

Chapter 7

λ_g^α -Quotient Maps in Topological Spaces

7.1 Introduction

The Extract of Mathematics lie in the construction of experiments in the respective definitions, one such study is the quotient space in which the identifications produce simple objects like circle, sphere, torus. Quotient maps are also known as identification maps or strong continuous maps because of the strong conditions to the continuity which have an intense base at the idea of gluing in the formation of topology. Lellis Thivagar [1991] introduced the concept of quotient maps and designed the way to study the properties of them. With the support from the previous study of quotient maps, Ravi et al. [2011], Umadevi I. Neeli [2012], Dunya Mohamed Hamed and Bushra Jaralla Tawfeeq [2013], Balamani and Parvathi [2017], Chidanand Badigar et al. [2020] introduced the concepts of $\alpha g s$ -quotient maps, $g \delta s$ -quotient maps, $rg\alpha$ -quotient maps, $\psi^* \alpha$ -quotient maps, rw -quotient maps respectively, which have created an interest to define the new type of quotient maps called λ_g^α -quotient maps, strongly λ_g^α -quotient maps and completely λ_g^α -quotient maps via λ_g^α -closed and λ_g^α -open maps. Moreover, in this chapter we have studied the fundamental properties and interrelations between them.

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7.2 λ_g^α -Quotient Maps and its Special Forms

This section provides the definition of λ_g^α -quotient map and investigates its properties. Later the special forms of λ_g^α -quotient maps such as, strongly λ_g^α -quotient maps

and completely λ_g^α -quotient maps are also defined and their interrelations are also accomplished.

Definition 7.2.1 A surjective map $u: (M, \mu) \rightarrow (N, \nu)$ is called a λ_g^α -quotient map if u is a λ_g^α -continuous map and $u^{-1}(T)$ is an open set in (M, μ) implies T is a λ_g^α -open set in (N, ν) .

Example 7.2.2 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i, j\}, M\}$, $\nu = \{\phi, \{i\}, \{j\}, \{i, j\}, N\}$. Then λ_g^α -open sets of (M, μ) and (N, ν) are $P(M)$ and $P(N)$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = j$, $u(j) = i$ and $u(k) = k$. Obviously u is λ_g^α -continuous and also $u^{-1}(\phi) = \phi$, $u^{-1}(\{i, j\}) = \{i, j\}$ and $u^{-1}(N) = M$ implies $\phi, \{i, j\}$ and N are λ_g^α -open sets in (N, ν) . Hence u is a λ_g^α -quotient map.

Proposition 7.2.3 If a map $u: (M, \mu) \rightarrow (N, \nu)$ is surjective, λ_g^α -continuous and λ_g^α -open then u is a λ_g^α -quotient map.

Proof: Let $u^{-1}(T)$ be an open set in (M, μ) . By Definition 7.2.1 it is enough to prove that T is a λ_g^α -open set in (N, ν) . Since u is a λ_g^α -open map, $u(u^{-1}(T))$ is a λ_g^α -open set and as u is a surjective map, $u(u^{-1}(T)) = T$. Therefore T is a λ_g^α -open set in (N, ν) . Hence u is a λ_g^α -quotient map.

Proposition 7.2.4 Every quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a λ_g^α -quotient map.

Proof: Follows from the fact that every continuous map is λ_g^α -continuous and every open set is λ_g^α -open.

The subsequent example shows that the converse of Proposition 7.2.4 may not hold good.

Example 7.2.5 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i, j\}, M\}$, $\nu = \{\phi, \{i\}, \{j\}, \{i, j\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = i$, $u(j) = k$ and $u(k) = j$. Then u is a λ_g^α -quotient map but not a quotient map, since for the open set $\{i\}$ in (N, ν) , $u^{-1}(\{i\}) = \{i\}$ is not an open set in (M, μ) .

Definition 7.2.6 Let $u: (M, \mu) \rightarrow (N, \nu)$ be a surjective map. Then u is called a **strongly λ_g^α -quotient map** provided a set T of (N, ν) is open in (N, ν) if and only if $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) .

Example 7.2.7 Let $M = \{i, j, k\}, \mu = \{\phi, \{i\}, \{j, k\}, M\}, N = \{p, q\}, \nu = \{\phi, \{p\}, \{q\}, N\}$, λ_g^α -open sets of (M, μ) and (N, ν) are $P(M)$ and $P(N)$ respectively. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = p$ and $u(j) = q = u(k)$. Then u is a strongly λ_g^α -quotient map.

