Chapter 4

Δ^*-Continuous Functions in Topological Spaces

4.1 Introduction

Functions are important tools for studying properties of spaces and for constructing new spaces from the existing spaces. The classical concept of continuity is that of metric continuity. This led to the notion of topological continuity. In the year 1970, Norman Levine initiated the idea of continuous functions. Noiri.T (1980) introduced δ-continuous. The generalised continuous (briefly, g-continuous) was studied by Balachandran.K (1991). Further many authors contributed their research towards continuity. In this chapter, a new class of continuous functions using Δ^*-closed sets is established. In literature, by changing some basic conditions various types of continuous functions namely, quasi continuity, perfectly continuity, totally continuity, strongly totally continuity, and contra continuity concepts are defined. A parallel study of these continuity concepts is carried out for Δ^*-continuity. The dependence relationships of Δ^*-continuous functions with various continuous functions are investigated in this chapter. Also the significant properties of Δ^*-continuous maps using specific spaces and important preservation of properties under composition mappings are established in this chapter.

4.2 Δ^*-Continuous Functions

In this section, Δ^*-continuous functions are defined and their properties as well as their association with various continuous functions are analyzed.

Definition 4.2.1 A mapping f : (X, τ) → (Y, σ) is said to be **Δ^*-continuous** if the inverse image of every closed set in (Y, σ) is Δ^*-closed in (X, τ).

Theorem 4.2.2 A mapping f : (X, τ) → (Y, σ) is Δ^*-continuous if and only if the inverse image of every open set in (Y, σ) is Δ^*-open in (X, τ).

Proof : (Necessary) : Let f : (X, τ) ⟶ (Y, σ) be Δ^*-continuous and U be any open set in (Y, σ). Then Y − U is closed in (Y, σ). Since f is Δ^*-continuous, $f^{-1}(Y - U) = X - f^{-1}(U)$ is closed in (X, τ) and hence $f^{-1}(U)$ is Δ^*-open in (X, τ).

(Sufficiency) : Assume that $f^{-1}(U)$ is Δ^*-open in (X, τ) for each open set U in (Y, σ). Let V be a closed set in (Y, σ). Then (Y − V) = U is open in (Y, σ). By hypothesis, $f^{-1}(U) = f^{-1}(Y - V) = X - f^{-1}(V)$ is Δ^*-open in (X, τ) which implies that $f^{-1}(V)$ is Δ^*-closed in (X, τ). Hence f is Δ^***-c**ontinuous.

Proposition 4.2.3 Every δ-continuous function (resp., δg^*-continuous function) is Δ^*-continuous but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a δ -continuous map (resp., δg^*-continuous map). Let V be any closed set in (Y, σ). Since f is a δ-continuous function (resp., δg^*-continuous function), $f^{-1}(V)$ is δ-closed (resp., δg^*-closed) in (X, τ). By proposition 2.2.2 (resp., proposition 2.2.4), every δ-closed set (resp., δg^*-closed set) is Δ^*-closed set which implies that $f^{-1}(V)$ is Δ^*-closed in (X, τ). Therefore f is Δ^*-continuous.

Counter Example 4.2.4 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}, {a, b}, {a, c}} and σ = {ϕ, Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*-continuous but not δ-continuous (resp., δg^*-continuous) since for the closed set {b} in (Y, σ), $f^{-1}(\{b\}) = \{b\}$ is Δ^*-closed but not δ-closed (resp., δg^*-closed) in (X, τ).

Proposition 4.2.5 Every totally continuous function is Δ^*-continuous but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be totally continuous map, Let V (Y, σ). Since f is totally continuous, f^{-1}(V) is clopen. By proposition 2.5.5, f^{-1}(V) is Δ^*-open in (X, τ). Hence f is Δ^*-continuous.

Counter Example 4.2.6 Let X = Y = {a, b, c} with τ = {ϕ, X, {a, b}} and σ = {ϕ, Y, {a}} and let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*-continuous as the inverse image of every closed set in (Y, σ) is Δ^*-closed set in (X, τ) but it is not totally continuous as for the open set {a} in (Y, σ), f^{-1}({a})={a} is not clopen in (X, τ).

Proposition 4.2.7 Every Δ^*-continuous function is gδ-continuous but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map. Let U be any closed set in (Y, σ). Then f^{-1}(U) is Δ^*-closed in (X, τ). By Proposition 2.2.8, every Δ^*-closed set is gδ-closed. This implies that f^{-1}(U) is gδ-closed in (X, τ). Hence f is gδ-continuous.

Counter Example 4.2.8 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}} and σ = {ϕ, X, {a}, {a, b}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is gδ-continuous as P(X) is the set of gδ-closed sets but not Δ^*-continuous since for the closed subset {c}, f^{-1}{(U)} = {c} is not Δ^*-closed in (X, τ).

Proposition 4.2.9 Every Δ^*-continuous function is rg-continuous but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map. Let V be any closed set in (Y, σ). Then f^{-1}{(V)} is Δ^*-closed set in (X, τ). By Proposition 2.2.11, we know that every Δ^*-closed set is rg-closed. Therefore f^{-1}{(V)} is rg-closed in (X, τ). Hence f is rg-continuous.

Counter Example 4.2.10 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}} and σ = {ϕ, X, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b Then f is rg-continuous but not Δ^*-continuous since for the closed subset {c} in (Y, σ), f^{-1}\{(c)\} = {b} is not a Δ^*-closed set in (X, τ).

Proposition 4.2.11 Every Δ^*-continuous function is a gpr continuous function but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map. Let V be any closed set in (Y, σ). Then f^{-1}(V) is Δ^*-closed in (X, τ). By Proposition 2.2.13, every Δ^*-closed set is gpr-closed in (X, τ) which implies that f^{-1}(V) is gpr-closed in (X, τ). Hence f is gpr-continuous.

Counter Example 4.2.12 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}} and σ = {ϕ, Y, {a, b}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is gpr-continuous but not Δ^*-continuous since for the closed subset {c} in (Y, σ), f^{-1}\{(c)\} = {c} is not Δ^*-closed in (X, τ).

