

Chapter 8

Bi-Contra λ_g^α -Continuous Maps in Topological Spaces

8.1 Introduction

Contra-continuous maps in topological spaces given by Dontchev [1996] encouraged the researchers to define the bi-contra continuous maps wherein the requirement of the negative cases on both sides are desirable which was introduced by Caldas et al. [2008 b]. The application in the field of bio-technology was established by using the concept of bi-contra continuity in topological spaces which were extended to nano topology by Lellis Thivagar et al. [2017]. He has provided the definition for bi-contra α -continuous maps and given comparisons, theorems related to the defined bi-contra continuous maps.

This chapter establishes the bi-contra λ_g^α -continuous maps, strongly bi-contra λ_g^α -continuous maps and completely bi-contra λ_g^α -continuous maps in topological spaces. Later the investigations regarding their properties and fundamental theorems are carried out which have their application in the field of bio-technology on analyzing the microbes.

8.2 Bi-Contra λ_g^α -Continuous Maps and its Special Forms

A very new form of contra continuous maps called bi-contra λ_g^α -continuous maps have been defined in this section. Further few of its variations like strongly bi-contra λ_g^α -continuous maps and completely bi-contra λ_g^α -continuous maps have also been defined and their properties are derived.

Definition 8.2.1 A surjective map $u: (M, \mu) \rightarrow (N, \nu)$ is called a **bi-contra λ_g^α -continuous map** if u is a contra λ_g^α -continuous map and $u^{-1}(T)$ is an open set in (M, μ) implies T is a λ_g^α -closed set in (N, ν) .

Example 8.2.2 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i, j\}, M\}$ and $\nu = \{\phi, \{i, j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = l, u(j) = j, u(k) = k$ and $u(l) = i$. Then u is a bi-contra λ_g^α -continuous map.

Proposition 8.2.3 Every bi-contra continuous map $u: (M, \mu) \rightarrow (N, \nu)$ is a bi-contra λ_g^α -continuous map.

Proof. Let T be a closed set in (N, ν) . Then by assumption $u^{-1}(T)$ is open in (M, μ) . By Proposition 2.4.3, $u^{-1}(T)$ is λ_g^α -open. Hence u is contra λ_g^α -continuous.

As u is bi-contra continuous, $u^{-1}(T)$ is open in (M, μ) implies T is closed in (N, ν) . By Proposition 2.2.5, T is λ_g^α -closed in (N, ν) . Hence u is a bi-contra λ_g^α -continuous map.

The subsequent example shows that the converse of Proposition 8.2.3 may not hold good.

Example 8.2.4 Consider M, N, μ, ν and u as in Example 8.2.2. Then the map u is a bi-contra λ_g^α -continuous map but not a bi-contra continuous map, since for the closed set $\{l\}$ in (N, ν) , $u^{-1}(\{l\}) = \{i\}$ is not an open set in (M, μ) implies u is not a contra continuous map and $u^{-1}(\{j, l\}) = \{i, j\}$ is an open set in (M, μ) but $\{j, l\}$ is not a closed set in (N, ν) . Hence u is not a bi-contra continuous map.

Definition 8.2.5 A surjective map $u: (M, \mu) \rightarrow (N, \nu)$ is called **bi-contra λ -continuous** if u is contra λ -continuous and $u^{-1}(T)$ is open in (M, μ) implies T is λ -closed in (N, ν) .

Example 8.2.6 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i\}, \{k\}, \{i, j\}, \{i, k\}, \{i, j, k\}, \{i, k, l\}, M\}$ and $\nu = \{\phi, \{i\}, \{j\}, \{i, j\}, \{i, j, k\}, \{i, j, l\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = l, u(j) = k, u(k) = j$ and $u(l) = i$. Then u is a bi-contra λ -continuous map.

Proposition 8.2.7 Every bi-contra λ -continuous map is a bi-contra λ_g^α -continuous map.

Proof. Evident from the Proposition 2.2.3.

The subsequent example shows that the converse of Proposition 8.2.7 may not hold good.

Example 8.2.8 Consider M, N, μ, ν and u as in Example 8.2.2. Then the map u is a bi-contra λ_g^α -continuous map but not a bi-contra λ -continuous map, since for the closed set $\{l\}$ in (N, ν) , $u^{-1}(\{l\}) = \{i\}$ is not a λ -open set in (M, μ) implies u is not a contra λ -continuous map and $u^{-1}(\{j, l\}) = \{i, j\}$ is an open set in (M, μ) but $\{j, l\}$ is not a λ -closed set in (N, ν) . Hence u is not a bi-contra λ -continuous map.

