CHAPTER 1

PRELIMINARY DEFINITIONS

This chapter consists of basic definitions related to first order fuzzy topological spaces, first order bipolar fuzzy topological spaces and second order fuzzy topological spaces for the development of this thesis.

In this thesis the terms 'fuzzy set' and 'first order fuzzy set' are used synonymously. Similarly, 'bipolar fuzzy set' and 'first order bipolar fuzzy set' are used synonymously.

Whenever a fuzzy set (bipolar fuzzy set) is considered without mentioning the order, it always refers to a first order fuzzy set (first order bipolar fuzzy set)

Similar terminology applies to all concepts related to first order fuzzy sets and first order bipolar fuzzy sets.

Fuzzy Topological Spaces:

Definition (Zadeh,1965):1.1

Let X be an arbitrary non-empty set. Let $I = [0,1]$. A **fuzzy set** in X is a mapping from X into I. For any two fuzzy sets f, g in I^X, we have

1. $f = g \Leftrightarrow f(x) = g(x)$, for every $x \in X$.
2. $f \leq g \Leftrightarrow f(x) \leq g(x)$, for every $x \in X$.

When $f \leq g$, the fuzzy set g is said to **contain** f.

The **union** $f \vee g$ and the **intersection** $f \wedge g$ are defined, respectively, by

$(f \vee g)(x) = \max\{f(x), g(x)\}$ and $(f \wedge g)(x) = \min\{f(x), g(x)\}$.

The **complement** f^c of a fuzzy set f is defined by $f^c(x) = 1 - f(x)$, for every $x \in X$.

For a family $\{f_\lambda / \lambda \in \Lambda\}$ of fuzzy sets defined on a set X, the **union** $\bigvee_{\lambda\in\Lambda} f_\lambda$ and the **intersection** $\bigwedge_{\lambda\in\Lambda} f_\lambda$ are defined, respectively, by

$(\bigvee_{\lambda\in\Lambda} f_\lambda)(x) = \bigvee_{\lambda\in\Lambda}(f_\lambda(x))$ and $(\bigwedge_{\lambda\in\Lambda} f_\lambda)(x) = \bigwedge_{\lambda\in\Lambda}(f_\lambda(x))$

For any $\alpha \in I$, **the constant fuzzy set $\boldsymbol{\alpha}$** in X is a fuzzy set in X defined by $\boldsymbol{\alpha}(x) = \alpha$, for every $x \in X$.

Support of any fuzzy set f, denoted by supp f, is defined as

$$\text{supp } f = \{x \in X \,/\, f(x) > 0\}.$$

Definition (Wong, 1974b):1.2

A **fuzzy point** x_t in a set X, with x in X and t in (0,1] is a fuzzy set in X defined by $x_t(y) = t$ for $y = x$ and $x_t(y) = 0$ for $y \neq x$. A fuzzy point x_t in X said to **belong** to a fuzzy set f, written as $x_t \in f$ iff $x_t(x) \leq f(x)$ that is $t \leq f(x)$. Hence $x_t \notin f \Leftrightarrow t > f(x)$. Two fuzzy points are said to be **distinct** if and only if their supports are distinct.

Definition (Chang, 1968):1.3

Let X be a non-empty set. A subset $\tau \subset I^X$ is called a **fuzzy topology** on X iff τ satisfies the following requirements:

(i) The constant fuzzy sets **0** and **1** belong to τ.

(ii) $f_\lambda \in \tau$ for each $\lambda \in \Lambda$ implies $\vee_{\lambda \in \Lambda} f_\lambda \in \tau$.

(iii) $f, g \in \tau$ implies $f \wedge g \in \tau$.

The pair (X, τ) is called a **fuzzy topological space**.

Definition (Lowen, 1976):1.4

Let X be a non-empty set. A subset $\tau \subset I^X$ is called a **fuzzy topology** on X iff the following conditions are satisfied:

(i) All constant fuzzy sets belong to τ.

(ii) $f_\lambda \in \tau$ for each $\lambda \in \Lambda$ implies $\vee_{\lambda \in \Lambda} f_\lambda \in \tau$.

(iii) $f, g \in \tau$ implies $f \wedge g \in \tau$.

The pair (X, τ) is called a **fuzzy topological space**. The elements in τ are called **open fuzzy sets** of the fuzzy topological space (X, τ). Complements of open sets are called **closed fuzzy sets** of (X, τ).

Definition (Gantner et al., 1978):1.5

A fuzzy topological space (X, τ) is said to be **fuzzy W-Hausdorff** if for every $x, y \in X$, $x \neq y$, there exist $f, g \in \tau$ such that $f(x) = 1, g(y) = 1$ and $f \wedge g = \mathbf{0}$.

Definition (Srivatsava et al., 1981):1.6

A fuzzy topological space (X, τ) is said to be **fuzzy S-Hausdorff** if for any pair of distinct fuzzy points x_t ,y_s in X, there exist $f, g \in \tau$ such that $x_t \in f$, $y_s \in g$ and $f \wedge g = \mathbf{0}$.