Proposition 4.2.13 Every Δ^*-continuous function is $\delta g^{\ddagger}$-continuous but not conversely.

Proof : Let f : (X, τ)→(Y, σ) be a Δ^*-continuous map. Let V be any closed set in (Y, σ). Then f^{-1}(V) is Δ^*-closed in (X, τ). By Proposition 2.2.6, every Δ^*-closed set is $\delta g^{\ddagger}$-closed in (X, τ) which implies that f^{-1}(V) is $\delta g^{\ddagger}$-closed in (X, τ). Hence f is $\delta g^{\ddagger}$-continuous function.

Counter Example 4.2.14 Let X = {a, b, c} = Y with τ = {ϕ, X, {a}} and σ = {ϕ, Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = c, f(c) = a. Then f is $\delta g^{\ddagger}$-continuous but not Δ^*-continuous since for the closed subset {c} in (Y, σ), f^{-1}\{(c)\} = {b} is not Δ^*-closed in (X, τ).

Proposition 4.2.15 Every Δ^*-continuous function is πg-continuous but not conversely.

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map. Let V be any closed set in (Y, σ). Then f^{-1}{(V)} is Δ^*-closed in (X, τ). By Proposition 2.2.19, every Δ^*-closed set is πg-closed in (X, τ) which implies that f^{-1}{(V)} is πg -closed in (X, τ). Hence f is πg -continuous function.

Counter Example 4.2.16 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}, {b, c}} and σ = {ϕ, Y, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = a, f(c) = b. Then f is πg-continuous but not Δ^*-continuous since for the closed subset {b, c} in (Y, σ), f^{-1}({b, c}) = {c} is not Δ^*-closed in (X, τ).

Similarly we can prove the following.

Proposition 4.2.17 Every Δ^*-continuous function is rwg-continuous, gspr-continuous, πgp-continuous, πgsp-continuous, πgs-continuous, πgα-continuous and πgb-continuous function. But the converses are not true.

Counter example 4.2.18 Let X = Y = {a, b, c, d} with τ = {ϕ, X, {a, b}} and σ = {ϕ, Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = d, f(c) = a, f(d) = b. Then f is rwg-continuous, gspr-continuous, πgp-continuous, πgsp-continuous, πgs-continuous, πgα-continuous and πgb-continuous function as for (X, τ) these closed sets form P(X) whereas f^{-1}({c, d}) = {a, b} which is not Δ^*-closed and hence f is not Δ^*-continuous .

Proposition 4.2.19 Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map and (X,τ) be a $\Delta^* T_\delta$-space. Then f is a δ-continuous function.

Proof : Let V be a closed set in (Y, σ). Then f^{-1}(V) is Δ^*-closed in (X, τ). Since (X, τ) is a $\Delta^* T_\delta$-space, we get f^{-1}(V) is δ-closed in (X, τ). Hence f is δ-continuous.

Proposition 4.2.20 Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map and (X, τ) be a $_{\Delta^*}T_{\delta g^*}$-space then f is δg^*-continuous.

Proof : Let V be any closed set in (Y, σ). Since f is Δ^*-continuous, $f^{-1}(V)$ is Δ^*-closed in (X, τ). Also (X, τ) is a $_{\Delta^*}T_{\delta g^*}$-space. Therefore we get $f^{-1}(V)$ is δg^*-closed in (X, τ) and hence f is δg^*-continuous.

Proposition 4..2.21 Let f : (X, τ) → (Y, σ) be a gδ-continuous function and (X, τ) be a $g\delta T_{\Delta^*}$-space then f is a Δ^*-continuous function.

Proof : Let V be any closed set in(Y, σ). Since f is gδ-continuous and (X, τ) is a $g\delta T_{\Delta^*}$-Space, we get $f^{-1}(V)$ is gδ-closed and hence Δ^*-closed in (X, τ). Therefore f is a Δ^*-continuous function.

Proposition 4.2.22 Let f : (X, τ)→(Y, σ) be a $\delta g^{\ddagger}$-continuous function and (X, τ) be a $\delta g^{\ddagger}T\Delta^*$-space then f is a Δ^*-continuous function.

Proof : Let V be a closed set in (Y, σ). Since f is a $\delta g^{\#}$-continuous function, $f^{-1}(V)$ is a $\delta g^{\ddagger}$-closed set in (X, τ). Also (X, τ) is a $\delta g^{\ddagger}T\Delta^*$-space. Therefore $f^{-1}(V)$ is Δ^*-closed in (X, τ). Hence f is a Δ^*-continuous function.

Proposition 4.2.23 Let f : (X, τ)→(Y, σ) be a Δ^*-continuous function where (X, τ) is a $_{\Delta^*}T_{\delta}$-space then f is δg^*-continuous.

Proof : Let V be any closed set in (Y, σ). Since f is Δ^*-continuous, $f^{-1}(V)$ is Δ^*-closed in (X, τ). Also since (X, τ) is a $_{\Delta^*}T_{\delta}$-space, $f^{-1}(V)$ is δ-closed in (X, τ). We know that every δ-closed is δg^*-closed (Sudha, 2012). Therefore $f^{-1}(V)$ is δg^*-closed in (X, τ). Hence f is δg^*-continuous.

Proposition 4.2.24 Let f : (X, τ)→(Y, σ) be a Δ^*-continuous where (X, τ) is a $_{\Delta^*}T_{\delta g^*}$-space and a $T_{3/4}$-space then f is δ-continuous.

Proof : Let V be any closed set in (Y, σ). Since f is Δ^*-continuous, $f^{-1}(V)$ is Δ^*-closed in (X, τ). Since (X, τ) is a $_{\Delta^*}T_{\delta g^*}$-space, $f^{-1}(V)$ is δg^*-closed in (X, τ). Also by the fact that every δg^*-closed set is δg-closed (Sudha, 2012) and by the assumption that (X, τ) is a $T_{3/4}$-space, $f^{-1}(V)$ is δ-closed in (X, τ). Hence f is δ-continuous.

