrival pattern describes the manner in which customers arrive and join the system. Arrivals may occur in single or in groups (batch or bulk arrival). It is specified by the probability of time between successive arrivals, that is, the interarrival time distribution. The batches may be of fixed size or of variable size. If the time between successive arrivals (interarrival time) is uncertain, the arrival pattern is measured by either mean arrival rate or mean inter arrival time.

Service Pattern

The service mechanism describes the way in which service is rendered. The customers may be served either singly or in batches. The time required for serving a unit or a batch is called service time. The service time is a random variable and represented by a probability distribution. The distribution describes the rate at which the customers are served per unit of time. The service time is conditioned on the fact that the system is not empty.

Service Channels

Queueing system may have several service channels to provide service. These service channels may be arranged in parallel or in series or in combination of both, depending on the design of the systems service mechanism. In parallel channels, each and every channel provides identical service facilities, so that several customers may be served simultaneously. In case of series of channels a customer must pass successively through the ordered channels before service is completed.

System Capacity

The number of customers in the queue and in service put together is called system capacity. A system may have a queue of finite capacity or effectively infinite capacity. A system with finite capacity can be viewed as one with forced balking of a customer arriving, when the system is in its full capacity. If the system capacity is not mentioned, it is assumed to be infinite.

Queue Discipline

The queue discipline is the mode by which the customers are selected for service from the set of customers waiting for service. The common disciplines observed in everyday life are (i) First In First Out (FIFO), also known as First Come First Served (FCFS), (ii) Last In First Out (LIFO), (iii) selection for Service In Random Order (SIRO) and (iv) a variety of priority schemes. The queue discipline followed in all the models of the present work is FIFO.

Kendall's Notation

Kendall (1953) has designed a very convenient universally accepted notation to denote a queueing system. A queueing model is described by the notation

A/B/C/X/Y where,

A - Represents the interarrival time distribution of the customers,

B - The service time distribution of the given service facility,

C - The number of parallel service channels,

X - The capacity of the system,

Y - The type of queue discipline

If the capacity of the system is infinite and the queue discipline is FCFS, the notations X and Y may be dropped. Whenever, the interarrival times and the service times are exponential (Markovian), then A and B are replaced by the symbols M and M.

This is because, exponential distribution has the memory-less or Markovian property and it is the only continuous time random variable which as the Markovian property.

Example :

1. $M^X/M/1$- is a queueing in which the arrival stream forms a Poisson process with some parameter λ and the actual number of customers in any arriving module is a random variable X, which may take on any positive integral value k less than ∞

with probability distribution $\{g_k = \Pr(X = k)\}$, k = 1, 2, 3, …, . The service time is exponential. There is a single server and the queue discipline is FIFO. The queue can accommodate infinitely many customers.

2. $M^X/G/1$-Model differs from $M^X/M/1$ only in service distribution. The service time of this model follows a general distribution.

Some of the characteristics of the queueing system considered in the present work are:

Multiple and Single Vacation Policy

The server leaves the system whenever the system becomes empty for a random amount of time which is termed as vacation. During vacations, the server will not be available in the system. After returning from a vacation to the system, if the server finds at least one customer, then he immediately starts to serve the customers. On the other hand, if he finds no one in the system, then he takes another vacation and repeats the vacations until the system becomes non empty. This policy is referred as **repeated or multiple vacation policy**. In single vacation, if the server finds no customers in the system after returning from vacation then he joins the system and stays idle in the system and starts his service as soon as a customer arrives. (i.e.,) the server will take only a **single vacation** between any two busy periods.

Bernoulli Vacation Policy

In both single and multiple vacation policies, servers take vacations only when the system becomes empty. But in some situations, especially when the service is done in two or more phases, the maintenance of the system may be required at the completion of each service and in such cases, the service may be stopped for maintenance and overhauling, or continued, if there is no fault in the system. The overhauling may be utilized as a vacation time. Recently several authors considered the vacation policy called **Bernoulli Schedule Vacation**, which is characterized by the feature that, at the completion of each service, the server may take a vacation with probability p or may continue to serve next unit if any, with probability (1 – p).

Queues with Server Breakdowns

Most of the existing research works have investigated classical queueing systems assuming a reliable machine or server, however in practice; the server may fail and can be repaired. These phenomena of server breakdowns, can be encountered in the area of computers, communication networks, flexible manufacturing systems etc. The performance of the system may be affected heavily by these breakdowns and limited repair capacity. Queueing systems with such unreliable stations are the topics of worth investigating from the performance prediction point of view.

Different types of Queueing Models

Markovian Queueing Models

Queueing models with exponential interarrival time and exponential service time are called Markovian queueing models. Markovian queueing models are solved by

(i) Difference differential equation method
(ii) Recursive computation approach
(iii) Laplace transform method and
(iv) Integral calculus approach

Non-Markovian Queueing Models

Queues in which inter-arrival and / or service time distributions are other than exponential are known as Non-Markovian queues. The following methods are generally used to study Non-Markovian models.

(i) Imbedded Markov Chain Technique
Non-Markovian queues are reduced to Markovian by a technique introduced by Kendall (1951).
(iii) Phase-Technique Method and
(iv) Matrix Geometric Method by Netus (1978,1980,1981)

Compound Poisson Process

There are many possible and unknown generalization of the Poisson – exponential process. The most common generalization is the one which relaxes the Poisson assumption that more than one occurrence in Δt time is $O(\Delta t)$. Instead, let

$$\Pr\{\text{i occurrences in } t + \Delta t\} = \lambda_i \Delta t + 0(\Delta t),\ i = 1, 2, \ldots, n \text{ with } \sum_{i=0}^{n} \lambda_i = \lambda$$

This is equivalent to allowing the event of i simultaneous occurrences in Δt with probability $\lambda_i \Delta t + 0(\Delta t)$ and each individual stream of occurrences of the same batch size i itself forms a Poisson process. If these sub streams are denoted by $N_i(t)$, then the total process is $N(t) = \sum i N_i(t)$, with probability function,

$$P_n(t) = P_r\{\text{n occurrence in } [0, t]\}$$

$$= \sum \frac{e^{-\lambda.t}(\lambda t)}{i!} g_n^{(i)} g_0^{(0)} = 1$$

where $g_n^{(i)}$ is the probability that i occurrences give a grand total of n (That is, the probability associated with the i-fold convolution of the batch size probabilities $\{\lambda_i / \lambda\}$). The process N(t) is known as the multiple Poisson and also has the stationary and independent-increment properties.

Transient and Steady-State Queueing System

A queueing system is said to be in transient state when its operating characteristics are dependent on time. Otherwise (independent of time), the system is said to be in steady-state or equilibrium state. Solution of a queueing system depending upon time is called transient solution and independent of time is called steady-state solution. Most of the analysis of queueing models is confined to steady-state results.

1.2 METHODOLOGY AND PRELIMINARIES

Supplementary Variable Technique

Supplementary Variable Technique introduced by Cox (1955) is the technique of introducing one or more supplementary variables to convert a Non-Markovian process in to Markovian process. The models discussed in the present work are Non-Markovian. These models are analysed using Supplementary variable technique. Remaining times of random variables are introduced as Supplementary variables to extract Markov process.

Probability Generating Function (P.G.F) of the Random Variable X

In dealing with integral-valued random variables, it is often of great convenience to apply the powerful tools of generating functions. Many stochastic processes that we come across involve non-negative integral valued random variables, and quite often we could use generating function in their studies. The principal advantage of its use is that a single function (generating function) may be used to represent a whole set of individual items.

Suppose that X is a random variable that assumes non-negative integral values 0,1, 2, …, and that $P_r\{X = k\} = P_k$, k = 0, 1, 2, …, $\sum_{i=0}^{n} P_k = 1$. Then the corresponding generating function $P(z) = \sum P_k z^k$ of the sequence of probabilities $\{P_k\}$ is known as the probability generating function of the random variable X.

We have P(1) = 1; the series P(z) converges for at least $-1 \le z \le 1$ and is infinitely differentiable. The function P(z) is defined by $\{p_k\}$ and in turn defines $\{p_k\}$ uniquely, that is, a probability generating function determines a distribution uniquely. The k^th^ moment of X . $E(X^k)$ is given by $E(X^k) = \sum_{n=1}^{\infty} n^k P_n$.

Laplace Transforms

Laplace transform is a generalization of generating function.

Definition :

Let f(t) be a function of a positive real variable t. Then the Laplace transform of f(t) is defined by, $L(f(t)) = \bar{f}(\theta) = \int_0^\infty e^{-\theta.t} f(t)dt$ for the range of values of θ for which the integral exists.

Laplace (Stieltjes) Transform of a Probability Distribution or of a Random Variable

Definition :

Let X be a non-negative random variable with distribution function,

$$F(X) = \Pr(X \leq x)$$

The Laplace – Stieltjes transform F*(θ) of this distribution is defined for θ ≥ 0 by,

$$F^*(\theta) = \int_0^\infty e^{-\theta.x} dF(X) \tag{1.1}$$

The Laplace Transform of the Distribution Function interms of the Density Function is

$$\overline{F(\theta)} = L\{F(X) = \int_0^\infty e^{-\theta.x} \left\{ \int_0^\infty f(t)dt \right\} dx$$

$$= L\left\{ \int_0^x f(t)dt \right\} = \frac{\overline{f(\theta)}}{\theta} = \frac{F^*(\theta)}{\theta}$$

Mean and Variance In terms of (Derivatives of) Laplace Transform

We note here that differentiation under the integral sign is valid for the Laplace transform given by (1.1), since the integrand is bounded and continuous. Differentiating (1.1) with respect to θ n-times, we get for n = 1, 2, 3,…,

$$\frac{d^n}{d\theta^n} F^*(\theta) = (-1)^n \int_0^\infty x^n e^{-\theta.x} f(x)d(x)$$

$$= (-1)^n E(X^n) \quad \text{at } \theta = 0$$

In particular, we have,

$$E(X)=\left(\frac{-d}{d\theta}(F^{*}(\theta))\right)_{\theta=0}$$

$$E(X^{2})=\left(\frac{-d^{2}}{d\theta^{2}}(F^{*}(\theta))\right)_{\theta=0}$$ and

$$\text{Var(X)}=\left(\frac{-d^{2}}{d\theta^{2}}(F^{*}(\theta))\right)_{\theta=0}-\left(\frac{-d}{d\theta}(F^{*}(\theta))\right)_{\theta=0}$$

Identities

The identities used in the present work are listed below:

1) $$\sum_{n=2}^{\infty}z^{n}\sum_{k=1}^{n}Q_{n-k}^{*}(\theta)g_{k}=\left(\sum_{n=0}^{\infty}Q_{n}^{*}(\theta)z^{n}\right)\left(\sum_{k=1}^{\infty}g_{k}z^{k}\right)$$

2) $$\sum_{n=2}^{\infty}z^{n}\sum_{k=1}^{n-1}P_{n-k,i}^{*}(\theta)g_{k}=\left(\sum_{n=1}^{\infty}P_{n}^{*}(\theta)z^{n}\right)\left(\sum_{k=1}^{\infty}g_{k}z^{k}\right)$$ i=1,2

3) $$\sum_{n=2}^{\infty}z^{n}\sum_{k=1}^{n-1}B_{n-k,i}^{*}(\theta)g_{k}=\left(\sum_{n=1}^{\infty}B_{n}^{*}(\theta)z^{n}\right)\left(\sum_{k=1}^{\infty}g_{k}z^{k}\right)$$ i=1,2

4) $$SOS^{*}(w_{x}(z))=S_{1}^{*}(w_{x}(z))(1-r+rS_{2}^{*}(w_{x}(z)))$$,

where $w_{x}(z)=\lambda(1-X(z))$ and $X(z)=\left(\sum_{k=1}^{\infty}g_{k}z^{k}\right)$, Then $\lim_{z\to 1}SOS^{*}(w_{x}(z))=1$

then,

4a) $$\lim_{z\to 1}\frac{d}{dz}SOS^{*}(w_{x}(z))=\lambda E(X)(E(S_{1})+rE(S_{2}))$$

4b) $$\lim_{z\to 1}\frac{d^{2}}{dz^{2}}SOS^{*}(w_{x}(z))=\lambda E(X(X-1))E(SOS)+(\lambda(E(X))^{2}E(SOS^{2})$$

where $E(SOS)=E(S_{1})+rE(S_{2})$

$$E(SOS^{2})=E(S_{1}^{2})+rE(S_{1}^{2})+2rE(S_{1})E(S_{2})$$

Let

5) $S^{*}_{B\mathbf{V}}(w_x(z)) = SOS^{*}(w_x(z)) + (V^{*}w_x(z) - 1)[p_2 r S^{*}_2 w_x(z) + (1-r)p_1](S^{*}_1(w_x(z)))$

Then, $\lim\limits_{z\to 1} S^{*}_{B\mathbf{V}}(w_x(z)) = 1$

5a) $\lim\limits_{z\to 1}\dfrac{d}{dz}S^{*}_{B\mathbf{V}}(w_x(z)) = \lambda E(X)(E(SOS) + \overline{P}E(V))$ where $\overline{P} = p_2 r + (1-r)p_1$

5b) $\lim\limits_{z\to 1}\dfrac{d^2}{dz^2}S^{*}_{B\mathbf{V}}(w_x(z)) = \lambda E(X(X-1))E(S_{BV}) + (\lambda E(X))^2 E(S^2_{BV})$

where $E(S_{BV}) = E(SOS) + \overline{P}E(V)$

$E(S^2_{BV}) = E(SOS^2) + \overline{P}E(V^2) + 2(\overline{P}E(S_1) + p_2 r E(S_2))E(V)$

6) Let $h_a(w_x(z)) = w_x(z) + a(1 + R^{*_1}(w_x(z)))$

then $\lim\limits_{z\to 1}(h_a(w_x(z))) = 1$

6a) $\lim\limits_{z\to 1}\dfrac{d}{dz}(h_a(w_x(z))) = -\lambda E(X)(1 + aE(R))$

7) $\lim\limits_{z\to 1}\dfrac{d}{dz}S^{*}(h_a(w_x(z))) = \lambda E(X)(1 + aE(R))E(S)$

8) $\lim\limits_{z\to 1}\dfrac{d^2}{dz^2}S^{*}(h_a(w_x(z))) = \lambda E(X(X-1))(1 + aE(R))E(S)$

$+ (\lambda E(X))^2(1 + aE(R))^2 E(S^2) + aE(R^2)E(S)$

9) If f(z)=g(z)=0 at z=1, then $\dfrac{d}{dz}\left(\dfrac{f(z)}{g(z)}\right) = \dfrac{g'(1)f''(1) - f'(1)g''(1)}{2(g'(1))^2}$

where single and double dashes denote the first and second derivatives of the respective functions

10) $\lim\limits_{z\to 1}\left(\dfrac{z-1}{z - S^{*}(w_x(z))}\right) = \dfrac{1}{(1 - \lambda E(X)E(S))}$

11) $$\lim_{z\to 1}\frac{d}{dz}\left(\frac{z-1}{z-S^*(w_x(z))}\right)=\frac{\left(\lambda E(X(X-1))E(S)+(\lambda E(X))^2E(S^2)\right)}{2(1-\lambda E(X)E(S))^2}$$

1.3 REVIEW OF LITERATURE

Second Optional Service Models

$M^X/(G_{SOS})/1$ with Second Optional Service

The class of queueing systems, where the service discipline involves more than one service has been receiving a lot of attention recently. There have been several contributions, considering queueing systems in which the service provided in two phases by a single server.

K.C. Madan (2000) was first to study the time dependent as well as steady state, behaviors of an M/G/1 queue with second optional service using the supplementary variable technique. In this model customers arrive according to a Poisson process and the server provides the first essential service to all the arriving customers. As soon as the first essential service of a customer is completed, the customer may leave the system with probability (1-r) or may immediately opt for a second optional service with probability r. In this model the service time for the first essential service follows the general distribution where as the second optional service has an exponential distribution. Gautam Choudhury (2003) has generalized the results of Madan (2000) by deriving the steady state queue size distribution at the stationary point of time for general second optional service times. He also derived the queue size distribution at departure point of time as a classical generalization of the well known Pollaczek Khinchin Formula. Finally he has obtained the Laplace Stieltjes transform of the waiting time distribution and some important performance measures which lead to remarkable simplification when solving other similar types of queueing models.