Theorem 8.2.9 If $u: (M, \mu) \rightarrow (N, \nu)$ is a bi-contra λ_g^α -continuous map and $p: (M, \mu) \rightarrow (K, \kappa)$ is a continuous map, where (K, κ) is a space which is constant on each set $u^{-1}(\{n\})$ for every $n \in N$, then p induces a contra λ_g^α -continuous map $w: (N, \nu) \rightarrow (K, \kappa)$ such that $(w \circ u) = p$.

Proof. As p is constant on $u^{-1}(\{n\})$ for every $n \in N$, the set $p(u^{-1}(\{n\}))$ is a one-point set in (K, κ) . If we let $w(n)$ denote this point, then we can define a map $w: (N, \nu) \rightarrow (K, \kappa)$ such that for each $m \in M$, $w(u(m)) = p(m)$.

To show that w is contra λ_g^α -continuous, let S be an open set in (K, κ) . Since p is a continuous map, $p^{-1}(S)$ is open in (M, μ) . Now $p^{-1}(S) = u^{-1}(w^{-1}(S))$ is open. Since u is bi-contra λ_g^α -continuous, $w^{-1}(S)$ is λ_g^α -closed in (N, ν) . Hence w is a contra λ_g^α -continuous map.

Definition 8.2.10 Let $u: (M, \mu) \rightarrow (N, \nu)$ be a surjective map. Then u is called a **strongly bi-contra λ_g^α -continuous map** provided u is a contra λ_g^α -continuous map and $u^{-1}(T)$ is an open set in (M, μ) if and only if T is a λ_g^α -closed set in (N, ν) .

Example 8.2.11 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{j, k\}, M\}$ and $\nu = \{\phi, \{i\}, N\}$. Then the identity map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly bi-contra λ_g^α -continuous map.

Proposition 8.2.12 Every strongly bi-contra λ_g^α -continuous map is a bi-contra λ_g^α -continuous map.

Proof. Obvious from the Definitions 8.2.1 and 8.2.10.

The subsequent example shows that the converse of Proposition 8.2.12 may not hold good.

Example 8.2.13 Consider M, N, μ, ν and u as in Example 8.2.2. Then u is a bi-contra λ_g^α -continuous map but not a strongly bi-contra λ_g^α -continuous map, since for the λ_g^α -closed set $\{j, k, l\}$ in (N, ν) , $u^{-1}(\{j, k, l\}) = \{i, j, k\}$ is not an open set in (M, μ) .

Definition 8.2.14 A surjective map $u: (M, \mu) \rightarrow (N, \nu)$ is called a **completely bi-contra λ_g^α -continuous map** if u is a contra λ_g^α -irresolute map and $u^{-1}(T)$ is a λ_g^α -open set in (M, μ) if and only if T is a closed set in (N, ν) .

Example 8.2.15 Let $M = \{i, j, k\}$, $N = \{p, q\}$, $\mu = P(M)$ and $\nu = P(N)$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = p = u(j)$ and $u(k) = q$. Then u is a completely bi-contra λ_g^α -continuous map.

Proposition 8.2.16 Every completely bi-contra λ_g^α -continuous map is a contra λ_g^α -irresolute map.

Proof. Follows from Definition 8.2.14.

The subsequent example shows that the converse of Proposition 8.2.16 may not hold good.

Example 8.2.17 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{j, k\}, M\}$ and $\nu = \{\phi, \{i\}, \{i, j\}, \{i, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = j$, $u(j) = i$ and $u(k) = k$. Then u is a contra λ_g^α -irresolute map but not a completely bi-contra λ_g^α -continuous map, since $u^{-1}(\{i, j\}) = \{i, j\}$ is a λ_g^α -open set in (N, ν) but $\{i, j\}$ is not a closed set in (M, μ) .

Proposition 8.2.18 Every completely bi-contra λ_g^α -continuous map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly bi-contra λ_g^α -continuous map.

