INTRODUCTION

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"The essence of mathematics is not to make simple things complicated, but to make complicated things simple."

S. Gudder

Closure spaces were studied as sets endowed with a grounded, extensive and monotone closure operator. In1970 generalized closed sets in topological spaces were introduced by Levine [49] in order to extend some important properties of closed sets to a larger family of sets.

In 1991 Balachandran, Sundaram and Maki [2] introduced the notion of generalized continuous maps (g-continuous maps) by using g-closed sets and studied some of their properties.

In 2008 Boonpok and Khampakdee [4] introduced a new class of closed sets called ∂-closed sets in closure spaces, which lies between the class of closed sets and the class of generalized closed sets. Using the concept of ∂- closed sets, Boonpok[4] in 2010 introduced $T^{*}_{1/2}$- spaces and $T^{**}_{1/2}$ - spaces.

In 1969 Kelly [46] introduced the notion of bitopological spaces, (i.e. spaces are equipped with two arbitrary topologies) and extended some of the standard results of separation axioms in topological spaces to bitopological spaces. There exist, large number of papers which generalize topological concepts to bitopological setting.

In 1968 closure spaces were introduced by Čech [23] in 1968 and studied by many authors Chvalina [33] in 1976, Šlapal [61] in 2008, etc., The notion of closure system and closure operators are very useful tools in several areas of classical mathematics. They play an important role in topological spaces, Boolean algebra, convex sets etc. This led several authors to investigate the closure operator in the frame work of fuzzy set theory.

In 1980 Gerla et al. [40] studied fuzzy closure operator and fuzzy closure system as extension of closure operator and closure system.

In 2010 Boonpok [11] introduced the notion of biclosure spaces. (ie spaces equipped with two arbitrary closure operators) and extended some of the standard results of separation axioms in closure spaces to biclosure spaces.

The aim of this thesis is to study some of the generalized separation axioms in biclosure spaces.

To study these concepts the following articles are chosen for our study:

(i) "∂-Closed sets in biclosure spaces" by Boonpok [6]

(ii) "∂-Closed maps in biclosure spaces" by Boonpok [19]

(iii) "∂- Continuous maps in biclosure spaces" by Boonpok [6]

(iv) "$T_{1/2}$ – biclosure spaces" by Boonpok [17]

(v) "$T^{*}_{1/2}$ - spaces and $T^{**}_{1/2}$ - spaces" by Boonpok [18]

(vi) "Hausdorff biclosure spaces" by Boonpok [11].

(vii) "Generalized Hausdorff biclosure spaces" by Boonpok [12]

(viii) "∂-Hausdorff biclosure spaces" by Boonpok [22]

(ix) "Pairwise closed sets in biclosure spaces" by Boonpok [20]

(x) "Bicontinuous maps in biclosure spaces" by Boonpok [21]

(xi) "On $-Closed sets in Bi-ˇ$c$ech closure spaces" by Santhiya, Saranya, Parvathi [59]

Chapter I deals with preliminary definitions and results on closure spaces, biclosure spaces and $T_{1/2}$ - biclosure spaces that are needed for our study.

Chapter II deals with ∂- Closed sets, ∂- Closed maps, ∂-Continuous maps, $T^{*}_{1/2}$- spaces and $T^{**}_{1/2}$ - spaces.

In the first section, the concepts of ∂- Closed sets in biclosure spaces and its properties are analyzed.

It is interesting to note that

(i) Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is ∂-open if and only if $F \subseteq X - u_2(X - A)$ for every g-closed subset F of (X, u_1) with $F \subseteq A$.

(ii) Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then G is a ∂-open subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $G \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

(iii) Let $\{(X_\alpha, u^1_\alpha, u^2_\alpha) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a ∂-closed subset of $(X_\beta, u^1_\beta, u^2_\beta)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod_{\alpha \in I} (X_\alpha, u^1_\alpha, u^2_\alpha)$.

In the second section, the properties and characterizations of ∂- Closed maps in biclosure spaces are analyzed.

