
CHAPTER 6

Chapter 6

Generalized π Closed Sets in Biminimal Structure Spaces

6.1 Introduction

Noiri and Popa (2009) investigated the concept of minimal structure by introducing the notions of m_X -open sets, m_X -closed sets and M -continuous functions. Boonpok (2010) enhanced the concept of biminimal structure space and studied the fundamental properties of $m_X^1 m_X^2$ -closed sets, $m_X^1 m_X^2$ -open sets and M -continuous functions. Boonpok (2011) introduced the concepts of generalized m -closed sets, regular generalized closed sets, weakly m -precontinuous functions and separation axioms in biminimal structure spaces.

In this chapter, we have introduced the concepts of $m_X^{(i,j)}$ - $g\pi$ -closed sets in biminimal structure spaces and tried to unify certain types of modifications of $g\pi$ -closed sets. The concepts of $m_X^{(i,j)}$ - $g\pi$ -continuous functions, $m_X^{(i,j)}$ - $g\pi$ -irresolute functions and $m^{(i,j)}$ - $g\pi$ $T_{1/2}$ spaces are introduced and their properties are studied.

6.2 $m_X^{(i,j)}$ - $g\pi$ closed sets

Definition 6.2.1 A subset A of a biminimal structure space (X, m_X^1, m_X^2) is said to be a $m_X^{(i,j)}$ - $g\pi$ closed set if $m_X^j \pi Cl(A) \subseteq U$, whenever $A \subseteq U$ and U is m_X^i -open in X , where $i, j = 1, 2$ and $i \neq j$. The complement of $m_X^{(i,j)}$ - $g\pi$ -closed is said to be $m_X^{(i,j)}$ - $g\pi$ open.

Example 6.2.1 (a) Let $X = \{a, b, c\}$. Consider two minimal structures

$$m_X^i = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\} \text{ and } m_X^j = \{X, \emptyset, \{a\}, \{b\}\}.$$

$$\text{Then } m_X^{(i,j)}\text{-}g\pi\text{-}C(X) = \{X, \emptyset, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$$

A subset A of a biminimal structure space (X, m_X^1, m_X^2) is called pairwise m - $g\pi$ -closed if A is $m_X^{(1,2)}$ - $g\pi$ -closed and $m_X^{(2,1)}$ - $g\pi$ -closed. The complement of pairwise m - $g\pi$ -closed is called pairwise m - $g\pi$ -open.

The family of all $m_X^{(i,j)}$ - $g\pi$ -closed (resp. $m_X^{(i,j)}$ - $g\pi$ -open) sets of (X, m_X^1, m_X^2) is denote by $m_X^{(i,j)}$ - $g\pi$ - $C(X)$ (resp. $m_X^{(i,j)}$ - $g\pi$ - $O(X)$), $i, j = 1, 2$ and $i \neq j$.

Theorem 6.2.2 If subsets A and B of a biminimal structure space (X, m_X^1, m_X^2) are $m_X^{(i,j)}$ - $g\pi$ -closed subsets of X , then $A \cup B$ is $m_X^{(i,j)}$ - $g\pi$ -closed, where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A and B be $m_X^{(i,j)}$ - $g\pi$ -closed subsets of X . Let U be m_X^i -open in X such that $A \cup B \subseteq U$. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are $m_X^{(i,j)}$ - $g\pi$ -closed, $m_X^j \pi cl(A) \subseteq U$ and $m_X^j \pi cl(B) \subseteq U$. Hence $m_X^j \pi cl(A \cup B) = m_X^j \pi cl(A) \cup m_X^j \pi cl(B) \subseteq U$. That is $m_X^j \pi cl(A \cup B) \subseteq U$. Therefore $A \cup B$ is an $m_X^{(i,j)}$ - $g\pi$ -closed set in X .

Remark 6.2.3 The intersection of two $m_X^{(i,j)}$ - $g\pi$ -closed sets need not be a $m_X^{(i,j)}$ - $g\pi$ -closed set in general as can be seen from the following example.

Example 6.2.4 Let $X = \{a, b, c\}$. Consider two minimal structures $m_X^1 = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ and $m_X^2 = \{X, \emptyset, \{a\}, \{b\}\}$. Then $\{a, b\}$ and $\{b, c\}$ are $m_X^{(i,j)}$ - $g\pi$ -closed but $\{a, b\} \cap \{b, c\} = \{b\}$ is not $m_X^{(i,j)}$ - $g\pi$ -closed.