Proposition 7.2.8 Every strongly λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a λ_g^α -open map.

Proof: Let T be an open set in (M, μ) . By Proposition 2.4.3, T is also a λ_g^α -open set in (M, μ) , i.e., $u^{-1}(u(T))$ is a λ_g^α -open set in (M, μ) . As u is a strongly λ_g^α -quotient map, $u(T)$ is open in (N, ν) and hence $u(T)$ is λ_g^α -open in (N, ν) . Therefore u is a λ_g^α -open map.

The subsequent example shows that the converse of Proposition 7.2.8 may not hold good.

Example 7.2.9 Let $M = N = \{i, j, k, l\}, \mu = \{\phi, \{i, j\}, M\}$ and $\nu = \{\phi, \{i, j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = j, u(j) = k, u(k) = i$ and $u(l) = l$. Then u is a λ_g^α -open map but not a strongly λ_g^α -quotient map, since for the open set $\{i, j, k\}$ in (N, ν) , $u^{-1}(\{i, j, k\}) = \{i, j, k\}$ is not a λ_g^α -open set in (M, μ) .

Proposition 7.2.10 Every strongly λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly λ_g^α -open map.

Proof: Let T be a λ_g^α -open set in (M, μ) , i.e., $u^{-1}(u(T))$ is a λ_g^α -open set in (M, μ) . As u is a strongly λ_g^α -quotient map, $u(T)$ is an open set in (N, ν) . By Proposition 2.4.3, $u(T)$ is a λ_g^α -open set in (N, ν) . Hence u is a strongly λ_g^α -open map.

The subsequent example shows that the converse of Proposition 7.2.10 may not hold good.

Example 7.2.11 Let $M = N = \{i, j, k, l\}, \mu = \{\phi, \{i\}, \{j\}, \{i, j\}, \{i, j, k\}, \{i, j, l\}, M\}$ and $\nu = \{\phi, \{i, j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = l, u(j) = k, u(k) = j$

and $u(l) = i$. Then u is a strongly λ_g^α -open map but not a strongly λ_g^α -quotient map, since $u^{-1}(\{i\}) = \{l\}$ is a λ_g^α -open set in (M, μ) , but the set $\{i\}$ is not an open set in (N, ν) .

Definition 7.2.12 A surjective map $u: (M, \mu) \rightarrow (N, \nu)$ is called a **completely λ_g^α -quotient map** if u is a λ_g^α -irresolute map and $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) implies T is an open set in (N, ν) .

Example 7.2.13 Let $M = \{i, j, k\}, \mu = \{\phi, \{i\}, \{j, k\}, M\}, N = \{p, q\}$ and $\nu = \{\phi, \{p\}, \{q\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = p$ and $u(j) = q = u(k)$. Then u is a completely λ_g^α -quotient map.

Proposition 7.2.14 Every completely λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a λ_g^α -irresolute map.

Proof: Follows from Definition 7.2.12.

Example 7.2.15 Let $M = \{i, j, k\}, \mu = \{\phi, \{i\}, M\}, N = \{p, q\}$ and $\nu = \{\phi, \{p\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = p, u(j) = q = u(k)$. Then u is a λ_g^α -irresolute map but not a completely λ_g^α -quotient map, since $u^{-1}(\{q\}) = \{j, k\}$ is a λ_g^α -open set in (M, μ) , but the set $\{q\}$ is not an open set in (N, ν) .

The subsequent example shows that the converse of Proposition 7.2.14 may not hold good.

Proposition 7.2.16 Every completely λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly λ_g^α -open map.

Proof: Let T be a λ_g^α -open set in (M, μ) , i.e., $u^{-1}(u(T))$ is a λ_g^α -open set in (M, μ) . As u is a completely λ_g^α -quotient map, $u(T)$ is an open set in (N, ν) . By Proposition 2.4.3, $u(T)$ is a λ_g^α -open set in (N, ν) . Therefore u is a strongly λ_g^α -open map.

The subsequent example shows that the converse of Proposition 7.2.16 may not hold good.

Example 7.2.17 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i\}, \{j\}, \{i, j\}, \{i, j, k\}, \{i, j, l\}, M\}$ and $\nu = \{\phi, \{i, j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = l$, $u(j) = k$, $u(k) = j$ and $u(l) = i$. Then u is a strongly λ_g^α -open map but not a completely λ_g^α -quotient map since $u^{-1}(\{i\}) = \{l\}$ is a λ_g^α -open set in (M, μ) but the set $\{i\}$ is not an open set in (N, ν) .