Some of the interesting results discussed are as follows:

(i) Let (X,u_1,u_2) be a biclosure space, $\{(Y_\alpha, v_\alpha^1, v_\alpha^2): \alpha \in I\}$ be a family of biclosure spaces and f: $X \to \prod_{\alpha\in I} Y_\alpha$ be a map. Then f: $(X,u_1,u_2) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂-closed if and only if $\pi_\alpha \circ f: (X,u_1,u_2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂-closed for each $\alpha \in I$.

(ii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2): \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2): \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let $f_\alpha: X_\alpha \to Y_\alpha$ be a surjection and let f: $\prod_{\alpha\in I} X_\alpha \to \prod_{\alpha\in I} Y_\alpha$ be defined by $f((x_\alpha)_{\alpha\in I})=(f_\alpha(x_\alpha))_{\alpha\in I}$. Then f: $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \to \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂-closed if and only if $f_\alpha: (X_\alpha, v_\alpha^1, v_\alpha^2) \to (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂-closed for each $\alpha \in I$.

In the third section, the concepts of ∂-Continuous maps in biclosure spaces and its properties are analyzed.

It is interesting to note that

(i) Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If $g \circ f: (X,u_1,u_2) \to (Z,w_1,w_2)$ is ∂-closed and $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is surjective and ∂-continuous, then $g: (Y,v_1,v_2) \to (Z,w_1,w_2)$ is ∂-closed.

(ii) Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. If a map $f: (X,u_1,u_2) \to (Y,v_1,v_2)$ is ∂-irresolute, then f is ∂-continuous.

(iii) Let (X, u_1, u_2) be a biclosure space. Then (X, u_1, u_2) is a $T^*_{1/2}$- biclosure space, if and only if every singleton subset of X is either a g-closed subset of (X, u_1) or an open subset of (X, u_2).

(iv) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then for each $\beta \in I$, the projection map $\pi_\beta: \prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2) \rightarrow (X_\beta, u_\beta^1, u_\beta^2)$ is continuous.

(v) Let (X, u_1, u_2) be a biclosure space and let $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Let f: $X \rightarrow \prod_{\alpha\in I} Y_\alpha$ be a map. If f: $(X, u_1, u_2) \rightarrow \prod_{\alpha\in I}(Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂- continuous, then $\pi_\alpha \circ f: (X, u_1, u_2) \rightarrow (Y_\alpha, v_\alpha^1, v_\alpha^2)$ is ∂- continuous for each $\alpha \in I$.

In the fourth section, properties and characterizations $T^*_{1/2}$- spaces and $T^{**}_{1/2}$ - spaces are discussed.

Some of the interesting results are discussed as follows:

(i) Let (X, u_1, u_2) be a biclosure spaces. If (X, u) is a $T^{**}_{1/2}$ - space, then every singleton subset of X is either a ∂-open subset of (X, u_2) or a closed subset of (X, u_1).

(ii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. If $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T^{**}_{1/2}$ - space, then $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T^{**}_{1/2}$ - space for each $\alpha \in I$.

(iii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha\in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T^*_{1/2}$ - space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a $T^*_{1/2}$ - space for each $\alpha \in I$.

Chapter III deals with Hausdorff biclosure spaces, Generalized Hausdorff biclosure spaces and δ- Hausdorff biclosure spaces.

In the first section, the concepts of Hausdorff biclosure spaces and its properties are analyzed.

It is interesting to note that

(i) A closed subspace of a Hausdorff biclosure space is a Hausdorff biclosure space.

(ii) A product of the family of biclosure spaces is a Hausdorff biclosure space if and only if the individual space is Hausdorff biclosure.

(iii) If (Y, v_1, v_2) is a Hausdorff biclosure space and if $f: (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is injective and continuous, then (X, u_1, u_2) is a Hausdorff biclosure space.

In the second section, properties and characterizations of g- Hausdorff biclosure spaces are discussed.

Some of the interesting results discussed are as follows:

(i) Every Hausdorff biclosure space is a g- Hausdorff biclosure space. The converse need not be true.

(ii) A product of the family of biclosure spaces is a g- Hausdorff biclosure space if and only if the individual space is g- Hausdorff biclosure.