Theorem 6.2.5. If A is a $m_X^{(i,j)}$ - $g\pi$ -closed set of (X, m_X^1, m_X^2) such that $A \subseteq B \subseteq m_X^j \pi cl(A)$, then B is $m_X^{(i,j)}$ - $g\pi$ -closed set, where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A be a $m_X^{(i,j)}$ - $g\pi$ -closed set and $A \subseteq B \subseteq m_X^j \pi cl(A)$. Let $B \subseteq U$ and U is m_X^i -open. Then $A \subseteq U$. Since A is $m_X^{(i,j)}$ - $g\pi$ -closed, $m_X^j \pi cl(A) \subseteq U$.

Since $B \subseteq m_X^j$ - π -Cl(A), then m_X^j - π -Cl(B) \subseteq m_X^j - π -Cl(A) \subseteq U. Hence, B is $m_X^{(i,j)}$ -g π -closed.

Theorem 6.2.6 If a subset A of a biminimal structure space (X, m_X^1, m_X^2) is both m_X^i -open and $m_X^{(i,j)}$ -g π -closed, then A is m_X^j - π closed, where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A be m_X^i open and $m_X^{(i,j)}$ -g π -closed, m_X^j - π -Cl(A) = A. Hence, A is m_X^j - π -closed.

Remark 6.2.9 $m_X^{(1,2)}$ -g π -C(X) is generally not equal to $m_X^{(2,1)}$ -g π -C(X) as can be seen from the following example.

Example 6.2.10 Let $X = \{a, b, c\}$. Consider two minimal structures

$$m_X^1 = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\} \text{ and } m_X^2 = \{X, \emptyset, \{a\}, \{b\}\}.$$

Then $m_X^{(1,2)}$ -g π -C(X) = $\{X, \emptyset, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$ and

$$m_X^{(2,1)}$$
-g π -C(X) = $\{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Thus $m_X^{(1,2)}$ -g π -C(X) \neq $m_X^{(2,1)}$ -g π -C(X)

Remark 6.2.11 Let m_X^1 and m_X^2 be m structures on X. If $m_X^1 \subseteq m_X^2$ then $m_X^{(1,2)}$ -g π C(X) \subseteq $m_X^{(2,1)}$ -g π C(X)

Theorem 6.2.12 For each element x of a biminimal structure space (X, m_X^1, m_X^2) , $\{x\}$ is m_X^i closed or $X - \{x\}$ is $m_X^{(i,j)}$ -g π -closed, where $i, j = 1, 2$ and $i \neq j$.

Proof: Let $x \in X$ and the singleton $\{x\}$ be not m_X^i -closed. Then $X - \{x\}$ is not m_X^i -open, and so X is the only m_X^i -open set which contains $X - \{x\}$. Hence $X - \{x\}$ is $m_X^{(i,j)}$ -g π -closed.

Theorem 6.2.13 Let A be a subset of a biminimal structure space (X, m_X^1, m_X^2) . If A is $m_X^{(i,j)}$ -g π -closed, then m_X^j - π -Cl(A) - A contains no nonempty m_X^i -closed set, where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A be a $m_X^{(i,j)}$ -g π -closed set and $F \neq \emptyset$ be a m_X^i -closed set such that $F \subseteq m_X^j$ - π -Cl(A) - A . Since $A \in m_X^{(i,j)}$ -g π -C(X), m_X^j - π -Cl(A) \subseteq X - F . Thus $F \subseteq m_X^j$ - π -Cl(A) \cap (X - m_X^j - π -Cl(A)) = \emptyset , this is a contradiction. Then m_X^j - π -Cl(A) - A contains no nonempty m_X^i -closed set.

Corollary 6.2.14 Let m_X^1 and m_X^2 be minimal structures on X satisfying property \mathfrak{B} . If A is $m_X^{(i,j)}$ -g π -closed in (X, m_X^1, m_X^2) , then A is m_X^j - π -closed if and only if m_X^j - π -Cl(A) - A is m_X^i -closed, where $i, j = 1, 2$ and $i \neq j$.

Proof: If A is m_X^j - π -closed, then m_X^j - π -Cl(A) = A . i.e. m_X^j - π -Cl(A) - A = \emptyset and hence m_X^j - π -Cl(A) - A is m_X^i -closed.

Conversely, if m_X^j - π -Cl(A) - A is m_X^i -closed, then by Theorem 6.2.13 m_X^j - π -Cl(A) - A = \emptyset , since A is $m_X^{(i,j)}$ -g π -closed. Therefore, A is m_X^j - π -closed.