The papers mentioned above are characterized by the common feature; the second phase of service is provided only to a portion of the original incoming customers. As related literature, there are some papers Doshi (1991), Krishna and Lee (1990), Selvam and Sivasankaran (1994) arising from distribution system control, where all customers received batch mode service in the first phase, followed by individual services in the second phase. Medhi (2002) considered an M/G/1 queueing system with an optional service channel, where a unit may depart from the system either after first essential service (FES) with probability (1-r) or at the end of the FES it may immediately go for a second optional service (SOS) with probability r ($0<\theta<1$). In fact, some aspects of this model were first studied by Madan (2000). He cited some important applications of this model in many real life situations. The case where both the FES and SOS are exponentially distributed random variables is also called Coxian distribution C_2. Bertsimas and Papa constantinou (1988) considered such a distribution to design a multi-server queue with application in a transportation system.

Vacation Queueing Models

Queueing systems with server vacations have been studied extensively since the late 70's. A considerable number of works in this area were completed in the early 80's and surveyed by Doshi (1986). As an extension to the classical queueing system, allowing idle servers to work on non-queueing jobs makes the vacation models more applicable in a variety of systems including flexible manufacturing or service and computer communication systems. Motivated by these applications, more studies on vacation models have been done during the late 80's and 90's and surveyed in the book by Takagi (1991) and the book by Tian and Zhang (2006). Ke,. et al. (2010b) provided a brief summary of the most recent research works on vacation queueing systems in the past 10 years.

The queueing model with server vacations has been well studied in the past three decades and successfully applied in many areas such as manufacturing/service and computer/communication network systems. These vacation queueing models can be classified according to the arrival processes, service processes, and the vacation policies. Excellent surveys on the earlier works of vacation models have been reported by Doshi (1986), Takagi (1991), Tian and Zhang (2006).

Modified vacation policy

Zhang and Tian (2001) investigated a Geo/G/1 queue with multiple adaptive vacations (MAV) where the server can take at most a certain number (J) of vacations continuously. Ke and Chu (2006) studied a batch arrival system with MAV. The MAV can be considered as a modified vacation policy. An MAV is reduced to the single or multiple vacation policy by setting the value of J to be one or infinity. Later, Ke (2007) extended the model in Ke and Chu (2006) to the case with customer balking behavior. Ke et al. (2010a) generalized the model to the case with N-policy. Furthermore, Ke et al. (2010b) investigated the threshold model of Ke et al. (2010a) with a randomized control policy. Later, more works on the models with the modified vacation policies were done. For example, Ke and Chang (2009) considered and M/G/1 retrial queue with modified vacation policy, customer balking, and feedbacks. Chang and Ke (2009) investigated an M^X/G/1 retrial queue with modified vacation policy by applying the supplementary variable technique. Ke and Chang (2009) extended Chang and Ke's model to more general cases with impatience customers and feedback behaviors.

Vacation concept was first introduced to utilize the idle time of the server when the system becomes empty. Vacation queues have been studied by numerous researchers including Keilson and Servi (1987), Doshi (1986), Shanthikumar (1988), Madan (1999). These vacation models deal with the condition of exhaustive service policies, which means that, the server leaves for a vacation of random length only when the system becomes empty. But later Keilson and Servi (1987) introduced an interesting vacation policy, in which, the server may go on a vacation with probability p as soon as he completes a service for a customer or may continue to attend the next service if any, with probability (1-p). An important merit of the Bernoulli schedule service discipline is the existence of the control parameters p. By adjusting the values of p the congestion of the system can be controlled. The examples of this kind of situation will be found in transportation system, in which a fairy driver or a locomotive deriver may like to go on vacation after every trip.

Various aspects of BSV models for single service queueing system have been studied by a good number of authors. M/G/1 queueing systems, under BSV have been studied by Keilson and Servi (1986, 1987), Servi (1987). Bernoulli Vacation policy is more effective, when the server operates more than one service. Queueing models where the service discipline involves more than one service receive a lot of attention recently. Different scenarios are often used in the literature. Madan (2000) introduced a M/G/1 queueing model, where the server provides FES to all the arriving customer. As soon as the FES of a customer is completed, it may leave the system with certain probability (1-r) or may immediately opt for SOS with probability r. The queueing system with two phase of heterogeneous service namely first phase of service followed by second phase of service is a particular case of the SOS policy introduced by Madan (2000).

Madan et al. (2003) analyzed and M/M/2 queue with a single Bernoulli schedule vacation policy and discussed two models under different conditions. Choudhury and Madan (2004) considered a batch arrival queueing system with two phase service and Bernoulli vacation. Choudhury and Madan (2005) further investigated the system with a modified Bernoulli vacation and N-policy. Tadj et al. (2006) studied a bulk service queueing system with random setup time under the Bernoulli vacation and N-policy. They developed an algorithm to determine the optimal policy. Later, Choudhury (2007) extended this model to a two-phase batch arrival retrial queueing system with Bernoulli vacations. Also Choudhury et al. (2007) examined an $M^X/G/1$ queue with two-phase service and Bernoulli vacation and multiple vacation policy. Choudhury (2008) analyzed an M/G/1 retrial queue with two-phase service and Bernoulli vacation schedule.

Recently, Kumar et al. (2009) studied and M/M/c retrial queueing system with Bernoulli vacations and obtained various system performance measures. Ke and Chang (2009) investigated and $M^X/G_1,G_2/1$ retrial queue under Bernoulli vacation schedules with general repeated attempts and starting failures.

Choudhury and Paul (2006) studied a single server Bernoulli Vacations, where the concept of multiple vacation policy is introduced. After two successive phases of service one after first vacation, the server may go for further vacation until he finds at least one customer in the system. Madan and Choudhury (2006) considered two stages of heterogeneous service batch arrival queue with a Bernoulli Single server vacation where the server either goes for vacation with probability p after completion of two stages of heterogeneous service in successive or may continue to serve the next customer with probability (1-p). Furthermore, they assumed the restricted admissibility policy of arriving batches introduced by Madan et al., (2004). According to this policy, not all batches are allowed to join the system at all times. It is assumed that after termination of a busy period, as soon as a customer or a batch of customers arrives the server needs a random setup time (or the warming up time) before actually starting service of the first customer. The steady state queue size distribution at a random epoch as well as at a departure point of time are analysed we also obtained some important performance measure of this model. Kailash, Madan and Gautam choudhury (2006) studied about steady state Analysis of an $M^X/G_1,G_2/1$ queue with Restricted Admissibility and Random Setup Time.

1.4 THESIS ORGANIZATION

Queueing systems with server classical vacation are characterized by the fact that the idle time of the server may be used for other secondary jobs. In multiple and single vacation queueing models, the servers will go for vacations only when the system becomes empty. But in some situations, especially when the service is done in two or more phases, the maintenance of the system may be required at the completion of each service and in such case, the service may be stopped for maintenance or overhauling or continued if there is no fault in the system. The overhauling may be utilized as a vacation. The vacation policy which is characterized by the fact that, at the completion of each service, the server may take a vacation with probability p or may continue to serve the customers with probability (1-p) is called Bernoulli Schedule Vacation (BSV) policy.

Various concepts of Bernoulli vacation queueing models for single service queueing systems have been studied by a good number of authors. In the present work, the author proposed to analyse Bernoulli vacation models where the server provides two phases of heterogeneous services. Chapter II deals with the batch arrival queueing system under Bernoulli Schedule Vacation in which the server is perfect and never fails during service time. In chapter III, it is assumed that the server may meet unpredictable breakdowns during service time.

The model considered in chapter II has the following specifications:

The customers arrive in batches according to the compound Poisson process with group arrival rate λ. Busy period initiates as soon as a batch of customers arrives. During busy period the server provides two stages of heterogeneous service of which one is optional. The service discipline assumed to be FCFS. The first phase of service is essential for all the units, which is referred as First Essential Service (FES). As soon as the FES of a customer is complete, the customer may leave the system with probability (1-r) or may immediately opt for a second phase of optional service (SOS) with probability r $(0 \leq r \leq 1)$. The service times S_1 of FES and S_2 of SOS are assumed to be mutually independent of each other having general distributions, with distribution functions, $S_i(x)$, $i=1,2$. It is assumed that the server can take vacation between any two services. If the customer leaves the system soon after completing the FES, then the server may take a vacation called BSV of random length V with probability p_1 ($0 \leq r \leq 1$) or may continue to serve the next customer, if any or stay idle in the system with probability ($1-p_1$). On the other hand, if a customer finishes FES and opts the SOS, then the server necessarily provides the SOS to the customer. After finishing the SOS, the server may take vacation with probability p_2 or stay in the system with probability ($1-p_2$). The vacation time in either case is a random variable and follows the same general distribution V(x) with finite moments. It is also assumed that the vacation is single vacation in the sense that, whenever the vacation period ends, then the server necessarily joins the system even if the system is empty at the vacation terminating epochs. These types of services and vacations will continue until the system becomes empty. This model is denoted by $M^X/G_{SOS}/1/SBV$ where SBV denotes single Bernoulli vacation.

The model of chapter III differs from the model of chapter II only in service interruptions. i.e., It is assumed that the server may undergo unpredictable breakdowns during the service time. The life time of the service is assumed to follow exponential distributions with mean $1/a_1$ in FES and $1/a_2$ in SOS. Whenever the breakdown occurs, the server is sent for repair immediately and the customer just being served waits in the service facility to complete the remaining service. Immediately after the server is fixed, the customer waiting for the completion of the remaining service is considered for service. The repair time distributions due to different service phases (FES and SOS) are arbitrarily distributed with probability distribution function $R_i(y)$, density functions $r_i(y)$ and the finite moment $E(R_i^k)$ i=1,2. Further, it is also assumed that the server is as good as new, after repair. Thus a cycle of the model is the sum of idle period, busy period, break down period and vacation period.

Since the models analyzed in the chapters (II and III) are non-Morkovian, they are analyzed by using Supplementary Variable Technique. The remaining service time, remaining vacation time, and the remaining repair time are introduced as supplementary variables to convert the non-Markovian process into a Markov process. The steady state system size equations are obtained for the queueing systems using the Supplementary variable technique of Cox (1955). The equations are transferred to linear equations by taking Laplace Steiltjes Transformation and the PGF of the system size at random epoch as well as departure epoch are obtained. The system performance measures such as the probability that the system in busy state, idle state, vacation state and breakdown are calculated, and the mean system size is also derived. Numerical results are obtained and the system performance measures are also investigated.

To fix the idea of Second optional service queueing model, the basic $M^X/G_{SOS}/1$ queuing system is presented in the following section by introducing remaining service time as Supplementary variables.

1.5 $M^X/G_{SOS}/1$ Queueing System with Second Optional Service

In day to day life, there are numerous examples of the queueing situations where all arriving customers require the main service and only some may require the subsidiary service provided by the server. For example at a barber shop everyone may need a haircut but only a part of the customers may need a shave after the haircut.

The other situations are:

1) In a small town one finds many shops which sell coffee beans and grains of various kinds. All such shop-keepers normally have a grinding machine. All customers coming to such a shop buy grains or coffee beans but only some of these customers want to utilize the grinding facility.
2) All passengers wish to travel to a big town or a metropolitan city in a particular airline but only a part of these customers take airline's further flight to an interior destination of tourist's interest.
3) All students joining a particular teaching department of a university want to complete their undergraduate program of study but only some of them may join the postgraduate program soon after completing the undergraduate program.
4) All ships arriving at a port may need unloading service on arrival but only some of them may require re-loading service soon after the unloading.
5) All clients who come to meet a lawyer discuss their cases with her/him but only some of them actually hire her/him to fight their cases in a court of law.

Model description:

The following assumptions are made to analyze the mathematical model. The customers arrive in batches and the arrival streams form a Poisson process with parameter (λ). i.e., The arrival occurs according to a Compound Poisson process with arrival size random variable X with Probability distribution $\Pr(X=k)=g_k$, k=1,2,3,… The server serves the units in two heterogeneous phases of service. The service discipline is assumed to be first come first served (FCFS). The first phase of service is essential for all the units, which is referred to as the first essential service (FES). As soon as the FES of a unit is complete it may leave the system with probability (1- r) or may immediately opt for a second phase of optional service (SOS) with probability r ($0 \le r \le 1$). It is assumed that the service times S_1 , S_2 are mutually independent of each other having general distributions, with distribution functions $S_i(x)$, density functions $S_i(x)$ and the k^th moment $E(S^k)$, ($k \ge 1$), i=1,2 (denoting the FES and SOS respectively). Further, it is also assumed that the same server serves both the phases of service.

The steady state system size Equations:

Supplementary variable technique is used to derive the steady state system size equations. For this the remaining service times of the First Essential Service (FES) and the Second Optional Service (SOS) are introduced as the supplementary variables.

The following notations are used to derive the equations.

N : threshold

λ : group arrival rate

X : group size random variable

g_k : $\Pr[X = k]$

X(z) : probability generating function (PGF) of X

$N_S(t)$: system size (including one in service) at time t

$S_1(t)$: remaining first essential service time (FES) at time t

$S_2(t)$: remaining second optional service time (SOS) at time t

$$Y(t) \; : \; \begin{cases} 0 & \text{If the system is idle at time t} \\ 1 & \text{If the system is busy with FES at time t} \\ 2 & \text{If the system is busy with SOS at time t} \end{cases}$$

Let

$R_0(t) = \Pr[N_S(t) = 0, Y(t) = 0]$,

$P_{n1}(x, t)\, dt = \Pr[N_S(t) = n, Y(t) = 1, x \le S_1^\circ(t) \le x + dt,] \; ; n \ge 1, x > 0$

$P_{n,2}(x,t)\, dt = \Pr\{N_s(t) = n, x \le S_2^0(t) \le x+dt, Y(t) = 2\} \qquad n\ge1$

Then

$R_0(t)$ denotes the probability that the system is empty at time t.

$P_{n1}(x, t)$ denotes the probability that there are n-customers in the system and the server is doing the first essential service and the remaining service time of the customer in FES is in (x, x+Δt).

$P_{n,2}(x,t)$ denotes the probability that there are n-customers in the system and the server is doing the second optional service and the remaining service time of the customer in SOS is in (x, x+Δt).