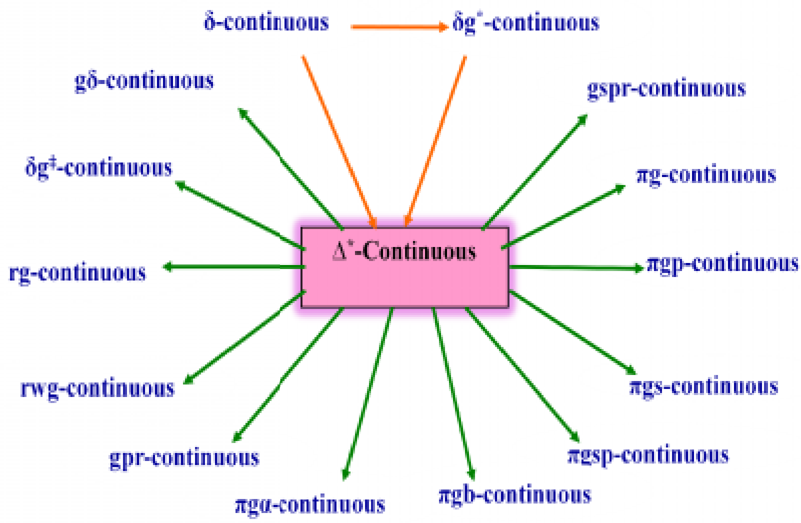
Proposition 4.2.25 Let f : (X, τ) → (Y, σ) be a gδ-continuous function where (X, τ) is a $_{g\delta}T_{\Delta^*}$-space and a $_{\Delta^*}T_{\delta g^*}$-space then f is δg^*-continuous.

Proof: Let V be any closed set in (Y, σ). Since f is gδ-continuous, $f^{-1}(V)$ is gδ-closed in (X, τ). Since (X, τ) is a $_{g\delta}T_{\Delta^*}$-space and also $_{\Delta^*}T_{\delta g^*}$-space, $f^{-1}(V)$ is Δ^*-closed and hence it is a δg^*-closed set in (X, τ). Therefore f is δg^*-continuous.

Proposition 4.2.26 Let f : (X, τ) → (Y, σ) be a gδ-continuous function where (X, τ) is a T_δ-space then f is Δ^*-continuous.

Proof : Let V be any closed set in (Y, σ). Since f is gδ-continuous, $f^{-1}(V)$ is gδ-closed in (X, τ). Since (X, τ) is a T_δ-space, $f^{-1}(V)$ becomes δ-closed in (X, τ). We know that every δ-closed is Δ^*-closed in (X, τ) (Proposition 2.2.2). Hence f is Δ^*-continuous.

Remark 4.2.27 The above results are picturized as follows.



Remark 4.2.28 The following counter examples show that Δ^*-continuous is independent from continuous function.

Counter Example 4.2.29 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = a, f(c) = c. Then f is Δ^*-continuous but not continuous since for the closed set {b, c} in (Y, σ), f^{-1}({b, c}) = {a, c} is Δ^*-closed but not closed in (X, τ).

Counter Example 4.2.30 Let X = {a, b, c, d}, Y = {a, b, c} with τ = {ϕ , X, {a}, {c}, {a, b}, {a, c}, {a, b, c}, {a, c, d}} and σ = {ϕ , Y, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = b, f(c) = a, f(d) = a. Then f is continuous but not Δ^*-continuous since for the closed set {b, c, d} in (Y, σ), f^{-1}({b, c, d}) = {b, c, d} is not Δ^*-closed in (X, τ).

Remark 4.2.31 The following counter examples show that Δ^*-continuity is independent from g-continuity function.

Counter Example 4.2.32 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a}}.

Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = a, f(c) = c. Then f is a g-continuous map but not Δ^*-continuous since for the closed set {b, c} in (Y, σ), f^{-1}({b, c}) = {a, c} is not Δ^*-closed in (X, τ).

Counter Example 4.2.33 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = a, f(c) = c. Then f is Δ^*-continuous but not g-continuous since for the closed set {b, c} in (Y, σ), f^{-1}({b, c}) = {a, c} is Δ^*-closed but not g-closed in (X, τ).

Remark 4.2.34 The following counter examples show that the notions of δg-continuity and Δ^*-continuity are independent.

Counter Example 4.2.35 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is δg-continuous but not Δ^*-continuous since for the closed set {b} in (Y, σ), f^{-1}({b}) = {b} is not Δ^*-closed in (X, τ).

Counter Example 4.2.36 Let X =Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = a, f(c) = c. Then f is Δ^*-continuous but not δg-continuous since for the closed set {b, c} in (Y, σ), f^{-1}({b, c}) = {a, c} is not δg-closed in (X, τ).

Remark 4.2.37 The following counter examples show that Δ^*-continuity is independent from g^*s-continuity.

Counter Example 4.2.38 Let X = Y = {a, b, c} with τ = {ϕ ,X, {a}, {a, b}} and σ = {ϕ , Y,{a}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = a, f(c) = c. Then f is Δ^*-continuous but not g^*s-continuous function since for the closed subset {c} in (Y, σ), f^{-1}{c} = {a, c} is not g^*s-closed in (X, τ).

Counter Example 4.2.39 Let X = {a, b, c} = Y with $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $\sigma = \{\phi, X, \{a\}, \{a, b\}\}$. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b. Then f is g^*s-continuous function but not Δ^*-continuous since for the closed subset {c} in (Y, σ), $f^{-1}\{c\}=\{b\}$ is not Δ^*-closed in (X, τ).

Remark 4.2.40 The following counter examples show that Δ^*-continuity is independent of αg-continuity.

Counter Example 4.2.41 Let X = Y = {a, b, c} with $\tau = \{\phi, X, \{a\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}\}$. Let f : (X, τ) → (Y, σ) be an identity map. Then f is αg-continuous but not Δ^*-continuous since for the closed set {c} in (Y, σ), $f^{-1}\{c\} = \{c\}$ is not Δ^*-closed in (X, τ).

