CHAPTER - VII

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ENTROPY BASED FUZZY ANALYTIC HIERARCHY PROCESS METHOD

Entropy method is introduced to measure the expected information content of certain message. In information theory, entropy is a criterion for the amount of information (or uncertainty) represented by a discrete probability distribution, P_i. A broad distribution represents more uncertainty than a narrowed distribution does. The entropy is represented by a probability distribution, and the terms ''entropy'' and ''uncertainty'' are considered as synonymous. Thus the principle of maximum uncertainty is called the principle of maximum entropy. The Shannon entropy, H, assumes the following form [34]

$$H(P_1, P_2, \cdots, P_h) = -g \sum_{l=1}^{h} P_1 \ln P_1$$

where $g = 1/\ln h$ is a positive constant which guarantees that $0 \leq H(P_1, P_2, \cdots, P_h) \leq 1$. The larger is the value of $H(P_1, P_2, \cdots, P_h)$, the less is the information contained in $P_1, P_2, \cdots, P_h$. In consequence, 0 entropy indicates that the maximum information (or uncertainty) is contained, and 1 indicates that the minimum information is contained. If all P_l are equal to each other for a given l, that is, $P_l = 1/h$, then $H(P_1, P_2, \cdots, P_h)$ takes on its maximum value.

In this entropy based Fuzzy AHP method, many fuzzy set theoretical concepts are applied. One of the most important concepts of fuzzy sets is the concept of an $\alpha - cut$. Defining the interval of confidence level α, the triangular fuzzy number $\widetilde{M} = (a, b, c)$ can be characterized as

$$\widetilde{M_\alpha} = [a^\alpha, c^\alpha] = [(b-a)\alpha + a, -(c-b)\alpha + c] \qquad \forall\, \alpha \in [0,1] \qquad (27)$$

The steps for the Fuzzy Analytic Hierarchy Process model with entropy method are summarized as follows:

Step 1: **Construct a Hierarchy Structure for any Problem.**

Decompose the problem in a hierarchical fashion into sub-problems that can be easily comprehended and evaluated.

Step 2: **Comparing the Performance Score**

Triangular fuzzy numbers $(\tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}, \tilde{9})$ are used to indicate the relative strength of each pair of elements in the same hierarchy. The linguistic values can be obtained from Table 8.

Fuzzy language	Quantitative value
Very important	$\tilde{9}$
Important	$\tilde{7}$
Equal important	$\tilde{5}$
Unimportant	$\tilde{3}$
Very unimportant	$\tilde{1}$

Table 8. Linguistic value table

Step 3: **Construct the Fuzzy Comparison Matrix**

By using triangular fuzzy numbers, via pairwise comparison, the fuzzy judgment matrix $\tilde{X} = (\widetilde{x_{ij}})$ is constructed as shown below

$$\tilde{X} = \begin{bmatrix} 1 & \widetilde{x_{12}} & \widetilde{x_{13}} & \cdots & \widetilde{x_{1n}} \\ \widetilde{x_{21}} & 1 & \widetilde{x_{23}} & \cdots & \widetilde{x_{2n}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \widetilde{x_{n1}} & \widetilde{x_{n2}} & \widetilde{x_{n3}} & \cdots & 1 \end{bmatrix}$$

where $\widetilde{x_{ij}}$ denote the relative impotence of the factor i to the factor j. Here

$$\widetilde{x_{ji}} = \begin{cases} \widetilde{x_{ij}}^{-1}, & i \neq j \\ 1, & i = j \end{cases}$$

Step 4: **Establish Fuzzy Weight Vector.**

The weight vector $\widetilde{w}$ is given by

$$\widetilde{w} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

where $w_i = \frac{a_i}{\sum_{i=1}^{n} a_i}$. Here $a_i = \sum_{j=1}^{n} \widetilde{x_{ij}}$

Step 5: **Establish the Total Fuzzy Judgment Matrix**

The total fuzzy judgment matrix, $\tilde{A}$ is established by multiplying the elements in fuzzy judgment matrix, $\tilde{X}$ with the elements in fuzzy weight vector $\widetilde{w}$.