By observing the changes of the states during (t, t+S(t)) at any time t, we obtain the following time dependent system size equations.

$$R_0(t+\Delta t) = R_0(t)(1-\lambda\Delta t) + (1-p_2)P_{1,2}(0,t)\Delta t + (1-r)(1-p_1)P_{1,1}(\theta,t)\Delta t$$

i.e., $$\frac{R_0(t+\Delta t) - R_0(t)}{\Delta t} = -\lambda R_0(t) + (1-p_2)P_{1,2}(0,t) + (1-r)(1-p_1)P_{1,1}(\theta,t)$$

As Δt → 0 the above equation becomes

$$R_0(t) = -\lambda R_0(t) + rP_{1,2}(0,t) + (1-r)P_{1,1}(0,t)$$

$$P_{1,1}(x-\Delta t, t+\Delta t) = P_{1,1}(x,t)(1-\lambda\Delta t) + (1-r)P_{2,1}(0,t)s_1(x)\Delta t + P_{2,2}(0,t)s_1(x)\Delta t$$
$$+\lambda R_0 g_1 s_1(x)\Delta t$$

$$\frac{P_{1,1}(x-\Delta t, t+\Delta t) - P_{1,1}(x,t)}{\Delta t} = -\lambda P_{1,1}(x,t) + (1-r)P_{2,1}(0,t)s_1(x) + P_{2,2}(0,t)s_1(x)$$
$$+\lambda R_0 g_n s_1(x)\Delta t$$

i.e., $$\frac{P_{1,1}(x-\Delta t,t+\Delta t)-P_{1,1}(x,t+\Delta t)+P_{1,1}(x,t+\Delta t)-P_{1,1}(x,t)}{\Delta t}$$

$$=-\lambda P_{1,1}(x,t)+(1-r)P_{2,1}(0,t)s_1(x)+P_{2,2}(0,t)s_1(x)+\lambda R_0 g_n s_1(x)\Delta t$$

As Δt → 0 the above equation becomes

$$-\frac{\partial}{\partial x}P_{1,1}(x,t)+\frac{\partial}{\partial t}P_{1,1}(x,t)=-\lambda P_{1,1}(x,t)+(1-r)P_{2,1}(0,t)s_1(x)+P_{2,2}(0,t)s_1(x)$$

$$+\lambda R_0 g_1 s_1(x)$$

Steady state system size equations:

At steady state as t → ∞, we have

$$R_0(t)=0;\ R_0(t)=R_0;$$

$$\frac{\partial}{\partial t}P_{n,i}(x,t)=0, P_{n,i}(x,t)=P_{n,i}(x) \quad \text{i=1,2} \quad \text{and} \quad \frac{\partial}{\partial x}P_{1,1}(x,t)=\frac{d}{dx}P_{1,1}(x);$$

Hence the first two equations at steady state are given by

$$\lambda R_0=rP_{1,2}(0)+(1-r)P_{1,1}(0) \tag{1.1}$$

$$-\frac{d}{dx}P_{1,1}(x)=-\lambda P_{1,1}(x)+(1-r)P_{2,1}(0)s_1(x)+P_{2,2}(0)s_1(x)+\lambda R_0 g_1 s_1(x) \tag{1.2}$$

Similarly, the other steady state equations are obtained as,

$$-\frac{d}{dx}P_{n,1}(x)=-\lambda P_{n,1}(x)+(1-r)P_{n+1,1}(0)s_1(x)+P_{n+1,2}(0)s_1(x)+\lambda R_0 g_n s_1(x)$$

$$+\lambda\sum_{k=1}^{n-1}P_{n-k,1}(x)g_k \qquad \text{n} \ge 2 \tag{1.3}$$

$$-\frac{d}{dx}P_{1,2}(x)=-\lambda P_{1,2}(x)+rP_{1,1}(0)s_2(x) \tag{1.4}$$

$$-\frac{d}{dx}P_{n,2}(x)=-\lambda P_{n,2}(x)+rP_{n,1}(0)s_2(x)+\lambda\sum_{k=1}^{n-1}P_{n-k,2}(x)g_k \qquad \text{n}\ge 2 \tag{1.5}$$

The Laplace-Stieltjes transform(LST) of the above equations are obtained by defining the following Laplace stieltjes transformations and using their properties.

$$P_{ni}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta.x} P_{ni}(x)dx \,, \qquad \text{i=1,2}$$

$$S_{i}^{*}(\theta) = \int_{0}^{\infty} e^{-\theta.x} dS_{i}(x) \,, \qquad \text{i=1,2}$$

Thus

The Laplace-Stieltjes transform (LST) of the above equations (1.1) to (1.5) are given by

$$(\theta-\lambda)P_{1,1}^{*}(\theta) = P_{1,1}(0) - (1-r)P_{2,1}(0)S_{1}^{*}(\theta) - P_{2,2}(0)S_{1}^{*}(\theta) - \lambda R_{0} g_{1} S_{1}^{*}(\theta) \qquad (1.6)$$

$$(\theta-\lambda)P_{n,1}^{*}(\theta) = P_{n,1}(0) - (1-r)P_{n+1,1}(0)S_{1}^{*}(\theta) - P_{n+1,2}(0)S_{1}^{*}(\theta) - \lambda R_{0} g_{n} S_{1}^{*}(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,1}^{*}(\theta) g_{k}$$

$$n \geq 2 \qquad (1.7)$$

$$(\theta-\lambda)P_{1,2}^{*}(\theta) = P_{1,2}(0) - rP_{1,1}(0)S_{2}^{*}(\theta)$$

$$(\theta-\lambda)P_{n,2}^{*}(\theta) = P_{n,2}(0) - rP_{n,1}(0)S_{2}^{*}(\theta) - \lambda \sum_{k=1}^{n-1} P_{n-k,2}^{*}(\theta) g_{k} - \lambda R_{0} g_{n} S_{1}^{*}(\theta) \qquad (1.9)$$

$$n \geq 2$$

Probability Generating Function

To analyse the system of the model, we define the following partial probability generation functions

$$P_{i}^{*}(z,\theta) \quad = \sum_{n=1}^{\infty} P_{n,i}^{*}(\theta) z^{n} \,, \qquad P_{i}(z,0) \quad = \sum_{n=1}^{\infty} P_{n,i}(0) z^{n} \qquad \text{i=1,2}$$

Thus $P_{i}^{*}(z,\theta)$ is gives the joint transform of the number of customers during busy period of FES and SOS accordingly i=1,2.

$P_{i}(z,0)$ is the generating function of the number of customers at departure epochs of FES and SOS accordingly, i=1,2.

To obtain the total probability generating function of the model, the expressions for the PGF's $P_{2}^{*}(z,\theta)$ and $P_{2}\,(z,0)$ are derived from equations (1.6) to (1.9).

Multiplying the equations (1.8) and (1.9) by the proper powers of z and then adding both the equations, we get

$$(\theta-\lambda)P_2^*(z,\theta) = P_2(z,0) - rP_1(z,0)S_2^*(\theta) - \lambda\sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} P_{n-k,2}^*(\theta)g_k \qquad (1.10)$$

Using the identity,

$$\sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} P_{n-k,1}(\theta)g_k = (\sum_{n=2}^{\infty} P_{n,1}^*(\theta)z^n)(\sum_{k=1}^{\infty} g_k z^k) \ = \ P^*(z,\theta)X(z), \text{ i=1,2}$$

The above equation becomes,

$$\left[\theta-\lambda+\lambda X(z)\right]P_2^*(z,\theta) = P_2(z,0) - rS_2^*(\theta)P_1(z,0) \qquad (1.11)$$

$P_2(z,0)$ can be obtained by letting $\theta = \lambda(1-X(z)) = w_x(z)$ in the above equation as

$$P_2(z,0) = rS_2^*(w_x(z))P_1(z,0) \qquad (1.12)$$

Substituting for $P_2(z,0)$ in (1.11),

$$P_2^*(z,\theta) = \frac{[r(S_2^*(w_x(z)) - S_2^*(\theta))]}{\theta - w_x(z)} P_1(z,0) \qquad (1.13)$$

Next to derive the expressions for the PGFS' $P_1^*(z,\theta)$ and $P_1(z,0)$ the equations (1.6) and (1.7) are used.

Multiplying the equations (1.6) and (1.7) by appropriate powers of z and then adding we have,

$$(\theta-\lambda)P_1^*(z,\theta) = -P_1(z,0) - \frac{(1-r)S_1^*(\theta)}{z}\left[P_1(z,0) - P_{1,1}(0)z\right] - \frac{S_1^*(\theta)}{z}[P_2(z,0) - P_{1,2}(0)z]$$

$$-\lambda\sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} P_{n-k,1}^*(\theta)g_k - \lambda R_0 S_1^*(\theta)\sum_{n=1}^{\infty} g_n z^n$$

Using the identity,

$$\sum_{n=2}^{\infty} z^n \sum_{k=1}^{\infty} P_{n-k,1}(\theta)g_k = (\sum_{n=2}^{\infty} P_{n,1}^*(\theta)z^n)(\sum_{k=1}^{\infty} g_k z^k) \ = \ P^*(z,\theta)X(z),$$

The above equation becomes,

$$(\theta - w_x(z))P_1^*(z,\theta) = P_1(z,0) \ - \frac{(1-r)S_1^*(\theta)}{z}\left[P_1(z,0)\right] - \frac{S_1^*(\theta)}{z}[P_2(z,0)]$$

$$+\lambda R_0 S_1^*(\theta) - \lambda R_0 S_1^*(\theta)X(z) \qquad (1.14)$$

Substituting for P$_2$(z,0) from (1.12),

$$(\theta - w_x(z))P_1^*(z,\theta) = P_1(z,0) - \frac{(1-r)S_1^*(\theta)}{z}\left[P_1(z,0)\right] - \frac{S_1^*(\theta)}{z} rS_2^*(w_x(z))(P_1(z,0))$$

$$+\lambda R_0 S_1^*(\theta)(1 - X(z))$$

$$(\theta - w_x(z))P_1^*(z,\theta) = P_1(z,0)\left(\frac{z - S_1^*(\theta)((1-r) + rS_2^*(w_x(z)))}{z}\right)P_1(z,0)$$

$$-\lambda R_0 S_1^*(\theta)(X(z)-1) \tag{1.15}$$

$P_1\ (z,0)$ can be obtained by letting $\theta = w_x(z)$ in the above equation as

$$P_1(z,0) = \frac{\lambda z S_1^*(w_x(z))R_0(X(z)-1))}{z - S_1^*(w_x(z))((1-r) + rS_2^*(w_x(z)))} \tag{1.16}$$

Thus equation (1.15) is reduced to

$$(\theta - w_x(z))P_1^*(z,\theta) = \frac{\lambda S_1^*(w_x(z))[z - S_1^*(\theta)((1-r) + rS_2^*(w_x(z)))]R_0(X(z)-1))}{z - S_1^*(w_x(z))((1-r) + rS_2^*(w_x(z)))}$$

$$-\lambda R_0 S_1^*(\theta)(X(z)-1)$$

$$P_1^*(z,\theta) = \frac{\lambda[S_1^*(w_x(z)) - S_1^*(\theta)]zR_0(X(z)-1)}{[z - S_1^*(w_x(z))((1-r) + rS_2^*(w_x(z)))](\theta - w_x(z))}$$

Therefore, when θ=0, the PGF of the system size probabilities when the server is busy with FES is obtained as,

$$P_1^*(z,0) = \frac{[S_1^*(w_x(z)) - 1]zR_0}{[z - S_1^*(w_x(z))((1-r) + rS_2^*(w_x(z)))]} \tag{1.17}$$

Using (1.16) in (1.13), we have

$$P_2^*(z,\theta) = \frac{r\lambda R_0(X(z)-1)[S_2^*(w_x(z)) - S_1^*(\theta)]zS_1^*(w_x(z))}{[z - S_1^*(w_x(z))((1-r) + rS_2^*(w_x(z)))](\theta - w_x(z))}$$

Therefore, at θ=0, we have, the PGF of the system size probabilities when the server is busy with SOS is given by,

$$P_2^*(z,0) = \frac{rR_0[S_2^*(w_x(z))-1]zS_1^*(w_x(z))}{[z-S_1^*(w_x(z))((1-r)+rS_2^*(w_x(z)))]} \tag{1.18}$$

Thus the total probability generating functions P(z) for the $M^X/G_{SOS}/1$ queue model is given by

$$P(z) = P_1^*(z,0) + P_2^*(z,0) + R_0 = \frac{R_0(z-1)[(1-r)+rS_2^*(w_x(z))-1)]S_1^*(w_x(z))}{[z-S_1^*(w_x(z))((1-r)+rS_2^*(w_x(z)))]} \tag{1.19}$$

Using the normalizing condition

$\lim\limits_{z\to 1} P(z) = 1$ implies $R_0 = \dfrac{z-1}{1-SOS^*(w_x(z))}$

where $SOS^*(w_x(z)) = S_1^*(w_x(z))(1-r+rS_2^*(w_x(z)))$

i.e., $R_0 = (1-\rho_{SOS})$ where $(1-\rho_{SOS}) = \lambda E(X)E(SOS)$

$E(SOS) = E(S_1) + rE(S_2)$

Theorem

The probability generating function of the stationary system size distribution for a $M^X/G_{SOS}/1$ queue under the stability condition $\rho_{SOS} < 1$ is given by

$$P(z) = \frac{(1-\rho_{SOS})(z-1)[(1-r)+rS_2^*(w_x(z))-1)]S_1^*(w_x(z))}{[z-S_1^*(w_x(z))((1-r)+rS_2^*(w_x(z)))]} \tag{1.20}$$

Chapter - II

CHAPTER-II

A BATCH ARRIVAL QUEUEING SYSTEM WITH A SECOND OPTIONAL SERVICE CHANNEL UNDER BERNOULLI SINGLE VACATION

Queueing systems with server's vacations have been studied extensively by many researchers in different frame works and applications of such vacation models are also elaborated. Vacation concept was first introduced to utilize the idle time of the server for ancillary works assigned to the server. The standard single and multiple vacation policies are framed under the exhaustive service discipline. That is, the server will take vacation only when all the customers are served and the system becomes empty.

In some practical situations, the server may need a random amount of time in between services to run the system smoothly. For example, when a single server operates multi-optional services, the server may require time to switch over from one mode of service to another at each service completion. If the server is a machine, it may have to undergo testing, overhauling or preventive maintenance work. This will extend the life time of the server and avoid frequent breakdowns of the server. The random amount of time required by the server to do the secondary jobs in between services is referred as a vacation. According to this vacation policy, the server on completion of a service to a customer, may take vacation with probability p or may continue to stay in the system to serve the next unit if any with probability (1 - p). Since the vacation is controlled by the parameter p, the congestion of the system may be controlled by adjusting the value of p. This type of vacation was originated and developed significantly by Keilson and Servi (1986) and it is termed as Bernoulli Schedule Vacation (BSV).

In this chapter, a bulk arrival queueing system with SOS facility is analysed under Bernoulli single vacation. The server provides two stages of service. The service in the first stage is single and all the customer will necessarily undergo the service. The service is in second stage is optimal. The customers after finishing the FES may either opt the SOS or leave the system after completing the FES. Because of the complexity, the authors who studied the two stage service queueing models with BSV (Madan and Choudhury (2004, 2005) and Madan et al. (2005)), allow the server to take vacation only after finishing the second stage of service. i.e., only a single parameter p is used to control the vacation policy. But in the model of this chapter, two different parameters are introduced so that the server either continues to serve the next unit or takes vacation after completing service to a customer.

2.1 MATHEMATICAL ANALYSIS OF THE SYSTEM

Model description :

The system has the Following specifications.

Arrival pattern

The customers arrive in batches in accordance with the time homogeneous Poisson process with group arrival rate λ. The batch size X is a random variable with probability distribution $Pr(X=k)=g_{k},$ k=1,2,3,… (i.e.,) the probability that the batch of k units arrive in an infinitesimal interval (t,t+h) is $\lambda g_k h+o(h)$. The arriving customers will join the system and form a single waiting line based on the order of the batches. The customers within a batch are pre-ordered for service. The customers are served one by one according to the First Come First Served queue discipline.

Initially the system is empty and the server is idle in the system. Busy period initiates as soon as a batch of customers arrives.

Service pattern

During busy period, the server provides single First Essential Service (FES) to all the arriving customers. As soon as the FES of a customer is completed the customer may either opt the Second Optional Service (SOS) with probability (r) or may leave the system with probability (1-r). It is assumed that the service time random variables S_1 and S_2 of FES and SOS respectively are mutually independent of each other having heterogeneous general law of distributions with distribution functions $S_1(x)$ and $S_2(x)$ respectively.

Bernoulli schedule vacation policy

After completing service to each customer, the server may take vacation or continue to serve the next service. (i.e.,) If a customer leaves the system after FES(without opting the SOS), then the server may take a Bernoulli single vacation of random length V with Probability $p_1(0\leq p_1\leq 1)$ or may continue to serve the next customer if any, or stay idle in the system with probability $(1- p_1)$. On the other hand, if a customer finishes FES and opts for the SOS, then the server can take vacation only after finishing SOS for the customer, with probability p_2 or continue to stay in the system with probability $(1- p_2)$.

The vacation time in either case is a random variable and follows the same general distribution V(x) of finite moments. It is assumed that the vacation is a single vacation. (i.e.,) Whenever the vacation period ends, the server joins the system irrespective of whether there are customers waiting in the system. This type of service continues until the system becomes empty.

The model is denoted by $\mathbf{M^X/G_{SOS}/1/SBV}$.

The system is analysed using supplementary variable technique by introducing remaining service time, remaining vacation time as supplementary variables. To obtain the steady state system size equations, the following notations are introduced.

λ : group arrival rate

X : group size random variable

g_k : Pr(X=k), k=1,2,3,…

X(z) : Probability generating function of X.