Counter Example 4.2.42 Let X = Y = {a, b, c} with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{b, c\}\}$. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = b, f(c) = c. Then f is Δ^*-continuous but not αg-continuous since for the closed set {c} in (Y, σ), $f^{-1}\{c\} = \{a, c\}$ is not αg-closed in (X, τ).

Remark 4.2.43 The following counter examples show that Δ^*-continuity is independent of gp-continuity.

Counter Example 4.2.44 Let X = Y = {a, b, c} with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = b, f(c) = c. Then f is Δ^*-continuous but not gp-continuous since for the closed set {c} in (Y, σ), $f^{-1}\{c\} = \{a, c\}$ is not gp-closed in (X, τ).

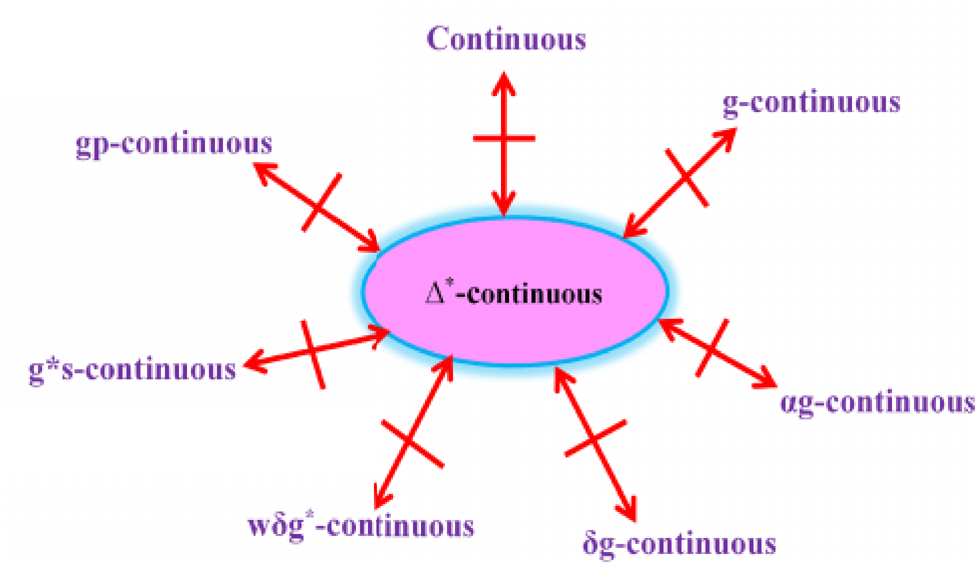
Counter Example 4.2.45 Let X = Y = {a, b, c} with $\tau = \{\phi, X, \{a\}, \{b, c\}\}$, $\sigma = \{\phi, Y, \{a\}, \{b, c\}\}$. Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = a, f(c) = c. Then f is gp-continuous but not Δ^*-continuous since for the closed set {a} in (Y, σ), $f^{-1}\{a\} = \{b\}$ is not Δ^*-closed set in (X, τ).

Remark 4.2.46 : The following counter examples show that Δ^*-continuity is independent of $w\delta g^*$-continuity.

Counter Example 4.2.47 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*-continuous but not $w\delta g^*$-continuous since for the closed set {c} in (Y, σ), f^{-1}\{c\} = {a, c} is not $w\delta g^*$-closed in (X, τ).

Counter Example 4.2.48 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {b, c}}, σ = {ϕ , Y, {a}, {b, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = b, f(b) = c, f(c) = a. Then f is $w\delta g^*$-continuous as $w\delta G^*C(X, \tau) = P(X)$ but not Δ^*-continuous since for the closed set {b, c} in (Y, σ), f^{-1}\{b, c\} = {a, b} is not Δ^*-closed set in (X, τ).

Remark 4.2.49 The above results are depicted by the following diagram.



Composition of mappings

Remark 4.2.50 The composition of two Δ^*-continuous functions need not be a Δ^*-continuous function as seen from the following example.

Counter Example 4.2.51 Let X = Y = {a, b, c} with τ = {ϕ, X, {a}}, σ ={ϕ, Y, {a, b}} and η = {ϕ, Z, {a}, {b}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = b, f(c) = b. Let g : (Y, σ) → (Z, η) be a map such that g(a) = b, g(b) = a, g(c) = c. Then the composition mapping (g ∘ f) : (X, τ) → (Z, η) is defined by (g ∘ f)(a) = b, (g ∘ f)(b) = a, (g ∘ f)(c) = a. Here both f and g are Δ^*-continuous but the composition mapping (g ∘ f) is not Δ^*-continuous since for the closed set {b, c} in (Z, η), $(g \circ f)^{-1}$({b, c})={a} is not Δ^*-closed in (X, τ).

Theorem 4.2.52 Let f : (X, τ) → (Y, σ) be Δ^*-continuous and g : (Y, σ) → (Z, η) also be a δ-continuous then (g ∘ f) : (X, τ) → (Z, η) is a Δ^*-continuous function.

Proof : Let V be any closed set in (Z, η). Since g is δ-continuous, $g^{-1}(V)$ δ-closed in (Y, σ). We know that every δ-closed set is closed. Therefore $g^{-1}(V)$ is closed in (Y, σ). Since f is Δ^*-continuous, $f^{-1}\{g^{-1}\{V\}\} = (g \circ f)^{-1}(V)$ is Δ^*-closed in (X, τ). Therefore (g ∘ f) is Δ^*-continuous.

Corollary 4.2.53 If f : (X, τ) → (Y, σ) is Δ^*-continuous and g : (Y, σ) → (Z, η) is continuous then (g ∘ f) : (X, τ) → (Z, η) is Δ^*-continuous.

Proof : Let V be any closed set in (Z, η). Since g is continuous, $g^{-1}(V)$ is closed in (Y, σ). Also f is Δ^*-continuous which implies that $f^{-1}\{g^{-1}\{V\}\} = (g \circ f)^{-1}(V)$ is Δ^*-closed in (X, τ). Hence (g ∘ f) is Δ^*-continuous.