Definition (Katsaras, 1981):1.7

A fuzzy topological space (X, τ) is said to be **fuzzy K-Hausdorff** if for every $x, y \in X$, $x \neq y$, there exists $f, g \in \tau$ such that $f(x) > 0$, $g(y) > 0$ and $f \wedge g = \mathbf{0}$.

Definition (Hazra et al., 1992):1.8

Let X be a non-empty set. A mapping $\mathcal{G}: I^X \to I$ is said to be a **gradation of openness** on X iff the following conditions are satisfied:

(G1) $\mathcal{G}(\mathbf{0}) = \mathcal{G}(\mathbf{1}) = 1$.

(G2) $\mathcal{G}(f_i) > 0$ for i = 1 to m $\Rightarrow \mathcal{G}(\wedge_{i=1}^{m} f_i) > 0$.

(G3) $\mathcal{G}(f_\lambda) > 0$ for $\lambda \in \Lambda \Rightarrow \mathcal{G}(\vee_{\lambda\in\Lambda} f_\lambda) > 0$.

The pair $(X, \mathcal{G})$is called a **gradation space.**

Definition (Hazra et al., 1992):1.9

Let $(X, \mathcal{G})$ be a gradation space. Then the fuzzy topology on X induced by $\mathcal{G}$ is given by $\tau(\mathcal{G}) = \{f \in I^X \,/\, \mathcal{G}(f) > 0\}$.

Definition (Hazra et al., 1992):1.10

Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two gradations of openness on X. Then $\mathcal{G}_1 \geq \mathcal{G}_2$ if $\mathcal{G}_1(f) \geq \mathcal{G}_2(f)$ for every $f \in I^X$.

Definition (Hazra et al., 1992):1.11

Let $(X, \mathcal{G}_1)$,$(X, \mathcal{G}_2)$ be two gradation spaces. Then a map $\theta: X \to Y$ is called

(i) a **gradation preserving map**, if $\mathcal{G}_2(f) \leq \mathcal{G}_1(\theta^{-1}(f))$ for each $f \in I^Y$.

(ii) a **strongly gradation preserving map**, if $\mathcal{G}_2(\mathrm{f}) = \mathcal{G}_1(\theta^{-1}(\mathrm{f}))$ for each $\mathrm{f} \in \mathrm{I}^{\mathrm{Y}}$.

(iii) a **weakly gradation preserving map**, if $\mathcal{G}_2(\mathrm{f}) > 0 \Rightarrow \mathcal{G}_1(\theta^{-1}(\mathrm{f})) > 0$ for each $\mathrm{f} \in \mathrm{I}^{\mathrm{Y}}$.

Remark (Hazra et al., 1992):1.12

Let $\theta: \mathrm{X} \to \mathrm{Y}$ be a map and let $\mathcal{G}_1$ be a gradation of openness on X. Then the largest gradation of openness $\mathcal{G}_2$ on Y which makes $\theta: (\mathrm{X}, \mathcal{G}_1) \to (\mathrm{X}, \mathcal{G}_2)$ a gradation preserving map is given by $\mathcal{G}_2(\mathrm{f}) = \mathcal{G}_1(\theta^{-1}(\mathrm{f}))$ for each $\mathrm{f} \in \mathrm{I}^{\mathrm{Y}}$.

Bipolar fuzzy topological space:

Definition (Kim et al., 2004):1.13

Let X be a non-empty set. Then a pair $\mathrm{A_{bp}} = (\mathrm{A_{bp}}^{+}, \mathrm{A_{bp}}^{-})$ is called a **bipolar-valued fuzzy set or bipolar fuzzy set** in X, where $\mathrm{A_{bp}}^{+}: \mathrm{X} \to [0,1]$ and $\mathrm{A_{bp}}^{-}: \mathrm{X} \to [-1,0]$. The set of all bipolar fuzzy set in X is denoted as BPF(X).

Definition (Kim et al., 2004):1.14

(i) The **bipolar fuzzy null set** denoted by $0_{\mathrm{bp}} = (0_{\mathrm{bp}}^{+}, 0_{\mathrm{bp}}^{-})$ is a bipolar fuzzy set in X defined as $0_{\mathrm{bp}}^{+}(\mathrm{x}) = 0$ and $0_{\mathrm{bp}}^{-}(\mathrm{x}) = 0$, for each $\mathrm{x} \in \mathrm{X}$.

(ii) The **bipolar fuzzy whole set** denoted by $1_{\mathrm{bp}} = (1_{\mathrm{bp}}^{+}, 1_{\mathrm{bp}}^{-})$ is a bipolar fuzzy set in X defined as $1_{\mathrm{bp}}^{+}(\mathrm{x}) = 1$ and $1_{\mathrm{bp}}^{-}(\mathrm{x}) = -1$, for each $\mathrm{x} \in \mathrm{X}$.