$N_s(t)$: The system size at time t

$$\text{Let } Y(t) = \begin{cases} 0 & \text{if the system is in vacation state} \\ 1 & \text{if the system is in FES state} \\ 2 & \text{if the system is in SOS state} \\ 3 & \text{if the system is in idle state} \end{cases}$$

The notations used for Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its k^{th} moments are listed below:

	RV	**CDF**	**Pdf**	**LST**	**kth moment**
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$
FES	S_1	$S_1(x)$	$S_1(x)$	$S_1^*(\theta)$	$E(S_1^k)$
SOS	S_2	$S_2(x)$	$S_2(x)$	$S_2^*(\theta)$	$E(S_2^k)$

Where $F^*(\theta) = \int_0^\infty e^{-\theta . t} dF(t)$

Let $V^0(t)$, $S_1^0(t)$ and $S_2^0(t)$ denote the remaining vacation time, remaining first essential service time and remaining second optional service time respectively at time t. Then the state space $\{N(t),\delta(t)\}$ where $\delta(t)\}=V^0(t)$, $S_1^0(t)$, $S_2^0(t)$,0 according as Y(t) = 0 to 3 respectively defines a bivariate Markov process.

Let

$Q_{n,B}(x,t)dt(Q_{n,I}(x,t)dt)$ = $\Pr\{N_s(t) = n, x \le V^0(t) \le x+dt, Y(t) = 0\}$ $n \ge 0$

$P_{n,1}(x,t)dt$ = $\Pr\{N_s(t) = n, x \le S_1^0(t) \le x+dt, Y(t) = 1\}$ $n \ge 1$

$P_{n,2}(x,t)dt$ = $\Pr\{N_s(t) = n, x \le S_2^0(t) \le x+dt, Y(t) = 2\}$ $n \ge 1$

$R_0(t)$ = $\Pr\{N_s(t) = 0, Y(t) = 3\}$

Thus,

$Q_{n,B}(x,t)$ denotes the probability that at time t there are n customers in the system, the server is on vacation during busy period and remaining vacation time is x.

$Q_{n,I}(x,t)$ denotes the probability that at time t there are n customers in the system, the server is on vacation during idle period and remaining vacation time is x.

$P_{n,1}(x,t)$ denotes the probability that at time t there are n customers in the system, the server is busy with FES and the remaining service time lies in the interval $(x, x+\Delta t)$.

$P_{n,2}(x,t)$ denotes the probability that at time t there are n customers in the system, the server is busy with SOS and the remaining service time lies in the interval (x, x+Δt).

$R_0(t)$ denotes the probability that the system is empty at time t.

Now using the supplementary variable technique (Cox (1955)) the following time dependent system equations are obtained for the proposed queueing model.

During idle period,

$$R_0(t+\Delta t) = R_0(t)(1-\lambda\Delta t)+(1-p_2)P_{1,2}(0,t)\Delta t+(1-r)(1-p_1)P_{1,1}(\theta,t)\Delta t+Q_0(0,t)\Delta t$$

i.e., $$\frac{R_0(t+\Delta t)-R_0(t)}{\Delta t}=-\lambda R_0(t)+(1-p_2)P_{1,2}(0,t)+(1-r)(1-p_1)P_{1,1}(\theta,t)+Q_0(0,t)$$

Taking limit as Δt→0, the above equation leads to

$$\frac{d}{dx}R_0(t)=-\lambda R_0(t)+(1-p_2)P_{1,2}(0,t)+(1-r)(1-p_1)P_{1,1}(0,t)+Q_0(0,t)$$

When the server is on vacation and the system is empty, then,

$$Q_{0,I}(x-\Delta t,t+\Delta t)=Q_{0,I}(x,t)(1-\lambda\Delta t)+p_2P_{1,2}(0,t)\Delta t+(1-r)p_1P_{1,1}(0,t)\Delta t$$

Adding and Subtracting $Q_{0,I}(x,t+\Delta t)$

$$\frac{Q_{0,I}(x-\Delta t,t+\Delta t)-Q_{0,I}(x,t+\Delta t)+Q_{0,I}(x,t+\Delta t)-Q_{0,I}(x,t)}{\Delta t}=-\lambda Q_{0,I}(x,t)$$

$$+p_2P_{1,2}(0,t)v(x)+(1-r)p_1P_{1,1}(0,t)v(x)$$

Taking limit as Δt→0

$$-\frac{\partial}{\partial x}Q_{0,I}(x,t)+\frac{\partial}{\partial t}Q_{0,I}(x,t)=-\lambda Q_{0,I}(x,t)+p_2P_{1,2}(0,t)v(x)+(1-r)p_1P_{1,1}(0,t)v(x)$$

Similarly,

The other set of equations namely,

$$Q_{n,I}(x-\Delta t,t+\Delta t)=Q_{n,I}(x,t)(1-\lambda\Delta t)+\lambda\sum_{k=1}^{n}Q_{n-k,I}(x,t)g_k\Delta t \qquad n\geq 1$$

$$Q_{1,B}(x-\Delta t,t+\Delta t)=Q_{1,B}(x,t)(1-\lambda\Delta t)+p_2P_{2,2}(0,t)v(x)\Delta t+(1-r)p_1p_{2,1}(0,t)v(x)\Delta t$$

$$Q_{n,B}(x-\Delta t,t+\Delta t)=Q_{n,B}(x,t)(1-\lambda\Delta t)+p_2P_{n+1,2}(0,t)v(x)\Delta t+(1-r)p_1P_{n+1,1}(0,t)v(x)\Delta t$$

$$+\lambda\sum_{k=1}^{n-1}Q_{n-k,B}(x,t)g_k\Delta t \qquad \text{n≥2}$$

$$P_{1,1}(x-\Delta t,t+\Delta t)=P_{1,1}(x,t)(1-\lambda\Delta t)+(1-r)(1-p_1)P_{2,1}(0,t)s_1(x)\Delta t+(1-p_2)P_{2,2}(0,t)s_1(x)\Delta t$$

$$+\lambda R_0g_1s_1(x)\Delta t+(Q_{1,B}(0,t)\Delta t+Q_{1,I}(0,t)\Delta t)s_1(x)$$

$$P_{n,1}(x-\Delta t,t+\Delta t)=P_{n,1}(x,t)(1-\lambda\Delta t)+(1-r)(1-p_1)P_{n+1,1}(0,t)S_1(x)\Delta t+P_{n+1,2}(0,t)(1-p_2)s_1(x)\Delta t$$

$$+\lambda R_0g_ns_1(x)\Delta t+\lambda\sum_{k=1}^{n-1}P_{n-k,1}(x,t)g_k\Delta t+Q_{n,B}(0,t)s_1(x)\Delta t+Q_{n,I}(0,t)s_1(x)\Delta t$$

$$\text{n≥2}$$

$$P_{1,2}(x-\Delta t,t+\Delta t)=P_{1,2}(x,t)(1-\lambda\Delta t)+rP_{1,1}(0,t)s_2(x)\Delta t$$

$$P_{n,2}(x-\Delta t,t+\Delta t)=P_{n,2}(x,t)(1-\lambda\Delta t)+rP_{n,1}(0,t)s_2(x)\Delta t+\lambda\sum_{k=1}^{n-1}P_{n-k,2}(x,t)g_k\Delta t \qquad \text{n≥2}$$

will lead to

$$-\frac{\partial}{\partial x}Q_{n,I}(x,t)+\frac{\partial}{\partial t}Q_{n,I}(x,t)=-\lambda Q_{n,I}(x,t)+\lambda\sum_{k=1}^{n}Q_{n-k,I}(x,t)g_k \qquad \text{n≥1}$$

$$-\frac{\partial}{\partial x}Q_{1,B}(x,t)+\frac{\partial}{\partial t}Q_{1,B}(x,t)=-\lambda Q_{1,B}(x,t)+p_2P_{2,2}(0,t)v(x)+(1-r)p_1p_{2,1}(0,t)v(x)$$

$$-\frac{\partial}{\partial x}Q_{n,B}(x,t)+\frac{\partial}{\partial t}Q_{n,B}(x,t)=-\lambda Q_{n,B}(x,t)+p_2P_{n+1,2}(0,t)v(x)+(1-r)p_1P_{n+1,1}(0,t)v(x)$$

$$+\lambda\sum_{k=1}^{n-1}Q_{n-k,B}(x,t)g_k \qquad \text{n≥2}$$

$$-\frac{\partial}{\partial x}P_{1,1}(x,t)+\frac{\partial}{\partial t}P_{1,1}(x,t)=-\lambda P_{1,1}(x,t)+(1-r)(1-p_1)P_{2,1}(0,t)s_1(x)+(1-p_2)P_{2,2}(0,t)s_1(x)$$

$$+\lambda R_0g_1s_1(x)+(Q_{1,B}(0,t)+Q_{1,I}(0,t))s_1(x)$$

$$-\frac{\partial}{\partial x}P_{n,1}(x,t)+\frac{\partial}{\partial t}P_{n,1}(x,t)=-\lambda P_{n,1}(x,t)+(1-r)(1-p_1)P_{n+1,1}(0,t)s_1(x)+P_{n+1,2}(0,t)(1-p_2)s_1(x)$$
$$+\lambda R_0 g_n s_1(x)+\lambda\sum_{k=1}^{n-1}P_{n-k,1}(x,t)g_k+Q_{n,B}(0,t)s_1(x)+Q_{n,I}(0,t)s_1(x) \quad n\geq 2$$

$$-\frac{\partial}{\partial x}P_{1,2}(x,t)+\frac{\partial}{\partial t}P_{1,2}(x,t)=-\lambda P_{1,2}(x,t)+rP_{1,1}(0,t)s_2(x)$$

$$-\frac{\partial}{\partial x}P_{n,2}(x,t)+\frac{\partial}{\partial t}P_{n,2}(x,t)=-\lambda P_{n,2}(x,t)+rP_{n,1}(0,t)s_2(x)+\lambda\sum_{k=1}^{n-1}P_{n-k,2}(x,t)g_k \qquad n\geq 2$$

Assuming that at steady state the probabilities are independent of time t, so that the steady state equations are obtained:

Under the steady state,

$$\lim_{t\to\infty}\frac{\partial}{\partial x}Q_n(x,t)=\frac{d}{dx}Q_n(x); \quad \lim_{t\to\infty}\frac{\partial}{\partial x}P_{n,i}(x,t)=\frac{d}{dx}P_{n,i}(x); \quad i=1,2$$

$$\lim_{t\to\infty}\frac{\partial}{\partial t}P_{n,i}(0,t)=\lim_{t\to\infty}\frac{\partial}{\partial t}Q_n(x,t)=0$$

$$\lim_{t\to\infty}P_{n,i}(0,t)=P_{n,i}(0); \qquad \lim_{t\to\infty}Q_n(x,t)=Q_n(x); \qquad \lim_{t\to\infty}P_{n,i}(x,t)=P_{n,i}(x); \ i=1,2$$

$$\lim_{t\to\infty}R_0(t)=R_0; \qquad \lim_{t\to\infty}Q_n(0,t)=Q_n(0);$$

Then the steady state system size equations are given by

Idle state

$$\lambda R_0=P_{1,2}(0)(1-p_2)+(1-r)(1-p_1)P_{1,1}(0)+Q_{0,I}(0) \qquad (2.0.0)$$

Vacation state during idle period

$$-\frac{d}{dx}Q_{0,I}(x)=-\lambda Q_{0,I}(x)+P_{1,2}(0)p_2v(x)+P_{1,1}(0)(1-r)p_1v(x) \qquad (2.0.1)$$

$$-\frac{d}{dx}Q_{n,I}(x)=-\lambda Q_{n,I}(x)+\lambda\sum_{k=1}^{n}Q_{n-k,I}(x)g_k \qquad n\geq 1 \qquad (2.0.2)$$

Vacation state during busy period

$$-\frac{d}{dx}Q_{1,B}(x)=-\lambda Q_{I,B}(x)+P_{2,2}(0)p_2v(x)+(1-r)p_1P_{2,1}(0)v(x) \qquad (2.0.3)$$

$$-\frac{d}{dx}Q_{n,B}(x)=-\lambda Q_{n,B}(x)+p_2P_{n+1,2}(0)v(x)+(1-r)p_1P_{n+1,1}(0)v(x)+\lambda\sum_{k=1}^{n-1}Q_{n-k,B}(x)g_k$$

$$n\geq 2 \qquad (2.0.4)$$

Busy state

Busy with FES

$$-\frac{d}{dx}P_{1,1}(x)=-\lambda P_{1,1}(x)+(1-r)(1-p_1)P_{2,1}(0)s_1(x)+(1-p_2)P_{2,2}(0)s_1(x)+\lambda R_0g_1s_1(x)$$

$$+(Q_{1,B}(0)+Q_{1,I}(0))s_1(x) \qquad (2.0.5)$$

$$-\frac{d}{dx}P_{n,1}(x)=-\lambda P_{n,1}(x)+(1-r)(1-p_1)P_{n+1,1}(0)s_1(x)+P_{n+1,2}(0)(1-p_2)s_1(x)+\lambda R_0g_ns_1(x)$$

$$+\lambda\sum_{k=1}^{n-1}P_{n-k,1}(x)g_k+(Q_{n,B}(0)+Q_{n,I}(0))s_1(x) \qquad n\geq 2 \qquad (2.0.6)$$

Busy with SOS

$$-\frac{d}{dx}P_{1,2}(x)=-\lambda P_{1,2}(x)+rP_{1,1}(0)s_2(x) \qquad (2.0.7)$$

$$-\frac{d}{dx}P_{n,2}(x)=-\lambda P_{n,2}(x)+rP_{n,1}(0)s_2(x)+\lambda\sum_{k=1}^{n-1}P_{n-k,2}(x)g_k \qquad n\geq 2 \qquad (2.0.8)$$

By taking LST on both sides for the steady state equations (2.0.0) to (2.0.7), we have

$$\lambda R_0 = P_{2,1}(0)(1-p_2)+(1-r)(1-p_1)P_{1,1}(0)+Q_{0,I}(0) \qquad (2.1)$$

$$\theta Q^*_{0,I}(\theta)-Q_{0,I}(0)=\lambda Q^*_{0,I}(\theta)-p_2P_{1,2}(0)V^*(\theta)-(1-r)p_1P_{1,1}(0)V^*(\theta) \qquad (2.2)$$

$$\theta Q^*_{n,I}(\theta)-Q_{n,I}(0)=\lambda Q^*_{n,I}(\theta)-\lambda\sum_{k=1}^{n}Q^*_{n-k,I}(\theta)g_k \qquad n\geq 1 \qquad (2.3)$$

$$\theta Q^*_{1,B}(\theta)-Q_{1,B}(0)=\lambda Q^*_{1,B}(\theta)-p_2P_{2,2}(0)V^*(\theta)-p_1(1-r)P_{2,1}(0)V^*(\theta) \qquad (2.4)$$

$$\theta Q^*_{n,B}(\theta)-Q_{n,B}(0)=\lambda Q^*_{n,B}(\theta)-p_2P_{n+1,2}(0)V^*(\theta)-p_1(1-r)P_{n+1,1}(0)V^*(\theta)$$

$$-\lambda\sum_{k=1}^{n-1}Q^*_{n-k,B}(\theta)g_k \quad n\geq 2 \qquad (2.5)$$

$$\theta P_{1,1}^*(\theta)-P_{1,1}(0)=\lambda P_{1,1}^*(\theta)-(1-p_1)(1-r)P_{2,1}(0)S_1^*(\theta)-(1-p_2)P_{2,2}(0)S_1^*(\theta)-\lambda R_0 g_1 S_1^*(\theta)$$
$$-(Q_{1,B}(0)+Q_{1,I}(0))S_1^*(\theta) \qquad (2.6)$$

$$\theta P_{n,1}^*(\theta)-P_{n,1}(\theta)=\lambda P_{n,1}^*(\theta)-P_{n+1,1}(0)(1-p_1)(1-r)S_1^*(\theta)-P_{n+1,2}(0)(1-p_2)S_1^*(\theta)$$
$$-\lambda R_0 g_n S_1^*(\theta)-\lambda\sum_{k=1}^{n-1}P_{n-k,1}^*(\theta)g_k-(Q_{n,B}(0)+Q_{n,I}(0))S_1^*(\theta) \qquad (2.7)$$

$$\theta P_{1,2}^*(\theta)-P_{1,2}(0)=\lambda P_{1,2}^*(\theta)-rP_{1,1}(0)S_2^*(\theta) \qquad (2.8)$$

$$\theta P_{n,2}^*(\theta)-P_{n,2}(0)=\lambda P_{n,2}^*(\theta)-P_{n,1}(0)rS_2^*(\theta)-\lambda\sum_{k=1}^{n-1}P_{n-k,2}^*(\theta)g_k \qquad n\geq 2 \qquad (2.9)$$