Theorem 4.2.54 Let f : (X, τ) → (Y, σ) be Δ^*-continuous in which (Y, σ) is a T_δ-space. Let g : (Y, σ) → (Z, η) be gδ-continuous then (g ∘ f) : (X, τ) → (Z, η) is Δ^*-continuous.

Proof : Let V be any closed set in (Z, η). Then $g^{-1}(V)$ is gδ-closed in (Y, σ). Since (Y, σ) is a T_δ-space, $g^{-1}(V)$ is δ-closed which implies that $g^{-1}(V)$ is closed. Since f is Δ^*-continuous $f^{-1}\{g^{-1}\{V\}\} = (g \circ f)^{-1}(V)$ is Δ^*-closed in (X, τ). Hence (g ∘ f) is Δ^*-continuous.

Theorem 4.2.55 If f : (X, τ) → (Y, σ) and g : (Y, σ) → (Z, η) are both Δ^*-continuous and (Y, σ) is a T_δ-space then (g ∘ f) : (X, τ) → (Z, η) is Δ^*-continuous.

Proof : Let V be closed in (Z, η). Since g is Δ^*-continuous, $g^{-1}(V)$ is Δ^*-closed in (Y, σ). Since every Δ^*-closed set is gδ-closed and (Y, σ) is a T_δ-space, $g^{-1}(V)$ is δ-closed which implies that $g^{-1}(V)$ is closed. Since f is Δ^*-continuous, $f^{-1}\{g^{-1}\{V\}\}=(g \circ f)^{-1}(V)$ is Δ^*-closed in (X, τ). Hence (g ∘ f) is Δ^*-continuous.

Theorem 4.2.56 Let f : (X, τ) → (Y, σ) be a Δ^*-continuous map. Then for every subset A of (X, τ), f(Δ^*cl(A)) □ cl(f(A)).

Proof : Let f : (X, τ) → (Y, σ) be a Δ^*-continuous function and A be any subset of (X, τ). Then cl(f(A)) is a closed set in (Y, σ). Since f is a Δ^*-continuous function, f^{-1}(cl(f(A)) is Δ^*-closed in (X, τ). Since f(A) □ cl(f(A)), A □ f^{-1}(cl(f(A)). That is f^{-1}(cl(f(A)) is a Δ^*-closed set containing A. By the definition of Δ^*-closure, we have Δ^*cl(A) □ f^{-1}(cl(f(A)) which implies that f(Δ^*cl(A)) □ cl(f(A).

Remark 4.2.56 (a) The converse of the above theorem is not true.

In counter example 4.2.8, f satisfies the condition in the theorem as Δ^*-closed sets coincide with the closed sets of (X, τ) but f is not Δ^*-continuous.

Remark 4.2.56 (b) If the condition in the above theorem holds then Δ^*cl(f^{-1}(A)) = f^{-1}(A) for every closed set A in (Y, σ).

Proof : The proof follows by the definition of Δ^*-continuous map and by the Proposition 2.4.2.

Corollary 4.2.57 Let f : (X, τ) → (Y, σ) be a δ-continuous (or Δ^*-continuous) function. Then for every subset A of (X, τ), f(Δ^*cl(A)) □ cl(f(A)).

Proof: Follows by the Proposition 4.2.3 and Theorem 4.2.56

Proposition 4.2.58 Let f : (X, τ) → (Y, σ) be a map. If for each point x □ X and each open set V in (Y, σ) containing f(x), there exists a Δ^*-open set U in (X, τ) containing x such that f(U) □ V then for every subset A of (X, τ), f(Δ^*cl(A)) □ cl(f(A)).

Proof : Let $y \in f(\Delta^* cl(A))$. Therefore $y = f(x)$ for some $x \in \Delta^* cl(A) \subset X$. Let V be any open set in (Y, σ) containing f(x). Then by hypothesis there exist a Δ^*-open set U in (X, τ) containing x such that $f(U) \subset V$. By proposition 2.5.18, $U \cap A \neq \phi$. Then $f(U \cap A) \neq \phi$ which implies that $V \cap f(A) \neq \phi$. Hence $y \in cl(f(A))$.

4.3 Quasi Δ^*-Continuous Functions and Perfectly Δ^*-Continuous Functions

Definition 4.3.1 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is called **quasi Δ^*-continuous** if the inverse image of every Δ^*-open set in (Y, σ) is open in (X, τ).

Theorem 4.3.2 A map $f : (X, \tau) \rightarrow (Y, \sigma)$ is quasi Δ^*-continuous if and only if the inverse image of every Δ^*-closed set in (Y, σ) is closed in (X, τ).

Proof (Necessary) : Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be quasi Δ^*-continuous function. Let V be a Δ^*-closed in (Y, σ) which implies that $(Y-V)$ is Δ^*-open in (Y, σ). Since f is quasi Δ^*-continuous, $f^{-1}(Y - V) = X - f^{-1}(V)$ is open in (X, τ) and hence $f^{-1}(V)$ is closed in (X, τ).

(Sufficiency) : Let U be Δ^*-open set in (Y, σ) which implies $(Y - U)$ is Δ^*-closed set in (Y, σ). By assumption $f^{-1}(Y - U) = X - f^{-1}(U)$ is open in (X, τ). Hence f is quasi Δ^*-continuous.

Proposition 4.3.3 If $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly continuous then it is quasi Δ^*-continuous but not conversely.

Proof : Let U be a Δ^*-open set in (Y, σ). Since $f : (X, \tau) \rightarrow (Y, \sigma)$ is strongly continuous, for any subset U, $f^{-1}(U)$ is clopen set in (X, τ). That is $f^{-1}(U)$ is open. Therefore f is quasi Δ^*-continuous function.

Counter Example 4.3.4 Consider $X = Y = \{a, b, c\}$ with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a map such that $f(a) = a$, $f(b) = a$, $f(c) = c$. Then f is quasi Δ^*-continuous but not strongly continuous since for the subset $\{a\}$ in (Y, σ), $f^{-1}\{a\} = \{a, b\}$ is open but not closed in (X, τ).