Proposition 7.2.18 Every strongly λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a λ_g^α -quotient map.

Proof: Let T be an open set in (N, ν) . Then $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) . Therefore, by Theorem 3.2.3, u is a λ_g^α -continuous map. Now let $u^{-1}(T)$ be an open set in (M, μ) , by Proposition 2.4.3, $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) . Since u is a strongly λ_g^α -quotient map, T is an open set in (N, ν) . Again, using Proposition 2.4.3, T is a λ_g^α -open set in (N, ν) . Hence u is a λ_g^α -quotient map.

The subsequent example shows that the converse of Proposition 7.2.18 may not hold good.

Example 7.2.19 Consider M, N, μ, ν and u as in Example 7.2.2. Then u is a λ_g^α -quotient map but not a strongly λ_g^α -quotient map since $u^{-1}(\{j, k\}) = \{i, k\}$ is λ_g^α -open in (M, μ) , but the set $\{j, k\}$ is not open in (N, ν) .

Proposition 7.2.20 Every completely λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly λ_g^α -quotient map.

Proof: Let T be an open set in (N, ν) . Then by Proposition 2.4.3, T is a λ_g^α -open set in (N, ν) . Then by assumption $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) . Moreover, if $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) , then T is open in (N, ν) as u is a completely λ_g^α -quotient map. Hence u is a strongly λ_g^α -quotient map.

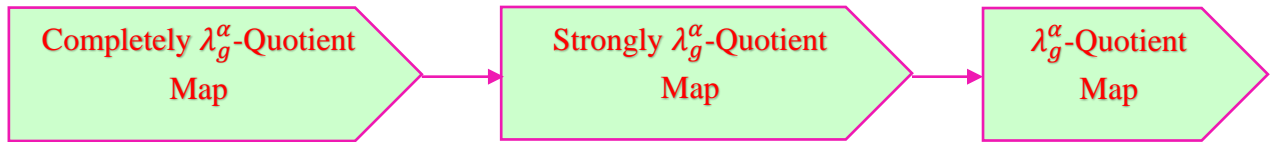
Proposition 7.2.21 Every completely λ_g^α -quotient map $u: (M, \mu) \rightarrow (N, \nu)$ is a λ_g^α -quotient map.

Proof: By assumption, u is a λ_g^α -irresolute map. By Proposition 3.3.4, u is a λ_g^α -continuous map. Let $u^{-1}(T)$ be an open set in (M, μ) , by Proposition 2.4.3, $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) . As u is a completely λ_g^α -quotient map, T is an open set in (N, ν) . Hence T is also a λ_g^α -open set in (N, ν) . Thus u is a λ_g^α -quotient map.

The subsequent example shows that the converse of Proposition 7.2.21 may not hold good.

Example 7.2.22 Consider M, N, μ, ν and u as in Example 7.2.2. Then u is a λ_g^α -quotient map but not a completely λ_g^α -quotient map, since $u^{-1}(\{i, k\}) = \{j, k\}$ is a λ_g^α -open set in (M, μ) , but the set $\{i, k\}$ is not open in (N, ν) .

Remark 7.2.23 From the above observations we obtain the following relations



7.3. Compositions of λ_g^α -Quotient Maps

This section provides the derivations regarding the compositions of λ_g^α -quotient maps and its special forms with at most accuracy.

Proposition 7.3.1 If a map $u: (M, \mu) \rightarrow (N, \nu)$ is open, surjective and λ_g^α -irresolute and $w: (N, \nu) \rightarrow (K, \kappa)$ is a λ_g^α -quotient map then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a λ_g^α -quotient map.

Proof: Let T be any open set in (K, κ) . Since w is a λ_g^α -quotient map, $w^{-1}(T)$ is a λ_g^α -open set in (N, ν) . As u is a λ_g^α -irresolute map $(w \circ u)^{-1}(T) = u^{-1}(w^{-1}(T))$ is a λ_g^α -open set in (M, μ) . Thus $(w \circ u)$ is a λ_g^α -continuous map.

Let T be any set in (K, κ) and suppose that $(w \circ u)^{-1}(T)$ is open in (M, μ) . Since u is open and surjective, $u((w \circ u)^{-1}(T)) = u(u^{-1}(w^{-1}(T)))$ is open in (N, ν) which

implies $w^{-1}(T)$ is open in (N, ν) . Since w is a λ_g^α -quotient map, T is a λ_g^α -open set in (K, κ) . Hence $(w \circ u)$ is a λ_g^α -quotient map.