(iii) If (Y, v_1, v_2) is a g- Hausdorff biclosure space and if $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ is injective and g- irresolute, then (X, u_1, u_2) is a g- Hausdorff biclosure space.

In third section, the concepts of ∂-Continuous maps in biclosure spaces and its properties are analyzed.

(i) Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is both a ∂-open subset of (X, u_1) and (X, u_2), then G∩Y is a ∂-open subset of (Y, v_1) and (Y, v_2).

(ii) Let (X, u_1, u_2) be a biclosure space and (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a ∂-Hausdorff biclosure space, then (Y, v_1, v_2) is a ∂-Hausdorff biclosure space.

(iii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂- Hausdorff biclosure space for each $\alpha \in I$.

In Chapter IV, the concepts of Pairwise closed sets and Bicontinuous maps in biclosure spaces are dicussed.

Section one deals with Pairwise closed sets in biclosure spaces and its properties. Some of the interesting results discussed are as follows:

(i) Let (X,u_1,u_2) be a biclosure space. If A and B are pairwise closed subsets of (X,u_1,u_2) , then A∩B is pairwise closed.

(ii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces and let $\beta \in I$. Then F is a pairwise closed subset of $(X_\beta, u_\beta^1, u_\beta^2)$ if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a pairwise closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2)$.

(iii) Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: (X,u_1,u_2) →(Y,v_1,v_2) and g: (Y,v_1,v_2) → (Z,w_1,w_2) are preserve pairwise closed maps, then g ο f: (X,u_1,u_2) →(Z,w_1,w_2) is a preserve pairwise closed.

Section two deals with Bicontinuous maps in biclosure spaces. Some of their properties are established.

The results are:

(i) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let f_α: $(X_\alpha, u_\alpha) \rightarrow (Y_\alpha, v_\alpha)$ be a map and let f: $\prod_{\alpha \in I} (X_\alpha, u_\alpha^1, u_\alpha^2) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v_\alpha^1, v_\alpha^2)$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is bicontinuous if and only if f_α is bicontinuous for each $\alpha \in I$.

(ii) Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ and $\{(Y_\alpha, v_\alpha^1, v_\alpha^2) : \alpha \in I\}$ be families of biclosure spaces. For each $\alpha \in I$, let f_α: $(X_\alpha, u_\alpha^1, u_\alpha^2) \rightarrow (Y_\alpha, v_\alpha^1, v_\alpha^2)$ be a surjection and let f: $\prod_{\alpha \in I} (X\alpha, u_\alpha^1, u_\alpha^2) \rightarrow \prod_{\alpha \in I} (Y_\alpha, v_\alpha^1, v_\alpha^2)$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. Then f is biclosed if and only if f_α is biclosed for each $\alpha \in I$.

(iii) Let (X,u_1,u_2), (Y,v_1,v_2) and (Z,w_1,w_2) be biclosure spaces. If f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ and g: $(Y,v_1,v_2) \rightarrow (Z,w_1,w_2)$ are maps. Then

(i) If $g \circ f$ is biclosed and f is a surjective 1-continuous, then g is biclosed.

(ii) If $g \circ f$ is biclosed and g is injective 2-continuous, then f is biclosed.

(iv) Let (X,u_1,u_2) and (Y,v_1,v_2) be biclosure spaces. Then f: $(X,u_1,u_2) \rightarrow (Y,v_1,v_2)$ is bicontinuous if and only if $u_1 f^{-1}(B) \subseteq f^{-1}(v_2 B)$ for every $B \subseteq Y$.

In Chapter V, the notions of \$-closed set in bi- ˇ Cech closure spaces are introduced by Santhiya, Saranya, Parvathi and characterizations are analyzed in section one.

It is interesting to note that:

(i) If A and B are (k_1, k_2)-\$ closed sets and so is a $A \cup B$.

(ii) If A is a (k_1, k_2) -\$ closed set, then $k\text{-}scl_1(x) \cap A \neq \phi$ holds for each $x \in k\text{-}scl_2(A)$

(iii) Let $A \subseteq Y \subseteq X$ and suppose that A is (k_1, k_2) -\$ closed in (X, k_1, k_2). Then A is (k_1, k_2)-\$ closed relative to Y.