Proposition 4.3.5 If f : (X, τ) → (Y, σ) is totally continuous and (Y, σ) is a $\Delta^{*}T_{\delta}$ - space then it is quasi Δ^{*}-continuous but not conversely.

Proof : Let U be any Δ^{*}-open set in (Y, σ). Since (Y, σ) is a $\Delta^{*}T_{\delta}$-space, U is a δ-open set which implies that U is open. Since f is totally continuous, $f^{-1}(U)$ is clopen set in (X, τ). That is $f^{-1}(U)$ is open in (X, τ). Hence f is quasi Δ^{*}-continuous.

Counter Example 4.3.6 Consider X = Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {b}, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = a, f(c) = c. Then f is quasi Δ^{*}-continuous but not totally continuous since for the subset {a} in (Y, σ), f^{-1}{a} = {a, b} is open but not closed in (X, τ).

Remark 4.3.7 The proposition 4.3.5 does not hold good if (Y, σ) is not a $\Delta^{*}T_{\delta}$-space.

Example 4.3.8 Consider X ={a, b, c}, Y={a, b, c, d} with τ={ϕ , X, {a}} and σ={ϕ , Y, {a, b, c}}. Here (Y, σ) is not a $\Delta^{*}T_{\delta}$-space. Let f : (X, τ) → (Y, σ) be the identity map. Then f is totally continuous but not quasi Δ^{*}-continuous as the inverse image of the Δ^{*}-open set {c, d} in (Y, σ) is not a open set in (X, τ).

Definition 4.3.9 A map f : (X, τ) → (Y, σ) is called **perfectly Δ^{*}-continuous** if the inverse image of every Δ^{*}-open set in (Y, σ) is clopen set in (X, τ).

Proposition 4.3.10 A map f : (X, τ) → (Y, σ) is perfectly Δ^{*}-continuous if and only if the inverse image of every Δ^{*}-closed set in (Y, σ) is clopen in (X, τ).

Proof : (Necessary) : Let f : (X, τ) → (Y, σ) be perfectly Δ^{*}-continuous and V be any Δ^{*}-closed set in (Y, σ). Then Y−V is Δ^{*}-open set in (Y, σ). Since f is perfectly Δ^{*}-continuous, $f^{-1}(Y-V) = X - f^{-1}(V)$ is clopen in (X, τ) which implies that $f^{-1}(V)$ is clopen in (X, τ).

(Sufficiency) : Let the inverse image of every Δ^*-closed set in (Y, σ) is clopen in (X, τ). Let U be any Δ^*-open set in (Y, σ). Then (Y − U) is Δ^*-closed set in (Y, σ). By our assumption, $f^{-1}(Y - U) = X - f^{-1}(U)$ is clopen in (X, τ) which implies that $f^{-1}(U)$ is clopen in (X, τ) and hence f is perfectlyΔ^*-continuous.

Proposition 4.3.11 If a map f : (X, τ) → (Y, σ) is perfectly Δ^*-continuous then it is quasi Δ^*-continuous but not conversely.

Proof : Follows from the definition of perfectly Δ^*-continuous and quasi Δ^*-continuous functions.

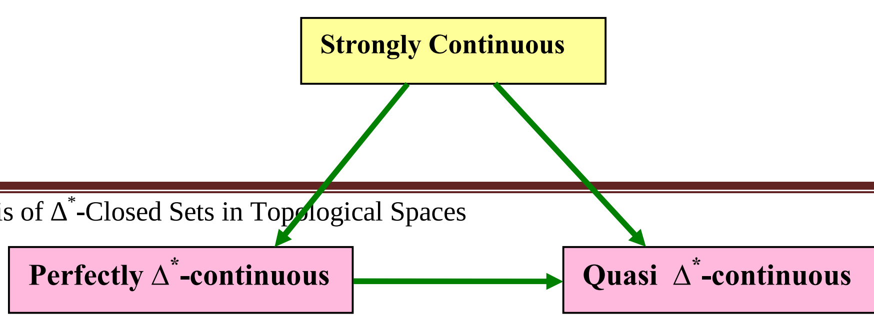
Counter Example 4.3.12 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = c. Then f is quasiΔ^*-continuous but not perfectly Δ^*-continuous since for the Δ^*-open set {a} in (Y, σ), f^{-1}{a}={a}is open but not closed in (X, τ).

Proposition 4.3.13 If a map f : (X, τ) → (Y, σ) is strongly continuous then it is perfectly Δ^*-continuous but not conversely.

Proof : Let V be any Δ^*-open set in (Y, σ). Since f is strongly continuous, for any subset V, $f^{-1}(V)$ is clopen set in (X, τ). Therefore f is perfectly Δ^*-continuous.

Counter Example 4.3.14 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {b, c}} and σ = {ϕ , Y, {a}, {b, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is perfectly Δ^*-continuous but not strongly continuous since for the subset {b} in (Y, σ), $f^{-1}(b) = b$ is not clopen set in (X, τ).

Remark 4.3.15 From the above results we have the following diagram.



Proposition 4.3.16 Let f : (X, τ) → (Y, σ) be a map in which (X, τ) is a discrete topological space and (Y, σ) is any topological space. Then the following are equivalent.

a) f is perfectly Δ^*-continuous.

b) f is quasi Δ^*-continuous.

Proof : (a) ⟹ (b) : Follows from the Proposition 4.3.11

(b) ⟹ (a) : Let U be any Δ^*-open set in (Y, σ). By hypothesis $f^{-1}(U)$ is open in (X, τ). Since (X, τ) is a discrete space, $f^{-1}(U)$ is also closed in (X, τ). That is $f^{-1}(U)$ is clopen in (X, τ). Hence f is perfectly Δ^*-continuous.