Let

$$Q_{n,I}^*(\theta)+Q_{n,B}^*(\theta)=Q_n^*(\theta) \qquad Q_{n,I}^*(0)+Q_{n,B}^*(0)=Q_n^*(0) \qquad (2.10)$$

$$Q_{0,I}^*(\theta)=Q_0^*(\theta) \qquad Q_{0,I}^*(0)=Q_0^*(0) \qquad n\geq 1 \qquad (2.11)$$

Using equations (2.10) and (2.11),

The equation (2.2) can be written as,

$$\theta Q_0^*(\theta)-Q_0(0)=\lambda Q_0^*(\theta)-p_2P_{1,2}(0)V^*(\theta)-p_1(1-r)P_{1,1}(0)V^*(\theta)$$ and

equations (2.2) to (2.5) can be rewritten as,

$$\theta Q_n^*(\theta)-Q_n(\theta)=\lambda Q_n^*(\theta)-(p_2P_{n+1,2}(0)+p_1(1-r)P_{n+1,1}(0))V^*(\theta)$$
$$-\lambda\left(\sum_{k=1}^{n-1}Q_{n-k}^*(\theta)g_k+Q_{0,I}^*(\theta)g_n\right) \qquad n\geq 1$$

Thus the equations corresponding to $Q_n^*(\theta)$ are given by

$$\theta Q_0^*(\theta)-Q_0(0)=\lambda Q_0^*(\theta)-p_2P_{1,2}(0)V^*(\theta)-p_1(1-r)P_{1,1}(0)V^*(\theta) \qquad (2.12)$$

$$\theta Q_n^*(\theta)-Q_n(\theta)=\lambda Q_n^*(\theta)-(p_2P_{n+1,2}(0)+p_1(1-r)P_{n+1,1}(0))V^*(\theta)-\lambda\sum_{k=1}^{n}Q_{n-k}^*(\theta)g_k$$
$$n\geq 1 \qquad (2.13)$$

Probability Generating Function

Now to obtain the partial PGFs of the number of customers in the system, the following partial PGF's are defined

$$Q^*(z,\theta) = \sum_{n=0}^{\infty} Q_n^*(\theta) z^n, \qquad Q(z,0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$P_i^*(z,\theta) = \sum_{n=1}^{\infty} P_{n,i}^*(\theta) z^n, \qquad P_i(z,0) = \sum_{n=1}^{\infty} P_{n,i}(0) z^n \qquad \text{i=1,2}$$

Multiplying the equations (2.12) and (2.13) by suitable powers of z^n and summing over n≥0, the partial generating function of the system size, when server is on vacation is obtained as

$$\theta Q^*(z,\theta) - Q(z,0) = \lambda Q^*(z,\theta) - \frac{V^*(\theta)}{z}(p_2 P_2(z,0) + p_1(1-r)P_1(z,0)) + \lambda X(z) Q^*(z,\theta) \tag{2.14}$$

{Since $\sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} Q_{n-k}^*(\theta) g_k = (\sum_{n=0}^{\infty} Q_n^*(\theta) z^n)(\sum_{k=1}^{\infty} g_k z^k) = Q^*(z,0)X(z)$ }

The equation (2.14) can be written as

$$[\theta - w_x(z)]Q^*(z,\theta) = Q(z,0) - \frac{V^*(\theta)}{z}\left[p_2 P_2(z,0) + (1-r)p_1 P_1(z,0)\right] \tag{2.15}$$

where $w_x(z) = \lambda(1 - X(z))$

At $\theta = w_x(z)$, Q(z,0) is obtained as,

$$Q(z,0) = \frac{V^*(w_x(z))}{z}\left[p_2 P_2(z,0) + (1-r)p_1 P_1(z,0)\right] \tag{2.16}$$

Substituting for Q(z,0) in (2.15),

$$Q^*(z,\theta) = \frac{\left[V^*(w_x(z)) - V^*(\theta)\right]}{z(\theta - w_x(z))}\left[p_2 P_2(z,0) + (1-r)p_1 P_1(z,0)\right] \tag{2.17}$$

Similarly,

multiplying the equations (2.8) and (2.9) by suitable powers of z^n and adding over n=1 to ∞, the partial generating function corresponding to the second optional service is calculated and given by,

$$\theta P_2^*(z,\theta) - P_2(z,0) = \lambda P_2^*(z,\theta) - rP_1(z,0)S_2^*(\theta) - \lambda \sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} P_{n-k,2}^*(\theta) g_k$$

Using the identity,

$$\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} P_{n-k,i}(\theta) g_k = (\sum_{n=0}^{\infty} P_{n,i}^*(\theta) z^n)(\sum_{k=1}^{\infty} g_k z^k) = P^*(z,\theta) X(z), (\text{ i=1,2}) \quad (2.18)$$

$P_2^*(z,\theta)$ is given by,

$$\left[\theta - w_x(z)\right] P_2^*(z,\theta) = P_2(z,0) - rS_2^*(\theta) P_1(z,0) \quad (2.19)$$

Evaluating P2(z,0) at $\theta = w_x(z)$, we get

$$P_2(z,0) = rS_2^*(w_x(z)) P_1(z,0) \quad (2.20)$$

Thus $P_2^*(z,\theta) = \dfrac{[r(S_2^*(w_x(z) - S_2^*(\theta))]}{\theta - w_x(z)} P_1(z,0)$ (2.21)

multiplying the equations (2.6) and (2.7) by z^n and adding over n=1 to ∞ and using the identity (2.18) the following equation is obtained.

$$\begin{aligned} \theta P_1^*(z,\theta) - P_1(z,0) = {} & w_x(z)(P_1^*(z,\theta)) \\ & - \frac{S_1^*(\theta)}{z} \left[((1-p_1)(1-r)P_1(z,0) - P_{1,1}(0)z) + ((1-p_2)(P_2(z,0) - P_{1,2}(0)z))\right] \\ & - S_1^*(\theta)[\lambda R_0 X(z) + Q(z,0) - Q_0(0)] \end{aligned}$$

Where $Q(z,0) = \sum_{n=0}^{\infty} (Q_{n,B}(0) + Q_{n,I}(0)) z^n$

Substituting for $P_2(z,0)$ and $Q(z,0)$ from equations (2.20) and (2.16), the above equation is simplified as

$$(\theta - w_x(z)) P_1^*(z,\theta) - P_1(z,0) =$$

$$\begin{aligned} & - \frac{S_1^*(\theta)}{z} \left[(1-p_1)(1-r) + (1-p_2) r S_2^* w_x(z) + V^* w_x(z)(p_2 r S_2^* w_x(z) + (1-r)p_1)\right] P_1(z,0) \\ & + S_1^*(\theta)[\lambda R_0 (1 - X(z))] \end{aligned}$$

(Since $\lambda R_0 = (1-p_1)(1-r)P_{1,1}(0) - P_{1,1}(0)z) + (1-p_2)P_{2,1}(0) + Q_0(0)$)

$$[\theta - w_x(z)]P_1^*(z,\theta) = \frac{P_1(z,0)}{z}(z-\phi(z)S_1^*(\theta)) + S_1^*(\theta)R_0 w_x(z) \tag{2.22}$$

Where $\phi(z) = (1-p_1)(1-r)+(1-p_2)rS_2^* w_x(z) + V^* w_x(z)(p_2 r S_2^* w_x(z) + (1-r)p_1)$

$$= (V^* w_x(z)-1)(p_2 r S_2^* w_x(z)+(1-r)p_1)+(1-r+rS_2^* w_x(z) \tag{2.23}$$

Thus $P_1(z,0)$ can be obtained by putting $\theta = w_x(z)$ as,

$$P_1(z,0) = \frac{-zS_1^*(w_X(z)R_0 w_X(z))}{z-S_{B\mathbf{V}}^*(w_X(z))} \tag{2.24}$$

Where $S_{B\mathbf{V}}^*(w_x(z) = S_1^* w_x(z)\phi(z)$

$$= [(V^* w_x(z)-1)(p_2 r S_2^* w_x(z)+(1-r)p_1)+(1-r+rS_2^* w_x(z))]S_1^* w_x(z)$$

which can be written as,

$$S_{B\mathbf{V}}^*(w_x(z) = SOS^*(w_x(z)) + (V^* w_x(z)-1)[p_2 r S_2^* w_x(z)+(1-r)p_1](S_1^* w_x(z)) \tag{2.25}$$

where $SOS^*(w_x(z)) = S_1^* w_x(z)+(1-r+rS_2^* w_x(z)$ (2.26)

Substituting for $P_1(z,0)$ in (2.22) and simplifying, $P_1^*(z,\theta)$ given by

$$P_1^*(z,\theta) = \frac{-zR_0 w_X(z)S_1^*(w_X(z)-S_1^*(\theta))}{z-S_{BV}^*(w_X(z))(\theta-w_x(z))} \tag{2.27}$$

Substituting for $P_2(z,0)$ and $P_1(z,0)$ from equations (2.20) and (2.24) in equation (2.17),

$$Q^*(z,\theta) = \left[\frac{V^*(w_x(z))-V^*(\theta)}{\theta-w_x(z)}\right][p_2 r S_2^*(w_x(z))+(1-r)p_1]\left(\frac{-S_1^*(w_x(z))R_0 w_x(z)}{z-S_{BV}^*(w_x(z))}\right) \tag{2.28}$$

Substituting for $P_1(z,0)$ from equation (2.24) in equation (2.21),

$$P_2^*(z,\theta) = \frac{[rS_2^*(w_X(z)) - S_2^*(\theta)]}{(\theta - w_x(z))}\left\{\frac{-zS_1^* w_x(z) w_x(z) R_0}{z - S_{BV}^*(w_X(z))}\right\} \tag{2.29}$$

Thus the partial generating function of the system size probabilities corresponding to different states at arbitrary epochs can be obtained from equations (2.27),(2.28) and (2.29).

Thus by putting $\theta = 0$ in equations (2.27),(2.28) and (2.29),the partial generating functions at arbitrary epochs are given by

$$P_1^*(z,0) = \left(\frac{z(S_1^*(w_x(z)) - 1)}{z - S_{BV}^*(w_x(z))}\right) R_0 \tag{2.30}$$

$$P_2^*(z,0) = \frac{rz\left[S_2^*(w_x(z)) - 1\right]}{z - S_{BV}^*(w_x(z))} S_1^*(w_x(z)) R_0 \tag{2.31}$$

$$Q^*(z,0) = (V^*(w_x(z)) - 1)[p_2 r S_2^*(w_x(z)) + (1-r)p_1] \left(\frac{S_1^*(w_x(z))}{z - S_{BV}^*(w_x(z))}\right) R_0 \tag{2.32}$$

Let P$_{busy}$(z) denote the Probability generating function of system size when the server is busy then

$$\text{P}_{\text{busy}}(\text{Z}) = P_1^*(z,0) + P_2^*(z,0) = \frac{z(SOS(w_x(z)) - 1)}{z - S_{BV}^*(w_x(z))} \tag{2.33}$$

Thus the total generating function of the system size for the model is given by

$$P_{BV}(z) = P_{busy}(z) + Q^*(z,0) + R_0 = \frac{R_0((z-1)SOS^*(w_x(z))}{z - S_{BV}^*(w_x(z))} \tag{2.34}$$

where $SOS^*(w_x(z))$ and $S_{BV}^*(w_x(z))$ are given in equations (2.26) and (2.25).

R$_0$ can be calculated using the normalizing condition P(1) = 1,

Thus $R_0\left(\lim_{z\to 1}\frac{(z-1)SOS^*(w_x(z))}{z - S_{BV}^*(w_x(z))}\right) = 1$

Using the L Hospital rule

$R_0 = 1-\rho_{BV}$, where $\rho_{BV} = \lambda E(X)E(S_{BV})$, $E(S_{BV}) = E(SOS) + \overline{P}E(V)$ and $\overline{P} = p_1(1-r) + rp_2$

{Since $\lim\limits_{z\to 1} SOS^*(w_x(z)) = 1$ and $\lim\limits_{z\to 1} \dfrac{(z-1)}{z - S^*_{BV}(w_x(z))} = \dfrac{1}{1-(\frac{d}{dz}(S^*_{BV}(w_x(z)))_{z=1}}$ }

It is easy to verify that

$\frac{d}{dz}(S^*_{BV}(w_x(z)))_{z=1} = \rho_{SOS} + \lambda E(X)E(V)(p_2 r + (1-r)p_1)$ from equations (2.26) and (2.25).

where $\rho_{SOS} = \lambda E(X)(E(S_{SOS})$

Substituting for R$_0$ in equation (2.34), we have

$$P_{BV}(z) = \frac{(1-\rho_{BV})(z-1)SOS^*(w_x(z)}{z - S^*_{BV}(w_x(z))} \tag{2.35}$$

2.2 PERFORMANCE MEASURES OF THE MODEL

The steady state system size probabilities and the mean number of customers when the system is in different states can be calculated from equations (2.30) to (2.32) and using the identities,

$$\lim_{z\to 1} \frac{S^*_i(w_x(z)) - 1}{z - S^*_{BV}(w_x(z)} = \frac{\lambda E(X)E(S_1)}{1-\rho_{BV}}$$

$$\lim_{z\to 1} \frac{S^*_V(w_x(z)) - 1}{z - S^*_{BV}(w_x(z)} = \frac{\lambda E(X)E(V)}{1-\rho_{BV}}$$

Let P$_V$, P$_{FES}$, PSOS and R$_0$ denote the steady state probability that when the server is in vacation state, busy with FES, busy with SOS and in idle respectively.

Then $P_V = \dfrac{\lambda E(X)(E(V)}{(1-\rho_{BV})}(p_2 r + (1-r)p_1)R_0$ (From (2.32)) (2.36)

(i.e.,) $P_V = \lambda E(X)(E(V)(p_2 r + (1-r)p_1)$

$$P_{FES} = \lambda E(X)(E(S_1) \qquad \text{(From (2.30))} \qquad (2.37)$$

$$P_{SOS} = \lambda E(X)(E(S_2) \qquad \text{(From (2.31))} \qquad (2.38)$$

Then probability that the server is busy in

$$P_{busy}(z) = \rho_{SOS} \qquad (2.39)$$

and $\quad R_0 = 1 - \rho_{BV}$

Mean system size

Let L denote the expected number of customers in the system at steady state then,

$$L_{BV} = \frac{d}{dz}(P_{BV}(z)) / z = 1$$

$$= (1-\rho_{BV})\left\{\left(\frac{d}{dz} SOS^*(w_x(z))\right)_{z=1} \lim_{z\to 1} \frac{(z-1)}{z - S^*_{BV}(w_x(z))} + \frac{d}{dz}\left(\frac{(z-1)}{z - S^*_{BV}(w_x(z))}\right)_{z=1}\right\}$$

$$L_{BV} = \rho_{SOS} + \left[\frac{\lambda E(X(X-1))E(S_{BV}) + (\lambda E(X))^2 E(S_{BV})^2}{2(1-\rho_{BV})}\right] \qquad (2.40)$$

where $E(S^2_{BV}) = E(SOS^2) + 2(\overline{P}E(S_1) + p_2 r E(S_2))E(V) + \overline{P}E(V^2)$

$$E(SOS^2) = E(S_1^2) + rE(S_2^2) + 2rE(S_1)(S_2)$$

2.3 PARTICULAR CASES

1. Let $M^X/G_1,G_2/1/SBV$ denote the Single Bernoulli Vacation model in which customers undergo both the types of service one after the other then the PGF P(z) and Expected system size L of the model can be obtained by putting r=1 in equations (2.35) and (2.40).