(iii) Let $\mathrm{A_{bp}}, \mathrm{B_{bp}} \in \mathrm{BPF(X)}$. $\mathrm{A_{bp}}$ is a **subset** of $\mathrm{B_{bp}}$, denoted by $\mathrm{A_{bp}} \subset \mathrm{B_{bp}}$ is defined as $\mathrm{A_{bp}}^{+}(\mathrm{x}) \le \mathrm{B_{bp}}^{+}(\mathrm{x})$ and $\mathrm{A_{bp}}^{-}(\mathrm{x}) \ge \mathrm{B_{bp}}^{-}(\mathrm{x})$, for each $\mathrm{x} \in \mathrm{X}$.

(iv) Let $\mathrm{A_{bp}} \in \mathrm{BPF(X)}$. The **complement** of $\mathrm{A_{bp}}$ is denoted by $\mathrm{A_{bp}}^{c} = ((\mathrm{A_{bp}}^{+})^{c}, (\mathrm{A_{bp}}^{-})^{c})$ is a bipolar fuzzy set in X defined as $(\mathrm{A_{bp}}^{+})^{c}(\mathrm{x}) = 1 - \mathrm{A_{bp}}^{+}(\mathrm{x})$ and $(\mathrm{A_{bp}}^{-})^{c}(\mathrm{x}) = -1 - \mathrm{A_{bp}}^{-}(\mathrm{x})$, for each $\mathrm{x} \in \mathrm{X}$.

(v) Let $\mathrm{A_{bp}}, \mathrm{B_{bp}} \in \mathrm{BPF(X)}$. The **intersection** of $\mathrm{A_{bp}}$ and $\mathrm{B_{bp}}$, denoted by $\mathrm{A_{bp}} \cap \mathrm{B_{bp}}$ is a bipolar fuzzy set in X defined as
$(\mathrm{A_{bp}} \cap \mathrm{B_{bp}})(\mathrm{x}) = (\mathrm{A_{bp}}^{+}(\mathrm{x}) \wedge \mathrm{B_{bp}}^{+}(\mathrm{x}),\ \mathrm{A_{bp}}^{-}(\mathrm{x}) \vee \mathrm{B_{bp}}^{-}(\mathrm{x}))$, for each $\mathrm{x} \in \mathrm{X}$.

(vi) Let $A_{bp}, B_{bp} \in BPF(X)$. The **union** of A_{bp} and B_{bp}, denoted by $A_{bp} \cup B_{bp}$, is a bipolar fuzzy set in X defined as

$$(A_{bp} \cup B_{bp})(x) = \left(A_{bp}{}^{+}(x) \vee B_{bp}{}^{+}(x)\ ,\ A_{bp}{}^{-}(x) \wedge B_{bp}{}^{-}(x)\right), \text{ for each } x \in X.$$

(vii) The **intersection** of $\left((A_{bp})_{\lambda}\right)_{\lambda\epsilon\Lambda}$ is a bipolar fuzzy set in X denoted by $\cap_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda}$ and is defined as

$$\left(\cap_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda}\right)(x) = \left(\wedge_{\lambda\epsilon\Lambda}\left((A_{bp}{}^{+})_{\lambda}\right)(x), \vee_{\lambda\epsilon\Lambda}\left((A_{bp}{}^{-})_{\lambda}\right)(x)\right), \text{ for each } x \in X.$$

(viii) The **union** of $\left((A_{bp})_{\lambda}\right)_{\lambda\epsilon\Lambda}$ is a bipolar fuzzy set in X denoted by $\cup_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda}$ and is defined as

$$\left(\cup_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda}\right)(x) = \left(\vee_{\lambda\epsilon\Lambda}\left((A_{bp}{}^{+})_{\lambda}\right)(x), \wedge_{\lambda\epsilon\Lambda}\left((A_{bp}{}^{-})_{\lambda}\right)(x)\right), \text{ for each } x \in X.$$

Definition (Kim et al., 2019):1.15

Let X be a non-empty set. A subset $\tau_{\mathfrak{B}} \subset BPF(X)$ is called a **bipolar fuzzy topology on X,** iff $\tau_{\mathfrak{B}}$ satisfies the following axioms:

(i) $0_{bp}\ ,\ 1_{bp} \in \tau_{\mathfrak{B}}$.

(ii) $A_{bp} \cap B_{bp} \in \tau_{\mathfrak{B}}$, for any $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$.

(iii) $\cup_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda} \in \tau_{\mathfrak{B}}$, for any $\left((A_{bp})_{\lambda}\right)_{\lambda\epsilon\Lambda} \in \tau_{\mathfrak{B}}$.

The pair $(X, \tau_{\mathfrak{B}})$ is called a bipolar fuzzy topological space.

Definition (Kim et al., 2019):1.16

Let X be a non-empty set. A subset $\tau_{\mathfrak{B}} \subset BPF(X)$ is called a **bipolar fuzzy topology on X,** if it satisfies the following axioms:

(i) All constant bipolar fuzzy sets belong to $\tau_{\mathfrak{B}}$.