Proposition 4.3.17 Let f : (X, τ) → (Y, σ) and g : (Y, σ) ⟶ (Z, η) be any two maps. Then their composition map (g ∘ f) : (X, τ) → (Z, η) is

a) Δ^*-continuous if g is strongly continuous and f is Δ^*-continuous.

b) Perfectly Δ^*-continuous if g is perfectly Δ^*-continuous and f is continuous.

Proof : a) Let g be strongly continuous and f is Δ^*-continuous. Let V be any closed set in (Z, η). Since g is strongly continuous, $g^{-1}(V)$ is a clopen set in (Y, σ). Since f is Δ^*-continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is Δ^*-closed in (Y, σ). Hence (g ∘ f) is Δ^*-continuous.

b) Let f : (X, τ) → (Y, σ) be continuous and g : (Y, σ) → (Z, η) be perfectly Δ^*-continuous. Let V be any Δ^*-closed in (Z, η). Since g is perfectly Δ^*-continuous, $g^{-1}(V)$ is a clopen set in (X,τ). Since f is continuous, $f^{-1}(g^{-1}(V)) = (g \circ f)^{-1}(V)$ is clopen in (X, τ). Hence (g ∘ f) is perfectly Δ^*-continuous.

Remark 4.3.18 Any totally continuous function need not be a perfectly Δ^*-continuous function as seen from the following example.

Counter example 4.3.19 Let X = Y = {a, b, c, d} with τ = {ϕ , X, {a}, {b, c}, {a, d}, {a, b, c}, {b, c, d}} and σ = {ϕ , Y, {a,b,c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = d, f(b) = a, f(c) = b, f(d) = c. Then f is totally continuous but not perfectly Δ^*-continuous since for the Δ^*-open set {a, d} in (Y, σ), f^{-1}(a, d) = {a, b} is not a clopen set in (X, τ).

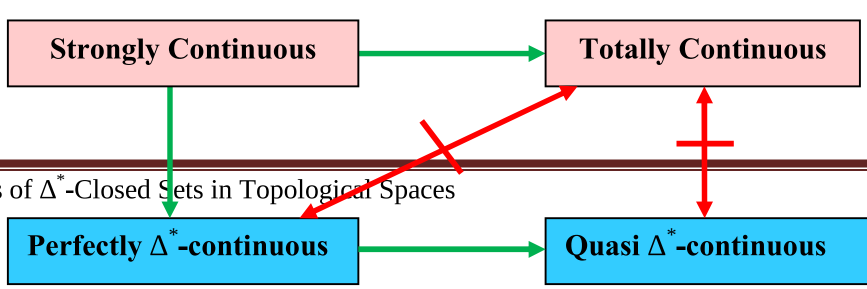
Proposition 4.3.20 If f : (X, τ) → (Y, σ) is totally continuous and (Y, σ) is a $_{\Delta^*}T_\delta$-space then f is perfectly Δ^*-continuous.

Proof : Let V be any Δ^*-open set in (Y, σ). Since (Y, σ) is a $_{\Delta^*}T_\delta$-space, V is δ-open and hence V is open. Since f is totally continuous, f^{-1}(V) is clopen in (X, τ). Hence f is perfectly Δ^*-continuous.

Remark 4.3.21 When (Y, σ) is not a $_{\Delta^*}T_\delta$-space, the above proposition does not hold good.

Counter example 4.3.22 Consider X = {a, b, c}, Y = {a, b, c, d} with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a, b, c}}. Here (Y, σ) is not a $_{\Delta^*}T_\delta$-space. Let f : (X, τ) → (Y, σ) be the identity map. Then f is totally continuous but not perfectly Δ^*-continuous as the inverse image of the Δ^*-open set {c, d} in (Y, σ) is not a open set in (X, τ).

Remark 4.3.23 The above results are depicted by the following diagram.



4.4 Totally Δ^*-Continuous, Strongly Totally Δ^*-continuous and Contra Δ^*-Continuous

Definition 4.4.1 A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be **totally Δ^*-continuous** if for every open set V of (Y, σ), the inverse image $f^{-1}(V)$ is both Δ^*-open set and Δ^*-closed set in (X, τ). i.e., $f^{-1}(V)$ is Δ^*-clopen set in (X, τ).

Proposition 4.4.2 Every totally Δ^*-continuous function is Δ^*-continuous but not conversely.

Proof : Follows from the definitions of totally Δ^*-continuous and Δ^*-continuous functions.

Counter Example 4.4.3 Let X = Y = {a, b, c} with $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$ and $\sigma = \{\phi, Y, \{a\}, \{a, b\}, \{a, c\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is Δ^*-continuous but not totally Δ^*-continuous since for the open subset {a} in (Y, σ), $f^{-1}(\{a\}) = \{a\}$ is Δ^*-open set but not Δ^*-closed set in (X, τ).

Proposition 4.4.4 Every totally Δ^*-continuous function need not be strongly continuous and totally continuous.

Proof : Follows from the definitions of totally Δ^*-continuous, strongly continuous and totally continuous functions.

Counter Example 4.4.5 Let X = Y = {a, b, c, d} with $\tau = \{\phi, X, \{a, b, c\}\}$ and $\sigma = \{\phi, Y, \{a, b, c\}\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is totally Δ^*-continuous but not strongly continuous and not totally continuous since for the open subset {a, b, c} in (Y, σ), f^{-1} ({a, b, c}) = {a, b, c} is not clopen in (X, τ).

Definition 4.4.6 A map f : (X, τ) → (Y, σ) is said to be **strongly totally Δ^*-continuous** if the inverse image of every subset of (Y, σ) is Δ^*-clopen set in (X, τ).

Proposition 4.4.7 Every strongly totally Δ^*-continuous function is totally Δ^*-continuous function but not conversely.

Proof : Follows from the definitions of strongly totally Δ^*-continuous and totally Δ^*-continuous function.

Counter Example 4.4.8 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {b, c}} and σ={ϕ , Y, {a}, {b, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is totally Δ^*-continuous but not strongly totally Δ^*-continuous since for the subset {b} in (Y, σ), f^{-1}({b}) = b is neither Δ^*-open nor Δ^*-closed in (X, τ).