   i.e., $P(z) = \dfrac{(1-\rho_{BV})(z-1)SOS^*_{G_1+G_2}(w_x(z)}{z - S^*_{BV}(w_x(z))}$

   $$L = \rho_{S_1+S_2} + \left[\frac{\lambda E(X(X-1))E(S_{BV}) + (\lambda E(X))^2 E(S_{BV})^2}{2(1-\rho_{BV})}\right]$$

where $\rho_{BV} = \lambda E(X)(E(S_1)+E(S_2)) + p_2 E(V)$

$$S^*_{G_1+G_2}(w_x(z)) = S_1^*(w_x(z))S_2^*(w_x(z))$$

$$S^*_{BV}(w_x(z)) = S^*_{G_1+G_2}(w_x(z)) + (V^*(w_x(z)-1))(S_1^*(w_x(z))S_2^*(w_x(z))$$

2. If r=0, then the results of single service Bernoulli vacation queueing system $M^X/G_{SOS}/1/SBV$ can be obtained.

$$P(z) = \frac{(1-\rho_{BV})(z-1)S_1^*(w_x(z)}{z - S^*_{BV}(w_x(z))}$$

where $S^*_{BV}(w_X(z) = S_1^*(w_x(z)) + (V^* w_x(z) - 1)p_1(S_1^* w_x(z))$

$$L = \rho_{S_1} + \left[\frac{\lambda E(X(X-1))E(S_{BV}) + (\lambda E(X))^2 E(S^2_{BV})}{2(1-\rho_{BV})}\right]$$

$$S^*_{BV}(w_X(z) = S_1^*(w_x(z)) + (V^* w_x(z) - 1)p_1(S_1^* w_x(z))$$

3. If $p_1 = p_2 = 0$ then the results of non-vacation $M^X/G_{SOS}/1$ queueing system can be obtained. as

$$P(z) = \frac{(1-\rho)(z-1)SOS^*(w_x(z)}{z - Sos^*(w_x(z))} \quad \text{where} \quad \rho = \lambda E(X)(E(S_1)+E(S_2))$$

Chapter III

CHAPTER –III

REPAIRABLE BATCH ARRIVAL QUEUEING SYSTEM WITH BERNOULLI SCHEDULE VACATION AND SECOND OPTIONAL SERVICE FACILITY

In many real systems, the server may meet unpredictable breakdowns. Therefore, queueing models with server breakdowns are more realistic representation of systems. Queueing models with server interruptions have proved to be a useful abstraction in situations, where a single server operates more than one service to the arriving customers. When an interruption occurs while a customer receiving service, the interruption is either disruptive or nondisruptive. Thus the repair time of the server is quite longer or shorter according to the nature of the interruptions. Accordingly, the customer in service may either like to remain in the service facility to complete the remaining service or to join the head of the queue to repeat the service as soon as the server is fixed. The server is subjected to breakdowns at any time while serving customers. The breakdown server is immediately sent for repair. The repair times follows arbitrary distributions.

In this chapter, a bulk arrival queueing system with SOS facility is analysed under Bernoulli single vacation. The server provides two stages of service. The service in the first stage is single and all the customer will necessarily undergo the service. The service is in second stage is optional. The customers after finishing the FES may either opt the SOS or leave the system after completing the FES. The customer who is being served during the server failure remains in the service facility for the return of the server in order to complete the remaining service. These models, have been receiving a lot of attention recently.

3.1 MATHEMATICAL ANALYSIS OF THE SYSTEM:

The system has the following Specifications:

Model description:

Arrival pattern:

Customers arrive in batches in accordance with the time homogeneous Poisson process with group arrival rate λ. The batch size X is a random variable with probability distribution $Pr(X=k) = g_{k,}$ $k=1,2,3,\ldots$

Service pattern and vacation policy

The service pattern and vacation policy adopted in this chapter are same as that of chapter II. During busy period, the server provides First Essential Service (FES) to the arriving customers in the first phase and Second Optional Service (SOS) in the second phase. As soon as the FES is completed, the customer may opt the SOS with probability r or leaves the system with probability (1-r) $(0\leq r\leq 1)$. The service time random variables $S_1$and S_2 of FES and SOS respectively are assumed to be mutually independent of each other having heterogeneous general law of distribution with distributions functions $S_i(x)$ $i=1,2$. When a customer completes FES and leaves the system, the server may take vacation of random length V with probability p_1 $(0\leq p_1\leq 1)$ or may continue to stay in the system either to serve the next customer if any or stay idle in the system expecting the customers to join the queue. Similarly if any customer leaves the system after completing the SOS, the server may take vacation with probability p_2 or continue in the system with probability $(1- p_2)$.

Break down period

The server may breakdown at any time while serving customers. It is assumed that the server's, lifetime follows exponential distribution with mean $1/a_1$ in the FES and the server fials at exponential rate a_2 in SOS. Whenever the breakdown occurs, the server is sent for repair immediately. If the server breaks down during the services then the customer just being served before server breakdown waits for the server in the service facility to complete the remaining service. The repair time distributions due to different service phases (FES or SOS) are arbitrarily distributed with probability distribution function $R_i(y)$ ($1 \le i \le 2$) respectively with density function $r_i(y)$ ($1 \le i \le 2$). Immediately after the server is fixed, the server starts to serve the customer who is waiting to complete the remaining service. It is assumed that the service time for a customer is cumulative and after repair the server is considered as good as new.

Thus the busy period, break down period, vacation period and idle period will constitute a cycle. The customers continue to arrive and join the system independent of the system states, following the compound Poisson process. Various Stochastic processes involved in the queueing system are assumed to be independent to each other. The system is denoted by $M^X/G_{SOS}/1/BSV$/Breakdown.

The following notations are used to obtain the steady- state system size equations.

λ : group arrival rate

X : group size random variable

g_k : Pr(X=k), k=1,2,3,…

X(z) : Probability generating function of X.

$N_s(t)$: The system size at time t

Y(t) : {0,1,2,3,4,5} according as the system is in vacation, idle, busy with FES, busy with SOS, repair in FES and repair in SOS state respectively.

The notations of Random Variables (RV), Cumulative Distribution Function (CDF), Probability Density Function (PDF), Laplace-Stieltjes Transform (LST) and its kth moments are listed as below:

	RV	**CDF**	**Pdf**	**LST**	**kth moment**
Vacation time	V	V(x)	v(x)	$V^*(\theta)$	$E(V^k)$
FES	S_1	$S_1(x)$	$s_1(x)$	$S_1^*(\theta)$	$E(S_1^K)$
SOS	S_2	$S_2(x)$	$s_2(x)$	$S_2^*(\theta)$	$E(S_2^K)$
Repair time(in FES)	R_1	$R_1(y)$	$r_1(y)$	$R_1^{*1}(\theta_1)$	$E(R^k)$
Repair time(in SOS)	R_2	$R_2(y)$	$r_2(y)$	$R_2^{*1}(\theta_1)$	$E(R_2^K)$

$$\text{Where}\quad F^*(\theta) = \int_0^\infty e^{-\theta.t}\,dF(t)$$

Let $V^0(t)$, $S_1^0(t)$ and $S_2^0(t)$, $R_1^0(t)$ and $R_2^0(t)$ denote the remaining vacation time FES time and SOS time, repair time due to FES and SOS respectively at time t. Then the state space $\{N(t),\delta(t)\}$ where $\delta(t)\}= V^0(t)$, 0, $S_1^0(t)$ and $S_2^0(t)$, $R_1^0(t)$ and $R_2^0(t)$ according as $Y(t) = 0$ to 5 respectively defines a bivariate Markov process.

Let

$Q_{n,B}(x,t)dt(Q_{n,I}(x,t)dt) = \Pr\{N_s(t) = n, x \le V^0(t) \le x+dt, Y(t) = 0\}$ $\quad n\ge 0$
$R_n(t) = \Pr\{N_s(t) = 0, Y(t) = 1\}$ $\quad 0\le n\le n-1$
$P_{n,1}(x,t)dt = \Pr\{N_s(t) = n, x \le S_1^0(t) \le x+dt, Y(t) = 2\}$ $\quad n\ge 1$
$P_{n,2}(x,t)dt = \Pr\{N_s(t) = n, x \le S_2^0(t) \le x+dt, Y(t) = 3\}$ $\quad n\ge 1$
$B_{n,1}(x,y,t)dt = \Pr\{N_s(t) = n, S_1^0(t) = x, y \le S_1^0(t) \le y+dt, Y(t) = 4\}$ $\quad n\ge 1$
$B_{n,2}(x,y,t)dt = \Pr\{N_s(t) = n, S_2^0(t) = x, y \le S_2^0(t) \le y+dt, Y(t) = 5\}$ $\quad n\ge 1$

$B_{n,i}(x,y,t)dt$, i=1,2 is the joint probability that at time t, there are n customers in the system, the remaining service time for the customer under service is equal to x, and the server is being repaired with the remaining repair time between y and y+dt, where $x\neq 0$, $n\ge 0$.

$B_{n,i}(x,0)$, i=1,2 denote the probability that there n customers in the system at the termination of repair period.

The remaining notations $P_{n,i}(x,t)dt$ and $P_{n,i}(0)$, i=1,2 and $Q_{n,i}(x,t)dt$ and $Q_n(0)$ for $n\ge 0$ and $R_0(t)$ are as in chapter II .

Steady state system size equations at random epoch

Following the argument of Cox (1955) and observing the changes of states during the interval $(t,t+\Delta t)$ for any time t, the steady state equations are obtained.

The Idle vacation state and state equations are exactly same as (2.0.0) to (2.0.4) and are listed below:

$$\lambda R_0 = P_{1,2}(0)(1-p_2) + (1-r)(1-p_1)P_{1,1}(0) + Q_{0,I}(0) \tag{3.0.0}$$

$$-\frac{d}{dx}Q_{0,I}(x) = -\lambda Q_{0,I}(x) + P_{1,2}(0)p_2 v(x) + P_{1,1}(0)(1-r)p_1 v(x) \tag{3.0.1}$$

$$-\frac{d}{dx}Q_{n,I}(x)=-\lambda Q_{n,I}(x)+\lambda\sum_{k=1}^{n}Q_{n-k,I}(x)g_k \qquad n\geq 1 \qquad (3.0.2)$$

$$-\frac{d}{dx}Q_{1,B}(x)=-\lambda Q_{I,B}(x)+P_{2,2}(0)p_2 v(x)+(1-r)p_1P_{2,1}(0)v(x) \qquad (3.0.3)$$

$$-\frac{d}{dx}Q_{n,B}(x)=-\lambda Q_{n,B}(x)+p_2P_{n+1,2}(0)v(x)+(1-r)p_1P_{n+1,1}(0)v(x)$$

$$+\lambda\sum_{k=1}^{n-1}Q_{n-k,B}(x)g_k \qquad n\geq 2 \qquad (3.0.4)$$

The equations corresponding to the busy states in FES, SOS and break down states are given by

$$-\frac{d}{dx}P_{1,1}(x)=-(\lambda+a_1)P_{1,1}(x)+(1-p_2)P_{2,2}(0)s_1(x)+(1-r)(1-p_2)P_{2,1}(0)s_1(x)$$

$$+B_{1,1}(x,0)+\lambda R_0 g_1 s_1(x)+(Q_{1,I}(0)+Q_{1,B}(0))s_1(x) \qquad (3.0.5)$$

$$-\frac{d}{dx}P_{n,1}(x)=-(\lambda+a_1)P_{n,1}(x)+(1-r)(1-p_1)P_{n+1,1}(0)S_1(x)+(1-p_2)P_{n+1,2}(0)s_1(x)+\lambda R_0 g_n S_1(x)$$

$$+\lambda\sum_{k=1}^{n-1}P_{n-k,1}(x)g_k+B_{n,1}(x,0)+(Q_{n,I}(0)+Q_{n,B}(0))s_1(x) \quad n\geq 2 \qquad (3.0.6)$$

$$-\frac{d}{dx}P_{1,2}(x)=-(\lambda+a_2)P_{1,2}(x)+rP_{1,1}(0)s_2(x)+B_{1,2}(x,0) \qquad (3.0.7)$$

$$-\frac{d}{dx}P_{n,2}(x)=-(\lambda+a_2)P_{n,2}(x)+rP_{n,1}(0)s_2(x)+\lambda\sum_{k=1}^{n-1}P_{n-k,2}(x)g_k+B_{n,2}(x,0) \quad n\geq 2 \qquad (3.0.8)$$

$$\frac{\partial}{\partial y}B_{1,1}(x,y)=-\lambda B_{1,1}(x,y)+a_1p_{1,1}(x)r_1(y) \qquad (3.0.9)$$

$$\frac{\partial}{\partial y}B_{n,1}(x,y)=-\lambda B_{n,1}(x,y)+a_1p_{n,1}(x)r_1(y)+\lambda\sum_{k=1}^{n-1}B_{n-k,1}(x,y)g_k \qquad n\geq 2 \qquad (3.0.10)$$

$$\frac{\partial}{\partial y}B_{1,2}(x,y)=-\lambda B_{1,2}(x,y)+a_2p_{1,2}(x)r_2(y) \qquad (3.0.11)$$

$$\frac{\partial}{\partial y}B_{n,2}(x,y)=-\lambda B_{n,2}(x,y)+a_2p_{n,2}(x)r_2(y)+\lambda\sum_{k=1}^{n-1}B_{n-k,2}(x,y)g_k \qquad n\geq 2 \qquad (3.0.12)$$

By taking LST on both sides of the equations (3.0.0) to (3.0.4) and using the equations

$$Q^*_{n,I}(\theta)+Q^*_{n,B}(\theta)=Q^*_n(\theta) \qquad n\geq 1$$

$$Q^*_{0,j}(\theta)=Q^*_0(\theta),$$

the transformed equations corresponding to the Idle and vacation state equations are given below:

Idle state

$$\lambda R_0 = P_{1,2}(0)(1-p_2)+(1-r)(1-p_1)P_{1,1}(0)+Q_0(0) \tag{3.0}$$

Vacation state

$$\theta Q_0^*(\theta)-Q_0(0)=\lambda Q_0^*(\theta)-p_2P_{1,2}(0)V^*(\theta)-(1-r)p_1P_{1,1}(0)V^*(\theta) \tag{3.1}$$

$$\theta Q_n^*(\theta)-Q_n(0)=\lambda Q_n^*(\theta)-p_2P_{n+1,2}(0)V^*(\theta)-p_1(1-r)P_{n+1,1}(0)V^*(\theta)$$

$$-\lambda\sum_{k=1}^{n-1}Q_{n-k}^*(\theta)g_k \qquad \text{n} \geq 1 \tag{3.2}$$

The LST of the remaining equations (3.0.5) to (3.0.8) lead to,

Busy with First Essential Service

$$\theta P_{1,1}^*(\theta)-P_{1,1}(0)= \ (\lambda+a_1)P_{1,1}^*(\theta)-(1-p_2)P_{2,2}(0)S_1^*(\theta)-(1-r)(1-p_1)P_{2,1}(0)S_1^*(\theta)$$

$$-B_{1,1}^*(\theta,0)-Q_1(0)S_1^*(\theta)-\lambda R_0 g_1 S_1^*(\theta) \tag{3.3}$$

$$\theta P_{n,1}^*(\theta)-P_{n,1}(0)= \ (\lambda+a_1)P_{n,1}^*(\theta)-(1-p_2)P_{n+1,2}S_1^*(\theta)-P_{n+1,1}(0)(1-r)(1-p_1)S_1^*(\theta)$$

$$-\lambda\sum_{k=1}^{n-1}P_{n-k,1}^*(\theta)g_k-B_{n,1}^*(\theta,0)-Q_n(0)S_1^*(\theta)-\lambda R_0 g_{n1} S_1^*(\theta) \tag{3.4}$$

Busy with Second Optional Service

$$\theta P_{1,2}^*(\theta)-P_{1,2}(0)=(\lambda+a_2)P_{1,2}^*(\theta)-rP_{1,1}(0)S_1^*(\theta)-B_{1,2}^*(\theta,0) \tag{3.5}$$

$$\theta P_{n,2}^*(\theta)-P_{n,2}(0)=(\lambda+a_2)P_{n,2}^*(\theta)-rP_{n,1}(0)S_1^*(\theta)-B_{n,2}^*(\theta,0)+\lambda\sum_{k=1}^{n-1}P_{n-k,2}^*(\theta)g_k \tag{3.6}$$