(ii) $A_{bp}, B_{bp} \in \tau_{\mathfrak{B}}$ implies $A_{bp} \cap B_{bp} \in \tau_{\mathfrak{B}}$.

(iii) $\left((A_{bp})_{\lambda}\right)_{\lambda\in\Lambda} \in \tau_{\mathfrak{B}}$ for each $\lambda \in \Lambda$ implies $\cup_{\lambda\epsilon\Lambda}(A_{bp})_{\lambda} \in \tau_{\mathfrak{B}}$.

The pair $(X, \tau_{\mathfrak{B}})$ is called a bipolar fuzzy topological space and each member of $\tau_{\mathfrak{B}}$ is called a bipolar fuzzy open set (BPFOS) in X. Complements of bipolar fuzzy open sets are called bipolar fuzzy closed sets of $(X, \tau_{\mathfrak{B}})$.

Definition (Kim et al., 2019):1.17

Let X and Y be two non-empty sets, let $A_{bp} \in BPF(X)$ and $B_{bp} \in BPF(Y)$ and let $\theta: X \to Y$ be a mapping. Then

(i) The **image** of A_{bp} under θ, denoted by $\theta(A_{bp}) = \left(\theta(A_{bp}^{+}), \theta(A_{bp}^{-})\right)$, is a bipolar fuzzy set in Y defined as follows: for each $y \in Y$,

$$\left(\theta(A_{bp}^{+})\right)(y) = \begin{cases} \bigvee_{x\in\theta^{-1}(y)} A_{bp}^{+}(x), & \text{if } \theta^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$\left(\theta(A_{bp}^{-})\right)(y) = \begin{cases} \bigwedge_{x\in\theta^{-1}(y)} A_{bp}^{-}(x), & \text{if } \theta^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

(ii) The **pre-image** of B_{bp} under θ, denoted by $\theta^{-1}(B_{bp}) = \left(\theta^{-1}(B_{bp}^{+}), \theta^{-1}(B_{bp}^{-})\right)$, is a bipolar fuzzy set in Y defined as follows: for each $x \in X$,

$$\left(\theta^{-1}(B_{bp}^{+})\right)(x) = B_{bp}^{+}(\theta(x)) \text{ and } \left(\theta^{-1}(B_{bp}^{-})\right)(x) = B_{bp}^{-}(\theta(x)).$$

Definition (Kim et al., 2019):1.18

Let $(X, \tau_{\mathfrak{B}_1})$ and $(Y, \tau_{\mathfrak{B}_2})$ be two bipolar fuzzy topological spaces. A mapping $\theta: (X, \tau_{\mathfrak{B}_1}) \to (Y, \tau_{\mathfrak{B}_2})$ is said to be **bipolar fuzzy continuous** if for all $A_{bp} \in \tau_{\mathfrak{B}_2}$, $\theta^{-1}(A_{bp}) \in \tau_{\mathfrak{B}_1}$.

Definition (Kim et al., 2019):1.19

Let $x \in X$, $(\alpha, \beta) \in (0,1] \times [-1,0)$ and let $A_{bp} \in BPF(X)$. Then

(i) $x_{(\alpha,\beta)}$ is called a **bipolar fuzzy point** in X with the value $(\alpha, \beta) \in (0,1] \times [-1,0)$ and the support $x \in X$, if for each $y \in X$,

$$[x_{(\alpha,\beta)}](y) = \begin{cases} (\alpha, \beta), & \text{if } y = x \\ (0,0), & \text{otherwise} \end{cases}$$

(ii) $x_{(\alpha,\beta)}$ is said to belong to A_{bp}, denoted by $x_{(\alpha,\beta)} \in A_{bp}$, if $A_{bp}^{+}(x) \geq \alpha$ and $A_{bp}^{-}(x) \leq \beta$.

The set of all bipolar fuzzy points in X is denoted as $BPF_P(X)$. It is clear that $A_{bp} = \bigcup\{x_{(\alpha,\beta)} \in BPF_P(X): x_{(\alpha,\beta)} \in A_{bp}\}$, for each $A_{bp} \in BPF(X)$.

Definition (Kim et al., 2019):1.20

Let X be a nonempty set. Then for any $A_{bp} \in BPF(X)$, the set $\mathbb{S}(A_{bp}) = \{x \in X : {A_{bp}}^{+}(x) > 0, {A_{bp}}^{-}(x) < 0\}$ is called the **support** of A_{bp}. If $\mathbb{S}(A_{bp})$ is finite, then A_{bp} is said to be finite.