Proposition 4.4.9 Every strongly totally Δ^*-continuous function is Δ^*-continuous but not conversely.

Proof : The proof follows from Proposition 4.4.7 and Proposition 4.4.2.

Counter example 4.4.10 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {a, b}, {a, c}} and σ = {ϕ , Y, {a}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be the identity map. Then f is Δ^*-continuous but not strongly totally Δ^*-continuous since for the subset {a} in (Y, σ), f^{-1} ({a}) = {a} is Δ^*-open set but not Δ^*-closed set in (X, τ).

Definition 4.4.11 A map f : (X, τ) → (Y, σ) is said to be **contra Δ^*-continuous** if the inverse image of every closed set of (Y, σ) is Δ^*-open in (X, τ).

Proposition 4.4.12 Every totally Δ^*-continuous map is contra Δ^*-continuous. But the converse is not true.

Proof : Let f : (X, τ) → (Y, σ) be a totally Δ^*-continuous map. Let V be any closed subset of (Y, σ). Then $f^{-1}(V)$ is Δ^*-clopen set in (X, τ). Hence f is contra Δ^*-continuous.

Counter Example 4.4.13 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = b, f(c) = a. Then the mapping f : (X, τ) → (Y, σ) is contra Δ^*-continuous but not totally Δ^*-continuous since for the closed set {c} in (Y, σ), $f^{-1}(\{c\})=\{a\}$ is Δ^*-open but not Δ^*-closed in (X, τ).

Proposition 4.4.14 A map f : (X, τ) → (Y, σ) is contra Δ^*-continuous if and only if $f^{-1}(V)$ is Δ^*-closed in (X, τ) for every open set V in (Y, σ).

Proof : (Necessary) : Let f : (X, τ) → (Y, σ) be contra Δ^*-continuous function and V be any open set in (Y, σ). Then (Y−V) is closed in (Y, σ). Since f is contra Δ^*-continuous, $f^{-1}(Y-V) = (X - f^{-1}(V))$ is Δ^*-open in (X, τ) which implies that $f^{-1}(V)$ is Δ^*-closed in (X, τ).

(Sufficiency) : Assume that $f^{-1}(V)$ is Δ^*-closed in (X, τ) for every open set V in (Y, σ). Let U be any closed set in (Y, σ) which implies that (Y − U) is open set in (Y, σ). By assumption, $f^{-1}(Y - U) = (X - f^{-1}(U))$ is Δ^*-closed in (X, τ). Therefore $f^{-1}(U)$ is Δ^*-closed in (X, τ). Hence f is contra Δ^*-continuous.

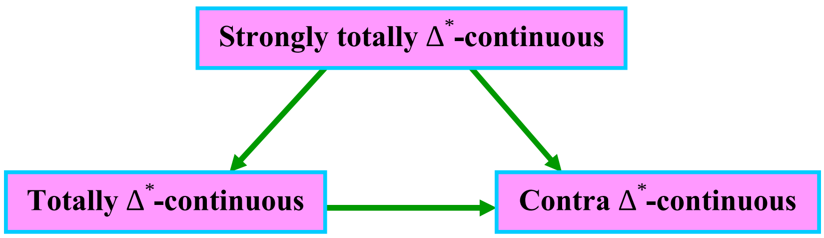
Proposition 4.4.15 Every strongly totally Δ^*-continuous function is contra Δ^*-continuous function but not conversely.

Proof : Follows from the definitions of strongly totally Δ^*-continuous and contra Δ^*-continuous function.

Counter Example 4.4.16 Let X = {a, b, c} = Y with τ = {ϕ , X, {a}, {a, b}} and σ = {ϕ , Y, {a, b}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c)=b.

Then f is contra Δ^*-continuous but not strongly totally Δ^*-continuous since for the closed set {c} in (Y, σ), $f^{-1}(\{c\}) = \{b\}$ is Δ^*-open but not Δ^*-closed in (X, τ).

Remark 4.4.17 From the above observations we have the following diagram.



Remark 4.4.18 The composition of two contra Δ^*-continuous functions need not be a contra Δ^*-continuous function as seen from the following example.

Counter Example 4.4.19 Let X = Y = Z = {a, b, c} with τ = {ϕ , X, {a}, {a, b}}, σ = {ϕ , Y, {a, b}} and η = {ϕ , Z, {a}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = a, f(b) = c, f(c) = b. Let g : (Y, σ) → (Z, η) be a map such that g(a) = b, g(b) = c, g(c) = a. Here both f and g are contra Δ^*-continuous but the composition mapping $(g \circ f) : (X, \tau) \to (Z, \eta)$ defined by $(g \circ f)(a) = b$, $(g \circ f)(b) = a$, $(g \circ f)(c) = c$ is not contra Δ^*-continuous function since for the closed set {b, c} in (Z, η), $f^{-1}(\{b, c\}) = \{a, c\}$ is not a Δ^*- open set in (X, τ).

Remark 4.4.20 The following counter examples show that quasi Δ^*-continuous and contra Δ^*-continuous are independent.

Counter example 4.4.21 Let X = Y = {a, b, c} with τ={ϕ , X, {a}} and σ ={ϕ , Y, {a }} Let f : (X, τ) → (Y, σ) be the identity map. Then f is quasi Δ^*-continuous but not contra

Δ^*-continuous since for the closed set {b, c}, $f^{-1}(\{b, c\}) = \{b, c\}$ is not a Δ^*-open set in (X, τ).

Counter example 4.4.22 Let X = Y = {a, b, c} with τ = {ϕ , X, {a}, {b}, {a, b}, {a, c}} and σ = {ϕ , X, {a}, {b}, {a, b}, {a, c}}. Let f : (X, τ) → (Y, σ) be a map such that f(a) = c, f(b) = b, f(c) = a. Then f is contra Δ^*-continuous but not quasi Δ^*-continuous since for the Δ^*-open set {a}, $f^{-1}(\{a\}) = \{c\}$ is not open in (X, τ).

Remark 4.4.23 The above observations are depicted by the following diagram.