Proof. By assumption u is a contra λ_g^α -irresolute map. Using Proposition 4.3.4, u is also a contra λ_g^α -continuous map. Let $u^{-1}(T)$ be open in (M, μ) . Using Proposition 2.4.3, $u^{-1}(T)$

is λ_g^α -open in (M, μ) . Since u is completely bi-contra λ_g^α -continuous map, $u^{-1}(T)$ is λ_g^α -open in (M, μ) if and only if T is closed in (N, ν) . Using Proposition 2.2.5, T is also a λ_g^α -closed set in (N, ν) . Hence u is a strongly bi-contra λ_g^α -continuous map.

The subsequent example shows that the converse of Proposition 8.2.18 may not hold good.

Example 8.2.19 Let $M = N = \{i, j, k\}$, $\mu = \{\phi, \{i\}, \{i, j\}, \{i, k\}, M\}$ and $\nu = \{\phi, \{i\}, \{j, k\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = i, u(j) = k$ and $u(k) = j$. Then u is a strongly bi-contra λ_g^α -continuous map but not a completely bi-contra λ_g^α -continuous map, since $u^{-1}(\{j\}) = \{k\}$ is a λ_g^α -open set in (M, μ) , but $\{j\}$ is not a closed set in (N, ν) .

Proposition 8.2.20 Every completely bi-contra λ_g^α -continuous map is a bi-contra λ_g^α -continuous map.

Proof: By Proposition 8.2.12 and Proposition 8.2.18 the proof follows.

The subsequent example shows that the converse of Proposition 8.2.20 may not hold good.

Example 8.2.21 Consider M, N, μ, ν and u as in Example 8.2.2. Then u is a bi-contra λ_g^α -continuous map but not a completely bi-contra λ_g^α -continuous map, since for the λ_g^α -closed set $\{k\}$ in (N, ν) , $u^{-1}(\{k\}) = \{k\}$ is not a λ_g^α -open set in (M, μ) .

Definition 8.2.22 A map $u: (M, \mu) \rightarrow (N, \nu)$ is called a **contra λ_g^α -open** (resp. **contra λ_g^α -closed**) map if the image of every open (resp. closed) set in (M, μ) is a λ_g^α -closed (resp. λ_g^α -open) set in (N, ν) .

Example 8.2.23 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i\}, M\}$ and $\nu = \{\phi, \{i, j\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = j, u(j) = i, u(k) = l$ and $u(l) = k$. Then u is a contra λ_g^α -open map.

Definition 8.2.24 A map $u: (M, \mu) \rightarrow (N, \nu)$ is called a **strongly contra λ_g^α -open** (resp. **strongly contra λ_g^α -closed**) map if the image of every λ_g^α -open (resp. λ_g^α -closed) set in (M, μ) is a λ_g^α -closed (resp. λ_g^α -open) set in (N, ν) .

Example 8.2.25 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i, j\}, M\}$ and $\nu = \{\phi, \{i\}, \{k\}, \{i, j\}, \{i, k\}, \{i, j, k\}, \{i, k, l\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = k, u(j) = l, u(k) = i$ and $u(l) = j$. Then u is a strongly contra λ_g^α -open map.

Proposition 8.2.26 Every strongly contra λ_g^α -open map $u: (M, \mu) \rightarrow (N, \nu)$ is a contra λ_g^α -open map.

Proof. Let T be an open set in (M, μ) . Then by Proposition 2.4.3, T is λ_g^α -open in (M, μ) . Since u is a strongly contra λ_g^α -open map, $u(T)$ is λ_g^α -closed in (N, ν) . Hence u is a contra λ_g^α -open map.

The subsequent example shows that the converse of Proposition 8.2.26 may not hold good.

Example 8.2.27 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i, j, k\}, M\}$ and $\nu = \{\phi, \{i, j\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = l, u(j) = j, u(k) = k$ and $u(l) = i$. Then u is a contra λ_g^α -open map but not a strongly contra λ_g^α -open map, since for the λ_g^α -open set $\{j, k, l\}$ in (M, μ) , $u(\{j, k, l\}) = \{i, j, k\}$ is not a λ_g^α -closed set in (N, ν) .

Theorem 8.2.28 If $u: (M, \mu) \rightarrow (N, \nu)$ is a surjective, contra λ_g^α -continuous and contra λ_g^α -open map, then u is bi-contra λ_g^α -continuous.