Definition (Kim et al., 2019):1.21

A bipolar fuzzy topological space $(X, \tau_{\mathfrak{B}})$ is said to be **bipolar fuzzy compact** if the following condition is satisfied:

Given a family of bipolar fuzzy open sets $\{(A_{bp})_{\lambda} / \lambda \in \Lambda\}$ with $\vee\{({A_{bp}}^{+})_{\lambda}(x)/\lambda \in \Lambda, A_{bp} \in \tau_{\mathfrak{B}}\} = 1$, $\wedge\{({A_{bp}}^{-})_{\lambda}(x)/\lambda \in \Lambda, A_{bp} \in \tau_{\mathfrak{B}}\} = -1$, for every $x \in X$ then there exists a finite subfamily $\Lambda_0(x) \subseteq \Lambda$ such that

$$\vee\left\{\left(({A_{bp}}^{+})_{\lambda}\right)(x) / \lambda \in \Lambda_0(x)\right\} = 1, \wedge\left\{\left(({A_{bp}}^{-})_{\lambda}\right)(x) / \lambda \in \Lambda_0(x)\right\} = -1, \text{ for every } x \in X$$

Then, $(X, \tau_{\mathfrak{B}})$ is said to be bipolar fuzzy compact space.

Definition (Kim et al., 2019):1.22

Let $(X, \tau_{\mathfrak{B}})$ be a bipolar fuzzy topological space. Then,

(i) $\mathfrak{B} \subset \tau_{\mathfrak{B}}$ is called a **base** for $\tau_{\mathfrak{B}}$, if for each $A_{bp} \in \tau_{\mathfrak{B}}$, $A_{bp} = 0_{bp}$ or there is $\mathfrak{B}' \subset \mathfrak{B}$ such that $A_{bp} = \bigcup \mathfrak{B}'$.

(ii) $\mathbb{S} \subset \tau_{\mathfrak{B}}$ is called a **subbase** for $\tau_{\mathfrak{B}}$, if the family of all finite intersections of members of $\mathbb{S}$ forms a base for $\tau_{\mathfrak{B}}$.

Second order fuzzy topological spaces

Definition (Kalaichelvi, 2007):1.23

Let X be an arbitrary non-empty set. A **second order fuzzy set** on X is a map $\hat{f} : X \rightarrow I^{I}$ where I is the closed unit interval [0,1]. The family of second order fuzzy sets is denoted by $(I^{I})^{X}$.

For any two second order fuzzy sets $\hat{f}, \hat{g}$ in $(I^{I})^{X}$,

(i) $\hat{f} \leq \hat{g}$ iff $\hat{f}(x) \leq \hat{g}(x)$, for every $x \in X$ that is iff $\hat{f}(x)(\alpha) \leq \hat{g}(x)(\alpha)$ for every $x \in X$ and for every $\alpha \in I$.

(ii) The **union** $\hat{f} \vee \hat{g}$ and the **intersection** $\hat{f} \wedge \hat{g}$ are defined, respectively, by

$(\hat{f} \vee \hat{g})(x) = \hat{f}(x) \vee \hat{g}(x)$ and

$(\hat{f} \wedge \hat{g})(x) = \hat{f}(x) \wedge \hat{g}(x)$.

(iii) For every $\alpha \in I$, **the constant second order fuzzy set** $\hat{\alpha}: X \to I^I$ is defined as follows:

For every $x \in X$, $\hat{\alpha}(x) =$ the constant fuzzy set $\boldsymbol{\alpha}$ on I.

In particular $\hat{0}(x) =$ the constant fuzzy set **0** on I and

$\hat{1}(x) =$ the constant fuzzy set **1** on I.

(iv) For a collection $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$ of second order fuzzy sets

$(\vee_{\lambda \in \Lambda} \hat{f}_\lambda)(x) = \vee_{\lambda \in \Lambda}(\hat{f}_\lambda(x))$ for every $x \in X$ and

$(\wedge_{\lambda \in \Lambda} \hat{f}_\lambda)(x) = \wedge_{\lambda \in \Lambda}(\hat{f}_\lambda(x))$ for every $x \in X$.

(v) A **second order fuzzy point** $\hat{x}_r$ for $x \in X$ and $r \in (0,1]$ is a second order fuzzy set in X defined as follows:

$\hat{x}_r(x) =$ the constant fuzzy set **r**, and

$\hat{x}_r(y) =$ the constant fuzzy set **0**, for $y \neq x$.

A second order fuzzy point $\hat{x}_r$ is said to **belong** to a second order fuzzy set $\hat{f}$ iff $\hat{f}(x)(\alpha) \geq r$, for every $\alpha \in I$. Two second order fuzzy points $\hat{x}_r$, $\hat{y}_s$ are said to be distinct iff $x \neq y$ or $r \neq s$.

(vi) For a second order fuzzy set $\hat{f}$ on X, the **complement** of $\hat{f}$ is defined in two different ways as follows:

a) $(\hat{f})_c(x)(\alpha) = \hat{f}(x)(1-\alpha)$, for every $x \in X$ and for every $\alpha \in I$.

b) $(\hat{f})^c(x)(\alpha) = 1 - \hat{f}(x)(\alpha)$, for every $x \in X$ and for every $\alpha \in I$.