The LST of the equations (3.0.9) to (3.0.12) with respect to x give:

$$\frac{\partial}{\partial y}B_{1,1}^*(\theta,y)=-\lambda B_{1,1}^*(\theta,y)+a_1P_{1,1}^*(\theta)r_1\ (y) \tag{3.7}$$

$$\frac{\partial}{\partial y}B_{n,1}^*(\theta,y)=-\lambda B_{n,1}^*(\theta,y)+a_1P_{n,1}^*(\theta)r_1\ (y)+\lambda\sum_{k=1}^{n-1}B_{n-k,2}^*(\theta,y)g_k \qquad \text{n} \geq 2 \tag{3.8}$$

$$\frac{\partial}{\partial y} B^{*}_{n,2}(\theta, y) = -\lambda B^{*}_{1,2}(\theta, y) + a_2 P^{*}_{1,2}(\theta) r_2 (y) \tag{3.9}$$

$$\frac{\partial}{\partial y} B^{*}_{n,2}(\theta, y) = -\lambda B^{*}_{n,2}(\theta, y) + a_2 P^{*}_{n,2}(\theta) r_2 (y) + \lambda \sum_{k=1}^{n-1} B^{*}_{n-k,2}(\theta, y) g_k \quad \text{n} \geq 2 \tag{3.10}$$

Taking the LST with respect to y, the equations (3.7) to (3.10) lead to

$$\theta_1 B^{**1}_{1,1}(\theta, \theta_1) - B^{*}_{1}(\theta, 0) = \lambda B^{**1}_{1,1}(\theta, \theta_1) - a_1 P^{*}_{1,1}(\theta) R^{*1}_{1}(\theta_1) \tag{3.11}$$

$$\theta_1 B^{**1}_{n,1}(\theta, \theta_1) - B^{*}_{n,1}(\theta, 0) = \lambda B^{**1}_{n,1}(\theta, \theta_1) - a_1 P^{*}_{n,1}(\theta) R^{*1}_{1}(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k,2}(\theta, \theta_1) g_k \tag{3.12}$$

$$\text{n} \geq 2$$

$$\theta_1 B^{**1}_{1,2}(\theta, \theta_1) - B^{*}_{1,2}(\theta, 0) = \lambda B^{**1}_{1,2}(\theta, \theta_1) - a_2 P^{**1}_{1,2}(\theta_1) R^{*1}_{2}(\theta_1) \tag{3.13}$$

$$\theta_1 B^{**1}_{n,2}(\theta, \theta_1) - B^{*}_{n,2}(\theta, 0) = \lambda B^{**1}_{n,2}(\theta, \theta_1) - a_2 P^{*}_{n,2}(\theta) R^{*1}_{2}(\theta_1) - \lambda \sum_{k=1}^{n-1} B^{**1}_{n-k,2}(\theta, \theta_1) g_k \tag{3.14}$$

$$\text{n} \geq 2$$

Probability Generating Function

To derive the partial generating functions of the system size the following Generating functions are introduced.

$$Q^{*}(z, \theta) = \sum_{n=0}^{\infty} Q^{*}_{n}(\theta) z^n, \qquad Q(z, 0) = \sum_{n=0}^{\infty} Q_n(0) z^n$$

$$P^{*}_{i}(z, \theta) = \sum_{n=1}^{\infty} P^{*}_{n,i}(\theta) z^n, \qquad P_i(z, 0) = P_i(z, 0) \qquad \text{for i=1,2}$$

$$B^{**1}_{i}(z, \theta, \theta_1) = \sum_{n=1}^{\infty} B^{**1}_{n,i}(\theta, \theta_1) z^n, \qquad B^{**1}_{i}(z, \theta, 0) = \sum_{n=1}^{\infty} B^{**1}_{n,i}(\theta, 0) z^n, \quad \text{for i=1,2}$$

Multiplying the equations (3.1) and (3.2) by suitable powers of z the partial probability generating functions of the system size, when the server is in vacation state is obtained as,

$$[\theta - w_x(z)] Q^{*}(z, \theta) = Q(z, 0) - \frac{V^{*}(\theta)}{z} [p_2 P_2(z, 0) + (1 - r) p_1 P_1(z, 0)]$$

where $w_x(z) = \lambda(1 - X(z))$

At $\theta = w_x(z)$,Q(z,0) is obtained as,

$$Q(z, 0) = \frac{V^{*}(w_x(z))}{z} [p_2 P_2(z, 0) + (1 - r) p_1 P_1(z, 0)] \tag{3.15}$$

Substituting for Q(z,0) in (2.15),

$$Q^*(z,\theta) = \frac{\left[V^*(w_x(z)) - V^*(\theta)\right]}{z(\theta - w_x(z))}\left[p_2 P_2(z,0) + (1-r)p_1 P_1(z,0)\right] \tag{3.16}$$

To get the generating functions corresponding to breakdown state equations (3.11) and (3.12) are used. Multiplying those equations by suitable powers of z and adding, we get

$$\theta_1 B_1^{**_1}(z,\theta,\theta_1) - B_1^*(z,\theta,0) \;=\; \lambda B_1^{**_1}(z,\theta,\theta_1) - a_1 P_1^*(z,\theta) R_1^{*_1}(\theta_1) - \lambda X(z) B_1^{**_1}(z,\theta,\theta_1)$$

(since $\sum_{n=2}^{\infty} z^n \sum_{k=1}^{n-1} B_{n-k,i}^{**_1}(\theta,\theta_1) g_k = \sum_{k=1}^{\infty} g_k z^k \sum_{n=1}^{\infty} B_{n,i}^{**_1}(\theta,\theta_1) z^n = B_i^{**_1}(z,\theta,\theta_1) X(z)$)

Simplifying further,

$$(\theta_1 - w_x(z)) B_1^{**_1}(z,\theta,\theta_1) \;=\; B_1^*(z,\theta,0) - a_1 P_1^*(z,\theta) R_1^{*_1}(\theta_1)$$

At $\theta_1 = w_x(z)$, $B_1^*(z,\theta,0) = a_1 P_1^*(z,\theta) R_1^{*_1}(w_x(z))$ and hence (3.17)

$$B_1^{**_1}(z,\theta,\theta_1) = a_1 P_1^*(z,\theta)\left(\frac{R_1^{*_1}(w_x(z)) - R_1^{*_1}(\theta_1)}{\theta_1 - w_x(z)}\right) \tag{3.18}$$

Similarly, equations (3.13) and (3.14) will imply

$B_2^*(z,\theta,0) \;=\; a_2 P_2^*(z,\theta) R_2^{*_1}(w_x(z))$ and hence (3.19)

$$B_2^{**_1}(z,\theta,\theta_1) = \frac{a_2 P_2^*(z,\theta)[R_2^{*_1}(w_x(z)) - R_2^{*_1}(\theta_1)]}{\theta_1 - w_x(z)} \tag{3.20}$$

Equations (3.5) and (3.6) give the Probability generating function corresponding to the SOS state as,

$$\theta P_2^*(z,\theta) - P_2(z,0) = (\lambda + a_2) P_1^*(z,\theta) - r P_1(z,0) S_2^*(\theta) - B_2^*(z,\theta,0) + \lambda X(z) P_2^*(z,\theta)$$

Substituting for $B_2^*(z,\theta,0)$ from equation (3.19), and simplifying,

$$[\theta - (\lambda(1 - X(z)) + a_2(1 - R_2^{*_1} w_x(z)))] P_2^*(z,\theta) = P_2(z,0) - r S_2^*(\theta) P_1(z,0)$$

i.e., $(\theta - ha_2(w_x(z))) P_2^*(z,\theta) = P_2(z,0) - r S_2^*(\theta) P_1(z,0)$ (3.21)

where $ha_2(w_x(z)) = w_x(z) + a_2(1 - R_2^{*_1}(w_x(z)))$

At $\theta = ha_2(w_x(z))$,

$$P_2(z,0) = r S_2^*(h_{a_2}(w_x(z))) P_1(z,0) \tag{3.22}$$

and $P_2^*(z,\theta)=rP_1(z,0)\dfrac{[S_2^*(h_{a_2}(w_x(z)))-S_2^*(\theta)]}{\theta-h_{a_2}(w_x(z))}$ (3.23)

Using equations (3.3) and (3.4) the partial generating function corresponding to the busy state during FES can be obtained.

Multiplying the equations (3.3) and (3.4) by suitable powers of z and adding, we get

$$\begin{aligned}\theta P_1^*(z,\theta)-P_1\ (z,0) &= (\lambda+a_1)P_1^*(z,\theta)-\frac{(1-p_2)}{z}(P_2\ (z,0)-zP_{1,2}(0))S_1^*(\theta)\\ &-\frac{(1-r)(1-p_1)}{z}(P_1\ (z,0)-zP_{1,1}(0))S_1^*(\theta)-B_1^*(z,\theta,0)\\ &-(Q(z,0)-Q_0(0))S_1^*(\theta)-\lambda X(z)P_1^*(z,\theta)-\lambda R_0X(z)S_1^*(\theta) \quad (3.24)\end{aligned}$$

Substituting for $P_2\ (z,0)$ from (3.22), $B_1^*(z,\theta,0)$ from (3.18), $Q(z,0)$ from (3.15) in the equation (3.24), and using equation (3.0.0) we have,

$$\begin{aligned}\theta P_1^*(z,\theta)-P_1\ (z,0) &= [\lambda(1-w_x(z))+a_1(1-R_1^{*_1}(w_x(z)))]P_1^*(z,\theta)-\lambda R_0X(z)S_1^*(\theta)\\ &-\frac{S_1^*(\theta)}{z}[(1-p_2)rS_2^*(h_{a_2}(w_x(z))+(1-r)(1-p_2)]P_1(z,0)\\ &+\lambda R_0S_1^*(\theta)--\frac{S_1^*(\theta)}{z}\left(\frac{V^*(w_x(z)}{z}\right)[p_2rS_2^*(h_{a_2}(w_x(z))+(1-r)p_1]P_1(z,0)\end{aligned}$$

i.e., $(\theta-h_{a_2}(w_x(z)))P_1^*(z,\theta)=\dfrac{P_1(z,0)}{z}[z-S_1^*(\theta)\varphi_{br}(z)]+\lambda R_0(1-X(z))S_1^*(\theta)$ (3.25)

Where

$$\begin{aligned}\phi_{br}(z) &= (1-p_1)(1-r)+(1-p_2)rS_2^*(h_{a_2}(w_x(z)))+V^*(w_x(z))(p_2rS_2^*(h_{a_2}(w_x(z)))+(1-r)p_1)\\ &=(V^*(w_x(z))-1)[p_2rS_2^*(h_{a_2}(w_x(z)))+(1-r)p_1]+[1-r+rS_2^*(h_{a_2}(w_x(z)))]\end{aligned}$$

At $\theta=h_{a_1}(w_x(z))$

$$P_1(z,0)=\frac{-zR_0w_x(z)S_1^*(h_{a_1}(w_x(z)))}{(z-S_{BV}^*(hw_x(z)))} \quad (3.26)$$

where $S_{BV}^*(hw_x(z))=S_1^*(h_{a_1}(w_x(z)))\varphi_{br}(z)$

i.e.,

$$S^*_{\mathbf{BV}}(hw_x(z) = SOS^{br}(hw_x(z)) + (V^*(w_x(z))-1)[p_2 r S_2^*(h_{a_2}(w_x(z))) + (1-r)p_1](S_1^* h_{a_1}(w_x(z))) \tag{3.26a}$$

where $SOS^{br}(hw_x(z)) = S_1^* h_{a_1} w_x(z) + (1-r+rS_2^* h_{a_2}(w_x(z)))$ (3.26b)

Substituting for $P_1(z,0)$ in (3.25)

$$(\theta - h_{a_1}(w_x(z)))P_1^*(z,\theta) = -R_0(w_x(z))\left(\frac{(z-S_1^*(\theta)\varphi_{br}(z))S_1^*(h_{a_1}(w_x(z)))}{(z-S^*_{BV}(hw_x(z)))} - S_1^*(\theta)\right)$$

$$(\theta - h_{a_1}(w_x(z)))P_1^*(z,\theta) = -R_0(w_x(z))z\left(\frac{S_1^*(h_{a_1}(w_x(z)))-S_1^*(\theta)}{(z-S^*_{BV}(hw_x(z)))}\right)$$

$$P_1^*(z,\theta)=\frac{-zR_0w_x(z)[S_1^*(h_{a_1}(w_x(z)))-S_1^*(\theta)]}{(\theta-h_{a_1}(w_x(z)))[z-S^*_{BV}(hw_x(z))]} \tag{3.27}$$

Substituting for $P_1(z,0)$ from (3.26) in equations (3.23) and (3.16)

the expressions for $P_2^*(z,\theta)$ and $Q^*(z,\theta)$ are obtained as,

$$P_2^*(z,\theta)=\frac{-zR_0S_1^*(h_{a_1}(w_x(z)))w_x(z)}{z-S^*_{BV}h(w_x(z))}\,\frac{r[S_2^*(h_{a_2}(w_x(z)))-S_2^*(\theta)]}{\theta-h_{a_2}(w_x(z))} \tag{3.28}$$

$$Q^*(z,\theta)=\left[p_2 r S_2^*(h_{a_2}(w_x(z)))+(1-r)p_1\right]\left(\frac{-R_0S_1^*(h_{a_1}(w_x(z)))w_x(z)}{z-S^*_{BV}(h(w_x(z)))}\right)\frac{\left[V^*(w_x(z))-V^*(\theta)\right]}{\theta-h_{a_1}(w_x(z))} \tag{3.29}$$

The equations (3.27), (3.28), (3.29), (3.18) and (3.20) at θ=0 give the partial Probability generating function of the system size when the server is in different states:

$$P_1^*(z,0)=\frac{(zR_0w_x(z))[S_1^*(h_{a_1}(w_x(z)))-1]}{h_{a_1}(w_x(z))[z-S^*_{BV}(h(w_x(z)))]} \tag{3.30}$$

$$P_2^*(z,0)=\frac{zR_0S_1^*(h_{a_1}(w_x(z)))w_x(z)}{z-S^*_{BV}(hw_x(z))}\,\frac{r[S_2^*(h_{a_2}(w_x(z)))-1]}{h_{a_2}(w_x(z))} \tag{3.31}$$

$$Q^*(z,0) = R_0(V^*(w_x(z)) - 1)[p_2 r S_2^*(h_{a_2}(w_x(z))) + (1-r)p_1]\left(\frac{S_1^*(h_{a_1}(w_x(z)))}{z - S_{BV}^*(hw_x(z))}\right)\left\{\frac{w_x(z)}{h_{a_1}(w_x(z))}\right\}$$

(3.32)

$$B_1^{**_1}(z,0,0) = \frac{a_1 P_1^*(z,\theta)[1 - R_1^{*_1}(w_x(z))]}{w_x(z)} \tag{3.33}$$

$$B_2^{**_1}(z,0,0) = \frac{a_2 P_2^*(z,\theta)[1 - R_2^{*_1}(w_x(z))]}{(w_x(z))} \tag{3.34}$$

Let $P_1(z)$, $P_2(z)$ denote probability generating functions of the system size when the server is either busy or in breakdown states during FES and SOS respectively.

Then $P_1(z) = P_1^*(z,0) + B_1^{**_1}(z,0,0) = (1 + a_1 R_1^{*_1}(w_x(z)))\dfrac{P_1^*(z,\theta)}{w_x(z)}$

(from equations (3.30) and (3.33))

and $P_2(z) = P_2^*(z,0) + B_2^{**_1}(z,0,0) = (1 + a_2 R_2^{*_1}(w_x(z)))\dfrac{P_2^*(z,\theta)}{w_x(z)}$

(from equations (3.31) and (3.34))

Substituting for $P_1^*(z,0)$ and $P_2^*(z,0)$ from the equations (3.30) and (3.31)

$P_1(z)$ and $P_2(z)$ are simplified as,

$$P_1(z) = \frac{R_0 z(S_1^*(h_{a_1}(w_x(z))) - 1)}{z - S_{BV}^*(hw_x(z))} \tag{3.35}$$

$$P_2(z) = \frac{R_0 r z S_1^*(h_{a_1}(w_x(z)))(S_2^*(h_{a_2}(w_x(z)) - 1)}{z - S_{BV}^*(hw_x(z))} \tag{3.36}$$

Then, $P_{comp}(z)$ = Probability generating function of the system size when the server is busy or in breakdown states.