Theorem 6.2.15 For a biminimal structure space (X, m_X^1, m_X^2) , where (m_X^1, m_X^2) have property \mathfrak{B} , m_X^i -O(X) \subseteq m_X^j - π -Cl(X) if and only if every subset of X is a $m_X^{(i,j)}$ -g π -closed set, where $i, j = 1, 2$ and $i \neq j$.

Proof: suppose that m_X^i -O(X) \subseteq m_X^j - π -Cl(X). Let A be a subset of X such that $A \subseteq U$, where $U \in m_X^i$ -O(X). Then, m_X^j - π -Cl(A) \subseteq m_X^j - π -Cl(U) \subseteq U and hence A is $m_X^{(i,j)}$ -g π -closed.

Conversely, suppose that every subset of X is $m_X^{(i,j)}$ -g π -closed. Let $U \in m_X^i$ -O(X). Since U is $m_X^{(i,j)}$ -g π -closed, m_X^j - π -Cl(U) \subseteq U . Therefore, $U \in m_X^j$ - π -Cl(X) and hence m_X^i -O(X) \subseteq m_X^j - π -Cl(X).

Theorem 6.2.16 A subset A of a biminimal structure space (X, m_X^1, m_X^2) is $m_X^{(i,j)}$ - $g\pi$ -open if and only if for every subset F of X , $F \subseteq m_X^j -\pi \text{Int}(A)$ whenever F is m_X^i closed and $F \subseteq A$, where $i, j = 1, 2$ and $i \neq j$.

Proof: Suppose that A is $m_X^{(i,j)}$ - $g\pi$ -open. Let $F \subseteq A$ and F is m_X^i closed. Then $X - A \subseteq X - F$ and $X - F$ is m_X^i open, $X - A$ is $m_X^{(i,j)}$ - $g\pi$ -closed, then $m_X^j \pi \text{Cl}(X - A) \subseteq X - F$. Thus $X - (m_X^j -\pi \text{Int}(A)) \subseteq X - F$ and hence $F \subseteq m_X^j -\pi \text{Int}(A)$.

Conversely, suppose that $F \subseteq m_X^j -\pi \text{Int}(A)$ whenever F is m_X^i - closed and $F \subseteq A$. Let $X - A \subseteq U$ and U is m_X^i - open. Then $X - U \subseteq A$ and $X - U$ is m_X^i -closed. By assumption, $X - U \subseteq m_X^j -\pi \text{Int}(A)$, and hence $X - (m_X^j -\pi \text{Int}(A)) \subseteq U$. Therefore, $m_X^j \pi \text{Cl}(X - A) \subseteq U$. Thus $X - A$ is $m_X^{(i,j)}$ - $g\pi$ -closed. Hence, A is $m_X^{(i,j)}$ - $g\pi$ -open.

Theorem 6.2.17 Let A and B be subsets of a biminimal structure spaces (X, m_X^1, m_X^2) such that $m_X^j -\pi \text{Int}(A) \subseteq B \subseteq A$. If A is $m_X^{(i,j)}$ - $g\pi$ -open, then B is $m_X^{(i,j)}$ - $g\pi$ -open, where $i, j = 1, 2$ and $i \neq j$.

Proof: Suppose that $m_X^j -\pi \text{Int}(A) \subseteq B \subseteq A$. Let F be m_X^i - closed such that $F \subseteq B$. Since A is $m_X^{(i,j)}$ - $g\pi$ -open, $F \subseteq m_X^j -\pi \text{Int}(A)$. Since $m_X^j -\pi \text{Int}(A) \subseteq B$, $m_X^j -\pi \text{Int}(m_X^j -\pi \text{Int}(A)) \subseteq m_X^j -\pi \text{Int}(B)$. Consequently, $m_X^j -\pi \text{Int}(A) \subseteq m_X^j -\pi \text{Int}(B)$. Hence, $F \subseteq m_X^j -\pi \text{Int}(B)$. Therefore, B is $m_X^{(i,j)}$ - $g\pi$ - open.

Theorem 6.2.18 Let A be a subset of a biminimal structure space (X, m_X^1, m_X^2) . If A is $m_X^{(i,j)}$ - $g\pi$ - closed, then $m_X^j \pi \text{Cl}(A) - A$ is $m_X^{(i,j)}$ - $g\pi$ -open, where $i, j = 1, 2$ and $i \neq j$.