Proof. Let u be the contra λ_g^α -continuous map and let $u^{-1}(T)$ be open in (M, μ) . Since u is contra λ_g^α -open, $u(u^{-1}(T)) = T$ is λ_g^α -closed in (N, ν) . Hence u is a bi-contra λ_g^α -continuous map.

Proposition 8.2.29 Every completely bi-contra λ_g^α -continuous map $u: (M, \mu) \rightarrow (N, \nu)$ is a strongly contra λ_g^α -open map.

Proof: Let T be a λ_g^α -open set in (M, μ) . Using the fact that $T = u^{-1}(u(T))$, $u^{-1}(u(T))$ is a λ_g^α -open set in (M, μ) . As u is a completely bi-contra λ_g^α -continuous map, $u(T)$ is a closed set in (N, ν) . By Proposition 2.2.5, $u(T)$ is λ_g^α -closed in (N, ν) . Hence u is a strongly contra λ_g^α -open map.

The subsequent example shows that the converse of Proposition 8.2.29 may not hold good.

Example 8.2.30 Let $M = N = \{i, j, k, l\}$, $\mu = \{\phi, \{i, j\}, M\}$ and $\nu = \{\phi, \{i\}, \{k\}, \{i, j\}, \{i, k\}, \{i, j, k\}, \{i, k, l\}, N\}$. Let $u: (M, \mu) \rightarrow (N, \nu)$ be a map defined by $u(i) = k, u(j) = l, u(k) = i$ and $u(l) = j$. Then u is a strongly contra λ_g^α -open map but not a completely bi-contra λ_g^α -continuous map, since for the λ_g^α -closed set $\{i, k, l\}$ in (N, ν) , $u^{-1}(\{i, k, l\}) = \{i, j, k\}$ is not λ_g^α -open in (M, μ) .

Theorem 8.2.31 Let $u: (M, \mu) \rightarrow (N, \nu)$ be an open, surjective, λ_g^α -irresolute map and $w: (N, \nu) \rightarrow (K, \kappa)$ be a bi-contra λ_g^α -continuous map. Then their composition $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a bi-contra λ_g^α -continuous map.

Proof. Let T be an open set in (K, κ) . Since w is bi-contra λ_g^α -continuous, $w^{-1}(T)$ is λ_g^α -closed in (N, ν) . As u is λ_g^α -irresolute, $u^{-1}(w^{-1}(T)) = (w \circ u)^{-1}(T)$ is λ_g^α -closed in (M, μ) . Hence $(w \circ u)$ is a contra λ_g^α -continuous map.

Let $u^{-1}(w^{-1}(T)) = (w \circ u)^{-1}(T)$ be an open set in (M, μ) . Since u is an open and surjective map, $w^{-1}(T)$ is an open set in (N, ν) which implies T is λ_g^α -closed in (K, κ) as w is bi-contra λ_g^α -continuous. Hence $(w \circ u)$ is a bi-contra λ_g^α -continuous map.

Theorem 8.2.32 Let $u: (M, \mu) \rightarrow (N, \nu)$ be a surjective, strongly λ_g^α -open and λ_g^α -irresolute map and $w: (N, \nu) \rightarrow (K, \kappa)$ be a completely bi-contra λ_g^α -continuous map. Then $(w \circ u): (M, \mu) \rightarrow (K, \kappa)$ is a completely bi-contra λ_g^α -continuous map.

Proof. Let T be a λ_g^α -open set in (K, κ) . Since w is completely bi-contra λ_g^α -continuous, $w^{-1}(T)$ is λ_g^α -closed in (N, ν) . Again u is λ_g^α -irresolute and $u^{-1}(w^{-1}(T)) = (w \circ u)^{-1}(T)$ is a λ_g^α -closed set in (M, μ) . Hence $(w \circ u)$ is a contra λ_g^α -irresolute map.

Let $(w \circ u)^{-1}(T) = u^{-1}(w^{-1}(T))$ be a λ_g^α -open set in (M, μ) . By assumption u is strongly λ_g^α -open and surjective, therefore $w^{-1}(T)$ is a λ_g^α -open set in (N, ν) . As w is completely bi-contra λ_g^α -continuous map, $w^{-1}(T)$ is λ_g^α -open if and only if T is closed in (K, κ) . Thus $(w \circ u)$ is a completely bi-contra λ_g^α -continuous map.