The equation is

$$\tilde{A} = \begin{bmatrix} \widetilde{x_{11}} \otimes w_1 & \widetilde{x_{12}} \otimes w_2 & \cdots & \widetilde{x_{1n}} \otimes w_n \\ \widetilde{x_{21}} \otimes w_1 & x_{22}\widetilde{\otimes} w_2 & \cdots & \widetilde{x_{2n}} \otimes w_n \\ \cdots & \cdots & \cdots & \cdots \\ \widetilde{x_{n1}} \otimes w_1 & \widetilde{x_{n2}} \otimes w_2 & \cdots & \widetilde{x_{nn}} \otimes w_n \end{bmatrix} \tag{28}$$

Step 6: **Establish the Total Fuzzy Judgment Matrix with $\alpha - cuts$**

The total fuzzy judgment matrix with $\alpha - cuts$ is obtained by performing fuzzy number multiplications and additions with the interval arithmetic and cuts. From Equation (27), Equation (28) can be simplified to

$$\widetilde{A_\alpha} = \begin{bmatrix} [a_{11l}^{\alpha}, a_{11u}^{\alpha}] & [a_{12l}^{\alpha}, a_{12u}^{\alpha}] & \cdots & [a_{1nl}^{\alpha}, a_{1nu}^{\alpha}] \\ [a_{21l}^{\alpha}, a_{21u}^{\alpha}] & [a_{22l}^{\alpha}, a_{22u}^{\alpha}] & \cdots & [a_{2nl}^{\alpha}, a_{2nu}^{\alpha}] \\ \cdots & \cdots & \cdots & \cdots \\ [a_{n1l}^{\alpha}, a_{n1u}^{\alpha}] & [a_{n2l}^{\alpha}, a_{n2u}^{\alpha}] & \cdots & [a_{nnl}^{\alpha}, a_{nnu}^{\alpha}] \end{bmatrix}$$

where $a_{ijl}^{\alpha} = w_{il}^{\alpha}\, x_{ijl}^{\alpha}$, $a_{iju}^{\alpha} = w_{iu}^{\alpha}\, x_{iju}^{\alpha}$, for $0 < \alpha \leq 1$ and for all i, j.

Step 7: **Establish the Total Fuzzy Judgment Matrix with $\alpha - cuts$ and the Degree of Satisfaction**

The degree of satisfaction of the judgment $\hat{A}$ will be estimated. When α is fixed, we will set the index of optimism λ by the degree of the optimism of a decision maker. A larger λ indicates a higher degree of optimism. The index of optimism is a linear convex combination, it is explained by

$$\widehat{a_{ij}^{\alpha}} = (1-\lambda)a_{ijl}^{\alpha} + \lambda a_{iju}^{\alpha} \qquad \forall \lambda \in [0,1]$$

Thus we have

$$\hat{A} = \begin{bmatrix} \widehat{a_{11}^{\alpha}} & \widehat{a_{12}^{\alpha}} & \cdots & \widehat{a_{1n}^{\alpha}} \\ \widehat{a_{21}^{\alpha}} & \widehat{a_{22}^{\alpha}} & \cdots & \widehat{a_{2n}^{\alpha}} \\ \cdots & \cdots & \cdots & \cdots \\ \widehat{a_{n1}^{\alpha}} & \widehat{a_{n2}^{\alpha}} & \cdots & \widehat{a_{nn}^{\alpha}} \end{bmatrix}$$

where $\hat{A}$ is a precise judgment matrix.

Step 8: **Calculate Entropy Weight**

The entropy must be first calculated by using the relative frequency of Equation (29) and the entropy formula of Equation (30), i.e.

$$\begin{bmatrix} \frac{\widehat{a_{11}^{\alpha}}}{s_1} & \frac{\widehat{a_{12}^{\alpha}}}{s_1} & \cdots & \frac{\widehat{a_{1n}^{\alpha}}}{s_1} \\ \cdots & \cdots & & \cdots \\ \frac{\widehat{a_{n1}^{\alpha}}}{s_n} & \frac{\widehat{a_{n2}^{\alpha}}}{s_n} & \cdots & \frac{\widehat{a_{nn}^{\alpha}}}{s_n} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix} \qquad (29)$$

where

$$s_k = \sum_{j=1}^{n} \widehat{a_{kj}^{\alpha}}$$

We can use this equation to calculate the entropy, i.e.

$$\left.\begin{aligned} H_1 &= -\sum_{j=1}^{n} f_{1j}\log_2(f_{1j}) \\ H_2 &= -\sum_{j=1}^{n} f_{2j}\log_2(f_{2j}) \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ H_n &= -\sum_{j=1}^{n} f_{nj}\log_2(f_{nj}) \end{aligned}\right\} \tag{30}$$

Here H_i is i^{th} entropy value.

The entropy weights can be determined by using Equation (31).

$$H_i = \frac{H_i}{\sum_{j=1}^{n} H_j} \tag{31}$$

Finally the alternative with highest entropy weight is selected as the best alternative