Proof: Suppose that A is $m_X^{(i,j)}$ - $g\pi$ -closed. We shall show that $m_X^j \pi \text{Cl}(A) - A$ is $m_X^{(i,j)}$ - $g\pi$ -open. Let $F \subseteq m_X^j \pi \text{Cl}(A) - A$ and F is m_X^i closed.

Since A is $m_X^{(i,j)}$ -g π -closed, m_X^j π -Cl(A) - A does not contain nonempty m_X^i closed set by Theorem 6.2.13. Consequently, $F = \emptyset$. Therefore, $\emptyset \subseteq m_X^j$ π -Cl(A) - A . Hence, m_X^j π -Cl(A) - A is $m_X^{(i,j)}$ -g π -open.

6.3 $m_X^{(i,j)}$ -g π continuous function

Definition 6.3 1. Let (X, m_X^1, m_X^2) and (Y, m_Y^1, m_Y^2) be biminimal structure space. A function $f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)$ is said to be a $m_X^{(i,j)}$ -g π continuous function if $f^{-1}(F)$ is $m_X^{(i,j)}$ -g π closed in X for every $m_Y^{(i,j)}$ -closed F of Y , where $i, j = 1, 2$ and $i \neq j$.

A function $f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)$ is $m_X^{(i,j)}$ -g π continuous if and only if $f^{-1}(U)$ is $m_X^{(i,j)}$ -g π -open in X for every $m_Y^{(i,j)}$ -open U of Y , where $i, j = 1, 2$ and $i \neq j$.

Definition 6.3.2. A biminimal structure space (X, m_X^1, m_X^2) is said to be $m^{(i,j)}$ -g π $T_{1/2}$ space if for every $m_X^{(i,j)}$ -g π -closed set is a $m_X^{(i,j)}$ -closed set, where $i, j = 1, 2$ and $i \neq j$.

Theorem 6.3.3. Let (X, m_X^1, m_X^2) be a $m^{(i,j)}$ -g π $T_{1/2}$ space and let (Y, m_Y^1, m_Y^2) be a biminimal structure space, where (m_Y^1, m_Y^2) have property \emptyset . For an injective function $f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)$, the following properties are equivalent:

- (1) f is $(m_X^{(i,j)})$ -g π -continuous.
- (2) For each $x \in X$ and for every $m_Y^{(i,j)}$ -open set V containing $f(x)$, there exists a $m_X^{(i,j)}$ -g π -open set U containing x such that $f(U) \subseteq V$.
- (3) $f(m_X^j$ π -Cl(A)) \subseteq m_Y^j π -Cl($f(A)$) for every subset A of X .
- (4) m_X^j π -Cl($f^{-1}(B)$) \subseteq $f^{-1}(m_Y^j$ π -Cl(B)) for every subset B of Y .

Proof: (1) \Rightarrow (2): Let $x \in X$ and V be a $m_Y^{(i,j)}$ -open subset of Y containing $f(x)$.

Then by (1), $f^{-1}(V)$ is $m_X^{(i,j)}$ - $g\pi$ -open of X containing x . If $U = f^{-1}(V)$, then $f(U) \subseteq V$.

(2) \Rightarrow (3): Let A be a subset of X and $f(x) \notin m_Y^j \pi\text{-Cl}(f(A))$. Then, there exists a $m_Y^{(i,j)}$ -open subset V of Y containing $f(x)$ such that $V \cap f(A) = \emptyset$. Then by (2), there exists a $m_X^{(i,j)}$ - $g\pi$ -open set U such that $f(x) \in f(U) \subseteq V$. Hence, $f(U) \cap f(A) = \emptyset$ implies $U \cap A = \emptyset$. Consequently, $x \notin m_X^j \pi\text{-Cl}(A)$ and $f(x) \notin f(m_X^j \pi\text{-Cl}(A))$.

(3) \Rightarrow (4): Let B be a subset of Y . By (3), we obtain $f(m_X^j \pi\text{Cl}(f^{-1}(B))) \subseteq m_Y^j \pi\text{Cl}(f(f^{-1}(B)))$. Thus $m_X^j \pi\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(m_Y^j \pi\text{Cl}(B))$.

(4) \Rightarrow (1): Let F be a $m_Y^{(i,j)}$ -closed subset of Y . Let U be a m_X^i -open subset of X such that $f^{-1}(F) \subseteq U$. Since $m_Y^j \pi\text{-Cl}(F) = F$ and by (4), $m_X^j \pi\text{-Cl}(f^{-1}(F)) \subseteq U$. Hence, f is $m_X^{(i,j)}$ - $g\pi$ continuous.