Proposition 7.3.2 Let $u: (M, \mu) \rightarrow (N, \nu)$ be a strongly λ_g^α -open, surjective and λ_g^α -irresolute map and $w: (N, \nu) \rightarrow (K, \kappa)$ be a strongly λ_g^α -quotient map. Then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a strongly λ_g^α -quotient map.

Proof: Let T be any open set in (K, κ) . Then $w^{-1}(T)$ is λ_g^α -open in (N, ν) , since w is a strongly λ_g^α -quotient map. As u is λ_g^α -irresolute, $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) which implies that $(w \circ u)^{-1}(T)$ is λ_g^α -open in (M, μ) .

Let $(w \circ u)^{-1}(T)$ is λ_g^α -open in (M, μ) , i.e., $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) . Since u is strongly λ_g^α -open and surjective $u(u^{-1}(w^{-1}(T)))$ is λ_g^α -open in (N, ν) implies $w^{-1}(T)$ is λ_g^α -open in (N, ν) . Since w is strongly λ_g^α -quotient map, T is open in (K, κ) . Hence $(w \circ u)$ is a strongly λ_g^α -quotient map.

Proposition 7.3.3 Let $u: (M, \mu) \rightarrow (N, \nu)$ be strongly λ_g^α -open, surjective and λ_g^α -irresolute map and $w: (N, \nu) \rightarrow (K, \kappa)$ be a completely λ_g^α -quotient map. Then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a completely λ_g^α -quotient map.

Proof: Since u and w are λ_g^α -irresolute, $(w \circ u)$ is λ_g^α -irresolute by Proposition 3.3.24. Suppose that $(w \circ u)^{-1}(T)$ is λ_g^α -open in (M, μ) for a subset T in (K, κ) , that is $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) . Since u is strongly λ_g^α -open and surjective, $u(u^{-1}(w^{-1}(T))) = w^{-1}(T)$ is λ_g^α -open in (N, ν) . As w is completely λ_g^α -quotient implies T is open in (K, κ) . Hence $(w \circ u)$ is a completely λ_g^α -quotient map.

Proposition 7.3.4 Let $u: (M, \mu) \rightarrow (N, \nu)$ be a strongly λ_g^α -quotient and λ_g^α -irresolute map and $w: (N, \nu) \rightarrow (K, \kappa)$ be a completely λ_g^α -quotient map. Then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a completely λ_g^α -quotient map.

Proof: Let T be a λ_g^α -open set in (K, κ) . Then $w^{-1}(T)$ is a λ_g^α -open set in (N, ν) , as w is completely λ_g^α -quotient. Since u is λ_g^α -irresolute, $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) which implies that $(w \circ u)^{-1}(T)$ is λ_g^α -open in (M, μ) . This shows that $(w \circ u)$ is λ_g^α -irresolute.

Let $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) for a subset $T \subseteq K$. Since u is strongly λ_g^α -quotient, $w^{-1}(T)$ is open in (N, ν) . This implies that $w^{-1}(T)$ is a λ_g^α -open set in (N, ν) . As w is completely λ_g^α -quotient, T is open in (K, κ) . Hence $(w \circ u)$ is a completely λ_g^α -quotient map.

Proposition 7.3.5 Let $u: (M, \mu) \rightarrow (N, \nu)$ and $w: (N, \nu) \rightarrow (K, \kappa)$ be completely λ_g^α -quotient maps. Then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a completely λ_g^α -quotient map.

Proof: Let T be λ_g^α -open set in (K, κ) . Since w is completely λ_g^α -quotient, then $w^{-1}(T)$ is λ_g^α -open in (N, ν) . As u is completely λ_g^α -quotient, $u^{-1}(w^{-1}(T)) = (w \circ u)^{-1}(T)$ is λ_g^α -open in (M, μ) . This shows that $(w \circ u)$ is λ_g^α -irresolute.

Let $(w \circ u)^{-1}(T)$ is a λ_g^α -open set in (M, μ) . Then $u^{-1}(w^{-1}(T))$ is λ_g^α -open in (M, μ) . Since u is completely λ_g^α -quotient, $w^{-1}(T)$ is open in (N, ν) . This implies $w^{-1}(T)$ is λ_g^α -open in (N, ν) . Since w is a completely λ_g^α -quotient map, T is open in (K, κ) . Hence $(w \circ u)$ is a completely λ_g^α -quotient map.