P$_{\text{comp}}$(z) = $P_1(z)+P_2(z)$

$$\text{P}_{\text{comp}}(\text{z}) = \frac{R_0 z(SOS^*(h\ w_x(z))-1)}{z-S_{BV}(hw_x(z))} \tag{3.37}$$

The total PGF P(z) of the model is given by,

$$P(z)=P_{comp}(z)+Q^*(z,0)+R_0$$

$$\text{i.e., } P_{BV}^{br}(z)=\frac{R_0(z-1)SOS^*(h\ w_x(z))}{z-S_{BV}^*(hw_x(z))} \tag{3.38}$$

where $SOS^*(h\ w_x(z))$ and $S_{BV}^*(hw_x(z))$ are given in equations (3.26a) and (3.26b)

Using the normalizing condition $P^{br}(z)=1$, R_0^{br} is calculated and is given by

$R_0^{br}=1-\rho_{BV}^{br}$

where $\rho_{BV}^{br}=\lambda E(X)E(S_{BV}^{br})$, $E(S_{BV}^{br})=E(SOS^{br})+\overline{P}E(V)$, $\overline{P}=p_1(1-r)+rp_2$.

and $E(SOS^{br})=[1+a_1E(R_1)]E(S_1)+r[1+a_2E(R_2)]E(S_2)$

Substituting for R$_0$ in (3.38), the total generating function for the model is given by

$$P_{BV}^{br}(z)=\frac{(1-\rho_{BV}^{br})(z-1)SOS^*(h\ w_x(z))}{z-S_{BV}^*(hw_x(z))} \tag{3.39}$$

3.2 PERFORMANCE MEASURES OF THE SYSTEM

Let $P_V, P_{FES}, P_{SOS}, R_0^{br}, P_{FES}^{br}, P_{SOS}^{br}$ denote the steady state system size probabilities that the server is on vacation state, busy with FES, busy with SOS, idle and in breakdown state corresponding to FES and SOS respectively. Then

$$P_V=\lambda E(X)E(V)(p_2r+(1-r)p_1) \tag{3.40}$$

$$P_{FES}=\lambda E(X)E(S_1) \tag{3.41}$$

$$P_{SOS}=\lambda E(X)E(S_2) \tag{3.42}$$

$$P_{FES}^{br} = \lambda E(X)E(S_1)a_1E(R_1) \tag{3.43}$$

$$P_{SOS}^{br} = \lambda E(X)rE(S_2)a_2E(R_2) \tag{3.44}$$

$$R_0^{br} = (1-\rho_{BV}^{br}) \tag{3.45}$$

Mean system size

Let L_{BV}^{br} denote the expected number of customers in the system at steady state

then $L_{BV}^{br} = \dfrac{d}{dz}(P_{BV}^{br}(z)) / z = 1$

$$= (1-\rho_{BV}^{br})\left(\frac{d}{dz}SOS^*(h\ w_x(z))\right)_{z=1} \lim_{z\to 1}\frac{(z-1)}{z-S_{BV}^*(hw_x(z))} + \left(\frac{d}{dz}\frac{(z-1)}{z-S_{BV}^*(hw_x(z))}\right)_{z=1}$$

$$L_{BV}^{br} = \rho_{SOS}^{br} + \left[\frac{\lambda E(X(X-1))E(\rho_{BV}^{br}) + (\lambda E(X))^2 E(\rho_{BV}^{br})^2}{2(1-\rho_{BV}^{br})}\right] \tag{3.46}$$

where $E(S_{BV}^{br^2}) = E(SOS^{br^2}) + 2(\overline{P}E(S_1) + E(S_2))E(V) + \overline{P}E(V^2)$

where $E(SOS^{br^2}) = (1+a_1E(R_1)^2)E(S_1^2) + r(1+a_2E(R_2)^2)E(S_2^2)$

$$+2(1+a_1E(R_1))E(S_1)[r(1+a_2E(R_2))E(S_2)] + a_1E(S_1)E(R_1^2) + ra_2E(S_2)E(R_2^2)$$

3.3 PARTICULAR CASES

1. When a_i=0, (i=1,2), the results of $M^X/G_{SOS}/1/SBV$ without breakdown can be deduced

   When a_i=0, i=1,2 $hw_x(z) = w_x(z)$ so that

   $P_{BV}^{br} = P_{BV}(z)$ from equation (3.39)

   $L_{BV}^{br} = L_{BV}$ from equation (3.46)

2. The results of $M^X/G_{SOS}/1/SBV$/Breakdown model in which each customer undergoes both types of services one after the other can be obtained by putting r=1.
3. The results of single service $M^X/G/1/SBV$/Breakdown model can be obtained by putting r=0.

3.4 NUMERICAL ANALYSIS

In this section, the effects of the system parameters,

1) Mean vacation time E(V)
2) The probability with which the server goes for vacation p_i (i=1,2)
3) Second optional service probability (r)
4) The mean batch size E(X)
5) Breakdown rates a_i ,(i=1,2)
6) Mean repair time E(R_i), (i=1,2), on the mean system size and the system size probabilities are studied.

Numerical values are presented for the mean queue length and the system size probabilities for the models of chapters II and III in Tables (3.1) to (3.4)

For computation, the following distributions are assumed for service time S_i, i=1,2 vacation time V, repair time R_i, i=1,2 and the batch size X.

Random variables (V)	**Distribution F(V)**	**Mean E(V)**	**Second order moments E(V^2)**
FES(S_1)	Two-stage hyper-exponential	$E(S)=\frac{b}{\mu_{1,1}}+\frac{1-b}{\mu_{1,2}}$	$E(S^2)=\frac{b}{\mu_{1,1}^2}+\frac{1-b}{\mu_{1,2}^2}$
SOS(S_2)	Erlang 3 type	$\frac{1}{\mu_2}$	$\frac{4}{3\mu_2^2}$
Vacation time(V)	Gamma (3,η) type	$\frac{3}{\eta}$	$\frac{12}{\eta^2}$
Repair in FES (R_1)	Erlang 2 type	$\frac{1}{\beta}$	$\frac{3}{2\beta^2}$
Repair in SOS (R_2)	Deterministic	$\frac{1}{\beta_1}$	$\frac{1}{\beta_3^2}$
Batch size X	Geometric	$\frac{1}{1-p}$	$\frac{2p}{(1-p)^2}=E(X(X-1))$

In all the tables L_{BV} (L^{br}_{BVbr}) denotes the mean system size and P_V, P_{busy}, R_0 ($P^{br}_V, P^{br}_{comp}, R^{br}_0$) denote the probabilities that the system is in vacation state, busy state and idle state for the $M^x/G_{SOS}/1$ queueing system without breakdown (with breakdown).

The values of Tables (3.1a) and (3.1b) show that for both the models $M^X/G_{SOS}/1$ without breakdown and with breakdown, L_{BV} and L^{br}_{BVbr} increase as the probability that the customers opt the SOS (r) or the mean arrival batch size E(X) increases.

Table (3.1a)

$(\eta, \lambda, p_1, p_2, E(SOS), E(S_{BV})) = (5, 0.48, 0.1, 0.1, 0.23, 0.1)$

E(X)	**R/P**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
1	**0**	0.134942	0.179447	0.226380	0.276025	0.328709	0.384816
3.3	**0.3**	0.230237	0.312622	0.402119	0.499361	0.607573	0.726911
4	**0.5**	0.720005	1.047328	1.450939	1.964437	2.644551	3.595174

Table (3.1b) $(\eta, \lambda, p_1, p_2) = (5, 0.48, 0.1, 0.1)$

$(a_1, a_2, E(R_1), E(R_2)\, E(SOS^{br}), E(S^{br}_{BV})) = (1, 2, 0.08, 0.1, 0.01, 0.8)$

E(X)	**r/p**	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
1	**0**	0.153192	0.177463	0.215315	0.256378	0.301078	0.349920
3.3	**0.3**	0.245333	0.313775	0.393987	0.484201	0.586411	0.703182
4	**0.5**	0.966830	1.082742	1.505916	2.0707726	2.862572	4.052779

The values of the Tables (3.2a) and (3.2b) show the effect of r and the batch size E(X) on the system size probabilities for the reliable and breakdown model respectively.

Table (3.2a)

$(\eta, \lambda, p_1, p_2, E(SOS), E(S_{BV})) = (5, 0.48, 0.1, 0.1, 0.23, 0.1)$

E(X)	**r** **p**		**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
1	**0**	P_V	0.04800	0.067200	0.086400	0.105600	0.124800	0.144000
		P_{busy}	0.054400	0.066400	0.078400	0.090400	0.102400	0.114400
		R_0	0.897600	0.866400	0.835000	0.804000	0.772800	0.741600
3.3	**0.3**	P_V	0.064286	0.090000	0.115714	0.141429	0.167143	0.192857
		P_{busy}	0.072857	0.088929	0.105000	0.121071	0.131743	0.153214
		R_0	0.862857	0.082101	0.779286	0.737500	0.695714	0.653929
4	**0.5**	P_V	0.117500	0.164500	0.211500	0.258500	0.305500	0.352500
		P_{busy}	0.133167	0.162542	0.191917	0.221292	0.250667	0.280042
		R_0	0.749333	0.672958	0.596583	0.520208	0.443833	0.367458

Table (3.2b)

$(\eta, \lambda, p_1, p_2) = (3, 0.48\ 0.3, 0.3\ 0.48)$

$(a_1, a_2, E(R_1), E(R_2)\, E(SOS^{br}), E(S^{br}_{BV})) = (1, 2, 0.08, 0.1, 0.01, 0.8)$

E(X)	**r** **p**		**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1.0**
1	**0**	P_V^{br}	0.048000	0.067200	0.086400	0.105600	0.124800	0.144000
		P_{comp}^{br}	0.060000	0.060000	0.060000	0.060000	0.060000	0.060000
		R_0^{br}	0.893067	0.859457	0.825867	0.792267	0.758667	0.725067
3.3	**0.3**	P_V^{br}	0.117500	0.264500	0.311500	0.358500	0.305500	0.352500
		P_{comp}^{br}	0.146875	0.146875	0.146875	0.146875	0.146875	0.146875
		R_0^{br}	0.738236	0.655986	0.573736	0.491486	0.409236	0.326986
4	**0.5**	P_V^{br}	0.064286	0.090000	0.115714	0.141429	0.167143	0.792857
		P_{comp}^{br}	0.080357	0.080357	0.080357	0.080357	0.080357	0.080357
		R_0^{br}	0.856786	0.811786	0.766786	0.721786	0.676786	0.631786

It is important to study the effects of the probabilities p_i, i=1,2 with which the server goes on vacation in between the services, on the mean system size. Tables (3.3a) and (3.3b) give the numerical values of the mean system size L_{BV}, L_{BVbr} for different values of p_i=i=1,2's arrival rates λ and mean vacation time E(V).

The table values of (3.3 a) and (3.3 b) clearly show that if p_i or λ increases L_{BV} and L_{BVbr} also increase. One can also note that the system size increases along with mean vacation time. The values corresponding to $p_1 = p_2 = 0$ give the expected system size for the non vacation $M^x/G_{SOS}/1$ queueing model.

Table (3.3a)

$$(\lambda, p, E(SOS), E(S_{BV})) = (0.48, 0.5, 0.23, 0.1)$$

E(x)	λ	p_1,p_2 / η	(0,0)	(0.4,0.3)	(0.5,0.5)	(0.8,0.85)	(1,1)
4	0.25	5	0.108743	0.152611	0.184722	0.247419	0.202200
		3	0.108745	0.191980	0.257167	0.396225	0.317277
2.9	0.35	5	0.178517	0.268212	0.337950	0.485249	0.560965
		3	0.178517	0.356873	0.512829	0.905407	1.149818

Table (3.3b)

$$(\lambda, p, a_1, a_2, E(R_1), E(R_2)\, E(SOS^{br}), E(S_{BV}^{br})) = (0.48, 0.5, 1, 2, 0.08, 0.1, 0.01, 0.8)$$

E(x)	λ	p_1,p_2 / η	(0,0)	(0.4,0.3)	(0.5,0.5)	(0.8,0.85)	(1,1)
4	0.25	5	0.084540	0.129305	0.162105	0.221185	0.187018
		3	0.084540	0.169446	0.236061	0.358518	0.303962
2.9	0.35	5	0.144252	0.236926	0.309179	0.462369	0.541420
		3	0.144252	0.328470	0.490392	0.901494	0.160004

The values of the tables (3.4a) and (3.4b) give the effect of the SOS probabilities p_1, p_2, on the system size probabilities for both the models $M^X/G_{SOS}/1$(without breakdown and with breakdown) for different values of the arrival rates.

Table (3.4a)

$(\lambda, p, E(SOS), E(S_{BV})) = (0.48, 0.5, 0.23, 0.1)$

E(x)	λ		**p_1,p_2** / η	**(0,0)**	**(0.4,0.3)**	**(0.5,0.5)**	**(0.8,0.85)**	**(1,1)**
4	**0.25**	P_V	5	0.000000	0.050000	0.083333	0.141667	0.120000
			3	0.000000	0.083333	0.138889	0.236116	0.200000
		P_{busy}	5	0.066204	0.066204	0.066204	0.066204	0.047667
			3	0.066204	0.066204	0.066204	0.066204	0.047667
		R_0	5	0.933796	0.883796	0.850463	0.792130	0.832333
			3	0.933798	0.850463	0.794907	0.697685	0.752333
2.9	**0.35**	P_V	5	0.000000	0.076829	0.128049	0.217683	0.256098
			3	0.000000	0.128049	0.213415	0.362805	0.426829
		P_{busy}	5	0.101728	0.101728	0.101728	0.101728	0.101728
			3	0.101728	0.101728	0.101728	0.101728	0.101728
		R_0	5	0.898272	0.821443	0.770224	0.680589	0.642175
			3	0.898272	0.770224	0.684858	0.535468	0.471443

Table (3.4b)

(λ, p ,a_1, a_2, $E(R_1)$, $E(R_2)$ $E(SOS^{br}), E(S_{BV}^{br})$) = (0.48, 0.5,1, 2, 0.08, 0.1, 0.01, 0.8)

E(x)	λ		**p_1,p_2** η	**(0,0)**	**(0.4,0.3)**	**(0.5,0.5)**	**(0.8,0.85)**	**(1,1)**
4	**0.25**	P_V^{br}	5	0.000000	0.050000	0.083333	0.141661	0.126000
			3	0.000000	0.083333	0.138889	0.236116	0.200000
		P_{comp}^{br}	5	0.039062	0.034722	0.034722	0.000000	0.025000
			3	0.034722	0.034722	0.034722	0.034722	0.025000
		R_0^{br}	5	0.924228	0.874228	0.840892	0.782562	0.825444
			3	0.924228	0.840895	0.785339	0.688117	0.745444
2.9	**0.35**	P_V^{br}	5	0.000000	0.076829	0.128049	0.217683	0.256098
			3	0.000000	0.128049	0.213415	0.362805	0.426829
		P_{comp}^{br}	5	0.053354	0.053354	0.053354	0.053354	0.053354
			3	0.053354	0.053354	0.053354	0.053354	0.053354
		R_0^{br}	5	0.883570	0.806741	0.755522	0.665887	0.627473
			3	0.883570	0.755522	0.670156	0.520766	0.456741

Conclusion

CONCLUSION

One of the important characteristics of queueing system is service process. Several queuing models belong to a class of systems where the service discipline involves more than one service and these models have been receiving a lot of attention recently. Various scenarios have been considered in the literature. In the present work, the author has considered that the server provides First essential service to all the arriving customers in the first phase. At the end of the First essential service, each customer may either choose the second optional service with probability r or depart the system with probability (1-r). Chapter II deals with a batch arrival queueing system $M^X/G/1$ with second optional service facility in which the server may take vacation at the completion of each service with probability p or may continue to stay in the system with probability (1-p).

In chapter III, the same model is analysed under the assumption that the server is subject to random breakdown during service and is sent for repair immediately. The server will resume his service as soon as he returns from the repair facility. The models are analysed using Supplementary variable technique and it is shown that, several other models can be deduced as special cases. Through numerical values it shown that the mean queue length increases with mean arrival rates, mean service time, mean vacation time, Second optional service probability and Bernoulli Vacation probability.

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