(vii) For a second order fuzzy set $\hat{f}$ on X, the support of $\hat{f}$ is defined in two different ways as follows:

$$\mathbb{S}_1(\hat{f}) = \{x \in X \,/\, \hat{f}(x)(\alpha) > 0 \text{ for some } \alpha \in I\} \text{ and}$$

$$\mathbb{S}_2(\hat{f}) = \{x \in X \,/\, \hat{f}(x)(\alpha) > 0 \text{ for every } \alpha \in I\}.$$

Definition (Kalaichelvi, 2007):1.24

For any two second order fuzzy sets $\hat{f}$, $\hat{g}$ on a set X,

(i) $\hat{f} \wedge_1 \hat{g}$ means given $x \in X$, either $\hat{f}(x) = \mathbf{0}$ or $\hat{g}(x) = \mathbf{0}$.

(ii) $\hat{f} \wedge_2 \hat{g}$ means given $x \in X$ and either $\alpha \in I$, either $\hat{f}(x)(\alpha) = 0$ or $\hat{g}(x)(\alpha) = 0$.

Definition (Kalaichelvi, 2007):1.25

Let X be a nonempty set. A collection $\hat{\tau}$ of second order fuzzy sets on X defines a **second order fuzzy topology** on X if the following conditions are satisfied:

(i) $\hat{0}, \hat{1} \in \hat{\tau}$.

(ii) $\hat{f}_\lambda \in \hat{\tau}$ for each $\lambda \in \Lambda$ implies $\left(\vee_{\lambda\in\Lambda} \hat{f}_\lambda\right) \in \hat{\tau}$.

(iii) $\hat{f}_i \in \hat{\tau}$ for i=1 to n implies $\wedge_{i=1}^{n} \hat{f}_i \in \hat{\tau}$.

The pair $(X, \hat{\tau})$ is called a **second order Chang fuzzy topological space**.

A **Second order Lowen fuzzy topology** $\hat{\tau}$ on X is defined by replacing axiom (i), in the above definition by axiom $(i)'$.

$(i)'$ All constant second order fuzzy sets $\in \hat{\tau}$.

The pair $(X, \hat{\tau})$ is called a **second order Lowen fuzzy topological space**.
The elements of $\hat{\tau}$ are called **second order fuzzy open sets.**

Definition (Kalaichelvi, 2007):1.26

Let $(X, \hat{\tau})$ be a second order fuzzy topological space. A family $\widehat{\mathfrak{B}} \subset \hat{\tau}$ is called a **base** for $\hat{\tau}$ iff for every $\hat{f} \in \hat{\tau}$, there exists a family $\{\hat{f}_j / j \in \mathcal{J}\}$ in $\widehat{\mathfrak{B}}$ such that $\hat{f} = \vee_{j\in\mathcal{J}} \hat{f}_j$.

A family $\hat{\mathcal{S}}$ of second order fuzzy open sets in $(X, \hat{\tau})$ is called a **subbase** for $\hat{\tau}$ iff the family of finite intersections of members of $\hat{\mathcal{S}}$ forms a base for $\hat{\tau}$.

Definition (Kalaichelvi, 2000):1.27

Let X and Y be two nonempty sets. Let θ be a function from X into Y. Let $\hat{f}$ be a second order fuzzy set in X. Then the **image** of $\hat{f}$ under θ, denoted by $\theta(\hat{f})$, is the second order fuzzy set in Y defined as follows:

$$\left(\theta(\hat{f})\right)(y) = \begin{cases} \vee_{\theta(x)=y}\left(\hat{f}(x)\right), & \text{if } \theta^{-1}(y) \text{is not empty} \\ 0, & \text{otherwise} \end{cases}$$

Let X and Y be two nonempty sets. Let θ be a function from X into Y. Let $\hat{g}$ be a second order fuzzy set in Y. Then the **inverse image** of $\hat{g}$ under θ, denoted by $\theta^{-1}(\hat{g})$, is the second order fuzzy set in X defined as follows:

$\left(\theta^{-1}(\hat{g})\right)(y) = \hat{g}\left(\theta(x)\right)$ for every $x \in X$.

Definition (Kalaichelvi, 2011a):1.28

Let $(X, \hat{\tau}_1)$ and $(Y, \hat{\tau}_2)$ be two second order fuzzy topological spaces. Then a function $\theta: X \to Y$ is said to be **2-f continuous** if the following conditions are satisfied:

$$\theta^{-1}(\hat{f}) \in \hat{\tau}_1 \text{ if } \hat{f} \in \hat{\tau}_2.$$

Definition (Kalaichelvi, 2012):1.29

Let $(X, \hat{\tau}_1)$ and $(Y, \hat{\tau}_2)$ be two second order fuzzy topological spaces. If $\hat{f}_1 \in \hat{\tau}_1$ and $\hat{f}_2 \in \hat{\tau}_2$, then the **star product** $\hat{f}_1 \hat{\times} \hat{f}_2$ on $X \times Y$ is defined as follows:

$(\hat{f}_1 \hat{\times} \hat{f}_2)(x, y)(\alpha) = \hat{f}_1(x)(\alpha) \wedge \hat{f}_2(y)(\alpha)$ for every $(x, y) \in X \times Y$ and for every $\alpha \in I$.