Theorem 6.3 4. Let (Y, m_Y^1, m_Y^2) be a $m^{(ij)}$ - $g\pi T_{1/2}$ space and let

$f : (X, m_X^1, m_X^2) \rightarrow (Y, m_Y^1, m_Y^2)$ and $g : (Y, m_Y^1, m_Y^2) \rightarrow (Z, m_Z^1, m_Z^2)$ be functions. If f and g are $m_X^{(i,j)}$ - $g\pi$ -continuous, then $g \circ f$ is $m_X^{(i,j)}$ - $g\pi$ -continuous.

Proof: Let F be a $m_Z^{(i,j)}$ -closed subset of Z . Since g is $m_Y^{(i,j)}$ - $g\pi$ -continuous, then $g^{-1}(F)$ is $m_Y^{(i,j)}$ - $g\pi$ -closed subset of Y . Since (Y, m_Y^1, m_Y^2) is $m^{(ij)}$ - $g\pi T_{1/2}$ space, $g^{-1}(F)$ is $m_Y^{(i,j)}$ -closed subset of Y . Since f is $m_X^{(i,j)}$ - $g\pi$ -continuous, then $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is $m_X^{(i,j)}$ - $g\pi$ -closed subset of X . Hence, $g \circ f$ is $m_X^{(i,j)}$ - $g\pi$ -continuous

6.4 $m_X^{(i,j)}$ - $g\pi$ - Dense Sets in Biminimal Structure Spaces

Definition 6.4.1. Let (X, m_X^1, m_X^2) be a biminimal structure space, A be a subset of X . A is called $m_X^{(i,j)}$ - $g\pi$ - dense set in X if $X = m_X^i \pi Cl (m_X^j \pi Cl(A))$, where $i, j = 1, 2$ and $i \neq j$.

Example 6.4.2. Let $X = \{a, b, c\}$. Define m -structures m_X^1 and m_X^2 on X as follows:

$m_X^1 = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ and $m_X^2 = \{X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\}\}$. Then $m_X^1 \pi Cl (m_X^2 \pi Cl(\{a, b\})) = X$ and $m_X^2 \pi Cl (m_X^1 \pi Cl(\{a, b\})) = X$. Hence $\{a, b\}$ is $m_X^{(1,2)}$ - $g\pi$ -dense set in X and $\{a, b\}$ is $m_X^{(2,1)}$ - $g\pi$ -dense set in X .

Theorem 6.4.3. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . If A is a $m_X^{(i,j)}$ - $g\pi$ -dense set in X then for any non-empty $m_X^{(i,j)}$ - π -closed subset F of X , where $i, j = 1, 2$ and $i \neq j$ such that $A \subseteq F$, we have $F = X$.

Proof: Let A be a $m_X^{(i,j)}$ - $g\pi$ -dense set in X and F be a $m_X^{(i,j)}$ - π -closed subset of X , where $i, j = 1, 2$ and $i \neq j$ such that $A \subseteq F$. Since A is $m_X^{(i,j)}$ - $g\pi$ dense set in X , $X = m_X^i \pi Cl (m_X^j \pi Cl(A))$. By assumption, F is $m_X^{(i,j)}$ - π closed and $A \subseteq F$, it follows that $X = m_X^i \pi Cl (m_X^j \pi Cl(A)) \subseteq m_X^i \pi Cl (m_X^j \pi Cl(A)) = F$. Hence $F = X$.

Note 6.4.4 By Theorem 6.4.3 if A is a $m_X^{(i,j)}$ - $g\pi$ -dense set in X , then only X is the $m_X^{(i,j)}$ - π -closed set in X containing A .

Remark 6.4.5. The Theorem 6.4.3 is not true if F is not $m_X^{(i,j)}$ - π -closed. We can be seen from the following example.

Example 6.4.6. Let $X = \{a, b, c\}$. Define m -structures m_X^1 and m_X^2 on X as follows:

$m_X^1 = \{ X, \emptyset, \{a, c\}, \{b, c\} \}$ and $m_X^2 = \{ X, \emptyset, \{a\}, \{b\}, \{a, c\}, \{b, c\} \}$. Then $m_X^1 \pi \text{Cl}(m_X^2 \pi \text{Cl}(\{a, b\})) = X$ and we have $\{a, b\}$ is $m_X^1 m_X^2$ - $g\pi$ -dense set in X . But $\{a, b\}$ is not $m_X^{(i,j)}$ - π -closed set in X .

Theorem 6.4.7. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . If $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = \emptyset$, then for any non-empty $m_X^{(i,j)}$ - π closed subset F of X , where $i, j = 1, 2$ and $i \neq j$ such that $A \subseteq F$, we have $F = X$.

Proof: Assume that $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = \emptyset$, and F is $m_X^{(i,j)}$ - π -closed subset of X , where $i, j = 1, 2$ and $i \neq j$ such that $A \subseteq F$. By assumption, we have $\emptyset = X \setminus m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A))$ and so

$$X = m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) \subseteq m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) = F. \text{ Consequently } F = X.$$

Theorem 6.4.8. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . If A is $m_X^{(i,j)}$ - $g\pi$ -dense set in X , then $G \cap A \neq \emptyset$ for any non empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A be a $m_X^{(i,j)}$ - $g\pi$ -dense set in X . Suppose that $G \cap A = \emptyset$ for some non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$. Thus $A \subseteq X \setminus G$. Then $X = m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) \subseteq m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(X \setminus G)) = X \setminus m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(G))$. Since G is $m_X^{(i,j)}$ - π -open, and by Theorem 6.4.7, $G = m_X^i \pi \text{int}(m_X^j \pi \text{int}(G)) = \emptyset$, this is a contradiction. Therefore $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$.

Theorem 6.4.9. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . If $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = \emptyset$. Then $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$.

Proof: Let $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = \emptyset$. Suppose that $G \cap A = \emptyset$ for some non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$. Thus

$A \subseteq X \setminus G$ and $X = m_X^i \pi \text{Cl} (m_X^j \pi \text{Cl}(A)) \subseteq m_X^i \pi \text{Cl} (m_X^j \pi \text{Cl}(X \setminus G)) = X \setminus G$.

Hence $G = \emptyset$. A contradiction. Therefore $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$.

Theorem 6.4.10. Let (X, m_X^1, m_X^2) be a biminimal structure space and A be a subset of X . If for any non-empty $m_X^{(i,j)}$ - π -closed subset F of X such that $A \subseteq F$, then $F = X$ if and only if $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$.

Proof: Let (X, m_X^1, m_X^2) be a biminimal structure space and $A \subseteq X$.

(\Rightarrow) Assume that for any non-empty $m_X^{(i,j)}$ - π -closed subset F of X such that $A \subseteq F$, then $F = X$. Suppose that $G \cap A = \emptyset$ for some a non-empty $m_X^{(i,j)}$ - π open subset G of X , where $i, j = 1, 2$ and $i \neq j$. Thus $A \subseteq X \setminus G$. Since G is $m_X^{(i,j)}$ - π -open, $X \setminus G$ is $m_X^{(i,j)}$ - π -closed. By assumption, we have $X \setminus G = X$. Therefore $G = \emptyset$, this is contradiction. Hence $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X , where $i, j = 1, 2$ and $i \neq j$

(\Leftarrow) Assume that $G \cap A \neq \emptyset$ for any non-empty $m_X^{(i,j)}$ - π -open subset G of X and F is a non-empty $m_X^{(i,j)}$ - π - closed subset of X , where $i, j = 1, 2$ and $i \neq j$ such that $A \subseteq F$. Suppose that $F \neq X$. Thus $X \setminus F$ is a non-empty $m_X^{(i,j)}$ - π open subset of X . By assumption, we have $(X \setminus F) \cap A \neq \emptyset$. This is a contradiction with $A \subseteq F$. Therefore $F = X$.

Theorem 6.4.11. Let (X, m_X^1, m_X^2) be a biminimal structure and A be a subset of X . $m_X^i \pi \text{int} (m_X^j \pi \text{int}(X \setminus A)) = \emptyset$ If and only if A is $m_X^{(i,j)}$ - $g\pi$ -dense set in X , where $i, j = 1, 2$ and $i \neq j$.

Proof: Let A be a subset of X .

(\Rightarrow) Assume that $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = \emptyset$. Thus $X \setminus m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) = \emptyset$, $m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) = X$. Therefore A is $m_X^{(i,j)}$ -g π -dense set in X, where $i, j = 1, 2$ and $i \neq j$.

(\Leftarrow) Suppose that A is $m_X^{(i,j)}$ -g π -dense set in X. Then $m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) = X$ and also $m_X^i \pi \text{int}(m_X^j \pi \text{int}(X \setminus A)) = X \setminus m_X^i \pi \text{Cl}(m_X^j \pi \text{Cl}(A)) = \emptyset$, where $i, j = 1, 2$ and $i \neq j$.