The **product topology** $\hat{\tau}_1 \hat{\times} \hat{\tau}_2$ on $X \times Y$ is the second order fuzzy topology having the collection $\{\hat{f}_1 \hat{\times} \hat{f}_2 \,/\, \hat{f}_1 \in \hat{\tau}_1, \hat{f}_2 \in \hat{\tau}_2\}$ as a basis.

Definition (Kalaichelvi, 2011b):1.30

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **second order fuzzy W-Hausdorff of type 1, denoted by $(\mathbf{W-H})_{\mathbf{1}}$** if for every $x, y \in X$, $x \neq y$, there exists $\hat{f}, \hat{g} \in \hat{\tau}$ such that $\hat{f}(x) =$ the constant function $\mathbf{1}$, $\hat{g}(y) =$ the constant function $\mathbf{1}$ and $\hat{f} \wedge_1 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2011b):1.31

A **second order fuzzy W-Hausdorff of type 2, denoted by** $(\mathbf{W-H})_{\mathbf{2}}$, is defined by replacing the condition $\hat{f} \wedge_1 \hat{g} = \hat{0}$ in the above definition by $\hat{f} \wedge_2 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2013):1.32

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **second order fuzzy S-Hausdorff of type 1, denoted by** $(\mathbf{S-H})_{\mathbf{1}}$, iff for any pair of distinct second order fuzzy points $\hat{x}_t, \hat{y}_s$ in X, there exists $\hat{f}, \hat{g} \in \hat{\tau}$ such that $\hat{x}_t \in \hat{f}, \hat{y}_s \in \hat{g}$ and $\hat{f} \wedge_1 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2013):1.33

A **second order fuzzy S-Hausdorff of type 2, denoted by** $(\mathbf{S-H})_{\mathbf{2}}$, is defined by replacing the condition $\hat{f} \wedge_1 \hat{g} = \hat{0}$ in the above definition by $\hat{f} \wedge_2 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2000):1.34

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **second order fuzzy K-Hausdorff of type 1, denoted by** $(\mathbf{K-H})_{\mathbf{1}}$, for every $x, y \in X$, $x \neq y$, there exists $\hat{f}, \hat{g} \in \hat{\tau}$ such that $\hat{f}(x) > \mathbf{0}$, $\hat{g}(y) > \mathbf{0}$ and $\hat{f} \wedge_1 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2000):1.35

A **second order fuzzy K-Hausdorff of type 2, denoted by** $(\mathbf{K-H})_{\mathbf{2}}$, is defined by replacing the condition $\hat{f} \wedge_1 \hat{g} = \hat{0}$ in the above definition by $\hat{f} \wedge_2 \hat{g} = \hat{0}$.

Definition (Kalaichelvi, 2011b):1.36

Let $\{(X_\lambda, \hat{\tau}_\lambda) | \lambda \in \Lambda\}$ be a family of second order fuzzy topological spaces and $X = \prod_{\lambda \in \Lambda} X_\lambda$. The **product topology** on X is the one with basic second order fuzzy open sets of the form $\prod_{\lambda \in \Lambda} \hat{f}_\lambda$ where $\hat{f}_\lambda \in \hat{\tau}_\lambda$ and $\hat{f}_\lambda = \hat{1}$ except for finitely many λ's. Here $\left(\prod_{\lambda \in \Lambda} \hat{f}_\lambda\right)((x_\lambda)_{\lambda \in \Lambda})(\alpha) = \wedge_{\lambda \in \Lambda} \hat{f}_\lambda(x_\lambda)(\alpha)$, for every $(x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_\lambda$ and for every $\alpha \in I$.

Definition (Kalaichelvi, 2000):1.37

Let X be a nonempty set. A mapping $\hat{\mathcal{G}}: (I^I)^X \to I$ is said to be a **second order gradation of openness** on X iff the following conditions are satisfied:

(SG1) $\hat{\mathcal{G}}(\hat{0}) = \hat{\mathcal{G}}(\hat{1}) = 1$.

(SG2) $\hat{\mathcal{G}}(\hat{f}_i) > 0$ for i = 1 to m $\Rightarrow \hat{\mathcal{G}}(\wedge_{i=1}^{m} \hat{f}_i) > 0$.

(SG3) $\hat{\mathcal{G}}(\hat{f}_\lambda) > 0$ for $\lambda \in \Lambda \Rightarrow \hat{\mathcal{G}}(\vee_{\lambda \in \Lambda} \hat{f}_\lambda) > 0$.

The pair $(X, \hat{\mathcal{G}})$ is called a **second order gradation space.**

Definition (Kalaichelvi, 2000):1.38

Let $(X, \hat{\mathcal{G}})$ be a second order gradation space. Then the **second order fuzzy topology induced by** $\hat{\mathcal{G}}$ is given by $\hat{\tau}(\hat{\mathcal{G}}) = \{\hat{f} \in (I^I)^X \,/\, \hat{\mathcal{G}}(\hat{f}) > 0\}$.

Definition (Kalaichelvi, 2000):1.39

Let $(X, \hat{\mathcal{G}}_1)$,$(X, \hat{\mathcal{G}}_2)$ be two second order gradation spaces. Then a map $\theta: X \to Y$ is called

(i) a **second order gradation preserving map**,

if $\hat{\mathcal{G}}_2(\hat{f}) \le \hat{\mathcal{G}}_1\left(\theta^{-1}(\hat{f})\right)$ for each $\hat{f} \in (I^I)^Y$.

(ii) a **second order strongly gradation preserving map**,

if $\hat{\mathcal{G}}_2(\hat{f}) = \hat{\mathcal{G}}_1\left(\theta^{-1}(\hat{f})\right)$ for each $\hat{f} \in (I^I)^Y$.

(iii) a **second order weakly gradation preserving map**,

if $\hat{\mathcal{G}}_2(\hat{f}) > 0 \Rightarrow \hat{\mathcal{G}}_1\left(\theta^{-1}(\hat{f})\right) > 0$ for each $\hat{f} \in (I^I)^Y$.

Definition (Kalaichelvi, 2000):1.40

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **2 fuzzy 1 – compact at x ∈ X** if the following condition is satisfied:

Given a family of second order fuzzy open sets $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$ with $\vee\{\hat{f}_\lambda\,(x) / \lambda \in \Lambda\} = \mathbf{1}$, there exists a finite subfamily $\Lambda_0(x) \subseteq \Lambda$ such that $\vee\{\hat{f}_\lambda(x) \,/ \lambda \in \Lambda_0(x)\} = \mathbf{1}$, for every $x \in X$. Then $(X, \hat{\tau})$ is said to be **2 fuzzy 1- compact space on X**, if it is 2 fuzzy 1-compact at every $x \in X$.

Definition (Kalaichelvi, 2000):1.41

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **2 fuzzy 1*– compact at X** if the following condition is satisfied:

Given a family of second order fuzzy open sets $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that if $\vee\{\hat{f}_\lambda\,(x)/\,\lambda \in \Lambda\} = \mathbf{1}$, for a given $x \in X$ then $\vee\{\hat{f}_\lambda(x)\,/\lambda \in \Lambda_0\} = \mathbf{1}$.

Definition (Kalaichelvi, 2000):1.42

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **2 fuzzy 2-compact at $\boldsymbol{\alpha} \in$ I** if the following condition is satisfied:

Given a family of second order fuzzy open sets $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$ with $\vee\{\hat{f}_\lambda\,(x)(\alpha)/\,\lambda \in \Lambda\} = 1$, for every $x \in X$, there exists a finite subfamily $\Lambda_0(\alpha) \subseteq \Lambda$ such that $\vee\{\hat{f}_\lambda(x)\,(\alpha)/\lambda \in \Lambda_0(\alpha)\} = 1$, for every $x \in X$. Then $(X, \hat{\tau})$ is said to be **2 fuzzy 2- compact space** if it is 2 fuzzy 2- compact at every $\alpha \in I$.

Definition (Kalaichelvi, 2000):1.43

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **2 fuzzy 2*– compact** if the following condition is satisfied:

Given a family of second order fuzzy open sets $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that if $\vee\{\hat{f}_\lambda\,(x)(\alpha)/\,\lambda \in \Lambda\} = 1$, for every $x \in X$ then $\vee\{\hat{f}_\lambda(x)(\alpha)\,/\lambda \in \Lambda_0\} = 1$, for every $x \in X$.

Definition (Kalaichelvi, 2000):1.44

A second order fuzzy topological space $(X, \hat{\tau})$ is said to be **2 fuzzy 3-compact** if the following condition is satisfied:

Given a family of second order fuzzy open sets $\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\}$ with $\vee\{\hat{f}_\lambda \,/\, \lambda \in \Lambda\} = \hat{1}$, there exists a finite subfamily $\Lambda_0 \subseteq \Lambda$ such that $\vee\{\hat{f}_\lambda/\lambda \in \Lambda_0\} = \hat{1}$, for every $x \in X$.

Fuzzy Matrix

Definition (Thomason, 1977):1.45

Let F be a matrix, $F = [F_{ij}]_{u\times v}$, where $[F_{ij}]_{u\times v} \in [0,1]$, where u and v represent the rows and columns of the matrix, $1 \leq i \leq u$ and $1 \leq j \leq v$, then F is called a fuzzy matrix.

First Order Bipolar Fuzzy Matrix

Definition (Thomason, 1977):1.46

Let X be a nonempty set. A **first order bipolar fuzzy matrix** (FOBPFM) $\left((A_{bp})_{ij}\right)_{p\times q}$ is defined as $\left((A_{bp})_{ij}\right)_{p\times q} = \left((A_{bp}{}^{+}, A_{bp}{}^{-})_{ij}\right)_{p\times q}$ where $(A_{bp}{}^{+})_{ij}(x) \in [0,1]$ and $(A_{bp}{}^{-})_{ij}(x) \in [-1,0]$, $\forall i, j$, for every $x \in X$ and p and q represent the rows and columns of the matrix.